Characterizing large cardinals via abstract logics

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Boise Extravaganza in Set Theory

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First-order logic

First-order logic lies at the foundation of modern mathematics.

What is a logic?

- Assigns a collection of formulas to every language.
- Assigns truth values to formulas for every model.

First-order logic $\mathbb{L}_{\omega,\omega}$

- Formulas: close atomic formulas under conjunctions, disjunctions, negations, quantifiers.
- Truth: Tarski's recursive definition.
- Notable properties:
  - ω-Compactness: every finitely satisfiable theory has a model.
  - A language has set-many formulas.
  - A formula can mention finitely much of a language.

First-order logic does not exist outside of mathematics.

A (fragment of a) set-theoretic background is necessary to interpret first-order logic.

- natural numbers
- recursion

Stronger logics require access to more of the set-theoretic background.

Infinitary logics

Infinitary logics allow transfinite conjunctions, disjunctions, and quantifier blocks of formulas.

Suppose $\gamma \leq \delta$ are cardinals.

Infinitary logics $\mathbb{L}_{\delta,\gamma}$

Close formulas under conjunctions and disjunctions of length $<\delta$ and quantifier blocks of length $<\gamma$.

- A language has set-many formulas.
- A formula can mention $<\delta$-much of a language.

Examples

- $\mathbb{L}_{\omega_1,\omega}$
  - There is a sentence expressing that the natural numbers are standard:
  $$n = 0 \vee n = 1 \vee n = 2 \vee \cdots$$
  - ω-Compactness fails.
- $\mathbb{L}_{\delta,\omega}$
  - For every ordinal $\xi < \delta$ and binary relation E, there is a formula $\varphi_\xi^{\mathrm{E}}(x)$ expressing that $(\{y \mid y \,\mathrm{E}\, x\}, \mathrm{E}) \cong (\xi, \in)$. Construct by transfinite recursion.

Infinitary logics (continued)

Examples (continued)

- $\mathbb{L}_{\omega_1,\omega_1}$
  - For every binary relation R there is a sentence $\varphi_{\mathrm{R}}^{\mathrm{WF}}$ expressing that R is well-founded:
$$\neg\exists\, x_0, x_1, \ldots, x_n, \ldots \quad \cdots x_n \,\mathrm{R}\, x_{n-1} \,\mathrm{R} \cdots \mathrm{R}\, x_1 \,\mathrm{R}\, x_0$$
  - For every unary relation I there is a sentence φ_I^{Inf} expressing that I is infinite:
$$\exists x_0, x_1, \ldots, x_n \ldots \bigwedge_{n,m<\omega} x_n \neq x_m$$
- $\mathbb{L}_{\omega_2,\omega_2}$
  - For every unary relation U there is a sentence φ_U expressing that U is uncountable:
$$\exists x_0, x_1, \ldots, x_\xi \ldots \bigwedge_{\xi,\eta<\omega_1} x_\xi \neq x_\eta$$

Not quite a logic

A quasi-logic $\mathbb{L}_{\text{Ord},\omega}$

Close formulas under set-length conjunctions and disjunctions.

- A language has class-many formulas.
- A formula can mention arbitrarily much of a language.

Second-order logic $\mathbb{L}^2$

Second-order logic requires access to the full powerset of the model.

Second-order quantifiers range over all relations over the model.

Expressive power

- A relation is well-founded: every subset has a least element.
- A relation is infinite: there is a bijection with a proper subset.
- (Magidor) For a binary relation E, $(\{y \mid y\,\mathrm{E}\,x\}, \mathrm{E}) \cong (V_\alpha, \in)$ for some α.
- A group F is free:
  - Suppose F has cardinality δ.
  - F is free if and only if there is a transitive model $M \models \mathrm{ZFC}^-$ of size δ with $F \in M$ which satisfies that F is free.
  - There is a relation E on F such that (F, E)
    - satisfies ZFC^-,
    - is well-founded,
    - has an element isomorphic to F,
    - satisfies that F is free.

$\mathbb{L}^2_{\delta,\gamma}$

Formulas are closed under conjunctions, disjunctions of length $<\delta$ and quantifier blocks of length $<\gamma$.

Sort logics $\mathbb{L}^{s,\Sigma_n}$

Sort logics were introduced by Väänänen and require access to Σ_n-truth in the set-theoretic universe.

$\mathbb{L}^{s,\Sigma_n}$

- $\mathbb{L}^2$
- Sort quantifiers $\tilde{\forall}$ and $\tilde{\exists}$
  - search the set-theoretic universe for a new structure such that there is a relation on the combination of the new and old structure satisfying a given formula.
  - at most *n*-alternations of sort quantifiers are allowed

Expressive power

- For every binary relation E there is a sentence $\varphi^n_{\mathrm{E}}(x)$ expressing that $(\{y \mid y \,\mathrm{E}\, x\}, \mathrm{E}) \cong (V_\alpha, \in)$ and $V_\alpha \prec_{\Sigma_n} V$ for some α.

Made to order logics

We can extend first-order logic to meet a particular need.

$\mathcal{L}^{\mathrm{WF}}$

Add a new quantifier $Q^{\mathrm{WF}}xy$ so that $Q^{\mathrm{WF}}xy\,\varphi(x,y)$ whenever $\varphi(x,y)$ is a well-founded relation.

$\mathcal{L}^{\mathrm{Unc}}$

Add a new quantifier $Q^{\mathrm{Unc}}x$ so that $Q^{\mathrm{Unc}}x\,\varphi(x)$ whenever $\{x \mid \varphi(x)\}$ is uncountable.

$\mathcal{L}^{\mathrm{vN}}$

Add a new formula $\varphi^{\mathrm{E}}_{\mathrm{vN}}$ for every binary relation E so that $\varphi^{\mathrm{E}}_{\mathrm{vN}}(x)$ holds whenever $(\{y \mid y\,\mathrm{E}\,x\}),\mathrm{E}) \cong (V_\alpha,\in)$ for some α.

Classes in set theory

First-order set theory ZFC

A class is a first-order definable (with parameters) collection of sets.

Second-order set theory

Separate variables and quantifiers for sets and classes.

Set axioms: ZFC

Class axioms:

- Global choice: there is a class well-order of sets.
- Replacement: every class function restricted to a set is a set.
- GBC Godel Bernays set theory
  - first-order comprehension: every first-order formula defines a class.
  - Every model of ZFC with a definable global well-order is a model of GBC.
- KM Kelley-Morse set theory
  - second-order comprehension: every second-order formula defines a class.
  - If κ is inaccessible, then V_κ together with all subsets of V_κ is a model of KM.

Languages

A language τ is a quadruple $(\mathfrak{F}, \mathfrak{R}, \mathfrak{C}, a)$ where:

- $\mathfrak{F}$ are the functions,
- $\mathfrak{R}$ are the relations,
- $\mathfrak{C}$ are the constants,
- $a : \mathfrak{F} \cup \mathfrak{R} \to \omega$ is the arity function.

A τ-structure is a set with interpretations for the functions, relations, and constants in τ.

A renaming $f : \tau \to \sigma$ between languages $\tau = (\mathfrak{F}, \mathfrak{R}, \mathfrak{C}, a)$ and $\sigma = (\mathfrak{F}', \mathfrak{R}', \mathfrak{C}', a')$ is an arity-preserving bijection between the functions, relations, and constants.

Given a renaming f, let f^* be the associated bijection between τ-structures and σ-structures.

What is a logic?

A logic is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ of classes satisfying the following conditions.

- $\mathcal{L}$ is a class function which takes a language τ to $\mathcal{L}(\tau)$: the set of all sentences in τ.
- $\models_{\mathcal{L}}$ is a sub-class of the class of all pairs (M, φ) where M is a τ-structure and $\varphi \in \mathcal{L}(\tau)$ which determines when M satisfies φ.
- If $\tau \subseteq \sigma$ are languages, then $\mathcal{L}(\tau) \subseteq \mathcal{L}(\sigma)$.
- If $\varphi \in \mathcal{L}(\tau)$, $\sigma \supseteq \tau$ are languages, and M is a σ-structure, then $M \models_{\mathcal{L}} \varphi$ if and only if the reduct $M \restriction \tau \models_{\mathcal{L}} \varphi$.
- If $M \cong N$ are τ-structures, then for all $\varphi \in \mathcal{L}(\tau)$ $M \models_{\mathcal{L}} \varphi$ if and only if $N \models_{\mathcal{L}} \varphi$.
- Every renaming $f : \tau \to \sigma$ induces a unique bijection $f_* : \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ such that for any τ-structure M and $\varphi \in \mathcal{L}(\tau)$

$$M \models_{\mathcal{L}} \varphi \text{ if and only if } f^*(M) \models_{\mathcal{L}} f_*(\varphi).$$

- There is a least cardinal κ, called the occurrence number of $\mathcal{L}$, such that for every sentence $\varphi \in \mathcal{L}(\tau)$, there is a sub-language τ^* of size less than κ such that $\varphi \in \mathcal{L}(\tau^*)$.

Note: Formulas are accommodated by introducing and interpreting constants.

Compactness

Suppose $\mathcal{L}$ is a logic and κ is a cardinal.

κ is a weak compactness cardinal for $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory of size κ has a model.

κ is a strong compactness cardinal for $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory has a model.

κ is a chain compactness cardinal for $\mathcal{L}$ if every $\mathcal{L}$-theory T, which is an increasing union $T = \bigcup_{\eta<\kappa} T_\eta$ of satisfiable theories, has a model.

Proposition: The quasi-logic $\mathbb{L}_{\mathrm{Ord},\omega}$ cannot have a weak compactness cardinal.

Proof: Suppose κ is a weak compactness cardinal for $\mathbb{L}_{\mathrm{Ord},\omega}$.

Let T be the theory in a language with a binary relation E and constant c.

- $\exists y\, (\varphi^{\mathrm{E}}_\kappa(y) \wedge c\mathrm{E}y)$: c is an ordinal below κ.
- $\exists y\{\varphi^{\mathrm{E}}_\xi(y) \wedge y\mathrm{E}c \mid \xi < \kappa\}$: $c > \xi$ for every ordinal $\xi < \kappa$.

T is $<\kappa$-satisfiable of size κ, but T cannot have a model. $\square$

Weakly compact and strongly compact cardinals

Theorem: (Tarski) A cardinal κ is weakly compact if and only if κ is a weak compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$.

Theorem: (Tarski) A cardinal κ is strongly compact if and only if κ is a strong compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$.

Compactness and measurable cardinals

Theorem: (Folklore) A cardinal κ is measurable if and only if it is a chain compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$.

Proof:

($\Longrightarrow$) Suppose κ is measurable and $T = \bigcup_{\eta<\kappa} T_\eta$ is an increasing union of satisfiable $\mathbb{L}_{\kappa,\kappa}(\tau)$-theories.

- Let $j : V \to \mathcal{M}$ be an elementary embedding with $\text{crit}(j) = \kappa$.
- Let $\vec{T} = \langle T_\eta \mid \eta < \kappa \rangle$.
- $j \text{ " } T \subseteq j(\vec{T})(\kappa)$ is a satisfiable $\mathbb{L}_{j(\kappa),j(\kappa)}(j(\tau))$-theory in $\mathcal{M}$ by elementarity.
- Let $M \models j(\vec{T})(\kappa)$ be a $j(\tau)$-structure in $\mathcal{M}$.
- $M \models j \text{ " } T$ in V by absoluteness of satisfaction for $\mathbb{L}_{\kappa,\kappa}$.
- $M \models T$ is a τ-structure via the renaming $j : \tau \to j \text{ " } \tau$.

($\Longleftarrow$) Suppose the compactness property and fix $\alpha > \kappa$.

- Language τ: binary relation $\in$ and constants $\{c_x \mid x \in V_\alpha\} \cup \{c\}$.
- Theory T: $\text{ED}_{\mathbb{L}_{\kappa,\kappa}}(V_\alpha, \in, c_x)_{x\in V_\alpha} \cup \{c_\xi < c < c_\kappa \mid \xi < \kappa\}$ ED: elementary diagram
- $T_\eta = \text{ED}_{\mathbb{L}_{\kappa,\kappa}}(V_\alpha, \in, c_x)_{x\in V_\alpha} \cup \{c_\xi < c < c_\kappa \mid \xi < \eta\}$
- Let $M \models T$.
  - Assume that M is transitive since M is well-founded.
  - $j : V_\alpha \to M$ with $\text{crit}(j) = \kappa$ since every ordinal below κ is $\mathbb{L}_{\kappa,\kappa}$-definable. $\square$

Compactness and supercompact cardinals

Supercompact cardinals κ are characterized in terms of a type of chain compactness together with omitting types in $\mathbb{L}_{\kappa,\kappa}$.

Suppose $\mathcal{L}$ is a logic and $\kappa < \delta$ are cardinals. An $\mathcal{L}(\tau)$-theory T is increasingly filtered by $\mathcal{P}_\kappa(\delta)$ if $T = \bigcup_{s\in\mathcal{P}_\kappa(\delta)} T_s$ such that whenever $s \subseteq s'$, then $T_s \subseteq T_{s'}$.

$\mathcal{P}_\kappa(\delta) = \{A \subseteq \delta \mid |A| < \kappa\}$

Theorem: (Benda, Boney) A cardinal κ is supercompact if and only if for every $\delta > \kappa$, if

- T is an $\mathbb{L}_{\kappa,\kappa}(\tau)$ theory that is increasingly filtered by $\mathcal{P}_\kappa(\delta)$, and
- $p^a(x)$ for $a \in A$ are types that are increasingly filtered by $\mathcal{P}_\kappa(\delta)$

such that every T_s has a model omitting all $p^a_s(x)$ for $a \in A$, then T has a model omitting all $p^a(x)$ for all $a \in A$.

$C^{(n)}$-extendible cardinals

A cardinal κ is extendible if for every $\alpha > \kappa$ there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

$C^{(n)}$ is the class of all cardinals α such that $V_\alpha \prec_{\Sigma_n} V$.

(Bagaria) A cardinal κ is $C^{(n)}$-extendible if for every $\kappa < \alpha \in C^{(n)}$ there an elementary embedding $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$, and $\beta \in C^{(n)}$.

Theorem: (folklore) We can drop the assumption $j(\kappa) > \alpha$.

Compactness and extendible cardinals

Theorem: (Magidor) A cardinal κ is extendible if and only if it is a strong compactness cardinal for $\mathbb{L}^2_{\kappa,\kappa}$.

Proof:

($\Longrightarrow$): Suppose κ is extendible and T is a $<\kappa$-satisfiable $\mathbb{L}^2_{\kappa,\kappa}(\tau)$-theory.

- Let $\alpha > \kappa$ with $T \in V_\alpha$.
- Let $j : V_\alpha \to V_\beta$ be an elementary embedding with $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$.
- $j(T)$ is a $<j(\kappa)$-satisfiable $\mathbb{L}^2_{j(\kappa),j(\kappa)}(j(\tau))$-theory by elementarity.
- $j\,"\,T$ is a $\mathbb{L}^2_{\kappa,\kappa}(j(\tau))$-theory of size $<j(\kappa)$ in V_β, and hence has a model.
- T has a model via the renaming $j : \tau \to j\,"\,\tau$.

($\Longleftarrow$): Suppose the compactness property holds and fix $\alpha > \kappa$.

- Language τ: binary relation $\in$, constants $\{c_x \mid x \in V_\alpha\} \cup \{d_\xi \mid \xi \leq \alpha\} \cup \{c\}$.
- Theory T
  - $\text{ED}_{\mathbb{L}_{\kappa,\kappa}}(V_\alpha, \in, c_x)_{x \in V_\alpha}$
  - $\{c_\xi < c < c_\kappa \mid \xi < \kappa\}$
  - $\{d_\xi < d_\eta < c_\kappa \mid \xi < \eta \leq \alpha\}$
  - $\Phi :=$ "I am V_β". □

Theorem: (Magidor) The least strong compactness cardinal for $\mathbb{L}^2$ is extendible.

Compactness and $C^{(n)}$-extendible cardinals

Theorem: (Boney) A cardinal κ is $C^{(n)}$-extendible if and only if it is a strong compactness cardinal for the sort logic $\mathbb{L}^{s,\Sigma_n}_{\kappa,\kappa}$.

Universal compactness and Vopěnka's Principle

Vopěnka's Principle: Every proper class of structures in the same language has two structures which elementarily embed.

Theorem: (Bagaria) The following are equivalent.

- Vopěnka's Principle
- For every $n<\omega$, there is a proper class of $C^{(n)}$-extendible cardinals.
- For every $n<\omega$, there is a $C^{(n)}$-extendible cardinal.

Theorem: (Makowsky) The following are equivalent.

- Vopěnka's Principle
- Every logic has a strong compactness cardinal.

Universal weak compactness

Assume GBC.

Ord is subtle if for every class club C and sequence $\langle A_\xi \mid \xi \in \mathrm{Ord}\rangle$ with $A_\xi \subseteq \xi$ there are $\alpha < \beta \in C$ such that $A_\beta \cap \alpha = A_\alpha$.

Theorem: Ord is subtle if and only if every logic has a stationary class of weak compactness cardinals.

Theorem: The following is consistent:

- Ord is subtle.
- Global choice fails.
- $\mathbb{L}^2$ doesn't have a weak compactness cardinal.

Proof sketch: Let $L[G]$ be the class forcing extension adding a Cohen subset to every regular cardinal. □

Theorem: (Brooke-Taylor, discussion with G. and Karagila) Vopěnka's Principle is consistent with the failure of Global Choice.

Virtual embeddings

Suppose M and N are structures in the same language. We call an elementary embedding $j : M \to N$ from a (set)-forcing extension $V[G]$ a virtual embedding.

Proposition: There is a virtual elementary embedding between structures M and N if and only if every $\mathrm{Coll}(\omega, |\mathrm{M}|)$-extension has such a virtual embedding.

Virtual large cardinals

A cardinal κ is:

- virtually measurable if for every $\alpha > \kappa$ there is a virtual elementary embedding $j : V_\alpha \to M$ with $\text{crit}(j) = \kappa$.
- virtually supercompact (remarkable) if for every $\alpha > \kappa$ there is a virtual elementary embedding $j : V_\alpha \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$, $M^\alpha \subseteq M$.
- weakly virtually extendible if for every $\alpha > \kappa$, there is a virtual elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$.
- virtually extendible if for every $\alpha > \kappa$, there is a virtual elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.
- weakly virtually $C^{(n)}$-extendible if for every $\kappa < \alpha \in C^{(n)}$, there is a virtual elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $\beta \in C^{(n)}$.
- virtually $C^{(n)}$-extendible if for every $\kappa < \alpha \in C^{(n)}$ above κ, there is a virtual elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$, and $\beta \in C^{(n)}$.
- virtually rank-into-rank if there is a virtual elementary embedding $j : V_\alpha \to V_\alpha$ with $\text{crit}(j) = \kappa$. There is no virtual Kunen Inconsistency: α can be much greater than the supremum of the critical sequence of j.

Virtual Vopěnka's Principle: Every proper class of structures in the same language has two structures which virtually elementarily embed.

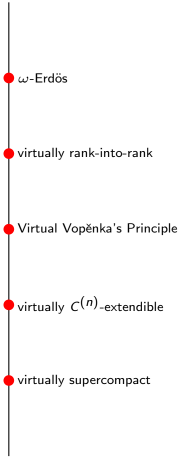
The virtual large cardinals hierarchy

Theorem: (Nielsen) Virtually measurable cardinals are equiconsistent with virtually supercompact cardinals.

Theorem: (G.) If a weakly virtually extendible cardinal is not virtually extendible, then it is virtually rank-into-rank.

Theorem: (G., Hamkins) Virtual Vopěnka's Principle holds if and only if for every $n < \omega$, there is a proper class of weakly virtually $C^{(n)}$-extendible cardinals.

Theorem: (G., Hamkins) It is consistent that Virtual Vopěnka's Principle holds, but there are no virtually supercompact cardinals.



Pseudo-models and pseudo-compactness cardinals

Suppose $\mathcal{L}$ is a logic, τ and τ^* are languages and δ is a cardinal.

A δ-forth system $\mathcal{F}$ from τ to τ^* is a collection of renamings $f : \sigma \to \sigma^*$ with $\sigma \subseteq \tau$, $\sigma^* \subseteq \tau^*$ of size less than δ such that:

- $\emptyset \in \mathcal{F}$.
- If $f \in \mathcal{F}$ and $\tau_0 \subseteq \tau$ has size less than δ, then there is $g \in \mathcal{F}$ with $f \subseteq g$ and $\tau_0 \subseteq \text{dom} g$.

A τ^*-structure M is a δ-pseudo-model for an $\mathcal{L}(\tau)$-theory T if there is a δ-forth system $\mathcal{F}$ from τ to τ^* such that for every $f : \sigma \to \sigma^*$ in $\mathcal{F}$ $M \models f_* " T \cap \mathcal{L}(\sigma)$.

"A δ-pseudo-model can be renamed to interpret small pieces of T and every renaming can be extended to enterpret a further small piece of T."

κ is a δ-pseudo-compactness cardinal for $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory has a δ-pseudo-model.

κ is a δ-pseudo-chain-compactness cardinal for $\mathcal{L}$ if every $\mathcal{L}$-theory T, which is an increasing union $T = \bigcup_{\eta<\kappa} T_\eta$ of satisfiable theories, has a δ-pseudo-model.

Proposition: A κ^+-pseudo-compactness cardinal κ for a logic $\mathcal{L}$ is a weak compactness cardinal for $\mathcal{L}$.

Pseudo-compactness and virtually measurable cardinals

Theorem: The following are equivalent for a cardinal κ.

- κ is virtually measurable.
- κ is a κ^+-pseudo-chain compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$.
- κ is an ω-pseudo-chain compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$.

Pseudo-compactness and virtually supercompact cardinals

Theorem: The following are equivalent for a cardinal κ.

- κ is virtually supercompact.
- For every $\delta > \kappa$, if T is an $\mathbb{L}_{\kappa,\kappa}(\tau)$-theory that is increasingly filtered by $\mathcal{P}_\kappa(\delta)$ and $p^a(x)$ for $a \in A$ are types increasingly filtered by $\mathcal{P}_\kappa(\delta)$ such that every T_s has a model omitting all $p^a_s(x)$, then there is a κ^+-pseudo-model of T omitting all $p^a(x)$.
- Replace κ^+-pseudo-model by ω-pseudo-model.

Characterizing virtual large cardinals via pseudo-compactness: virtually extendible

Theorem: The following are equivalent for a cardinal κ.

- κ is virtually extendible.
- κ is a κ^+-pseudo-compactness cardinal for $\mathbb{L}^2_{\kappa,\kappa}$.
- κ is an ω-pseudo-compactness cardinal for $\mathbb{L}^2_{\kappa,\kappa}$.

Theorem: If there are no measurable cardinals, then the least κ^+-pseudo-compactness cardinal κ for $\mathbb{L}^2$ is virtually extendible.

Theorem: The following are equivalent for a cardinal κ.

- κ is weakly virtually extendible.
- κ is a κ^+-pseudo-chain compactness cardinal for $\mathbb{L}^2_{\kappa,\kappa}$.
- κ is an ω-pseudo-chain compactness cardinal for $\mathbb{L}^2_{\kappa,\kappa}$.

Theorem: The least ω-pseudo-chain compactness cardinal κ for $\mathbb{L}^2$ is weakly virtually extendible.

Universal pseudo-compactness

Theorem: The following are equivalent.

- For every $n < \omega$, there is a virtually $C^{(n)}$-extendible cardinal.
- Every logic has a κ^+-pseudo-compactness cardinal κ.
- Every logic has an ω-pseudo-compactness cardinal κ.

Theorem: The following are equivalent.

- Virtual Vopěnka's Principle
- Every logic $\mathcal{L}$ has a κ^+-pseudo-chain compactness cardinal κ.
- Every logic $\mathcal{L}$ has an ω-pseudo-chain compactness cardinal κ.