MODERN CLASS FORCING

CAROLIN ANTOS AND VICTORIA GITMAN

Abstract. We survey recent developments in the theory of class forcing formalized in the second-order set-theoretic setting.

1. Introduction

Set theorists started forcing with class partial orders to modify global properties of the universe very soon after general forcing techniques were developed. Once Cohen showed that CH could fail, and that indeed the continuum could assume any reasonable value, it was natural to ask whether, for instance, the GCH can fail unboundedly often, or more generally what global patterns were possible for the continuum function. Since a set partial order can only affect a set-sized chunk of the continuum function, a class partial order was required to modify the GCH pattern unboundedly. Easton used a class product of set-forcing notions, with what became known as Easton support, to show that the GCH can fail at every regular cardinal, and that, in fact, any reasonable pattern on the continuum function was consistent with ZFC [Eas70]. Since then class products and iterations, usually with Easton support, have been used, for example, to globally kill large cardinals, to make all supercompact cardinals indestructible, or to code the universe into the continuum function.

There are two approaches to handling class forcing in first-order set theory. We can use a generic filter G for a class forcing notion $\mathbb{P}$ to interpret the $\mathbb{P}$-names and obtain the forcing extension $V[G]$, and then throw G away. We can also work in the structure $(V[G], \in, G)$ with a predicate for G. While both approaches were used by set theorists to handle specific instances of class forcing, neither approach could provide a robust framework for the study of general properties of class forcing because it relegated properties of classes to the meta-theory.

Important early results on class forcing, due to Friedman and Stanley, showed that class partial orders satisfying 'niceness' conditions, such as pretameness, behave like set partial orders. But despite the pervasive use of class forcing in set theoretic constructions, a general theory of class forcing was not fully developed until recently. These recent results demonstrated that properties of class forcing depend on what other classes exist around them, establishing that the theory of class forcing can only be properly investigated in a second-order set theoretic setting. The mathematical framework of second-order set theory has objects for both sets and classes, and allows us to move the study of classes out of the meta-theory.

Class forcing becomes even more important in the context of second-order set theory, where it can be used to modify the structure of classes. With class forcing, we can, for instances, add a global well-order or shoot class clubs with desirable properties. Both these forcing notions leave the first-order part of the model intact because they do not add sets. Such forcing constructions over models of second-order set theory can actually be used to prove theorems about models of ZFC. One of the most important results in the model theory of set theory is the Keisler-Morley extension theorem asserting that every countable model of ZFC has an elementary top-extension (adding only sets of rank above the ordinals of the model) [KM68]. Given a countable model M of ZFC, the top-extension is built as an ultrapower of M by a special ultrafilter, which we can construct provided that M can be expanded to a model of GBC. If M doesn't already have a definable global well-order, we use class forcing to add it without altering M itself.

In many ways class forcing does not behave as nicely as set forcing. Class forcing can destroy ZFC; even the atomic forcing relation may fail to be definable for a class forcing notion; densely embedded class forcing notions need not be forcing equivalent; most class forcing notions don't have Boolean completions; the intermediate model theorem fails badly for class forcing, etc. At the same time, conditions on class forcing, such as pretameness, guarantee that these pathologies are avoided, and these conditions hold for most familiar class forcing notions used by set theorists, such as progressively closed Easton support products and iterations.

In this article, we survey most of the recent results developing a general theory of class forcing in the second-order setting and establishing surprising connections between properties of class forcing and the structure of second-order set theories.

2. Second-order set theories

The most natural setting for investigating class forcing is second-order set theory, where classes are not restricted to definable collections. Second-order set theory is formalized in a two-sorted logic with separate variables and quantifiers for sets and classes. Following standard convention, we will use lower-case letters for set variables and upper-case letters for class variables. Models of second-order set theory are triples $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$, where V consists of the sets, $\mathcal{C}$ consists of the classes, and $\in$ is the membership relation between sets and between sets and classes. Each element of $\mathcal{C}$ is viewed as a sub-collection of V consisting of those elements which are $\in$-related to it. The sets with their membership relation comprise the first-order part of the model and the classes form the second-order part. We will say that a formula is first-order, or Σ^1_0, if it has only set quantifiers, and that a formula is Σ^1_n or Π^1_n if it has n-alternations of second-order quantifiers.

Axiomatizations of second-order set theory have axioms for sets, axioms for classes, and axioms for how the two types of objects interact. Let ZF^- be the theory ZF without the powerset axiom and with collection instead of replacement (see [GHJ16] for the significance of this), and let ZFC^- be ZF^- together with the axiom of choice. Let GB^- be the second-order theory whose set axioms consist of ZF^- and class axioms consist of: (1) extensionality for classes, (2) the class replacement axiom asserting that every class function restricted to a set is a set, and (3) a weak comprehension scheme asserting that every definable collection is a class. More precisely, the weak comprehension scheme consists of assertions for every first-order formula $\varphi(x, A)$, with a class parameter A, that there is a class $C = \{x \mid \varphi(x, A)\}$. We will call GBc^- the theory we get when we replace the set axioms in GB^- by ZFC^-, we will call GB the theory we get when we replace the set axioms in GB^- by ZF, and finally if we replace the set axioms by ZFC, we will call the resulting theory GBc. Clearly, a model of each of the first-order theories ZF^-, ZFC^-, ZF, or ZFC together with all its definable (with parameters) collections is a model of the corresponding second-order theory. If together with GBc, we additionally require that (4) there is a global well-order - a class well-order of the sets - then we get the familiar Gödel-Bernays set theory GBC. Since, (if ZFC is consistent) there are models of ZFC without a definable global well-order, not every model of ZFC together with just its definable collections is a model of GBC. However, we can force with the partial order $\mathrm{Add}(\mathrm{Ord}, 1)$ over any model of GBc to add a global well-order class without adding sets and while preserving GBc (see Section 3 for details). This means, in particular, that GBC is *conservative* over ZFC, anything we can prove about sets in GBC, can already be proven in ZFC.

We can get stronger theories by increasing the amount of available comprehension. Let Σ^1_n-CA denote the comprehension scheme for all Σ^1_n-formulas (this also gives comprehension for all Π^1_n-formulas). The theory $\mathrm{GBC} + \Sigma^1_1$-CA is already much stronger than GBC. To explain where some of this strength comes from and its relationship to class forcing, we need the important second-order theoretic notion of meta-ordinals.

A *meta-ordinal* is simply a class well-order such as Ord, $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \times \omega$, etc. In sufficiently strong second-order set theories such as $\mathrm{GBC}+\Sigma^1_1$-CA, meta-ordinals behave very much like ordinals: any two meta-ordinals are comparable, we can perform transfinite recursions along meta-ordinals, pretame forcing (see Section 3 for definition) does not add meta-ordinals (the last is a result of [HW18]). The form of transfinite recursion implied by $\mathrm{GBC}+\Sigma^1_1$-CA is the principle *elementary transfinite recursion*, ETR, which asserts that every first-order definable recursion on classes along a meta-ordinal has a solution. A solution to such a recursion would have to be a class function from sets to classes, but this is easily possible. Given a class C, for every set a, C naturally codes a class associated to a, namely the a-th slice $C_a = \{x \mid \langle a, x\rangle \in C\}$. The formal statement of ETR is that given a first-order formula $\varphi(x, Y, A)$, with a class parameter A, and a meta-ordinal $(R, <)$, there is a class S (the solution to the recursion φ) such that for every $r \in R$,

$$S_r = \{x \mid \varphi(x, S \restriction r, A)\},$$

where $S \restriction r$ is our shorthand for the class $\{\langle s, x\rangle \in S \mid s < r\}$. We can stratify ETR by permissible recursion length, so that for example, ETR_ω allows only recursions of length ω and $\mathrm{ETR}_{\mathrm{Ord}}$ allows recursions of length at most Ord. Allowing recursions of longer length gives a properly stronger principle [Wil18]. Over GBC, many consequences of Σ^1_1-CA are already consequences of ETR or its fragments. It is not difficult to see that in $\mathrm{GBC} + \mathrm{ETR}$ any two meta-ordinals are comparable (it is not known whether meta-ordinal comparability holds in the absence of ETR). $\mathrm{GBC} + \mathrm{ETR}_\omega$ implies that there is a truth predicate for the first-order part of the model $(V, \in)$ because we can define the truth predicate using Tarskian recursive definition of truth, which is a recursion on classes of length ω. Thus, in particular, $\mathrm{GBC} + \mathrm{ETR}$ implies Con(ZFC), Con(Con(ZFC)), and much more. Remarkably, over GBC, the principle $\mathrm{ETR}_{\mathrm{Ord}}$ is equivalent to the class forcing theorem, the assertion that every class forcing notion has definable forcing relations (see Section 4) [GHH+]. On the other hand, it is not known whether $\mathrm{GBC} + \mathrm{ETR}$ implies that pretame forcing, or, in fact, even set forcing, does not add meta-ordinals.

The theories $\mathrm{GBC} + \Sigma^1_n$-CA, for $n < \omega$, form a proper hierarchy by strength, culminating in Kelley-Morse set theory KM, which consists of GBC together with comprehension for all second-order assertions. Natural models of Kelley-Morse are rank initial segments $\mathscr{V} = (V_\kappa, \in, V_{\kappa+1})$ of a model of ZFC with an inaccessible cardinal κ.

We can strengthen Kelley-Morse further by adding a certain very useful choice principle for classes. Let the *class choice principle* CC consist of assertions for every second-order formula $\varphi(x, X, A)$, with class parameter A, that if for every set x, there is a class X witnessing $\varphi(x, X, A)$, then there is a single class Z choosing a class witness for every set x on its x-th slice so that $\varphi(x, Z_x, A)$ holds. The principle CC, even when restricted to first-order assertions, is independent of KM [GH], but the two theories KM and KM+CC are equiconsistent because the appropriately defined constructible universe of a model of KM satisfies KM+CC (see [Wil18] for details). Also, the natural models $\mathscr{V} = (V_\kappa, \in, V_{\kappa+1})$ for an inaccessible κ are satisfy CC. The theory KM + CC is bi-interpretable with a familiar first-order theory, which we call here ZFC^-_I [Mar73]. The theory ZFC^-_I consists of ZFC^- together with the assertion that there is a largest cardinal κ which is inaccessible, meaning that $P(\alpha)$ exists for every $\alpha < \kappa$ and has size less than κ (this implies, in particular, that V_α exists for every $\alpha \leq \kappa$). Natural models of ZFC^-_I are H_{κ^+}, all sets with transitive closure of size less than κ, for an inaccessible cardinal κ. A model $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{KM} + \mathrm{CC}$ can build a model of ZFC^-_I, which we will call its companion model $M_{\mathscr{V}}$, by defining the obvious equivalence relation and membership relation on all its extensional well-founded class relations. The largest cardinal of the companion model $M_{\mathscr{V}}$ is $\kappa = \mathrm{Ord}^V$, we have $V_\kappa^{M_{\mathscr{V}}} = V$ and $\{A \in M_{\mathscr{V}} \mid A \subseteq V_\kappa^{M_{\mathscr{V}}}\} = \mathcal{C}$ (all this obviously modulo isomorphism). In the other direction, given a model $M \models \mathrm{ZFC}^-_I$ with largest cardinal κ, the structure $\mathscr{V} = (V_\kappa^M, \in, \mathcal{C})$, where $\mathcal{C} = \{A \in M \mid A \subseteq V_\kappa^M\}$ satisfies KM + CC, and its companion model $M_{\mathscr{V}}$ is (isomorphic to) M. The bi-interpretability of the two theories allows us to investigate second-order properties in a familiar first-order setting. In second-order set theory, class choice CC is used to prove, among other principles, Fodor's Lemma for class clubs [GHK] and the Łoś Theorem for ultrapowers [GH]. In the theory of class forcing, class choice is used to prove a version of the intermediate model theorem [AFG] (see Section 5) and to provide a theory of hyperclass forcing [AF17] (see Section 7).

We will say that a *hyperclass* in a model $\mathscr{V} = (V, \in, \mathcal{C})$ of second-order set theory is a definable (with class parameters) collection of classes. Hyperclasses constitute the third-order part of a model and play the role which classes play in first-order set theory. Observe finally that every hyperclass of a model of KM + CC is a definable class over its companion model.

3. Class forcing preliminaries

First, we will explain how to form a forcing extension of a second-order model. For the discussion in this section, we fix a model $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{GBC}$ in which we will work.

Let $\mathbb{P} \in \mathcal{C}$ be a class partial order. Since a forcing extension by $\mathbb{P}$ also has to be a model of second-order set theory, we need a notion of class $\mathbb{P}$-names that will turn into classes of the forcing extension. We define that a class Γ is a *class $\mathbb{P}$-name* if it consists of pairs $\langle \tau, p\rangle$, where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. The second-order forcing language of $\mathbb{P}$ consists of all second-order assertions with $\mathbb{P}$-names and class $\mathbb{P}$-names. A filter G on $\mathbb{P}$ is said to be *$\mathscr{V}$-generic* if it meets every dense sub-class of $\mathbb{P}$ in $\mathcal{C}$. Given a $\mathscr{V}$-generic filter G, we can form the forcing extension $\mathscr{V}[G] = (V[G], \in, \mathcal{C}[G])$ consisting of interpretations of $\mathbb{P}$-names for the first-order part and interpretations of class $\mathbb{P}$-names for the second-order part. An interpretation Γ_G of a class $\mathbb{P}$-name Γ is the collection $\{\tau_G \mid \langle \tau, p\rangle \in \Gamma \text{ and } p \in G\}$.

Set forcings preserve the theories $\mathrm{GBC} + \Sigma^1_n$-CA, for $n < \omega$, KM, and KM + CC to the forcing extension (it is still open whether set forcings preserves ETR). Class partial orders, on the other hand, can easily destroy (even just the definable) replacement. Consider, for instance, the class partial order $\mathrm{Coll}(\omega, \mathrm{Ord})$ whose conditions are finite functions from ω into Ord ordered by extension. Forcing with $\mathrm{Coll}(\omega, \mathrm{Ord})$ clearly destroys class replacement but, in fact, the forcing adds a bijection between ω and every ordinal, and so also destroys the replacement scheme (for details, see Lemma 2.2 in [HKL+16]).

Friedman [Fri00] isolated properties of class partial orders which are precisely equivalent to preserving GB^- and GB to the forcing extension.

Definition 3.1. A class partial order $\mathbb{P} \in \mathcal{C}$ is *pretame* if for every class sequence $\langle D_x \mid x \in a\rangle \in \mathcal{C}$ of dense classes of $\mathbb{P}$, indexed by elements of a set a, and condition $p \in \mathbb{P}$, there is a condition $q \leq p$ and a sequence $\langle d_x \mid x \in a\rangle$ of subsets of $\mathbb{P}$ such that each $d_x \subseteq D_x$ is pre-dense below q in $\mathbb{P}$.

Intuitively, a notion of forcing is pretame, if we can reduce dense classes to pre-dense sets. In particular, every Ord-cc (having set-sized antichains) partial order is obviously pretame.

We will say, following Friedman, that a pair of sub-classes $\langle D_0, D_1\rangle$ of a class partial order $\mathbb{P}$ is a *pre-dense partition below* $q \in \mathbb{P}$ if $D_0 \cup D_1$ is pre-dense below q and every condition in D_0 is incompatible with every condition in D_1. Given two sequences $\{\langle D^x_0, D^x_1\rangle \mid x \in a\}$ and $\{\langle E^x_0, E^x_1\rangle \mid x \in a\}$ of pre-dense partitions below q, we will say that they are *equivalent below* q if for each $x \in a$, the class

$$\{p \in \mathbb{P} \mid \exists r \in D^x_0\, p \leq r \longleftrightarrow \exists s \in E^x_0\, p \leq s\}$$

is dense below q in $\mathbb{P}$.

Definition 3.2. $\mathbb{P}$ is *tame* if it is pretame and for every $p \in \mathbb{P}$, there is $q \leq p$ and ordinal α such that whenever $\vec{D} = \{\langle D^x_0, D^x_1\rangle \mid x \in a\} \in \mathcal{C}$, for a set a, is a sequence of pre-dense partitions below q, then the class

$$\{r \in \mathbb{P} \mid \vec{D} \text{ is equivalent below } r \text{ to some partition } \vec{E} \in V_\alpha\}$$

is dense below q.

Intuitively, the significance of pre-dense partitions indexed by elements of a set a is that each such partition gives rise to a function $f : a \to 2$ in the forcing extension defined by $f(x) = 0$ whenever the generic filter G meets the 0-th side of the partition at x, and conversely a function $f : a \to 2$ in a forcing extension has a natural partition associated with it determined by conditions deciding its values.

Friedman showed that pretame forcing notions are precisely those which preserve GB^- to the forcing extension and tame forcings are precisely those which preserve GB [Fri00] (namely the powerset axiom).

Theorem 3.3 (Friedman)**.** *A class forcing notion in a model of* GB^- *preserves* GB^- *if and only if it is pretame. A class forcing notion in a model of* GB *preserves* GB *if and only if it is tame.*

Since forcing in general preserves both the axiom of choice and the existence of a global well-order class, pretame forcings preserve GBc^-, and tame forcings preserve

GBc, and GBC. Tame forcings also preserve the stronger theories GBC+Σ^1_n-CA for $n < \omega$, KM, and KM + CC.

We will say that an Ord-length forcing product $\Pi_{\alpha\in\mathrm{Ord}}\mathbb{P}_\alpha$ is *progressively closed* if for every ordinal β, $\mathbb{P}_\alpha$ is $<\beta$-closed for a tail of α's. We will also say that an Ord-length forcing iteration $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha, \beta \in \mathrm{Ord}\}$ is *progressively closed* if for every ordinal β, on a tail of α's, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is $<\beta$-closed. Progressively closed products and iterations which use Easton-support are tame (see [Rei06] for details), and such class forcing notions are amongst the most commonly used by set-theorists.

Common class forcing notions used to add classes without adding sets are also usually tame. Friedman showed that every $<\mathrm{Ord}$-distributive forcing is pretame. We will see in Section 4 that such forcing notions have definable forcing relations, so that standard distributivity arguments can then be used to show that they don't add sets. Thus, $<\mathrm{Ord}$-distributive forcing notions trivially preserve powerset, and are therefore also tame.

We end this section with a discussion of the class partial order $\mathrm{Add}(\mathrm{Ord}, 1)$, whose conditions are functions from the ordinals into 2 ordered by extension. The forcing $\mathrm{Add}(\mathrm{Ord}, 1)$ is clearly $<\mathrm{Ord}$-distributive, and thus, the arguments given above show that forcing over a model of GBc with $\mathrm{Add}(\mathrm{Ord}, 1)$ does not add sets and preserves GBc. Finally, it is not difficult to see that the generic for $\mathrm{Add}(\mathrm{Ord}, 1)$ adds a global well-order. We can code every set in V by a subset of an ordinal and this subset must appear somewhere on the generic class function by density. We can then well-order the sets in the forcing extension by comparing the least locations where codes for them occur on the generic function.

4. The Class Forcing Theorem

In the context of set forcing, the forcing theorem is best understood as a theorem about countable models of ZFC. Given a countable model $M \models \mathrm{ZFC}$ and a partial order $\mathbb{P} \in M$, we define that a condition p forces a statement $\varphi(\tau)$ in the forcing language if whenever G is M-generic for $\mathbb{P}$ with $p \in G$, then the forcing extension $M[G] \models \varphi(\tau_G)$. The forcing theorem then says that for each fixed formula $\varphi(x)$, the relation $p \Vdash \varphi(\tau)$ is definable. The forcing relation for atomic formulas is defined by a transfinite recursion on name rank, and the definition is then extended to all formulas by induction on complexity. More precisely, the forcing relation for formulas $\sigma \in \tau$, $\sigma = \tau$, and $\sigma \subseteq \tau$ is given simultaneously by a solution to the recursion satisfying the rules:

(1) $p \Vdash \sigma \in \tau$ if and only if there are densely many conditions $q \leq p$ for which there is $\langle \rho, r\rangle \in \tau$ with $q \leq r$ and $q \Vdash \tau = \rho$.
(2) $p \Vdash \tau = \sigma$ if and only if $p \Vdash \tau \subseteq \sigma$ and $p \Vdash \sigma \subseteq \tau$.
(3) $p \Vdash \sigma \subseteq \tau$ if and only if whenever $\langle \rho, p\rangle \in \sigma$ and $q' \leq p, r$, there is $q \leq q'$ with $q \Vdash \rho \in \tau$.

This recursion has length Ord, but for a class partial order $\mathbb{P}$, it is a recursion on classes. Therefore it might be expected that in the absence of ETR, the solution to such a recursion may sometimes fail to exist.

Definition 4.1. The *class forcing theorem* for a class partial order $\mathbb{P}$ is the assertion that the solution to the above recursion exists for $\mathbb{P}$.

Clearly the statement of the class forcing theorem is expressible in the language of second-order set theory. If the class forcing theorem holds for a class partial order $\mathbb{P}$, we take the solution of the recursion to be the definition of the forcing relation for atomic formulas. We can then definably extend the forcing relation to all formulas of the second-order forcing language by induction on complexity of formulas as follows.

Suppose Γ is a class $\mathbb{P}$-name and suppose φ and ψ are assertions in the second-order forcing language.

(1) $p \Vdash \sigma \in \Gamma$ if and only if there are densely many $q \leq p$ for which there is $\langle \tau, r\rangle \in \Gamma$ with $q \leq r$ and $q \Vdash \sigma = \tau$.
(2) $p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$.
(3) $p \Vdash \neg\varphi$ if and only if there is no $q \leq p$ with $q \Vdash \varphi$.
(4) $p \Vdash \forall x\, \varphi(x)$ if and only if $p \Vdash \varphi(\tau)$ for every $\mathbb{P}$-name τ.
(5) $p \Vdash \forall X\, \varphi(X)$ if and only if $p \Vdash \varphi(\Delta)$ for every class $\mathbb{P}$-name Δ.

The class forcing theorem is a statement internal to a model of second-order set theory and makes no reference to the existence of generic filters. However, in the case that the model in question is countable, we can interpret it to say that the forcing relations are indeed definable.

It was shown by Stanley that in GB^-, the class forcing theorem holds for all pretame forcing notions, so that, in particular, all ZFC-preserving forcing notions have definable forcing relations (see [HKS18]).

Theorem 4.2 (Stanley)**.** *In* GB^-*, the class forcing theorem holds for all pretame forcing notions.*

It was shown in [HKL$^+$16] that GBC alone cannot prove the class forcing theorem, namely, there is a definable forcing notion for which the recursion to define the forcing relation for atomic formulas fails to have a solution in any model of GBC having no truth predicate for the first-order part of the model.

Theorem 4.3 ([HKL$^+$16])**.** *There is a definable class forcing notion* $\mathbb{P}$ *for which the class forcing theorem fails in any model of* GBC *without a truth predicate for its first-order part. In particular, the class forcing theorem fails in any model of* GBC *arising from a model of* ZFC *with its definable collections.*

The 'in, particular' assertion is true because truth predicates can never be definable by Tarski's theorem on the undefinability of truth. The forcing notion $\mathbb{P}$ of Theorem 4.3 was introduced by Friedman and codes the $\in$-relation on V as a subset of ω of the forcing extension. So given the atomic forcing relation for $\mathbb{P}$, we can use it to define a truth predicate for $(V, \in)$. It follows first that the forcing $\mathbb{P}$ must always destroy replacement, and second, that the class giving the forcing relation for atomic formulas for $\mathbb{P}$, whenever it exists, can never be definable because the truth predicate never is.

Since the recursion for obtaining the atomic forcing relation has length Ord, in $\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$, the class forcing theorem holds for all class partial orders. In particular, it already holds in models of $\mathrm{GBC}+\Sigma^1_1$-CA. Indeed, $\mathrm{ETR}_{\mathrm{Ord}}$ is precisely equivalent to the class forcing theorem over GBC.

Theorem 4.4 ([GHH$^+$])**.** *Over* GBC*, the class forcing theorem is equivalent to* $\mathrm{ETR}_{\mathrm{Ord}}$.

5. Class forcing pathologies

Class forcing fails to have many nice properties of set forcing. We already saw that class forcing can destroy replacement, and in weaker second-order set theories, such as GBC, class forcing notions may fail to have definable forcing relations. Many other pathologies can occur with class forcing even in strong second-order theories. Class forcing notions which densely embed may not have the same forcing extensions. A class forcing notion may fail to have nice names for sets of ordinals. Only Ord-cc class partial orders have (weak) Boolean completions. The intermediate model theorem, asserting that every intermediate model between a universe and its forcing extension is itself a forcing extension, can fail badly for class partial orders. Ground model definability can also fail in various ways.

5.1. **Dense embeddings.** Recall that $\mathrm{Coll}(\omega, \mathrm{Ord})$ is the forcing notion whose conditions are finite functions from ω into Ord, ordered by extension. Now consider the slightly altered partial order $\mathrm{Coll}_*(\omega, \mathrm{Ord})$ whose conditions are functions $f : n \to \mathrm{Ord}$, ordered by extension. Thus, the difference between the two partial orders is that in $\mathrm{Coll}(\omega, \mathrm{Ord})$ the domain of a condition can be any finite subset of ω, but in $\mathrm{Coll}_*(\omega, \mathrm{Ord})$, the domain of a condition must be an initial segment. Clearly $\mathrm{Coll}_*(\omega, \mathrm{Ord})$ is a dense sub-partial order of $\mathrm{Coll}(\omega, \mathrm{Ord})$. However, the two forcing notions have very different forcing extensions. Both forcing notions clearly add a class bijection between ω and Ord, and hence destroy class replacement. But while forcing with $\mathrm{Coll}(\omega, \mathrm{Ord})$ adds a bijection between every ordinal and ω, destroying ZFC, forcing with $\mathrm{Coll}_*(\omega, \mathrm{Ord})$ does not add sets, and hence preserves ZFC. [HKL$^+$16]. These pathologies can occur however only with non-pretame forcing notions.

Theorem 5.1 ([HKS18]). *Suppose $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{GBC}^-$. If $\mathbb{P} \in \mathcal{C}$ is a pretame class forcing notion and $\mathbb{P}$ densely embeds into another class forcing notion $\mathbb{Q} \in \mathcal{C}$, then $\mathbb{P}$ and $\mathbb{Q}$ have the same forcing extensions.*

5.2. **Nice names.** We can use the partial order $\mathrm{Coll}(\omega, \mathrm{Ord})$ to demonstrate another pathology of class forcing. With a set forcing $\mathbb{P}$, we always have nice $\mathbb{P}$-names for sets of ordinals in the forcing extension. Recall that a *nice $\mathbb{P}$-name for a subset of ordinals* has the form $\bigcup_{\xi<\alpha}\{\check{\xi}\} \times A_\xi$, where α is an ordinal and each A_ξ is an antichain of $\mathbb{P}$. One may not expect nice names to necessarily exist for class partial orders because the required antichains A_ξ may not be sets. In the case of the partial order $\mathrm{Coll}(\omega, \mathrm{Ord})$, the subset of ω in the forcing extension consisting of those natural numbers mapped to 0 by the generic function does not have a nice name [HKS18]. Nice names do exist for all Ord-cc partial orders in GBC^-, and over $\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$, a partial order has nice names if and only if it is pretame.

Theorem 5.2 ([HKS18]). *In GBC^-, if a class partial order $\mathbb{P}$ has the Ord-cc, then every subset of ordinals in the forcing extension has a nice $\mathbb{P}$-name. Over $\mathrm{GBC}+\mathrm{ETR}_{\mathrm{Ord}}$, a class partial order $\mathbb{P}$ has nice $\mathbb{P}$-names for every subset of ordinals if and only if it is pretame.*

In [HKS18], the second part of the theorem used the stronger assumption KM, but since the proof only needs the class forcing theorem, the assumption can be reduced to $\mathrm{ETR}_{\mathrm{Ord}}$.

5.3. **Boolean completions.** There are two ways in which we can define what a Boolean completion means for a class partial order $\mathbb{P}$. We can require that all set-sized suprema exist or that all class-sized suprema exists. Let us say that a Boolean algebra is *set-complete* if it has suprema for all its subsets and that it is *class-complete* if it has suprema for all its sub-classes. Correspondingly, we will say that a class partial order $\mathbb{P}$ has a *Boolean set-completion* if it densely embeds into a set-complete Boolean algebra and that $\mathbb{P}$ has a *Boolean class-completion* if it densely embeds into a class-complete Boolean algebra. It turns out that a Boolean algebra is class-complete only if it has the Ord-cc. The intuition here is that once there is a proper class antichain, the collection of all suprema is too large to be a class. Thus, a class partial order has a Boolean class-completion if and only if it has the Ord-cc. A class partial order has a Boolean set-completion if and only if the class forcing theorem holds for it, but the completion is only unique if it is also class-complete. Thus, only Ord-cc partial orders have unique Boolean set-completions.

Theorem 5.3 ([HKL$^+$16]). *Assume* GBC *holds.*

(1) *A class partial order has a Boolean set-completion if and only if the class forcing theorem holds for it.*
(2) *A class partial order has a Boolean class-completion if and only if it has the* Ord*-cc.*
(3) *A class partial order has a unique Boolean set-completion is unique if and only if it has the* Ord*-cc.*

It follows that in $\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$ every class partial order has a Boolean set-completion, which will be unique only in the case that the partial order has the Ord-cc. But no matter how strong the theory, no Boolean algebra with a proper class antichain can be class-complete. We will discuss in Section 6 that in sufficiently strong second-order set theories, every class partial order has a hyperclass Boolean completion with desired properties. Existence of hyperclass Boolean completions can be used to show a version of the intermediate model theorem for class partial orders.

5.4. **Intermediate model theorem.** The intermediate model theorem in first-order set theory asserts that every intermediate model $W \models \mathrm{ZFC}$ between a model $V \models \mathrm{ZFC}$ and its set-forcing extension $V[G]$, $V \subseteq W \subseteq V[G]$, is also a set-forcing extension of V [Gri75]. The assertion of the intermediate model theorem fails in various ways in the absence of the axiom of choice, so it is crucial that we specify what theory the models satisfy. Given a second-order set theory T, let the *intermediate model theorem* for T be the assertion that every intermediate model $\mathscr{W} = (W, \in, \mathcal{C}^*) \models T$ between a model $\mathscr{V} = (V, \in, \mathcal{C}) \models T$ and its class-forcing extension $\mathscr{V}[G] = (V[G], \in, \mathcal{C}[G]) \models T$ is itself a class-forcing extension of $\mathscr{V}$. To be *intermediate* here means that $V \subseteq W \subseteq V[G]$ and $\mathcal{C} \subseteq \mathcal{C}^* \subseteq \mathcal{C}[G]$. The intermediate model theorem fails for every second-order theory discussed in this article, including the strongest theory KM+CC, but a natural weaker version of it holds for KM+CC. Given a model $\mathscr{V} = (V, \in, \mathcal{C})$, we will say that a model $\mathscr{W} = (W, \in, \mathcal{C}^*)$ is a *simple extension* of $\mathscr{V}$ if $V \subseteq W$, $\mathcal{C} \subseteq \mathcal{C}^*$, and the classes of $\mathcal{C}^*$ are generated by $\mathcal{C}$ together with a single class A in the sense that every element of $\mathcal{C}^*$ is definable over W with parameters some $C_1, \ldots, C_n \in \mathcal{C}$ and A. Clearly every forcing extension is a simple extension, and so in particular, the failure of the intermediate model theorem follows from the existence of intermediate models that are not simple extensions. We will say that the *simple intermediate model theorem* holds for a theory T if whenever $\mathscr{V} \models T$ and $\mathscr{V}[G] \models T$ is a forcing extension of $\mathscr{V}$, then every intermediate model $\mathscr{W} \models T$ between $\mathscr{V}$ and $\mathscr{V}[G]$ that is a simple extension of $\mathscr{V}$ is a class-forcing extension of $\mathscr{V}$. Hamkins and Reitz showed that the simple intermediate model theorem can fail in GBC even for Ord-cc partial orders [HR17], but we showed in [AFG], using the existence of hyperclass Boolean completions, that it holds in KM+CC.

Theorem 5.4 ([HR17])**.** *In* GBC*, the simple intermediate model theorem can fail for an* Ord*-cc partial order.*

Theorem 5.5 ([AFG])**.** *Every model* $\mathscr{V} \models \mathrm{KM} + \mathrm{CC}$ *has a forcing extension* $\mathscr{V}[G] \models \mathrm{KM} + \mathrm{CC}$ *with an intermediate model of* $\mathrm{KM} + \mathrm{CC}$ *that is not a simple extension of* $\mathscr{V}$*. Therefore the intermediate model theorem fails for* $\mathrm{KM} + \mathrm{CC}$.

Theorem 5.6 ([AFG])**.** *There is a model* $\mathscr{V} \models \mathrm{KM} + \mathrm{CC}$ *with a forcing extension* $\mathscr{V}[G]$ *by* Ord*-cc partial order that has an intermediate model of* KM *that is not a simple extension of* $\mathscr{V}$*. Therefore the intermediate model theorem can fail for* KM *for an* Ord*-cc partial order.*

Theorem 5.7 ([AFG])**.** *In* KM+CC*, the simple intermediate model theorem holds.*

5.5. **Ground model definability.** Laver, and independently Woodin, showed that a ground model is always definable (with a ground model parameter) in its set-forcing extensions [Lav07], [Woo04]. Both first-order and second-order versions of ground model definability can fail for class forcing.

Working in the first-order setting, one can ask whether the first-order part $(V, \in)$ of the ground model is definable over the first-order part $(V[G], \in)$ of the forcing extension. Consider the class product $\mathbb{P} = \Pi_{\kappa \in \mathrm{Reg}} \mathrm{Add}(\kappa, 1)$ with Easton support, adding a Cohen subset to every regular cardinal. Clearly, $\mathbb{P}$ is isomorphic to the product $\Pi_{\kappa \in \mathrm{Reg}} \mathrm{Add}(\kappa, 1) \times \mathrm{Add}(\kappa, 1)$, which adds two Cohen subsets to every regular cardinal. By using an automorphism of the later forcing, which switches the two copies of $\mathrm{Add}(\kappa, 1)$, it is not difficult to show that V is not definable in $V[G]$, a forcing extension by $\mathbb{P}$. The forcing $\mathbb{P}$ is a progressively closed Easton-support product and therefore, in particular, tame.

Theorem 5.8 ([Ant18])**.** *Suppose* $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{GBC}$ *and* $\mathscr{V}[G] = (\mathscr{V}[G], \in, \mathcal{C}[G])$ *is a forcing extension by* $\mathbb{P}$ *(defined as above). Then* $(V, \in)$ *is not definable (even with parameters from* $V[G]$*) in* $(V[G], \in)$*.*

In the second-order setting, it does not make sense to ask whether the first-order part $(V, \in)$ is definable in the forcing extension $\mathscr{V}[G]$ because $(V, \in)$ is a class of $\mathscr{V}$, and hence also a class of any forcing extension $\mathscr{V}[G]$. The appropriate analogue of first-order ground model definability here is to have the ground model classes $\mathcal{C}$ be a hyperclass of the forcing extension $\mathscr{V}[G]$. Results in [GJ14] show that there are models of ZFC^-_I, namely models of the form H_{κ^+} for an inaccessible cardinal κ, which are not definable in their forcing extensions by $\mathrm{Add}(\kappa, 1)$, even with a parameter from the extension. Translating this via the bi-interpretability, we get a counter-example to second-order ground model definability for models of KM+CC.

Theorem 5.9. *There is a model* $\mathscr{V} = (V, \in, \mathcal{C})$ *of* $\mathrm{KM} + \mathrm{CC}$ *such that* $\mathcal{C}$ *is not a hyperclass of its forcing extensions* $\mathscr{V}[G] = (V[G], \in, \mathcal{C}[G])$ *by* $\mathrm{Add}(\mathrm{Ord}, 1)$*, even with a parameter from* $\mathcal{C}[G]$.

It should be mentioned that ground model definability can fail in the second-order setting even for set forcing. Again, this follows from a counter-example to ground model definability for ZFC^-_I. A construction of Woodin (see [GJ14] for details) can be used to show that assuming I_0 (the axiom that there is a nontrivial elementary embedding $j : L(V_{\kappa+1}) \to L(V_{\kappa+1})$), there is a model of ZFC^-_I, again of the form H_{κ^+} for an inaccessible κ, which is not definable in its forcing extensions by $\mathrm{Add}(\omega, 1)$, even with a parameter from the extension. It follows, via the bi-interpretability, that assuming I_0, there is a model $\mathscr{V} = (V, \in, \mathcal{C})$ of $\mathrm{KM} + \mathrm{CC}$ such that $\mathcal{C}$ is not a hyperclass of its forcing extension $V[G]$ by $\mathrm{Add}(\omega, 1)$.

6. Hyperclass Boolean completions

Given a set partial order $\mathbb{P}$, the complete Boolean algebra into which it densely embeds consists of the regular open subsets of $\mathbb{P}$ with the natural Boolean operations. For a class partial order $\mathbb{P}$, the Boolean algebra resulting from this construction is a definable collection of classes with definable Boolean operations on it, in other words, a hyperclass Boolean algebra. We will call this hyperclass object $\mathbb{B}_{\mathbb{P}}$. There are three levels of completeness which we can consider for a hyperclass Boolean algebra: it can be *set-complete*, having suprema for sub-collections indexed by sets, *class-complete*, having suprema for sub-collections indexed by classes, or *hyperclass-complete*, having suprema for definable sub-collections. In GBC, for any class partial order $\mathbb{P}$, the hyperclass Boolean algebra $\mathbb{B}_{\mathbb{P}}$ is class-complete, but we would like more to be true. The purpose of having Boolean completions is to be able to force with them, and for this a Boolean algebra needs to be fully complete, which in the case of hyperclass Boolean algebras means being hyperclass-complete. In KM, for every class partial order $\mathbb{P}$, the Boolean algebra $\mathbb{B}_{\mathbb{P}}$ is hyperclass-complete, and surprisingly the statement reverses. Any model $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{GBC}$ in which there is a non-Ord-cc class partial order $\mathbb{P} \in \mathcal{C}$ for which its hyperclass completion $\mathbb{B}_{\mathbb{P}}$ is hyperclass-complete must already be a model of KM. Given a class partial order with a proper class antichain, we can code all instances of comprehension into instances of existence of suprema for its hyperclass Boolean completion.

Theorem 6.1 ([AFG])**.** *Suppose* $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{GBC}$ *and* $\mathbb{P} \in \mathcal{C}$ *is a class partial order with a proper class antichain. If the hyperclass Boolean algebra* $\mathbb{B}_{\mathbb{P}}$ *is hyperclass-complete, then* $\mathscr{V} \models \mathrm{KM}$.

We have the intuition that although $\mathbb{B}_{\mathbb{P}}$ is a hyperclass, since a class $\mathbb{P}$ densely embeds into it, its antichains must all be small, namely class-sized. More precisely, we would like it to be the case that every hyperclass antichain of $\mathbb{B}_{\mathbb{P}}$ can be indexed by a class, and this holds true in KM.

Theorem 6.2 ([AFG])**.** *In* KM*, for every class partial order* $\mathbb{P}$*, every antichain of the hyperclass completion* $\mathbb{B}_{\mathbb{P}}$ *is indexed by a class.*

Even with all these desired properties, it is still not clear how one would force with $\mathbb{B}_{\mathbb{P}}$ or more generally with a hyperclass partial order. In order to develop a theory of forcing with hyperclass partial orders, we need to move to the theory $\mathrm{KM} + \mathrm{CC}$.

7. Hyperclass forcing

The first author and Friedman developed the theory of hyperclass forcing over models of $\mathrm{KM} + \mathrm{CC}$ in [AF17]. Suppose $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{KM} + \mathrm{CC}$ and $\mathbb{P}$ is hyperclass partial order defined over $\mathscr{V}$. We will say that $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic if it meets every dense sub-hyperclass of $\mathbb{P}$. To make sense of forcing with $\mathbb{P}$, we move to the companion model $M_{\mathscr{V}} \models \mathrm{ZFC}^-_I$ of $\mathscr{V}$ in which $\mathbb{P}$ is a definable class partial order, and so all the class forcing machinery applies. We will say that $\mathscr{V}$ has a *hyperclass* forcing extension by $\mathbb{P}$ only in the special case that forcing with $\mathbb{P}$ over $M_{\mathscr{V}}$ preserves ZFC^-_I to the forcing extension with $\kappa = \mathrm{Ord}^V$ remaining the largest cardinal. So suppose that $\mathbb{P}$ preserves ZFC^-_I over $M_{\mathscr{V}}$ and $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic, and hence also $M_{\mathscr{V}}$-generic. Let $\mathcal{C}^* = \{A \in M_{\mathscr{V}}[G] \mid A \subseteq V_\kappa^{M_{\mathscr{V}}[G]}\}$. Then we define that the forcing extension $\mathscr{V}[G] = (V_\kappa^{M_{\mathscr{V}}[G]}, \in, \mathcal{C}^*)$. Note that the companion model of the forcing extension $\mathscr{V}[G]$ is $M_{\mathscr{V}}[G]$. By Theorem 3.3, all pretame forcing notions preserve ZFC^- to the forcing extension, but even set forcing can collapse the largest cardinal κ of $M_{\mathscr{V}}$. Thus, a hyperclass forcing extension exists precisely in the case that the forcing $\mathbb{P}$ is pretame and κ-preserving over $M_{\mathscr{V}}$. In particular, any $\mathrm{Ord}^{M_{\mathscr{V}}}$-cc partial order $\mathbb{P}$ has both these properties.

Suppose $\mathbb{P}$ is a class partial order and $\mathbb{B}_{\mathbb{P}}$ is the hyperclass Boolean completion of $\mathbb{P}$. By Theorem 6.2, $\mathbb{B}_{\mathbb{P}}$ has the $\mathrm{Ord}^{M_{\mathscr{V}}}$-cc in $M_{\mathscr{V}}$. Since $\mathbb{P}$ densely embeds into $\mathbb{B}_{\mathbb{P}}$, which is pretame, the two partial orders have the same forcing extensions by Theorem 5.1. Using this, we can show that the hyperclass forcing extension of $\mathscr{V}$ obtained by forcing with $\mathbb{B}_{\mathbb{P}}$ is precisely the forcing extension by $\mathbb{P}$, so that, just as in the set forcing case, we can choose to force either with the class partial order or its hyperclass Boolean algebra.

Surprisingly, hyperclass forcing arises even in connection with properties of models of first-order set theory. In [Wel19], models of the form $L[C]$ for a proper class club of uncountable cardinals are characterized as hyperclass forcing extensions of a truncated iterate of a mouse with large cardinals [Wel19]. More precisely, if $m_1^\#$ is a mouse which is a sharp for a proper class of measurable cardinals, then there is an Ord-length iterate of $m_1^\#$ such that if M is the truncation of the iterate at Ord, then $L[C]$ is a hyperclass forcing extension of M by a Prikry-type forcing singularizing all its measurable cardinals. This theorem was generalized in [FGM] to characterize models $L[C_1, \ldots, C_n]$ for specially nested clubs C_i of uncountable cardinals as hyperclass forcing extensions of a truncated iterate of a mouse with stronger large cardinals.

8. Open questions

While $\mathrm{GBC} + \mathrm{ETR}$ easily implies that any two meta-ordinals are comparable, it is not known whether this holds in weaker second-order set theories. The assertion may indeed turn out to be equivalent to ETR.

Question 8.1. Does GBC imply that any two meta-ordinals are comparable?

Question 8.2. Does GBC together with the principle that any two meta-ordinals are comparable imply ETR?

Pretame forcing preserves most robust second-order set theories such as $\mathrm{GBC} + \Sigma^1_n$-CA for any $n < \omega$, KM, and KM+CC, but it is still not known whether even set forcing preserves ETR.

Question 8.3. Does pretame forcing preserve ETR?

Over $\mathrm{GBC} + \Sigma^1_1$-CA and stronger theories, pretame forcing does not add meta-ordinals [HW18], but it is not known whether this result holds for weaker theories, in particular, ETR.

Question 8.4. Can pretame forcing add meta-ordinals over a model of GBC or a model of $\mathrm{GBC} + \mathrm{ETR}$?

The intermediate model theorem can fail for GBC and KM even for Ord-cc partial orders, but while it is known that the intermediate model theorem can fail for $\mathrm{KM} + \mathrm{CC}$, it is not known whether there is an Ord-cc instance of failure.

Question 8.5. Does the intermediate model theorem fail in KM+CC for an Ord-cc partial order?

References

[AF17] Carolin Antos and Sy-David Friedman. Hyperclass forcing in Morse-Kelley class theory. *J. Symb. Log.*, 82(2):549–575, 2017.

[AFG] Carolin Antos, Sy David Friedman, and Victoria Gitman. Boolean-valued class forcing. *Submitted*.

[Ant18] Carolin Antos. Class forcing in class theory. In *The hyperuniverse project and maximality*, pages 1–16. Birkhäuser/Springer, 2018.

[Eas70] William Bigelow Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.

[FGM] Sy-David Friedman, Victoria Gitman, and Sandra Müller. Some properties of the Stable Core. *In preparation*.

[Fri00] Sy D. Friedman. *Fine structure and class forcing*, volume 3 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2000.

[GH] Victoria Gitman and Joel David Hamkins. Kelley-Morse set theory and choice principles for classes. In preparation.

[GHH+] Victoria Gitman, Joel David Hamkins, Peter Holy, Philipp Schlicht, and Kameryn Williams. The exact strength of the class forcing theorem. Submitted.

[GHJ16] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone. What is the theory ZFC without power set? *MLQ Math. Log. Q.*, 62(4-5):391–406, 2016.

[GHK] Victoria Gitman, Joel David Hamkins, and Asaf Karagila. Kelley-morse set theory does not prove the class Fodor theorem. *Submitted*.

[GJ14] Victoria Gitman and Thomas A. Johnstone. On ground model definability. In *Infinity, Computability, and Metamathematics: Festschrift in honour of the 60th birthdays of Peter Koepke and Philip Welch*, Series:Tributes. College publications, London, GB, 2014.

[Gri75] Serge Grigorieff. Intermediate submodels and generic extensions in set theory. *Ann. Math. (2)*, 101:447–490, 1975.

[HKL+16] Peter Holy, Regula Krapf, Philipp Lücke, Ana Njegomir, and Philipp Schlicht. Class forcing, the forcing theorem and Boolean completions. *J. Symb. Log.*, 81(4):1500–1530, 2016.

[HKS18] Peter Holy, Regula Krapf, and Philipp Schlicht. Characterizations of pretameness and the Ord-cc. *Ann. Pure Appl. Logic*, 169(8):775–802, 2018.

[HR17] Joel David Hamkins and Jonas Reitz. The set-theoretic universe V is not necessarily a class-forcing extension of HOD. *ArXiv e-prints*, September 2017. manuscript under review.

[HW18] Joel David Hamkins and W. Hugh Woodin. Open class determinacy is preserved by forcing. *ArXiv e-prints*, pages 1–14, 2018. under review.

[KM68] H. Jerome Keisler and Michael Morley. Elementary extensions of models of set theory. *Israel J. Math.*, 6:49–65, 1968.

[Lav07] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Ann. Pure Appl. Logic*, 149(1-3):1–6, 2007.

[Mar73] W. Marek. On the metamathematics of impredicative set theory. *Dissertationes Math. (Rozprawy Mat.)*, 98:40, 1973.

[Rei06] Jonas Reitz. *The ground axiom*. ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–City University of New York.

[Wel19] Philip Welch. Closed unbounded classes and the Härtig quantifier model. *ArXiv e-prints*, March 2019. Manuscript under review.

[Wil18] Kameryn J. Williams. *The Structure of Models of Second-Order Set Theories*. ProQuest LLC, Ann Arbor, MI, 2018. Thesis (Ph.D.)–City University of New York.

[Woo04] Hugh Woodin. Recent developments on Cantor's Continuum Hypothesis. In *Proceedings of the continuum in Philosophy and Mathematics*. Carlsberg Academy, Coppenhagen, 2004.

(C. Antos) Universitt Konstanz, Zukunftskolleg Postfach 216, 78457 Konstanz Germany

E-mail address: `carolin.antos-kuby@uni-konstanz.de`

URL: `http://www.logic.univie.ac.at/~antos/`

(V. Gitman) The City University of New York, CUNY Graduate Center, Mathematics Program, 365 Fifth Avenue, New York, NY 10016

E-mail address: `vgitman@nylogic.org`

URL: `http://boolesrings.org/victoriagitman`