BOOLEAN-VALUED CLASS FORCING

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Abstract. We show that the Boolean algebras approach to class forcing can be carried out in the theory Kelley-Morse plus the Choice Scheme ($\mathrm{KM}+\mathrm{CC}$) using hyperclass Boolean completions of class partial orders. Applying the Boolean algebras approach, we show that every intermediate model between a model $\mathscr{V} \models \mathrm{KM}+\mathrm{CC}$ and one of its class forcing extensions is itself a class forcing extension if and only if it is simple - generated by the classes of $\mathscr{V}$ together with a single new class. We show that a model of $\mathrm{KM}+\mathrm{CC}$ and its class forcing extension may have non-simple intermediate models, and thus, the Intermediate Model Theorem can fail for models of $\mathrm{KM}+\mathrm{CC}$.

1. Introduction

There are two standard approaches to carrying out the forcing construction over a model of ZFC set theory: with partial orders or with complete Boolean algebras. The two approaches yield the same forcing extensions because every partial order densely embeds into a complete Boolean algebra, and when a partial order densely embeds into another partial order, the two have the same forcing extensions. Although the partial order construction can be viewed as more straightforward, the complete Boolean algebras approach offers some advantages. For instance, there are theorems about forcing which have no known proofs without the use of Boolean algebras. One such fundamental result is the Intermediate Model Theorem, which states that if a universe $V \models \mathrm{ZFC}$ and $W \models \mathrm{ZFC}$ is an intermediate model between V and one of its set-forcing extensions, then W is itself a forcing extension of V. The theorem makes a fundamental use of the Axiom of Choice, but a weaker version of it still holds for models of ZF. If $V \models \mathrm{ZF}$ and $V[a] \models \mathrm{ZF}$, with $a \subseteq V$, is an intermediate model between V and one of its set-forcing extensions, then $V[a]$ is itself a set-forcing extension of V. Grigorieff in [Gri75] attributes the Intermediate Model Theorem to Solovay.

The standard Boolean algebras approach is not available in the context of class forcing because most class partial orders cannot be densely embedded into a sufficiently complete class Boolean algebra. Set forcing uses complete Boolean algebras, those which have suprema for all their subsets, because completeness is required for assigning Boolean values to formulas in the forcing language. With a class Boolean algebra, which has suprema for all its subsets, we can still define the Boolean values of atomic formulas, but the definition of Boolean values for existential formulas needs to take suprema of subclasses, meaning that we must require a Boolean algebra to have those in order to be able to construct the Boolean-valued model. However, as shown in [HKL+], a class Boolean algebra with a proper class antichain can never have this level of completeness. Thus, only ORD-cc Boolean

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algebras (having set-sized antichains) can potentially have suprema for all their subclasses. Indeed, it is shown in [HKL+] that every ORD-cc partial order densely embeds into a complete Boolean algebra.

In this article we nevertheless show that the Boolean algebras approach to class forcing can be carried out in sufficiently strong second-order set theories, for example, the theory Kelley-Morse plus the Choice Scheme, using hyperclass Boolean completions. We apply the Boolean algebras approach to show that the ZF-analogue of the Intermediate Model Theorem holds for models of Kelley-Morse plus the Choice Scheme.

Let us call an extension $\mathscr{W}$ of a model $\mathscr{V}$ of second-order set theory *simple* if it is generated by the classes of $\mathscr{V}$ together with a single new class (for a precise definition see Section 5). In particular, every forcing extension of $\mathscr{V}$ is simple. We show that every simple intermediate model between a model of $\mathrm{KM} + \mathrm{CC}$ and one of its class forcing extensions is itself a forcing extension, so that the Intermediate Model Theorem holds for simple extensions. We also show that an intermediate model between a model of $\mathrm{KM}+\mathrm{CC}$ and one of its class forcing extensions need not be simple, and thus the Intermediate Model Theorem can fail. For models of KM, the Intermediate Model Theorem can fail even where the forcing has the ORD-cc because a model of KM and its forcing extension by ORD-cc forcing can have non-simple intermediate models. We don't know whether this can happen for models of $\mathrm{KM} + \mathrm{CC}$. Finally, we show that if an intermediate model $\mathscr{W}$ between a model $\mathscr{V} \models \mathrm{KM} + \mathrm{CC}$ and its forcing extension $\mathscr{V}[G]$ has a definable global well-ordering of classes, and we have additionally that $\mathscr{W}$ is definable in $\mathscr{V}[G]$, then $\mathscr{W}$ must be a simple extension of $\mathscr{V}$.

2. A hierarchy of second-order set theories

The formal framework in which we shall investigate the properties of class partial orders is second-order set theory. Second-order set theory is formalized in a two-sorted logic with separate objects and quantifiers for sets and classes. Following convention, we will use upper-case letters to denote class variables and lower-case letters to denote set variables. Models of second-order set theory are triples $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$ where V consists of the sets and $\mathcal{C}$ consists of the classes, and $\in$ is a relation between sets as well as between sets and classes. The weakest reasonable axiomatization of second-order set theory is Gödel-Bernays set theory GBC[1]. The theory GBC consists of the ZFC axioms for sets, Extensionality axiom for classes, Replacement axiom stating that the restriction of a class function to a set is a set, the assertion that there is a class well-ordering of sets (global well-order), and the comprehension scheme for first-order assertions. The comprehension scheme is a collection of set existence assertions stating for every first-order formula (with class parameters) that there is a class whose elements are exactly the sets satisfying the formula. A much stronger second-order set theory is the Kelley-Morse set theory KM, which strengthens GBC by extending the comprehension scheme to all second-order assertions, so that every second-order formula defines a class.

We can further strengthen KM by adding a scheme of assertions called the Choice Scheme, which can be thought of as a choice principle or alternatively as a collection principle for classes. The *Choice Scheme* is a scheme of assertions, which states for every second-order formula $\varphi(x, X, A)$ that if for every set x, there is a class X

[1]This theory is also referred to in the literature as NBG.

witnessing $\varphi(x, X, A)$, then there is a single class Y making a choice of a witness for every set x in the sense that for every set x, $\varphi(x, Y_x, A)$ holds, where $Y_x = \{y \mid \langle x, y\rangle \in Y\}$ is the x-th slice of Y. It is not difficult to see that an assertion of the Choice Scheme for a formula $\varphi(x, X, A)$ is equivalent to the collection assertion that there is a single class Y collecting witnesses for every set x in sense that for every set x, there is some set z such that $\varphi(x, Y_z, A)$ holds. Given a collecting class Y, we define a class $\bar{Y}$ such that for every set x, $\bar{Y}_x = Y_z$ where z is least according to the global well-order for which $\varphi(x, Y_z, A)$ holds. We will call $\mathrm{KM} + \mathrm{CC}$ the resulting theory consisting of KM together with the Choice Scheme, where CC is meant to stand for "class choice" or "class collection" depending on one's viewpoint. It was shown in [GHJ] that the Choice Scheme is independent of KM. Indeed, even the weakest instances of the Choice Scheme, those for a first-order assertion and just ω-many choices, can fail in a model of KM. At the same time the theories KM and $\mathrm{KM} + \mathrm{CC}$ are equiconsistent. Analogously to how the constructible universe of a model of ZF satisfies the Axiom of Choice, L together with the *constructible* classes of a model of KM is a model of $\mathrm{KM}+\mathrm{CC}$. In a model $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}$, we can continue the L-construction beyond ORD, along any class well-order, such as $\mathrm{ORD} + \mathrm{ORD}$, $\mathrm{ORD} \cdot \omega$, etc. We define a class well-order Γ to be *constructible* if there is a larger class well-order Δ such that L_Δ, the union of the L-construction along Δ, can well-order ORD in order-type Γ. We should think of the constructible class well-orders as those appearing in "L_{ORD^+}". We then define a class in $\mathcal{C}$ to be *constructible* if it is an element of L_Γ for some constructible Γ.[2]

Before we continue our overview of second-order set theories, let's establish some terminology. We will call a definable (with class parameters) collection of classes in a model of second-order set theory a *hyperclass*. Given a class C and a set a, we will denote by $C_a = \langle x \mid \langle a, x\rangle \in C\rangle$ the a-th *slice* of C. We will say that a hyperclass defined by a formula $\varphi(X, A)$ is *coded* in the model $\mathscr{V}$ if there is a class S such that $\{S_\xi \mid \xi \in \mathrm{ORD}\}$ is exactly the collection of classes satisfying $\varphi(X, A)$, saying in essence that there are "class-many" classes satisfying φ. In this case, we will say that S *codes* the hyperclass.

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$ is a model of $\mathrm{KM} + \mathrm{CC}$. Consider the collection of all extensional well-founded binary relations on ORD in $\mathcal{C}$ modulo isomorphism. Among these are relations coding $\mathrm{ORD} + \mathrm{ORD}$, $\mathrm{ORD} \cdot \omega$, $V \cup \{V\}$, etc. We can view such relations as coding transitive sets which sit above the sets of $\mathscr{V}$. A natural membership relation on the equivalence classes of the relations gives us a first-order set-theoretic structure, which we will denote by $M_{\mathscr{V}}$ and refer to as the *companion* model of $\mathscr{V}$. For more details on the construction of $M_{\mathscr{V}}$, see [AF17]. It turns out the model $M_{\mathscr{V}}$ satisfies a relatively strong and well-understood first-order set theory, which we will call here ZFC^-_I. The theory ZFC^-_I consists of the axioms of ZFC without the powerset axiom (with Collection instead of Replacement) $(-)$, together with the axiom that there is a largest cardinal, which is inaccessible (I). Since it may not be clear what inaccessibility means in the absence of powerset, let's be more precise by saying that there is the largest cardinal κ which is regular, and that for every $\alpha < \kappa$, $P(\alpha)$ exists and $|P(\alpha)| < \kappa$. It follows that for all $\alpha < \kappa$, V_α exists and $|V_\alpha| < \kappa$, and hence V_κ exists as well. Natural models of the theory ZFC^-_I are H_{κ^+}, the collection of all sets of hereditary size at most κ, for an

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[2]The notion of a constructible universe of a second-order model first appeared in Tharp's dissertation [Tha65]. Details of the construction can be found in [GH16].

inaccessible cardinal κ. The collection of sets V is isomorphic to the V_κ of $M_{\mathscr{V}}$ and each class in $\mathcal{C}$ corresponds to a subset of V_κ in $M_{\mathscr{V}}$. Conversely now suppose that M is any model of ZFC^-_I. Then $\mathscr{V} = \langle V^M_\kappa, \in, \{A \in M \mid A \subseteq V_\kappa\}\rangle$ is a model of $\text{KM} + \text{CC}$ and its companion model $M_{\mathscr{V}} \cong M$. Thus, the theories $\text{KM} + \text{CC}$ and ZFC^-_I are bi-interpretable.

We can further strengthen the theory $\text{KM} + \text{CC}$ by adding the ω-Dependent Choice Scheme, which is the analogue of the Axiom of Dependent Choices for classes, calling the resulting theory $\text{KM} + \text{CC} + \text{DC}_\omega$. The *$\omega$-Dependent Choice Scheme* states for every second-order formula $\varphi(X, Y, A)$ that if for every class X there is a class Y such that $\varphi(X, Y, A)$ holds, so that the relation φ has no terminal nodes, then there is an ω-sequence of dependent choices according to φ, a class Z such that $\varphi(Z_n, Z_{n+1}, A)$ holds for all $n \in \omega$. The ω-Dependent Choice Scheme in a model $\mathscr{V} \models \text{KM} + \text{CC}$ translates into a version of the axiom of Dependent Choices for definable relations in the companion model $M_{\mathscr{V}}$. More precisely, if $\mathscr{V} \models \text{KM} + \text{CC} + \text{DC}_\omega$, then the companion model $M_{\mathscr{V}}$ satisfies ZFC^-_I together with the assertion that we can make ω-many dependent choices along any definable relation without terminal nodes.

The ω-Dependent Choice Scheme is equivalent, over $\text{KM} + \text{CC}$, to a reflection principle for classes. The natural analogue of the Lévy-Montague reflection in ZFC, which states that every first-order formula is reflected by a transitive set (namely an element of the V_α-hierarchy), for classes is the Class Reflection Principle. The *Class Reflection Principle* is a scheme of assertions stating that every second-order formula is reflected by a coded hyperclass, so that given a second-order formula $\varphi(X, A)$, there is a class S such that $\langle V, \in, \{S_\xi \mid \xi < \text{ORD}\}\rangle$ reflects $\varphi(X, A)$. A model $\mathscr{V} \models \text{KM} + \text{CC}$ satisfies the Class Reflection Principle if and only if its companion model $M_{\mathscr{V}}$ satisfies the usual Lévy-Montague reflection (not necessarily having the V_α-hierarchy) because coded hyperclasses become sets in the companion model. To see that the two principles are equivalent, observe in one direction that given a second-order formula $\psi(X, A)$, every class S coding a hyperclass can be extended to a class S^* coding a hyperclass that is closed under witnesses for all existential subformulas of ψ, and so the relation $\varphi(X, Y, A)$ which holds of pairs $X = S$ and $Y = S^*$ has no terminal nodes. In the other direction, given a relation $\varphi(X, Y, A)$ without terminal nodes, we can reflect the statement $\forall X \exists Y \varphi(X, Y, A)$ to a coded hyperclass and use the fact its classes are well-ordered (by definition) to make ω-many dependent choices along φ.

We can also consider strengthening the ω-Dependent Choice Scheme to the α-Dependent Choice Scheme DC_α, for an uncountable regular cardinal α, asserting for every second-order formula $\varphi(X, Y, A)$ that if for every class X, there is a class Y such that $\varphi(X, Y, A)$ holds, then there is a single class Z making α-many dependent choices along φ so that for all $\beta < \alpha$, $\varphi(Z \upharpoonright \beta, Z_\beta, A)$ holds, or indeed to the full ORD-Dependent Choice Scheme DC_{ORD}. The principle α-Dependent Choice Scheme can be reformulated as a reflection principle, the *α-closed Class Reflection Principle*, stating that every second-order assertion $\psi(X, A)$ can be reflected to a coded hyperclass that is closed under $<\alpha$-sequences. This means that if S codes the hyperclass, then for every $\beta < \alpha$, whenever there is a definable function $f : \beta \to \{S_\xi \mid \xi \in \text{ORD}\}$, then the class B with $B_\xi = f(\xi)$ for $\xi < \beta$ is in the hyperclass. In the companion model $M_{\mathscr{V}}$ of $\mathscr{V} \models \text{KM} + \text{CC}$, the α-closed Class Reflection Principle translates to the statement that every formula can be reflected to a transitive set closed under $<\alpha$-sequences. Similarly the ORD-Dependent Choice Scheme is equivalent to the ORD-*closed Class Reflection Principle*.

It is not known whether the ω-Dependent Choice Scheme can fail in a model of $\mathrm{KM} + \mathrm{CC}$ or whether the α-Dependent Choice Schemes and the ORD-Dependent Choice Scheme form a hierarchy over $\mathrm{KM} + \mathrm{CC}$. However a recent result of the second and third authors showing that the Dependent Choice Scheme can fail in a model of full second-order arithmetic Z_2 together with the Choice Scheme [FGK], strongly suggests that ω-Dependent Choice will turn out to be independent of KM+ CC. In either case, all these theories are equiconsistent because the constructible classes of a model of KM is a model of $\mathrm{KM} + \mathrm{CC} + \mathrm{DC}_{\mathrm{ORD}}$.

The Choice Scheme and the ORD-Dependent Choice Scheme are superseded by the assumption that a model $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}$ has a hyperclass well-ordering of classes. But this property as just stated does not appear to be second-order expressible. For this reason, we will consider a very specific kind of hyperclass well-order whose existence is second-order definable. If the companion model $M_{\mathscr{V}}$ has the form $L[A]$ for some $A \subseteq \kappa$ where κ is the largest cardinal, then it has a definable global well-order which translates into a hyperclass well-order of $\mathcal{C}$ definable from the class A. The statement that the companion model $M_{\mathscr{V}}$ has the form $L[A]$ is second-order expressible over $\mathscr{V}$. In the case that this property holds, we will say that $\mathscr{V}$ has a *canonical hyperclass well-order of classes*. Although the assertion that there exists such a well-order appears to be both quite strong and restrictive, it is indeed the case that any model $\mathscr{V}$ of $\mathrm{KM}+\mathrm{CC}+\mathrm{DC}_{\mathrm{ORD}}$ has a "forcing extension" to a model of KM together with the assertion that there exists a canonical hyperclass well-order of classes with the same sets, but possibly new classes. The extension is obtained by forcing over the companion model $M_{\mathscr{V}}$, and then taking the second-order model obtained from the forcing extension. The forcing over the companion model $M_{\mathscr{V}}$ is done in three steps. The first step is a class forcing to add a Cohen subclass to $\mathrm{ORD}^{M_{\mathscr{V}}}$ with bounded conditions, which in particular adds a global well-order of $M_{\mathscr{V}}$. The second step of the forcing "reshapes" B into B' having the right properties for the third step which is the almost disjoint coding forcing to code B into a subset A of κ. The final forcing extension is a model $M = L[A]$ for $A \subseteq \kappa$ such that $V_\kappa^M = V_\kappa^{M_{\mathscr{V}}}$. The ORD-Dedepent Choice Scheme is required to show that the forcing to add a Cohen subclass to $\mathrm{ORD}^{M_{\mathscr{V}}}$ is $<\alpha$-distributive for every cardinal α. For details of the forcing constructions, see [AF17].

3. Class partial orders and class Boolean algebras

For the remainder of the article, whenever we say partial order, we will mean a *separative* partial order. Recall that a set Boolean algebra is said to be *complete* if every one of its subsets has a supremum. It is a standard fact that every set partial order densely embeds into a complete Boolean algebra. Given a set partial order $\mathbb{P}$, a complete Boolean algebra embedding $\mathbb{P}$ is obtained by putting a natural Boolean operations structure on the regular cuts of $\mathbb{P}$, and the Boolean algebra constructed in this way is the unique up to isomorphism complete Boolean algebra into which $\mathbb{P}$ densely embeds (see, for instance, [Jec03]). To distinguish the relevant levels of completeness for a class Boolean algebra, we will say that a class Boolean algebra is *set-complete* if all its subsets have suprema and that it is *class-complete* if all its subclasses have suprema.

The theory GBC cannot even prove that every class partial order densely embeds into a set-complete class Boolean algebra. It is shown in [HKL$^+$] that, in a model of GBC, a class partial order $\mathbb{P}$ densely embeds into a set-complete class Boolean algebra if and only if $\mathbb{P}$ satisfies the Forcing Theorem, the statement that the forcing relation for atomic formulas is a class[3], and there are models of GBC having class partial orders for which the Forcing Theorem fails. A slightly stronger theory GBC together with the principle $\mathrm{ETR}_{\mathrm{ORD}}$ proves that the Forcing Theorem holds for all class partial orders, and therefore that every class partial order densely embeds into a set-complete class Boolean algebra. The principle $\mathrm{ETR}_{\mathrm{ORD}}$, which states that every first-order definable recursion of length ORD whose stages are classes has a solution, follows from $\mathrm{GBC} + \Sigma^1_1$-Comprehension[4] (for details, see [GHH$^+$]). In particular, in a model of KM, every class partial order densely embeds into a set-complete class Boolean algebra. However, even when a class partial order can be embedded into a set-complete class Boolean algebra, the completion is not unique unless the partial order has only set-sized antichains [HKS18]. As we already mentioned in the introduction, no Boolean algebra with proper class antichains can be class-complete [HKS18], so that there is no hope of embedding every class partial order into a class-complete Boolean algebra. Class Boolean algebras are simply "too small" to have suprema for all their subclasses. Thus, we are naturally led to consider hyperclass Boolean algebras.

We will say that a hyperclass Boolean algebra is *class-complete* whenever all its coded sub-hyperclasses have suprema, so that it has suprema for collections consisting of "class-many" of its elements. Using the analogue of the regular cuts construction for set partial orders, we will now argue that every class partial order densely embeds into a class-complete hyperclass Boolean algebra.

Proposition 3.1. *In a model of* GBC*, every class partial order densely embeds into a class-complete hyperclass Boolean algebra.*

Proof. Working in a model $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$, fix a class partial order $\mathbb{P} \in \mathcal{C}$. The hyperclass Boolean algebra $\mathbb{B}_{\mathbb{P}}$ is constructed completely analogously to the set case. Define that a *cut* U of $\mathbb{P}$ is a subclass of $\mathbb{P}$ that is closed downward so that whenever $p \in U$ and $q \leq p$, then $q \in U$. Given a condition $p \in \mathbb{P}$, let $U_p = \{q \in \mathbb{P} \mid q \leq p\}$ be the cut of all elements in the cone below p. We say that a cut U is *regular* if whenever $p \notin U$, then there is $q \leq p$ such that $U \cap U_q = \emptyset$. Given any cut U of $\mathbb{P}$, define $\overline{U} = \{p \in \mathbb{P} \mid \forall q \leq p\, U \cap U_q \neq \emptyset\}$, and note that $\overline{U}$ is a regular cut. If $p \in U$, then $U_p \subseteq U$, so $U \subseteq \overline{U}$. Also, clearly if W is a regular cut and $U \subseteq W$, then $\overline{U} \subseteq W$. So $\overline{U}$ is the least regular cut containing U. The Boolean structure on the regular cuts of $\mathbb{P}$ is defined precisely as in the set case (see [Jec03]), for instance, $U + W$ is defined to be $\overline{U \cup W}$. It is easy to see that $\mathbb{B}_{\mathbb{P}}$ is class-complete. Fix a class S whose slices S_ξ for $\xi \in \mathrm{ORD}$ are elements of $\mathbb{B}_{\mathbb{P}}$. Then the supremum of all S_ξ is the regular cut $U = \overline{\bigcup_{\xi \in \mathrm{ORD}} S_\xi}$. □

[3] The Forcing Theorem for atomic formulas implies the Forcing Theorem for all formulas: if the forcing relation for atomic formulas is a class, then the forcing relation for any fixed first-order formula (with a class name parameter) is a class, and the forcing relation for any fixed second-order formula is (second-order) definable.

[4] Indeed $\mathrm{ETR}_{\mathrm{ORD}}$ already follows from GBC plus Δ^1_1-Comprehension given some induction, namely Σ^1_1-induction. Thus, for β-models, transitive models that are moreover correct about well-foundedness of class relations, $\mathrm{ETR}_{\mathrm{ORD}}$ already follows from GBC plus Δ^1_1-Comprehension.

We will call $\mathbb{B}_{\mathbb{P}}$ the *hyperclass Boolean completion* of $\mathbb{P}$.

Next, let us say that a hyperclass Boolean algebra is *fully complete* if all its sub-hyperclasses have suprema. Full completeness is required in the usual Boolean-valued model construction to define the Boolean values of existential assertions. If $\mathscr{V} \models \mathrm{KM}$, then it is clear that a hyperclass Boolean completion $\mathbb{B}_{\mathbb{P}}$ of a class partial order $\mathbb{P}$ is fully complete because the supremum of a sub-hyperclass of $\mathbb{B}_{\mathbb{P}}$ given by a (second-order) formula $\varphi(X, A)$ is the regular cut $\overline{U}$ obtained from the union U of all regular cuts satisfying $\varphi(X, A)$, which exists by full comprehension. Now we would like to argue that if in a model $\mathscr{V} \models \mathrm{GBC}$, there is a partial order $\mathbb{P}$ with a proper class antichain whose hyperclass Boolean completion $\mathbb{B}_{\mathbb{P}}$ is fully complete, then indeed $\mathscr{V} \models \mathrm{KM}$.

Theorem 3.2. *In a model $\mathscr{V} \models \mathrm{GBC}$, a partial order $\mathbb{P}$ with a proper class antichain has a fully complete Boolean completion $\mathbb{B}_{\mathbb{P}}$ if and only if $\mathscr{V} \models \mathrm{KM}$.*

Proof. We already argued for the easy direction above. So suppose that $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is a partial order with a proper class antichain, call it A, and let's assume that the hyperclass Boolean completion $\mathbb{B}_{\mathbb{P}}$ is fully complete. Fix a second-order formula $\varphi(x, B)$. We would like to argue that the collection of all sets x satisfying $\varphi(x, B)$ in $\mathscr{V}$ is a class. Fix a bijection $f : A \xrightarrow[\text{onto}]{1-1} V$, and consider the definable antichain $\bar{A} = \{p \in A \mid \varphi(f(p), B)\}$ of $\mathbb{P}$. By our assumption, the hyperclass antichain of $\mathbb{B}_{\mathbb{P}}$ consisting of all U_p with $p \in \bar{A}$ has a supremum, call it U. Clearly for every $p \in \bar{A}$, we have $p \in U$. Now we would like to argue that if $q \in A$ but $q \notin \bar{A}$, then q cannot be in U. So suppose that for some $q \in A \setminus \bar{A}$, $q \in U$. Consider the class $W = \{p \in U \mid p \text{ is incompatible to } q \text{ in } \mathbb{P}\}$. Clearly W is a cut because if $p \in W$ and $p' \leq p$, then $p' \in U$ and p' is incompatible to q, which means that $p' \in W$. Also, W is regular because if $p \notin W$, then either $p \notin U$ or $p \in U$ is compatible to q, in which case, we can pick $p' \leq p, q$, and check that $W \cap U_{p'} = \emptyset$. Finally, observe that $p \in W$ for every $p \in \bar{A}$ since it is incompatible to q. So W is a regular cut above all the U_p, which is below U, contradicting that U was the supremum. Now we have that a set x satisfies $\varphi(x, B)$ if and only if $x = f(p)$ for some $p \in \bar{A}$ if and only if $p \in A \cap U$, which is a first-order definition. Thus, the collection of all sets x satisfying $\varphi(x, B)$ is a class. □

We would also like to argue that since $\mathbb{P}$ densely embeds into $\mathbb{B}_{\mathbb{P}}$ all the antichains of $\mathbb{B}_{\mathbb{P}}$ should be "class-sized", meaning that they are coded hyperclasses.

Theorem 3.3. *Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}$ and $\mathbb{P} \in \mathcal{C}$ is a class partial order. Let $\mathbb{B}_{\mathbb{P}}$ be the hyperclass Boolean completion of $\mathbb{P}$. Then every antichain of $\mathbb{B}_{\mathbb{P}}$ is a coded hyperclass.*

Proof. Fix a hyperclass antichain of $\mathbb{B}_{\mathbb{P}}$ given by a (second-order) formula $\varphi(X, A)$. Given a regular cut U such that $\varphi(U, A)$ holds, let p_U be the least element of $\mathbb{P}$ (according to some fixed global well-order) such that $p_U \in U$. Note that if $U \neq W$ are such that $\varphi(U, A)$ and $\varphi(W, A)$ holds, then p_U must be incompatible to p_W, in particular, $p_U \neq p_W$. So using the full comprehension of KM, we can define the class whose slices are indexed by elements of $\mathbb{P}$ such that U sits on the slice indexed by p_U. □

Theorem 3.2 gives that KM is the weakest second-order theory in which hyperclass Boolean completions $\mathbb{B}_{\mathbb{P}}$ of class partial orders $\mathbb{P}$ have all the desired properties.

But even in KM it is not clear how to perform the forcing construction with a hyperclass object. For instance, the forcing names would themselves have to be classes. So our strategy will be to further expand our theory to $\mathrm{KM}+\mathrm{CC}$, and then work in the companion model $M_{\mathscr{V}}$ in which $\mathbb{B}_{\mathbb{P}}$ is a very nice class forcing notion.

4. Boolean-valued class forcing in $\mathrm{KM}+\mathrm{CC}$

Suppose that $\mathscr{V}=\langle V,\in,\mathcal{C}\rangle$ is a model of KM+CC, and let $M_{\mathscr{V}}$ be the companion model of ZFC^-_I with a largest cardinal κ. Let $\mathbb{P}\in\mathcal{C}$ be a class partial order, and let $\mathbb{B}_{\mathbb{P}}$ be the hyperclass Boolean completion of $\mathbb{P}$. Let's now pass to the model $M_{\mathscr{V}}$. In $M_{\mathscr{V}}$, $\mathbb{P}$ is a set and $\mathbb{B}_{\mathbb{P}}$ is a definable Boolean algebra that has the ORD-cc and is class-complete. Since $\mathbb{B}_{\mathbb{P}}$ has the ORD-cc, it is pretame, and therefore forcing with it preserves ZFC^- to the forcing extension (although of course it may not preserve the inaccessibility of κ, a case that would correspond to $\mathbb{P}$ not preserving KM over $\mathscr{V}$) by a theorem of Stanley (see [HKS18] for details).

We can define the collection $M_{\mathscr{V}}^{\mathbb{B}_{\mathbb{P}}}$ of Boolean-valued names as usual by a recursion on name rank (measuring the depth of a $\mathbb{B}_{\mathbb{P}}$-name). The Boolean values of atomic formulas are defined by the usual recursion, which has a solution because it is set-like (since to determine the Boolean value of a formula with names τ and σ we only need to know the Boolean values of formulas with names in the domain of τ and σ).[5]

$$[[\tau\in\sigma]]=\bigvee_{\langle\nu,b\rangle\in\sigma}[[\nu=\tau]]\cdot b$$
$$[[\tau=\sigma]]=[[\tau\subseteq\sigma]]\cdot[[\sigma\subseteq\tau]]$$
$$[[\tau\subseteq\sigma]]=\bigwedge_{\nu\in\mathrm{dom}(\tau)}[[\nu\in\tau]]\to[[\nu\in\sigma]]$$

The Boolean values are extended to all formulas by the usual recursion on formula complexity. Note that we can define the Boolean value of an existential formula by the class completeness of $\mathbb{B}_{\mathbb{P}}$.

$$[[\exists x\varphi(x,\nu)]]=\bigvee_{\tau\in M_{\mathscr{V}}^{\mathbb{B}_{\mathbb{P}}}}[[\varphi(\tau,\nu)]]$$

So we have everything we need to define the Boolean-valued model.

Finally, let's argue that the Boolean-valued model is full.

Proposition 4.1. *The Boolean valued model $M_{\mathscr{V}}^{\mathbb{B}_{\mathbb{P}}}$ is full.*

Proof. Let $b=[[\exists x\varphi(x,\sigma)]]=\bigvee_{\tau\in M_{\mathscr{V}}^{\mathbb{B}_{\mathbb{P}}}}[[\varphi(\tau,\sigma)]]$. Let $D=\{p\in\mathbb{P}\mid\exists\tau\, p\leq[[\varphi(\tau,\sigma)]]\}$. Observe that D is dense below b. So let A be a maximal antichain of D. It is easy to see that $\bigvee A=b$. Now for each $a\in A$, using Collection, we can choose some τ_a such that $a\leq[[\varphi(\tau_a,\sigma)]]$. Let μ be the mixed name such that $a\leq[[\mu=\tau_a]]$ for every $a\in A$. It follows that for each $a\in A$, $a\leq[[\mu=\tau_a]]\cdot[[\varphi(\tau_a,\sigma)]]$, and so $a\leq[[\varphi(\mu,\sigma)]]$. So $b\leq[[\varphi(\mu,\sigma)]]$, and hence $b=[[\varphi(\mu,\sigma)]]$. $\square$

While set partial orders which densely embed always produce the same forcing extensions, this is not necessarily the case in class forcing. In our special case, however, $\mathbb{P}$ and $\mathbb{B}_{\mathbb{P}}$ do produce the same forcing extensions.

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[5] In contrast, the definition of the forcing relation for atomic formulas for a class partial order is given by a recursion which may not be set-like, and therefore the principle $\mathrm{ETR}_{\mathrm{ORD}}$ may be required to prove the existence of a solution.

Theorem 4.2 ([HKS18]). *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are class partial orders such that $\mathbb{P}$ has the* ORD*-cc and $\mathbb{P}$ densely embeds into $\mathbb{Q}$. Then for every $\mathbb{Q}$-name σ, there is a $\mathbb{P}$-name $\bar{\sigma}$ such that $\mathbb{1}_{\mathbb{Q}} \Vdash \sigma = \bar{\sigma}$.*

Next, we would like to determine the relationship between forcing extensions of $\mathscr{V}$ and forcing extensions of $M_{\mathscr{V}}$. To do that, let's first define precisely how our forcing extensions are constructed. Since even a transitive model of KM, if it is wrong about the well-foundedness of its class relations, may have an ill-founded companion model, we will give a general construction of a forcing extension that works for ill-founded models.

Let us say that a class $\Gamma \in \mathcal{C}$ is a *class $\mathbb{P}$-name* if it consists of pairs $\langle \tau, p\rangle$ where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. Suppose $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic, meaning that it meets all dense classes of $\mathbb{P}$ in $\mathcal{C}$. The elements of the first-order part $V[G]$ of the forcing extension are equivalence classes $[\tau]_G$ for $\mathbb{P}$-names $\tau \in V$ of the equivalence relation $\tau \sim \sigma$ whenever there is $p \in G$ with $p \Vdash \tau = \sigma$. The elements of the classes $\mathcal{C}[G]$ of the forcing extension are equivalence classes $[\Gamma]_G$ for class $\mathbb{P}$-names $\Gamma \in \mathcal{C}$ of the equivalence relation $\Gamma \sim \Delta$ whenever there is $p \in G$ with $p \Vdash \Gamma = \Delta$. Define that $[\sigma]_G \in [\tau]_G$ whenever there is $p \in G$ such that $p \Vdash \sigma \in \tau$, and similarly for the membership relation between sets and classes. Note that $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic if and only if it is also $M_{\mathscr{V}}$-generic. So we can analogously define $M_{\mathscr{V}}[G]$ to consist of the equivalence classes $[\tau]_G$ for $\mathbb{P}$-names $\tau \in M_{\mathscr{V}}$.

For notational purposes, given a model $M \models \mathrm{ZFC}^-_I$ with the largest cardinal κ, we will call $V^M_{\kappa+1}$ the collection, which may not be a set in M, of all subsets of V^M_κ in M.

Recall that $M_{\mathscr{V}}[G]$ must be a model of ZFC^-. It is not difficult to see, using that we have the same $\mathbb{P}$-names and the same forcing relation, that $V[G] \subseteq V_\kappa^{M_{\mathscr{V}}[G]}$ and $\mathcal{C}[G] \subseteq \mathscr{V}_{\kappa+1}^{M_{\mathscr{V}}[G]}$ (modulo appropriate isomorphisms). Note that we cannot in general expect even $V[G] = V_\kappa^{M_{\mathscr{V}}[G]}$ because if $\mathbb{P} = \mathrm{Coll}(\alpha, \mathrm{ORD})$ for some $\alpha \in \mathrm{ORD}^V$, so that it becomes $\mathrm{Coll}(\alpha, \kappa)$ in $M_{\mathscr{V}}$, then $M_{\mathscr{V}}[G]$ has a new subset of α, which cannot have a name in $V_\kappa^{M_{\mathscr{V}}}$. In the special case that $\mathbb{P}$ preserves KM+CC over $\mathscr{V}$, we will have that $V[G] \cong V_\kappa^{M_{\mathscr{V}}[G]}$, $\mathcal{C}[G] \cong \mathscr{V}_{\kappa+1}^{M_{\mathscr{V}}[G]}$ and indeed that $M_{\mathscr{V}}[G]$ is precisely the companion model of $\mathscr{V}[G]$.

A class partial order $\mathbb{P}$ preserves KM if and only if $\mathbb{P}$ is tame (see [Ant18] for details). Indeed, tame forcing notions also preserve the Choice Scheme.

Proposition 4.3. *The theories* KM+CC*,* KM+CC+DC_α*, and* KM+CC+$\mathrm{DC}_{\mathrm{ORD}}$ *are all preserved by tame forcing.*

Proof. Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM} + \mathrm{CC}$. Let $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G]\rangle$ be a forcing extension by a class forcing $\mathbb{P}$. Using, the Choice Scheme, we will argue that whenever p is a condition in $\mathbb{P}$ and $p \Vdash \exists X \varphi(X, \dot{A})$, then there is a $\mathbb{P}$-name $\dot{X}$ such that $p \Vdash \varphi(\dot{X}, \dot{A})$. Let D be the dense class of conditions q below p for which there is a class $\mathbb{P}$-name $\dot{X}_q$ such that $q \Vdash \varphi(\dot{X}_q, \dot{A})$. The class D exists by comprehension. Let A be a maximal antichain of D. Now, using the Choice Scheme, we can pick for every $q \in A$, a class $\mathbb{P}$-name $\dot{X}_q$ such that $q \Vdash \varphi(\dot{X}_q, \dot{A})$. After this, we do the usual mixing argument to build the name $\dot{X}$, that is $\dot{X} = \bigcup_{q\in A}\{(\tau, r) \mid r \le q, r \Vdash \tau \in \dot{X}_q, \tau \in \mathrm{dom}(\dot{X}_q)\}$.

Now suppose that the class forcing $\mathbb{P}$ is tame. By tameness, $\mathscr{V}[G]$ is a model of KM. Suppose $\mathscr{V}[G]$ satisfies $\forall\alpha\exists X\varphi(\alpha, X, A)$. So there is a condition $p \in \mathbb{P}$ such that $p \Vdash \forall\alpha\exists X\varphi(\alpha, X, \dot{A})$, where $\dot{A}$ is a class $\mathbb{P}$-name for A. Fix an ordinal α. By the argument above, we can build a class $\mathbb{P}$-name $\dot{X}_\alpha$ such that $p \Vdash \varphi(\check{\alpha}, \dot{X}_\alpha, \dot{A})$. Again using the Choice Scheme, we pick for every α, a class $\mathbb{P}$-name $\dot{X}_\alpha$ and put them all together to form a name for the sequence of choices in $\mathscr{V}[G]$.

Next, suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \text{KM+CC+DC}_\alpha$ and $\mathbb{P} \in \mathcal{C}$ is a tame class forcing. Let $\mathscr{V}[G]$ be a forcing extension by $\mathbb{P}$. Suppose $\mathscr{V}[G]$ satisfies $\forall X\exists Y\varphi(X, Y, A)$. So there is a condition $p \in \mathbb{P}$ such that $p \Vdash \forall X\exists Y\varphi(X, Y, \dot{A})$, where $\dot{A}$ is a class $\mathbb{P}$-name for A. Define a relation $\mathcal{R}$ on the classes of V such that $X\,\mathcal{R}\,Y$ whenever Y is a $\mathbb{P}$-name and if X, viewed as a sequence $\langle X_\xi \mid \xi \in \text{ORD}\rangle$, has a tail consisting of $\mathbb{P}$-names, then $p \Vdash \varphi(\bar{X}, Y, \dot{A})$, where $\bar{X}$ is a class $\mathbb{P}$-name for the sequence given by the tail of X. Let's argue that $\mathcal{R}$ has no terminal nodes. Fix a class X. If the tail of X is not a sequence of $\mathbb{P}$-names, then any $\mathbb{P}$-name Y is $\mathcal{R}$ related to X. Otherwise, by the argument made above, we can build a witnessing $\mathbb{P}$-name Y such that $p \Vdash \varphi(\bar{X}, Y, \dot{A})$. Thus, using the existing level of the Dependent Choice in $\mathscr{V}$, there is a sequence of appropriate length of witnessing names for classes. □

Proposition 4.4. *Suppose $\mathscr{V} \models \text{KM} + \text{CC}$, $\mathbb{P} \in \mathcal{C}$ is a tame partial order and $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic. If $M_{\mathscr{V}}$ is the companion model of $\mathscr{V}$ with the largest cardinal κ, then $V_\kappa^{M_{\mathscr{V}}[G]} \cong V[G]$, $V_{\kappa+1}^{M_{\mathscr{V}}[G]} \cong \mathcal{C}[G]$ and $M_{\mathscr{V}}[G]$ is the companion model of $\mathscr{V}[G]$.*

Proof. First, let's argue that κ remains inaccessible in $M_{\mathscr{V}}[G]$. Observe that every subset of κ in $M_{\mathscr{V}}[G]$ has a $\mathbb{P}$-name in $V_{\kappa+1}^{M_{\mathscr{V}}}$, and hence in $\mathcal{C}$. Thus, by tameness of $\mathbb{P}$ in $\mathscr{V}$, $M_{\mathscr{V}}[G]$ cannot have a cofinal $f : \alpha \to \kappa$ for $\alpha < \kappa$. Also, fixing $\alpha < \kappa$, $P(\alpha)$ must exist in $V[G]$, and hence, since $M_{\mathscr{V}}[G]$ cannot have any additional subsets of α, $P(\alpha)^{M_{\mathscr{V}}[G]} \in V_\kappa^{M_{\mathscr{V}}[G]}$. In particular, it follows that every element of $V_{\kappa+1}^{M_{\mathscr{V}}[G]}$ can be coded by a subset of κ. This immediately gives that $\mathcal{C}[G] \cong V_{\kappa+1}^{M_{\mathscr{V}}[G]}$. To see that $V_\kappa^{M_{\mathscr{V}}[G]} = V[G]$, suppose that $a \in V_\kappa^{M_{\mathscr{V}}[G]}$. By inaccessibility of κ, we can assume without loss that a is a subset of α for some $\alpha < \kappa$, and hence a has a name in $\mathcal{C}$. But since $\mathscr{V}[G] \models \text{KM}$, a must already have a name in V. Finally, to see that $M_{\mathscr{V}}[G]$ is the companion model of $\mathscr{V}[G]$, suppose that a is any element of $M_{\mathscr{V}}[G]$. Since κ is the largest cardinal of $M_{\mathscr{V}}[G]$, there must be $A \subseteq \kappa$ coding a. But then $A \in \mathcal{C}[G]$, and therefore $a \in M_{\mathscr{V}[G]}$.

□

5. Intermediate Model Theorem

Recall from the introduction that the Intermediate Model Theorem states that if a universe $V \models \text{ZFC}$ and $W \models \text{ZFC}$ is an intermediate model between V and one of its set-forcing extensions, then W is itself a forcing extension of V. Indeed, if a partial order $\mathbb{P} \in V$ densely embeds into a complete Boolean algebra $\mathbb{B}$, then every intermediate model W between V and its forcing extension $V[G]$ by $\mathbb{B}$ has the form $V[\mathbb{D} \cap G]$ for some complete subalgebra $\mathbb{D}$ of $\mathbb{B}$ from V. The ZF-version of the Intermediate Model Theorem states that if $V \models \text{ZF}$ and $V[a]$, with $a \subseteq V$, is an intermediate model between V and one of its set-forcing extensions, then $V[a]$ is itself a set-forcing extension of V.

We would like to formulate and consider the statement of the Intermediate Model Theorem in the context of class forcing. We start by giving a precise definition of the notion of a simple extension of a model $\mathscr{V}$ of a second-order set theory, which is generated by the classes of $\mathscr{V}$ together with a single new class. So suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$. We say that $\mathscr{W} = \langle W, \in, \mathcal{C}^*\rangle$ is a *simple* extension of $\mathscr{V}$ if there is a class $A \in \mathcal{C}^*$ with $A \subseteq V$ such that $W = V[A]$, namely the structure consisting of the union of $L^V_\alpha(x, A \cap x)$ over all $\alpha \in \mathrm{ORD}^V$ and $x \in V$, and $\mathcal{C}^*$ consists precisely of the classes first-order definable over W from $\mathcal{C} \cup \{A\}$. In particular, of course, every forcing extension is a simple extension.

Given a second-order set theory T, we will say that the *Intermediate Model Theorem for* T holds if whenever $\mathscr{V} \models T$ and $\mathscr{W} \models T$ is an intermediate model between $\mathscr{V}$ and one of its class forcing extensions $\mathscr{V}[G] \models T$, then $\mathscr{W}$ is itself a forcing extension of $\mathscr{V}$. Note that the Intermediate Model Theorem for T implies, in particular, that every intermediate model between a model $\mathscr{V} \models T$ and its forcing extension satisfying T is a simple extension of $\mathscr{V}$. We will say that the *simple Intermediate Model Theorem for* T holds if the Intermediate Model Theorem for T holds for all simple intermediate models.

The second author showed in [Fri99] that the simple Intermediate Model Theorem for GBC can fail in a very strong way. There is a model $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$ of a very strong second-order set theory, at least $\mathrm{KM} + \mathrm{CC} + \mathrm{DC}_{\mathrm{ORD}}$, and a class B in the forcing extension $\mathscr{V}[G]$ such that $\langle V[B], A, B\rangle$ is not a forcing extension of $\langle V[A], A\rangle$ for any class $A \in \mathcal{C}$. Hamkins and Reitz showed that there is a class B in an ORD-cc forcing extension $V[G]$ such that $\langle V[B], B\rangle$ is not a forcing extension of $\langle V, \emptyset\rangle$ [HR17], and thus the simple Intermediate Model Theorem for GBC can fail even where the forcing has the ORD-cc. Whether the result of [Fri99] can be done with an ORD-cc forcing remains open.

We will show that the simple Intermediate Model Theorem for $\mathrm{KM} + \mathrm{CC}$ holds, but the full Intermediate Model Theorem fails.

Theorem 5.1. *The simple Intermediate Model Theorem for* $\mathrm{KM} + \mathrm{CC}$ *holds.*

Proof. Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM} + \mathrm{CC}$, $\mathbb{P} \in \mathcal{C}$ is a class partial order and $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic. Let $\mathscr{W} = \langle W, \in, \mathcal{C}^*\rangle \models \mathrm{KM} + \mathrm{CC}$ be a simple intermediate model between $\mathscr{V}$ and $\mathscr{V}[G]$, so that $\mathcal{C}^*$ is generated by $\mathcal{C}$ together with a class $A \in C^*$. It should be clear that $M_{\mathscr{V}} \subseteq M_{\mathscr{W}} \subseteq M_{\mathscr{V}[G]}$ is the relationship between the companion models. Observe also that $M_{\mathscr{W}} = M_{\mathscr{V}}[A]$, where we view A as being a set in $M_{\mathscr{V}}$. By the Intermediate Model Theorem for models of ZFC^- Theorem 5.2, proved below, we have that $M_{\mathscr{W}} = M_{\mathscr{V}}[H]$ is a forcing extension of $M_{\mathscr{V}}$ by a set partial order, call it $\mathbb{Q} \in M_{\mathscr{V}}$. We can assume without loss that $\mathbb{Q} \subseteq V_\kappa^{M_{\mathscr{V}}}$, so that we can think of it as an element of $\mathcal{C}$. But then by Proposition 4.4, $M_{\mathscr{V}[H]} \cong M_{\mathscr{W}}$, which means, by the bi-interpretability, that $\mathscr{W} = \mathscr{V}[H]$. □

Theorem 5.2. *Suppose* $M \models \mathrm{ZFC}^-$, $\mathbb{P} \in M$ *is a partial order, and* $G \subseteq \mathbb{P}$ *is* M-*generic. If* $a \in M[G]$ *with* $a \subseteq M$, *and* $M[a] \models \mathrm{ZFC}^-$, *then* $M[a]$ *is a set-forcing extension of* M.

Proof. We will assume that the powerset of $\mathbb{P}$ does not exist in M because the other case is even easier. By the arguments of Section 4, we can embed $\mathbb{P}$ into a definable class complete ORD-cc Boolean algebra $\mathbb{B}_{\mathbb{P}}$, for which we can define the Boolean-valued model.

Since $M[a] \models \mathrm{ZFC}^-$, there is some ordinal $\alpha \in M$ such that we can recover a from the Mostowski collapse of a subset $\bar{a}$ of α. It follows that $M[a] = M[\bar{a}]$, and so we can assume without loss that $a \subseteq \alpha$ for some ordinal α. Let $\dot{a}$ be a $\mathbb{P}$-name for a

such that $\mathbb{1}_{\mathbb{P}} \Vdash \dot{a} \subseteq \alpha$. Let $X \in M$ be the set of all Boolean values $b = [[\check{\xi} \in \dot{a}]] \in \mathbb{B}_{\mathbb{P}}$ for $\xi < \alpha$.

Now we will explain how X can be used to generate a complete Boolean subalgebra $\mathbb{D}$ of $\mathbb{B}_{\mathbb{P}}$. Let us say that a well-founded tree $T \in M$ of elements of $\mathbb{B}_{\mathbb{P}}$ is an *X-tree* if the leaves of T are elements of X obeying the following rules. If an element $b \in T$ has a single successor, then it is $-b$, if it has multiple successors, then it is the join of them. Now let $\mathbb{D}$ consist of all $b \in \mathbb{B}_{\mathbb{P}}$ such that there is an X-tree with b as the root. It is easy to see that $\mathbb{D}$ is a complete subalgebra of $\mathbb{B}_{\mathbb{P}}$.

Let's first argue that $M[\mathbb{D} \cap G] \subseteq M[a]$. Suppose $y \in M[\mathbb{D} \cap G]$. Let $y = \tau_G$ for a $\mathbb{D}$-name τ. So there is a set $s \in M$ of elements of $\mathbb{D}$ such that we can construct y from τ together with $s \cap G$ (namely all the elements $b \in \mathbb{D}$ appearing hereditarily in τ). Since $\mathbb{D}$ is generated by X, there is a set $\bar{s} \in M$ with $\bar{s} \subseteq X$ such that we can compute $s \cap G$ from $\bar{s} \cap G$. Now observe that $b = [[\check{\xi} \in \dot{a}]]$ is in $\bar{s} \cap G$ if and only if $\xi \in a$. Thus, $\bar{s} \cap G$ is in $M[a]$, and hence $y \in M[a]$ as well.

To see that $M[a] \subseteq M[\mathbb{D} \cap G]$, it suffices to show that $a \in M[\mathbb{D} \cap G]$, but this is straightforward because $\xi \in a$ if and only if $[[\check{\xi} \in \dot{a}]] \in X \cap G$. Note that $X \cap G$ is an element of $M[\mathbb{D} \cap G]$ since $\langle M[\mathbb{D} \cap G], G\rangle \models \mathrm{ZFC}^-$ by the pretameness of $\mathbb{B}_{\mathbb{P}}$.

Finally, note that since the class $\mathbb{B}_{\mathbb{P}}$ has the dense subset $\mathbb{P}$, any complete subalgebra $\mathbb{D}$ of $\mathbb{B}_{\mathbb{P}}$ also has a dense subset consisting of the infima $\bigwedge\{d \in \mathbb{D} \mid p \leq d\}$ for $p \in \mathbb{P}$. So $M[\mathbb{D} \cap G]$ is actually a set-forcing extension. $\square$

Basically, the same argument gives the following stronger version of the Intermediate Model Theorem for models of ZFC^-.

Theorem 5.3. *Suppose $M \models \mathrm{ZFC}^-$, $\mathbb{B}$ is an ORD-cc Boolean algebra definable in M, and $G \subseteq \mathbb{B}$ is M-generic. If $A \subseteq M$ is definable in the structure $\langle M[G], \in, G\rangle$ and $M[A] \models \mathrm{ZFC}^-$, then $M[A] = M[\mathbb{D} \cap G]$ is a forcing extension of M by a complete subalgebra $\mathbb{D}$ of $\mathbb{B}$ definable in M.*

Proof. Suppose that A is defined by the formula $\varphi(x, a, G)$ in $\langle M[G], \in, G\rangle$. Let $\dot{a}$ be a name for a. Let X be the class in M consisting of all Boolean values $[[\varphi(\check{x}, \dot{a}, \dot{G})]]$ for $x \in M$. To argue that $M[A] \subseteq M[\mathbb{D} \cap G]$, we show that for every $y \in M$, we have $y \cap A \in M[\mathbb{D} \cap G]$. $\square$

Corollary 5.4. *Suppose $M \models \mathrm{ZFC}^-$, $\mathbb{B}$ is an ORD-cc Boolean algebra definable in M and $G \subseteq \mathbb{B}$ is M-generic. If $M \subseteq N \subseteq M[G]$ is an intermediate model of ZFC^- with a definable global well-order such that M is definable in N and N is definable in $\langle M[G], \in, G\rangle$, then $N = M[\mathbb{D} \cap G]$ is a forcing extension of M by a complete subalgebra $\mathbb{D}$ of $\mathbb{B}$ in M.*

Proof. Using the definable global well-order and that M is definable, we can argue that N has a definable $A \subseteq \mathrm{ORD}$ coding all its subsets of $\mathbb{B}$. Observe that $N = M[A]$ since obviously $M[A] \subseteq N$ and every $y \in N$ has the form τ_G, and so can be constructed from τ and a subset of $\mathbb{B}$ coded in A. Also, A is definable in $\langle M[G], \in, G\rangle$ because N is definable. $\square$

Note that if $\mathbb{B} = \mathbb{B}_{\mathbb{P}}$ is a Boolean completion of a set partial order $\mathbb{P}$, then we don't need the assumption that M is definable in N. In this case, we can let $\mathbb{B}^*$ be the Boolean completion of $\mathbb{P}$ in N and observe that $\mathbb{B} \subseteq \mathbb{B}^*$. So if we code all subsets of $\mathbb{B}^*$, we will in particular code all subsets of $\mathbb{B}$.

Translated back to models of KM + CC via companion models, Corollary 5.4 gives the following sufficient conditions for an intermediate model to be a simple extension.

Corollary 5.5. *Suppose an intermediate model $\mathscr{W} \models \mathrm{KM} + \mathrm{CC}$ between a model $\mathscr{V} \models \mathrm{KM} + \mathrm{CC}$ and its forcing extension $\mathscr{V}[G]$ has a definable global well-ordering of classes, and we have additionally that $\mathscr{W}$ is definable in $\mathscr{V}[G]$. Then $\mathscr{W}$ must be a simple extension of $\mathscr{V}$*

Next, we show that the full Intermediate Model Theorem for KM + CC, and indeed for $\mathrm{KM} + \mathrm{CC} + \mathrm{DC}_{\mathrm{ORD}}$, fails in a very strong sense.

Theorem 5.6. *Every model $\mathscr{V} \models \mathrm{KM} + \mathrm{CC}$ has a forcing extension $\mathscr{V}[G]$ with a non-simple intermediate model $\mathscr{W} \models \mathrm{KM} + \mathrm{CC}$, and if additionally $\mathscr{V} \models \mathrm{DC}_{\mathrm{ORD}}$, then we can have $\mathscr{W} \models \mathrm{DC}_{\mathrm{ORD}}$ as well.*

Proof. Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM} + \mathrm{CC}$. Let $\mathbb{P} = \Pi_{n<\omega} \mathrm{Add}(\mathrm{ORD}, 1)$ be the finite-support ω-product of $\mathrm{Add}(\mathrm{ORD}, 1)$, the forcing to add a Cohen subclass to ORD. Clearly $\mathbb{P} \in \mathcal{C}$. Let $\mathbb{P}_n$ be the restriction of $\mathbb{P}$ to the first n-many coordinates of the product. Observe that each $\mathbb{P}_n$ is $<\mathrm{ORD}$-closed, and therefore does not add any sets. Suppose $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic, and let G_n be the restriction of G to $\mathbb{P}_n$. Also, let $\mathbb{P}^{(n)}_{\mathrm{tail}}$ denote the tail forcing after $\mathbb{P}_n$. Let $\mathcal{C}_n$ be the classes of $\mathscr{V}[G_n]$, so that $\mathscr{V}[G_n] = \langle V, \in, \mathcal{C}_n\rangle$.

We will now argue that $\mathscr{W} = \langle V, \in, \bigcup_{n<\omega} \mathcal{C}_n\rangle$ is a model of KM + CC. Observe that the model $\mathscr{W}$ is definable in $\mathscr{V}[G]$ and that the same definition over $\mathscr{V}[G_n]$ for $\mathbb{P}^{(n)}_{\mathrm{tail}}$ also gives $\mathscr{W}$. Replacement holds in $\mathscr{W}$ because it holds in each $\mathscr{V}[G_n]$.

Next, let's verify comprehension in $\mathscr{W}$. Suppose $\varphi(x, A)$ is a second-order assertion with a class parameter $A \in \mathcal{C}_n$. We will argue that the collection $C = \{x \mid \mathscr{W} \models \varphi(x, A)\}$ belongs to $\mathcal{C}_n$. If $\mathscr{W} \models \varphi(x, A)$, then it must be forced by $\mathbb{1}_{\mathbb{P}^{(n)}_{\mathrm{tail}}}$ that $\varphi^{\mathscr{W}}(\check{x}, \check{A})$ holds because $\mathbb{P}^{(n)}_{\mathrm{tail}}$ has for any two condition p and q an automorphism π such that $\mathscr{W}$ is invariant under π and $\pi(p)$ is compatible to q. Thus, we can define C in $\mathscr{V}[G_n]$ as the collection of all those x such that $\mathbb{1}_{\mathbb{P}^{(n)}_{\mathrm{tail}}} \Vdash \varphi^{\mathscr{W}}(\check{x}, \check{A})$.

Next, let's verify that the Choice Scheme holds in $\mathscr{W}$. Suppose $\forall\alpha\exists X\varphi(\alpha, X, A)$ holds in $\mathscr{W}$ with a class parameter $A \in \mathcal{C}_n$. Fix α. First, we will argue that there is X in $\mathscr{V}[G_{n+1}]$ such that $\mathscr{W} \models \varphi(\alpha, X, A)$ holds. By assumption, there is X in some $\mathcal{C}_m$ such that $\mathscr{W} \models \varphi(\alpha, X, A)$. Let $\dot{X}$ be a $\mathbb{P}^{(n)}_{\mathrm{tail}}$-name for X and let p be some condition forcing $\varphi^{\mathscr{W}}(\check{\alpha}, \dot{X}, \check{A})$. Let π be an automorphism of $\mathbb{P}^{(n)}_{\mathrm{tail}}$ which combines the coordinates up to m into a single coordinate in such a way that $\pi(p) \in G$. Note that the definition of $\mathscr{W}$ is invariant under π. Conditions in the new name $\pi(\dot{X})$ reference only the first coordinate of $\mathbb{P}^{(n)}_{\mathrm{tail}}$ and $\pi(p) \Vdash_{\mathbb{P}^{(n)}_{\mathrm{tail}}} \varphi^{\mathscr{W}}(\check{\alpha}, \pi(\dot{X}), \check{A})$. Thus, $\mathscr{W} \models \varphi(\alpha, \pi(\dot{X})_G, A)$ with $\pi(\dot{X})_G \in \mathcal{C}_{n+1}$. Now we move to $\mathscr{V}[G_{n+1}]$, where we have just shown that for every α, there is a class X such that $\mathbb{1}_{\mathbb{P}^{(n+1)}_{\mathrm{tail}}} \Vdash \varphi^{\mathscr{W}}(\check{\alpha}, \check{X}, \check{A})$. Since $\mathbb{P}_{n+1}$ is $<\mathrm{ORD}$-closed, it is in particular tame [Fri00], and therefore $\mathscr{V}[G_{n+1}]$ satisfies KM + CC by Proposition 4.3. Thus, $\mathscr{V}[G_{n+1}]$ can collect the witnesses into a single class.

It remains to show that if $\mathrm{DC}_{\mathrm{ORD}}$ holds in $\mathscr{V}$, then it also holds in $\mathscr{W}$. So suppose that $\mathscr{W} \models \forall X\exists Y\varphi(X, Y, A)$ with a class parameter $A \in \mathcal{C}_n$. Let's consider the extension $\mathscr{V}[G_{n+1}] = \mathscr{V}[G_n][g]$ with $G_{n+1} = G_n * g$. For the moment, we will view $\mathscr{V}[G_{n+1}]$ as an extension of $\mathscr{V}[G_n]$ by the bounded-support ORD-length product of $\mathrm{Add}(\mathrm{ORD},1)$, with g_α being the restriction of g to the first α-many coordinates of the product. Now suppose that $X \in \mathscr{V}[G_{n+1}]$ is an element of some $\mathscr{V}[G_n][g_\alpha]$. Let $\dot{X}$ be a $\mathbb{P}^{(n)}_{\text{tail}}$-name for X with conditions referencing only the first coordinate of the product and leaving unboundedly much space in that coordinate (because g_α is bounded). By assumption, there is some $m > n$ and $Y \in \mathcal{C}_m$ such that $\mathscr{W} \models \varphi(X, Y, A)$. Let $\dot{Y}$ be a $\mathbb{P}^{(n)}_{\text{tail}}$-name for Y, and let $p \in \mathbb{P}^{(n)}_{\text{tail}}$ force that $\varphi^{\mathscr{W}}(\dot{X}, \dot{Y})$ holds. Using a coordinate-combining automorphism as above, to move conditions used in $\dot{Y}$ to conditions on the space unused by $\dot{X}$, we can in fact assume that $\dot{Y}$ references only conditions in $\mathbb{P}_{n+1}$. So $\mathscr{V}[G_{n+1}]$ satisfies that for every X in $\mathscr{V}[G_n][g_\alpha]$, there is Y such that $\mathbb{1}_{\mathbb{P}^{(n)}_{\text{tail}}} \Vdash \varphi^{\mathscr{W}}(\check{X}, \check{Y}, \check{A})$. Consider now a new relation $\psi(X, Y, A)$ which holds of pairs X and Y whenever Y belongs to $V[G_n][g_\alpha]$ for some α and if X, viewed as a sequence $\langle X_\xi \mid \xi \in \mathrm{ORD}\rangle$, has a tail $X_{\text{tail}} \in V[G_n][g_\alpha]$ for some α, then $\mathbb{1}_{\mathbb{P}^{(n)}_{\text{tail}}} \Vdash \varphi^{\mathscr{W}}(\check{X}_{\text{tail}}, \check{Y}, \check{A})$. Now observe that any sequence of ORD-many dependent choices along ψ gives ORD-many dependent choices along φ by just ignoring the first element.

Finally, observe that $\mathscr{W}$ cannot be a simple extension of $\mathscr{V}$ because no single class G_m suffices to generate all the remaining G_n. $\square$

For the theory KM, it is consistent that there are non-simple extensions even between a model and its forcing extension by ORD-cc forcing.

Theorem 5.7. *There is a model $\mathscr{V} \models \mathrm{KM} + \mathrm{CC}$ and an ORD-cc forcing extension $\mathscr{V}[G]$ of $\mathscr{V}$ with an intermediate model $\mathscr{W}$ such that $\mathscr{W} \models \mathrm{KM}$ but not the Choice Scheme and is not a simple extension of $\mathscr{V}$.*

Proof. The argument uses a construction of Gitman and Hamkins from [GHJ], which we will briefly review here. Suppose V is a model of ZFC and κ is inaccessible in V. We can force to add an ω-sequence $\langle T_n \mid n < \omega\rangle$ of κ-Souslin trees with the following properties. Each tree T_n is homogeneous. The full-support product forcing $\Pi_{n<\omega}T_n$ has the κ-cc and it is $<\kappa$-distributive. In particular, forcing with the product $\Pi_{n<\omega}T_n$ preserves the inaccessibility of κ. Forcing with any initial segment $\Pi_{n<m}T_n$ of the product does not add branches to any T_k with $k \geq m$. So by passing to a forcing extension if necessary we can assume that the sequence $\langle T_n \mid n < \omega\rangle$ already exists in V. Since κ is inaccessible, $\mathscr{M} = \langle V_\kappa, \in, V_{\kappa+1}\rangle$ is a model of KM. Each T_n is a class of $\mathscr{M}$ and so is the full-support product $\Pi_{n<\omega}T_n$.

Let $V[G]$ be a forcing extension by the full-support product $\Pi_{n<\omega}T_n$. Clearly $\langle V_\kappa^{V[G]}, \in, V_{\kappa+1}^{V[G]}\rangle = \mathscr{M}[G]$. Now let N be the symmetric submodel of $V[G]$ determined by the group $\mathcal{G}$ of coordinate-respecting automorphisms and the filter $\mathcal{F}$ on the subgroups of $\mathcal{G}$ generated by the subgroups H_n fixing the first n-many coordinates. Thus, we have that a name τ is symmetric if it is fixed by all elements of some subgroup H_n. The elements of N are interpretations τ_G of hereditarily symmetric names τ. Using the homogeneity of T_n, it can be shown that a set of ordinals of $V[G]$ is in N if and only if it is added by some initial segment $\Pi_{n<m}T_n$ of the product forcing. Thus, while every T_n has a branch in N, the model N does not have a sequence collecting a branch from every T_n because no such sequence can be added by an initial segment of the product forcing. Let $\mathscr{N} = \langle V_\kappa^N, \in, V_{\kappa+1}^N\rangle$, which is a model of KM, but as we just argued cannot be a model of $\mathrm{KM} + \mathrm{CC}$ because every tree T_n has a branch but the model cannot collect them. $\square$

We do not know whether a model $\mathscr{V} \models \mathrm{KM} + \mathrm{CC}$ and one of its ORD-cc forcing extensions $\mathscr{V}[G]$ can have non-simple intermediate models of $\mathrm{KM} + \mathrm{CC}$. Does the Intermediate Model Theorem for $\mathrm{KM} + \mathrm{CC}$ hold for ORD-cc forcing extensions?

Finally, let's observe that the Intermediate Model Theorem holds for KM with the existence of a canonical global well-order. First, note that tame forcing extensions preserve the existence of a canonical global well-order because if a companion model $M_{\mathscr{V}}$ of $\mathscr{V} \models \mathrm{KM}$ has the form $L[A]$ for some $A \subseteq \kappa$, then its forcing extension $M_{\mathscr{V}}[G]$ by some $\mathbb{P} \subseteq V_\kappa$ has the form $L[\bar{A}]$ where $\bar{A}$ codes A together with G. The Intermediate Model Theorem now follows by Theorem 5.2 because if an intermediate model between $M_{\mathscr{V}}$ and $M_{\mathscr{V}}[G]$ arises as the companion model of an intermediate model of $\mathscr{V}$ and $\mathscr{V}[G]$ having a canonical global well-order, then it must have the form $L[B]$ for some $B \subseteq \kappa$.

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