A model of second-order arithmetic with the choice scheme in which Π^1_2-dependent choice fails

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Class choice principles for Kelley-Morse set theory

Kelley-Morse set theory: (KM)

- First-order: ZFC
- Second-order:
  - Replacement
  - full second-order comprehension
  - Global Choice

Class choice scheme: (Σ^1_∞-AC) For every second-order formula $\varphi(x, X, A)$,

$$\forall x \exists X\, \varphi(x, X, A) \longrightarrow \exists Y \forall x\, \varphi(x, Y_x, A),$$

where $Y_x = \{y \mid \langle x, y\rangle \in Y\}$.

Class choice scheme over sets: (Set-Σ^1_∞-AC) For every second-order formula $\varphi(x, X, A)$ and set a,

$$\forall x \in a\, \exists X\, \varphi(x, X, A) \longrightarrow \exists Y \forall x \in a\, \varphi(x, Y_x, A).$$

Class dependent choice scheme: (Σ^1_∞-DC) For every second-order formula $\varphi(X, Y, A)$,

$$\forall X \exists Y\, \varphi(X, Y, A) \longrightarrow \exists Z \forall n\, \varphi(Z \restriction n, Z_n, A).$$

Applications of class choice schemes

Theorem: (Marek, Mostowski) The theory $\mathrm{KM}+\Sigma^1_\infty\text{-}\mathrm{AC}$ is bi-interpretable with the theory ZFC^-_I:

- ZFC^-,
- there is a largest cardinal κ,
- κ is strongly inaccessible.

Theorem: (G., Hamkins, Johnstone) Over KM, Set-Σ^1_∞-AC is equivalent to the Łoś Theorem for second-order ultrapowers.

Theorem: (G., Hamkins, Johnstone) Over $\mathrm{KM}+\Sigma^1_\infty\text{-}\mathrm{AC}$, $\Sigma^1_\infty\text{-}\mathrm{DC}$ is equivalent to second-order reflection. For every second-order formula $\varphi(X)$, there is a class A whose slices $\{A_\xi \mid \xi\in\mathrm{Ord}\}$ reflect $\varphi(X)$.

(Antos, Friedman) In $\mathrm{KM}+\Sigma^1_\infty\text{-}\mathrm{AC}+\Sigma^1_\infty\text{-}\mathrm{DC}$ we can formalize hyperclass forcing.

Theorem: The "L" of a model of KM satisfies $\mathrm{KM}+\Sigma^1_\infty\text{-}\mathrm{AC}+\Sigma^1_\infty\text{-}\mathrm{DC}$.

A class is *constructible* if it is an element of L_Γ for a meta-ordinal Γ.

Independence of class choice schemes

Theorem: (G., Hamkins, Johnstone) It is relatively consistent that:

- There is a model of KM in which Set-Σ^0_1-AC fails.
- There is a model of KM in which Set-Σ^1_∞-AC holds, but Σ^0_1-AC fails.

Question: Does Σ^1_∞-AC imply Σ^1_∞-DC over KM?

Conjecture: It is relatively consistent that there is a model of KM in which Σ^1_∞-AC holds, but Σ^1_∞-DC fails.

Strategy: Prove an analogous result for models of second-order arithmetic and generalize.

Choice principles for models of second-order arithmetic

Full second-order arithmetic: (Z_2)

- Analogue of KM.
- First-order: PA
- Second-order:
  - Induction
  - full second-order comprehension

Choice scheme: (Σ^1_∞-AC) For every second-order formula $\varphi(n, X, A)$,

$$\forall n \exists X\, \varphi(n, X, A) \longrightarrow \exists Y \forall n\, \varphi(n, Y_n, A).$$

Dependent choice scheme: (Σ^1_∞-DC) For every second-order formula $\varphi(X, Y, A)$,

$$\forall X \exists Y\, \varphi(X, Y, A) \longrightarrow \exists Z \forall n\, \varphi(Z \upharpoonright n, Z_n, A).$$

Independence of the choice scheme over Z_2

Lemma: Z_2 proves Σ^1_2-AC.

Proof: Suppose $\mathcal{M} \models \mathrm{Z}_2$ and $\mathcal{M} \models \forall n \exists X \varphi(n, X)$, where φ is Σ^1_2.

(We ignore parameters for simplicity.)

- If α is an ordinal coded by a set in $\mathcal{M}$, then $\mathcal{M}$ has a set coding L_α.
- $\mathcal{M}$ has its own constructible universe $L^{\mathcal{M}}$!
- $\mathcal{M}$ satisfies Shoenfield Absoluteness:
  If ψ is a Σ^1_2-assertion, then
  $$\mathcal{M} \models \psi \text{ iff } L^{\mathcal{M}} \models \psi.$$
  In $L^{\mathcal{M}}$, ψ is interpreted as an assertion about numbers and sets of numbers.
- $L^{\mathcal{M}}$ has a witness to every Σ^1_2-assertion $\exists X \varphi(n, X)$.
- Choose the $L^{\mathcal{M}}$-least X and use comprehension to collect! $\square$

Theorem: (Feferman-Lévy) It is relatively consistent that there is a model of Z_2 in which Π^1_2-AC fails.

The symmetric model argument

The Feferman-Lévy model: $N \models \mathrm{ZF}$ is a symmetric submodel of a forcing extension of L by the finite-support product $\Pi_{n<\omega}\mathrm{Coll}(\omega, \aleph_n)$. In N:

- Each $\aleph_n^L$ is countable.
- $\aleph_\omega^L = \aleph_1^N$ is the first uncountable cardinal.

Proof: Let $\mathcal{M} \models \mathrm{Z}_2$ with second-order part $P^N(\omega)$.

- Every $L_{\aleph_n}$ is coded in $\mathcal{M}$, but $L_{\aleph_\omega}$ is not coded in $\mathcal{M}$.
- We cannot collect the (codes of) $L_{\aleph_n}$.
- The assertion
$$\forall n \exists X = L_{\aleph_n} \to \exists Z \forall n\, Z_n = L_{\aleph_n}$$
fails in $\mathcal{M}$.
- The assertion $X = L_{\aleph_n}$ is Π^1_2 (The assertion that a set of numbers codes an ordinal is Π^1_1). $\square$

Theorem: (Mansfield, Simpson) Z_2 implies Σ^1_2-DC.

Independence of the dependent choice scheme over $\mathrm{Z}_2 + \Sigma^1_\infty$-$\mathrm{AC}$

Theorem: (Friedman, G.) It is relatively consistent that there is a model of $\mathrm{Z}_2 + \Sigma^1_\infty$-$\mathrm{AC}$ in which Π^1_2-DC fails.

History: Simpson claimed to have proved the result in 1973, but his proof is lost.

Strategy:

- Construct a symmetric submodel N of some forcing extension such that in N:
  - AC_ω holds,
  - DC fails for a Π^1_2-definable relation on the reals.
- Let $\mathcal{M} \models \mathrm{Z}_2$ with second-order part $P(\omega)^N$.

Classical model of $\mathrm{AC}_\omega + \neg\mathrm{DC}$ (Jensen)

The forcing $\mathbb{P}$

- Adds a collection of Cohen subsets of ω_1 indexed by nodes of the tree ${}^{<\omega}\omega_1$.
- Conditions: $p : D_p \to 2$ for some countable $D_p \subseteq {}^{<\omega}\omega_1 \times \omega_1$.
- Order: $p \leq q$ if $D_p \supseteq D_q$ and for all $s \in D_q$, $p(s) \leq q(s)$.
- $\mathbb{P}$ is countably closed.

Automorphisms of $\mathbb{P}$

- Every automorphism π of the tree ${}^{<\omega}\omega_1$ extends to an automorphism π^* of $\mathbb{P}$.
- $H = \{\pi^* \mid \pi \text{ is an automorphism of } {}^{<\omega}\omega_1\}$ is a group automorphisms of $\mathbb{P}$.

The symmetric model N

- A countable tree $T \subseteq {}^{<\omega}\omega_1$ is good if it has no infinite branch.
- Given a good tree T, let H_T be the group of all π^* with π point-wise fixing T.
- Given a $\mathbb{P}$-name σ, let $\mathrm{sym}(\sigma)$ be the group of all π^* fixing σ.
- A $\mathbb{P}$-name σ is symmetric if $\mathrm{sym}(\sigma) \supseteq H_T$ for some good tree T.
- HS is the collection of all hereditarily symmetric $\mathbb{P}$-names.
- Let $G \subseteq \mathbb{P}$ be V-generic.
- $N = \{\sigma_G \mid \sigma \in \mathrm{HS}\}$ is an inner model of $V[G]$ satisfying ZF.

Classical model of $\mathrm{AC}_\omega + \neg\mathrm{DC}$ (continued)

Preliminaries

- Let $\dot{\mathscr{T}}$ be the canonical $\mathbb{P}$-name for the tree of Cohen subsets of ω_1 added by $\mathbb{P}$.
- $\dot{\mathscr{T}}$ is hereditarily symmetric, and hence $\mathscr{T} = (\dot{\mathscr{T}})_G \in N$.

Lemma: DC fails in N.

Proof sketch:

- Suppose that $b \in N$ is an infinite branch through $\mathscr{T}$.
- Let $\sigma \in HS$ be a $\mathbb{P}$-name for b, witnessed by a good tree T.
- Use that eventually b lies outside of T to derive a contradiction. □

Lemma: AC_ω holds in N.

Proof sketch:

- Let $F = \{F_n \mid n < \omega\} \in N$ be a family of non-empty sets.
- Let $\sigma \in \mathrm{HS}$ be a $\mathbb{P}$-name for F, witnessed by a good tree S.
- Build a descending sequence of conditions $p_0 \geq p_1 \geq \cdots \geq p_i \geq \cdots$ such that:
  - $p_i \Vdash \tau_i \in \sigma(i)$ for some $\tau_i \in \mathrm{HS}$, witnessed by a good tree T_i.
  - For $i < j$, $T_i \cap T_j = S$.
- Let $\tau \in \mathrm{HS}$ be a $\mathbb{P}$-name for the sequence of the τ_i, as witnessed by $T = \bigcup_{i<\omega} T_i$.
- Let $p \leq p_i$ for all $i < \omega$.
- $p \Vdash$ "τ is a choice function for σ". □

Obstacle: $\mathscr{T}$ is not a tree of reals.

A variation on the classical model (Friedman, G.)

The forcing $\mathbb{P}$

- Adds a collection of Cohen subsets of ω indexed by nodes of the tree ${}^{<\omega}\omega_1$.
- Conditions: $p : D_p \to 2$ for some finite $D_p \subseteq {}^{<\omega}\omega_1 \times \omega$.
- Order: $p \leq q$ if $D_p \supseteq D_q$ and for all $s \in D_q$, $p(s) \leq q(s)$.
- $\mathbb{P}$ has the ccc.

The symmetric model N

- Constructed analogously.
- DC fails in N.
- AC_ω holds in N (use ccc instead of countable closure).

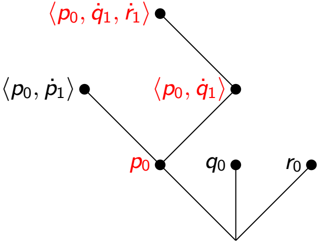
Obstacle: Why is $\mathscr{T}$ definable over $P^N(\omega)$?

- domain
- order

Strategy

The forcing $\mathbb{P}$

- Let $\langle \mathbb{P}_n \mid n < \omega\rangle$ be a sequence of forcing iterations such that:
  - $\mathbb{P}_n$ is an iteration of length n,
  - a generic filter for $\mathbb{P}_n$ is determined by an n-length sequence of reals,
  - for $m > n$, $\mathbb{P}_m \upharpoonright n = \mathbb{P}_n$,
  - The collection of all generic n-length sequences of reals for $\mathbb{P}_n$ is Π^1_2-definable.
- Conditions: $p : D_p \to \bigcup_{n<\omega} \mathbb{P}_n$ such that:
  - D_p is a finite subtree of ${}^{<\omega}\omega_1$,
  - for all $s \in D_p$, $p(s) \in \mathbb{P}_{\text{len}(s)}$,
  - for $s \subseteq t$ in D_p, $p(s) = p(t) \upharpoonright \text{len}(s)$.
- Order: $p \leq q$ if $D_p \supseteq D_q$ and for all $s \in D_q$, $p(s) \leq q(s)$.
- $\mathbb{P}$ is an "iteration along the tree ${}^{<\omega}\omega_1$".
- Suppose $G \subseteq \mathbb{P}$ is V-generic.
- An n-length sequence of reals in $V[G]$ is V-generic for $\mathbb{P}_n$ if and only if it comes from a node of the tree added by G.
- $\mathbb{P}$ has the ccc.



Strategy (continued)

The symmetric model N

- Constructed analogously.
- DC fails in N.
- Using that $\mathbb{P}$ has the ccc, it follows that AC_ω holds in N.

The tree $\mathcal{T}$

- Domain: Π^1_2-definable.
- Order: extension.

Obstacle: Find $\langle \mathbb{P}_n \mid n < \omega \rangle$ with desired properties.

Jensen's poset $\mathbb{P}$: overview

Properties

- $\mathbb{P} \in L$.
- Subposet of Sacks forcing.
- $\mathbb{P}$ has the ccc.
- Adds a unique generic real over L.
- In any model V, the collection of all L-generic reals for $\mathbb{P}$ is Π^1_2-definable.

Finite-support products

Let $\mathbb{P}^{<\omega}$ be the finite-support ω-length product of $\mathbb{P}$.

Theorem: (Lyubetsky, Kanovei) If $G \subseteq \mathbb{P}^{<\omega}$ is L-generic, then the only L-generic reals for $\mathbb{P}$ in $L[G]$ are those on the coordinates of G.

Perfect posets

Perfect trees

- A tree $T \subseteq {}^{<\omega}2$ is perfect if every node in T has a splitting node above it.
- The meet $T \wedge S$ of perfect trees T and S:
  - the largest perfect tree contained in $T \cap S$, if T and S are compatible.
  - $\emptyset$ otherwise.

Perfect posets

A perfect poset $\mathbb{P}$ has the following properties:

- $\mathbb{P}$ is a subposet of Sacks forcing.
- $({}^{<\omega}2)_s \in \mathbb{P}$ for every $s \in {}^{<\omega}2$.
- If $T, S \in \mathbb{P}$ are compatible, then $T \wedge S \in \mathbb{P}$ (closed under meets).
- If $T, S \in \mathbb{P}$, then $T \cup S \in \mathbb{P}$ (closed under unions).

We associate to $\mathbb{P}$, the fusion poset $\mathbb{Q}(\mathbb{P})$:

- Conditions: (T, n) with $T \in \mathbb{P}$ and $n \in \omega$.
- Order: $(S, m) \leq (T, n)$ if $S \subseteq T$, $m \geq n$, and $T \cap {}^n2 = S \cap {}^n2$.
- If $G \subseteq \mathbb{Q}(\mathbb{P})$ is V-generic, then $\mathcal{T} = \bigcup_{(T,n)\in G} T \cap {}^n2$ is a perfect tree.

Growing perfect posets

Preliminaries

- A countable transitive model M is suitable if
  - $M = L_\alpha$,
  - $M \models$ "$\mathrm{ZFC}^- + P(\omega)$ exists".
- Suppose $\mathbb{P} \in M$ is a perfect poset.
- Let $\mathbb{Q}(\mathbb{P})^{<\omega}$ be the finite-support ω-length product of $\mathbb{Q}(\mathbb{P})$.
- Let $\langle \dot{\mathcal{T}}_n \mid n < \omega \rangle$ be the canonical names for the perfect trees added by $\mathbb{Q}(\mathbb{P})^{<\omega}$.
- Let $G \subseteq \mathbb{Q}(\mathbb{P})^{<\omega}$ be M-generic and let $\mathcal{T}_n = (\dot{\mathcal{T}}_n)_G$.

Inside $M[G]$

- Let $\mathbb{U} = \{\mathcal{T}_n \wedge S \mid S \in \mathbb{P} \text{ and } \mathcal{T}_n \wedge S \neq \emptyset\}$.
- Let $\mathbb{P}^*$ be the closure under unions of $\mathbb{P}$ and $\mathbb{U}$.

Lemma:

(1) $\mathbb{P}^*$ is a perfect poset.

(2) $\langle \mathcal{T}_n \mid n < \omega \rangle$ is a maximal antichain of $\mathbb{P}^*$.

(3) Every maximal antichain of $\mathbb{P}$ from M remains maximal in $\mathbb{P}^*$.

(4) Every maximal antichain of $\mathbb{P}^{<\omega}$ from M remains maximal in $\mathbb{P}^{*<\omega}$.

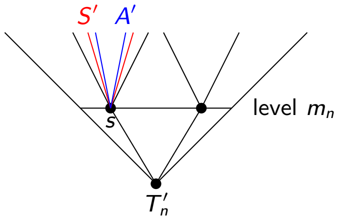
Proof: **(1)** Easy. **(2)**

- By density, if $n \neq m$, then $\mathcal{T}_n \cap \mathcal{T}_m$ is finite.
- By density, for every $T \in \mathbb{P}$, there is n such that $\mathcal{T}_n \leq T$.

Growing perfect posets (continued)

Proof of (3):

- Fix a maximal antichain $\mathcal{A} \in M$ of $\mathbb{P}$.
- Fix a condition $\mathcal{T}_n \wedge S \in \mathbb{U}$ and let $p \in G$ force $\dot{\mathcal{T}}_n \wedge \check{S} \neq \emptyset$.
- The set of conditions $\langle (T_i, m_i) \mid i < \omega \rangle$ with a node s on level m_n such that $(T_n)_s \leq A, S$ for some $A \in \mathcal{A}$ is dense below p.
- Fix a condition $\langle (T_i', m_i) \mid i < \omega \rangle$ below p.



- There is a node s on level m_n of T_n' such that $S' = (T_n')_s \wedge S \neq \emptyset$.
- There is $A \in \mathcal{A}$ such that $A' = S' \wedge A \neq \emptyset$.
- Let T_n be the result of thinning out $(T_n')_s$ to A'. $\square$

Jensen's poset $\mathbb{P}$: definition

Preliminaries

- Fix a canonical $\diamondsuit$-sequence $\langle S_\alpha \mid \alpha < \omega_1\rangle$.

Construction: $\mathbb{P}$ will be the union of an increasing sequence of countable perfect posets

$$\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \cdots \subseteq \mathbb{P}_\alpha \subseteq \cdots$$

of length ω_1.

Stage 0: $\mathbb{P}_0$ is the smallest perfect poset.

Stage limit λ: $\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$.

Stage $\alpha+1$:

- Suppose S_α codes a well-founded binary relation $E \subseteq \alpha\times\alpha$ and $E \cong M_\alpha$ such that:
  - M_α is suitable,
  - $\omega_1^{M_\alpha} = \alpha$,
  - $\mathbb{P}_\alpha \in M_\alpha$.

  Then
  - Let G_α be the L-least M_α-generic filter for $\mathbb{Q}(\mathbb{P}_\alpha)^{<\omega}$.
  - Let $\mathbb{P}_{\alpha+1} = (\mathbb{P}^*_\alpha)^{M_\alpha[G_\alpha]}$.
- Otherwise, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha$.

Jensen's poset $\mathbb{P}$: properties

Preliminaries

- The models M_α form an increasing sequence. If $\alpha < \beta$, then $S_\alpha \in M_\beta$.
- If $\mathcal{A} \in M_\alpha$ is a maximal antichain of $\mathbb{P}_\alpha$, then $\mathcal{A}$ remains maximal in $\mathbb{P}$.

Lemma: $\mathbb{P}$ (and $\mathbb{P}^{<\omega}$) have the ccc.

Proof: Fix a maximal antichain $\mathcal{A}$ of $\mathbb{P}$.

- Let $X \prec L_{\omega_2}$ such that $|X| = \omega_1$ and $\mathcal{A} \in X$.
- Let

$$X_0 \prec X_1 \prec \cdots \prec X_\xi \prec \cdots \prec X$$

  be a continuous increasing chain of length ω_1.
- By properties of $\diamondsuit$, there is α such that
  - S_α codes X_α,
  - $X_\alpha \cap \omega_1 = \alpha$,
  - $\mathcal{A} \in X_\alpha$.
- $\pi : M_\alpha \cong X_\alpha$, $\pi(\mathbb{P}) = \mathbb{P}_\alpha$, $\pi(\mathcal{A}) = \bar{\mathcal{A}}$.
- $\mathbb{P}_{\alpha+1} = (\mathbb{P}^*_\alpha)^{M_\alpha[G_\alpha]}$.
- $\bar{\mathcal{A}}$ remains maximal in $\mathbb{P}$.
- $\bar{\mathcal{A}} = \mathcal{A}$ is countable. $\square$

Jensen's poset $\mathbb{P}$: properties (continued)

Preliminaries

- M is a suitable model, $\bar{\mathbb{P}} \in M$ is a perfect poset, $\bar{G} \subseteq \mathbb{Q}(\bar{\mathbb{P}})^{<\omega}$ is M-generic.
- $\dot{x}_i$ is the canonical name for the i-th generic real added by $\bar{\mathbb{P}}^{<\omega}$.

Uniqueness Lemma: (Lyubetsky, Kanovei) Suppose $\dot{r}$ is a $\bar{\mathbb{P}}^{<\omega}$-name for a real such that

$$\text{for all } i < \omega,\ \mathbb{1}_{\bar{\mathbb{P}}^{<\omega}} \Vdash \dot{r} \neq \dot{x}_i.$$

Then for every n, conditions forcing that $\dot{r} \notin [\check{\mathcal{T}}_n]$ are dense in $\bar{\mathbb{P}}^{*<\omega}$.

Theorem: (Lyubetsky, Kanovei) Suppose $G \subseteq \mathbb{P}^{<\omega}$ is L-generic. If $r \in L[G]$ is L-generic for $\mathbb{P}$, then $r = (\dot{x}_i)_G$ for some $i < \omega$.

Proof: Suppose $r \in L[G]$ and $r \neq (\dot{x}_i)_G$ for any $i < \omega$.

- Let $\dot{r}$ be a nice $\mathbb{P}^{<\omega}$-name for r such that for all $i < \omega$, $\mathbb{1}_{\mathbb{P}^{<\omega}} \Vdash \dot{r} \neq \dot{x}_i$.
- Let $X \prec L_{\omega_2}$ such that $|X| = \omega_1$ and $\dot{r} \in X$.
- Let $X_0 \prec X_1 \prec \cdots \prec X_\xi \prec \cdots \prec X$ be a continuous increasing chain of length ω_1.
- By properties of $\diamondsuit$, there is α such that S_α codes X_α, $X_\alpha \cap \omega_1 = \alpha$, $\dot{r} \in X_\alpha$.
- $\pi : M_\alpha \cong X_\alpha$, $\pi(\mathbb{P}) = \mathbb{P}_\alpha$, $\pi(\dot{r}) = \dot{r}$.
- $\mathbb{P}_{\alpha+1} = (\mathbb{P}^*_\alpha)^{M_\alpha[G_\alpha]}$.
- $\langle \mathcal{T}_n \mid n < \omega \rangle \in M_\alpha[G_\alpha]$ remains maximal in $\mathbb{P}^{<\omega}$.
- For every $n < \omega$, $\mathbb{P}_{\alpha+1}$ has a maximal antichain $\mathcal{A}_n$ of conditions forcing $\dot{r} \notin \check{\mathcal{T}}_n$.
- Each $\mathcal{A}_n$ remains maximal in $\mathbb{P}^{<\omega}$, so $r \notin [\mathcal{T}_n]$. $\square$

Iterations of perfect posets (Abraham)

- $\mathbb{P}_n = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \cdots * \dot{\mathbb{Q}}_{n-1}$ is an n-length iteration of perfect posets:
  - $\mathbb{Q}_0$ is a perfect poset,
  - for $1 \leq i < n$, $\mathbb{P}_i \Vdash \dot{\mathbb{Q}}_i$ is a perfect poset.
- $\vec{P} = \langle \mathbb{P}_i \mid 1 \leq i < \omega \rangle$ is an ω-iteration of perfect posets:
  - for all $i < \omega$, $\mathbb{P}_i$ is an i-length iteration of perfect posets,
  - for $i < j$, $\mathbb{P}_j \upharpoonright i = \mathbb{P}_i$.

We associate to $\mathbb{P}_n$, the fusion poset $\mathbb{Q}(\mathbb{P}_n)$:

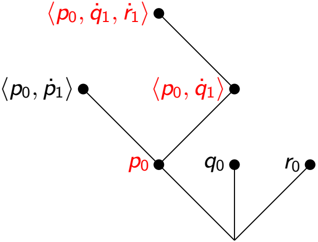
- Conditions: (p, F) with $p \in \mathbb{P}_n$ and $F : n \to \omega$.
- Order: $(p', F') \leq (p, F)$ if $p' \leq p$ and for all $i < n$,
  - $F'(i) \geq F(i)$,
  - $p' \upharpoonright i \Vdash p'(i) \cap {}^{F(i)}2 = p(i) \cap {}^{F(i)}2$.
- $\mathbb{Q}(\mathbb{P}_n)$ is not an iteration!

Tree iterations of perfect posets

Suppose $\vec{P} = \langle \mathbb{P}_i \mid 1 \leq i < \omega \rangle$ is an ω-iteration of perfect posets.

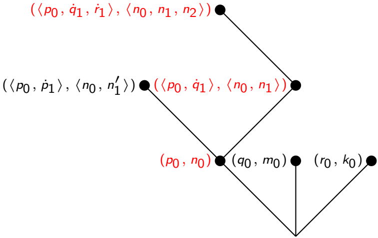
$\mathbb{P}(\vec{P}, {}^{<\omega}\omega)$ is a tree iteration of perfect posets:

- Conditions: $p : D_p \to \bigcup_{i<\omega} \mathbb{P}_i$ such that:
  - D_p is a finite subtree of ${}^{<\omega}\omega$,
  - for all $s \in D_p$, $p(s) \in \mathbb{P}_{\text{len}(s)}$,
  - for $s \subseteq t$ in D_p, $p(s) = p(t) \restriction \text{len}(s)$.
- Order: $p \leq q$ if $D_p \supseteq D_p$ and for all $s \in D_q$, $p(s) \leq q(s)$.



Let $\mathbb{Q}(\vec{P}, {}^{<\omega}\omega)$ be the tree iteration of $\mathbb{Q}(\mathbb{P}_n)$:

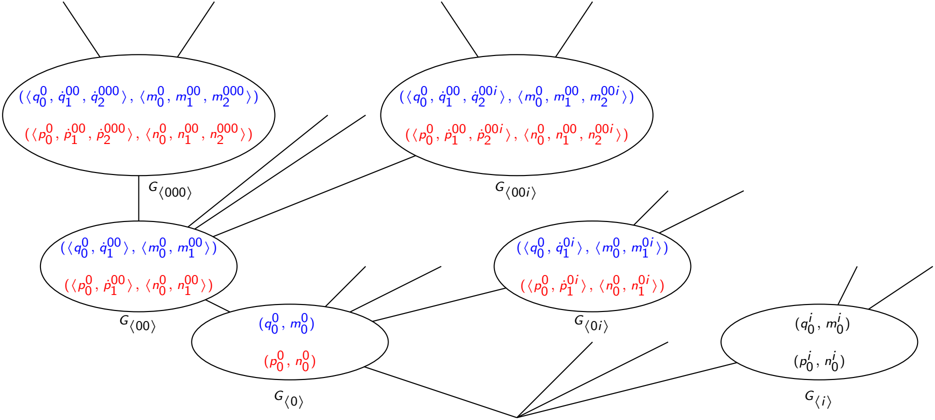
- Conditions: $p : D_p \to \bigcup_{i<\omega} \mathbb{Q}(\mathbb{P}_i)$ such that:
  - D_p is a finite subtree of ${}^{<\omega}\omega$,
  - for all $s \in D_p$, $p(s) \in \mathbb{Q}_{\text{len}(s)}$,
  - for $s \subseteq t$ in D_p, if $p(t) = (q_p, F_p)$, then $p(s) = (q_p \restriction \text{len}(s), F_p \restriction \text{len}(s))$.
- Order: $p \leq q$ if $D_p \supseteq D_p$ and for all $s \in D_q$, $p(s) \leq q(s)$.



Growing iterations of perfect posets

Preliminaries

- M is a suitable model.
- $\vec{P} = \langle \mathbb{P}_i \mid 1 \leq i < \omega \rangle \in M$ is an ω-iteration of perfect posets.
- $G \subseteq \mathbb{Q}(\vec{P}, {}^{<\omega}\omega)$ is M-generic.



Growing iterations of perfect posets (continued)

The perfect poset $\mathbb{Q}_0^*$

- The finite-support product $\Pi_{i<\omega} G_{\langle i\rangle}$ of the $G_{\langle i\rangle}$ is M-generic for $\mathbb{Q}(\mathbb{Q}_0)^{<\omega}$.
- For $i<\omega$, let $\mathcal{T}_i^0$ be the perfect tree constructed from $G_{\langle i\rangle}$.
- $\mathbb{Q}_0^*$ is constructed from $\mathbb{Q}_0$ and $\langle \mathcal{T}_i^0 \mid i<\omega\rangle$.
- $\langle \mathcal{T}_i^0 \mid i<\omega\rangle$ is a maximal antichain of $\mathbb{Q}_0^*$.
- Every maximal antichain of $\mathbb{Q}_0$ from M remains maximal in $\mathbb{Q}_0^*$.

The $\mathbb{Q}_0^*$-name $\dot{\mathbb{Q}}_1^*$ for a perfect poset

- Fix a node $\langle i\rangle \in {}^{<\omega}\omega$.
- Suppose $H^*\subseteq \mathbb{Q}_0^*$ is V-generic. Inside $V[H^*]$:
  - Let $\mathbb{Q}_1=(\dot{\mathbb{Q}}_1)_{H^*}$.
  - Let $G_{i,j}=\{(p(1)_{H^*},F(1))\mid (p,F)\in G_{\langle ij\rangle}\}\subseteq \mathbb{Q}(\mathbb{Q}_1)$.
  - Let $\dot{G}_{i,j}$ be the canonical name for $G_{i,j}$.
- **Theorem**: (Abraham) If there is $p\in\mathbb{Q}_0^*$ such that

$$p\leq q(0) \text{ for all } (q,F)\in G_{\langle ij\rangle},$$

then

$$p\Vdash \text{“The finite-support product } \Pi_{j<\omega}\dot{G}_{i,j} \text{ is } M[\dot{H}]\text{-generic for } \mathbb{Q}(\dot{\mathbb{Q}}_1)^{<\omega}.\text{”},$$

where $\dot{H}$ is the restriction of $\dot{H}^*$ to $\mathbb{Q}_0$.

Growing iterations of perfect posets (continued)

The $\mathbb{Q}_0^*$-name $\dot{\mathbb{Q}}_1^*$ for a perfect poset (continued)

- $\mathcal{T}_i^0 \leq q(0)$ for every $(q, F) \in G_{\langle ij \rangle}$.
- $\mathcal{T}_i^0 \Vdash$ "The finite-support product $\Pi_{j<\omega} \dot{G}_{i,j}$ is $M[\dot{H}]$-generic for $\mathbb{Q}(\dot{\mathbb{Q}}_1)^{<\omega}$."
- By mixing over the antichain $\langle \mathcal{T}_i^0 \mid i < \omega \rangle$:
    - Let $\dot{\mathcal{T}}_j^1$ for $j < \omega$ be names for perfect trees added by $\dot{G}_{i,j}$.
    - Let $\dot{\mathbb{Q}}_1^*$ be a $\mathbb{Q}_0^*$-name for the perfect poset constructed from $\dot{\mathbb{Q}}_1$ and $\langle \dot{\mathcal{T}}_j^1 \mid j < \omega \rangle$.
- $\{\langle \mathcal{T}_i^0, \dot{\mathcal{T}}_j^1 \rangle \mid i, j < \omega\}$ is a maximal antichain of $\mathbb{P}_2^*$.
- Every maximal antichain of $\mathbb{P}_2$ from M remains maximal in $\mathbb{P}_2^*$.

The $\mathbb{P}_n^*$-name $\dot{\mathbb{Q}}_{n+1}^*$ for a perfect poset

- Fix a node $s \in {}^{<\omega}\omega$.
- Suppose $H^* \subseteq \mathbb{P}_n^*$ is V-generic. Inside $V[H^*]$:
    - Let $\mathbb{Q}_n = (\dot{\mathbb{Q}}_n)_{H^*}$.
    - Let $G_{s,j} = \{(p(n)_{H^*}, F(n)) \mid (p, F) \in G_{s^\frown j}\}$.
- $\langle T_{s(0)}^0, \mathcal{T}_{s(1)}^1, \ldots, \mathcal{T}_{s(n-1)}^0 \rangle \leq q \restriction n$ for every $(q, F) \in G_{s^\frown j}$.
- $\langle T_{s(0)}^0, \mathcal{T}_{s(1)}^1, \ldots, \mathcal{T}_{s(n-1)}^0 \rangle \Vdash$ "The finite-support product $\Pi_{j<\omega} \dot{G}_{s,j}$ is $M[\dot{H}]$-generic for $\mathbb{Q}(\dot{\mathbb{Q}}_n)^{<\omega}$."
- etc.

The ω-iteration $\vec{P}$: definition

Preliminaries

- Fix a canonical $\diamondsuit$-sequence $\langle S_\alpha \mid \alpha < \omega_1\rangle$.

Construction: Completed in ω_1-many steps.

Stage 0: $\vec{P}_0 = \langle \mathbb{P}^0_i \mid 1 \leq i < \omega\rangle$

- $\mathbb{Q}^0_0$ is the smallest perfect poset.
- $\dot{\mathbb{Q}}^0_i = \check{\mathbb{Q}}^0_0$.

Stage limit λ: $\vec{P}_\lambda = \langle P^\lambda_i \mid 1 \leq i < \omega\rangle$.

- $\mathbb{Q}^\lambda_0 = \bigcup_{\xi<\lambda} \mathbb{Q}^\xi_0$.
- $\dot{\mathbb{Q}}^\lambda_i$ is the name for the union of $\dot{\mathbb{Q}}^\xi_i$ for $\xi < \lambda$.

Stage $\alpha + 1$:

- Suppose S_α codes a well-founded binary relation $E \subseteq \alpha \times \alpha$ and $E \cong M_\alpha$ such that:
  - M_α is suitable,
  - $\omega_1^{M_\alpha} = \alpha$,
  - $\vec{P}_\alpha \in M_\alpha$.

  Then
  - Let G_α be the L-least M_α-generic filter for $\mathbb{Q}(\vec{P}_\alpha, {}^{<\omega}\omega)$.
  - Let $\vec{P}_{\alpha+1} = (\vec{P}^*_\alpha)^{M_\alpha[G_\alpha]}$.
- Otherwise, $\vec{P}_{\alpha+1} = \vec{P}_\alpha$.

Wrapping up

Preliminaries

- M is a suitable model, $\vec{P}_\alpha \in M$ is an ω-iteration of perfect posets, $\bar{G} \subseteq \mathbb{Q}(\vec{P}_\alpha, {}^{<\omega}\omega)$ is M-generic.
- $\dot{x}_s$ is the canonical name for the sequence of generic reals added on node s of $\bar{G}$.

Uniqueness Lemma: (Friedman, G.) Suppose that $\dot{r}$ is a $\mathbb{P}(\vec{P}_\alpha, {}^{<\omega}\omega)$-name for an n-length sequence of reals such that for all s on level n of ${}^{<\omega}\omega$,

$$\mathbb{1} \Vdash_{\mathbb{P}(\vec{P}_\alpha, {}^{<\omega}\omega)} \dot{r} \neq \dot{x}_s.$$

Then for every s on level n of ${}^{<\omega}\omega$, conditions forcing that for some $i < n$, $\dot{r}(i) \notin [\dot{T}^i_{s(i)}]$ are dense in $\mathbb{P}(\vec{P}^*_\alpha, {}^{<\omega}\omega)$.

Theorem: (Friedman, G.,) $\mathbb{P}(\vec{P}, {}^{<\omega}\omega)$ has the ccc.

Theorem (Friedman, G.) Suppose $G \subseteq \mathbb{P}(\vec{P}, {}^{<\omega}\omega)$ is L-generic. If an n-length sequence of reals $r \in L[G]$ is L-generic for $\mathbb{P}_n$, then $r = (\dot{x}_s)_G$ for some node s on level n of ${}^{<\omega}\omega$.

Corollary: All results carry over to the tree iteration $\mathbb{P}(\vec{P}, {}^{<\omega}\omega_1)$ over ${}^{<\omega}\omega_1$.

Thank you!