Class forcing in its rightful setting

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Class forcing (forcing with a class partial order) is ubiquitous in set theory.

- Make $\text{GCH}$ fail at every regular cardinal.
  - $\prod_{\alpha \in \text{Reg}} \text{Add}(\alpha, \alpha^{++})$ with Easton support.
- Make $V = \text{HOD}$ hold by coding all sets into the continuum function.
- Make all supercompact cardinals Laver indestructible.
- Add a global well-order (bijection $F : V \xrightarrow{1-1} \text{Ord}$).
  - $\text{Add}(\text{Ord}, 1)$: conditions are partial functions $f : \alpha \to \text{Ord}$ for $\alpha \in \text{Ord}$ ordered by extension.
  - $<\text{Ord}$-closed.
  - Does not add sets.
- Shoot a class club through a fat stationary class.
  - Conditions are closed sets of ordinals from the stationary class ordered by end-extension.
  - $<\text{Ord}$-distributive.
  - Does not add sets.
The right setting for class forcing

Class forcing is fundamentally about classes.

- A generic filter for a class partial order is a subclass of the partial order meeting all dense subclasses.
- The properties of the partial order depend on which classes exist around it.

First-order set theory

- Classes are definable (with parameters) collections of sets.
- Classes are objects in the meta-theory.

Second-order set theory

- Classes are elements of the model.
- We can quantify over classes.
- We can study general properties of classes.
- The theory determines which classes exist.

We should not expect class forcing to be as nice as set forcing because classes even in the strongest second-order set theories do not behave as nicely as sets.
Second-order set theory

Second-order set theory has two sorts of objects: sets and classes.

Syntax: Two-sorted logic

- Separate variables and quantifiers for sets and classes
- Convention: upper-case letters for classes, lower-case letters for sets
- Notation:
  - $\Sigma^0_n$ - first-order $\Sigma_n$-formula
  - $\Sigma^1_n$ - $n$-alternations of class quantifiers followed by a first-order formula

Semantics: A model is a triple $\mathcal{V} = \langle V, \in, C \rangle$.

- $V$ consists of the sets.
- $C$ consists of the classes.
- Every set is a class: $V \subseteq C$.
- $C \subseteq V$ for every $C \in C$. 
Weakest second-order theories

Set axioms

- $\text{ZF}^-$: no choice, no powerset, collection instead of replacement
- $\text{ZFC}^-$: no powerset, collection instead of replacement, well-ordering principle instead of axiom of choice\(^1\)

\(^1\)Different formulations of the axiom of choice are not equivalent over $\text{ZF}^-$. 
Weakest second-order set theories

**Second-order theories:** axioms for sets and classes

- **GB**
  - set axioms: **ZF**
  - **Replacement:** If $F$ is a function and $a$ is a set, then $F \upharpoonright a$ is a set.
  - **First-order comprehension:** If $\varphi(x, A)$ is a first-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.
  - Every model of ZF with together with its definable collections is a model of GB.

- **GBc:** GB with ZF replaced by ZFC.
  - Every model of ZFC together with its definable collections is a model of GBc.

- **GBC:** Gödel-Bernays set theory
  - GBc
  - **Global well-order:** there is a bijection $F : V \xrightarrow{1-1 \text{onto}} \text{Ord}$.²
  - Every model of ZFC, with a definable global well-order, together with its definable collections is a model of GBC, e.g. $L$.
  - Every model of ZFC has a class forcing extension with the same sets satisfying GBC: force with $\text{Add(Ord, 1)}$.
  - Conservative over ZFC.

- **GB⁻, GBc⁻, GBC⁻:** ZF⁻ instead of ZF.

²Different formulations of global choice are not equivalent over GBc⁻.
Hyperclasses

**Definition:** In a model of second-order set theory, a hyperclass is a definable (with parameters) collection of classes.
- Analogue of classes in second-order.
- Third-order part of the model.

Suppose $\mathcal{V} = \langle V, \in, C \rangle \models \text{GBC}$. Given a set $a$ and a class $A$, the $a$-th slice of $A$ is the class

$$\{ x \mid \langle a, x \rangle \in A \},$$

where $\langle , \rangle$ is the Gödel pairing function.

**Definition:** A hyperclass, given by a formula $\varphi(X, A)$, is coded by a class $C$ if for every class $B$, $\varphi(B, A)$ holds if and only if there is a set $b$ such that $C_b = B$.

“A coded hyperclass has class-many classes.”
Transfinite recursion on classes along meta-ordinals

Suppose $\mathcal{V} = \langle V, \in, C \rangle \models GBC$.

**Definition:** A meta-ordinal is a well-order $(\Gamma, \leq) \in C$.

- **Examples:** $\text{Ord}$, $\text{Ord} + \text{Ord}$, $\text{Ord} \cdot \omega$.
- **Notation:** For $a \in \Gamma$, $\Gamma \upharpoonright a$ is the restriction of the well-order to $\leq$-predecessors of $a$.
- Meta-ordinals may not have unique representations.

**Question:** Does $GBC$ prove that any two meta-ordinals are comparable?

**Definition:** Suppose $\Gamma \in C$ is a meta-ordinal. A solution along $\Gamma$ to a first-order recursion rule $\varphi(x, b, F)$ is a class $S$ such that for every $b \in \Gamma$, $S_b = \{ x \mid \varphi(x, b, S \upharpoonright b) \}$.

- $S_0 = \{ x \mid \varphi(x, 0, \emptyset) \}$
- $S_1 = \{ x \mid \varphi(x, 1, \langle S_0 \rangle) \}$
- $S_2 = \{ x \mid \varphi(x, 2, \langle S_0, S_1 \rangle) \}$
- etc.
**ETR**: Elementary transfinite recursion

**ETR**: Every first-order recursion on classes along a meta-ordinal has a solution.

**ETR\(_\Gamma\)**: **ETR** restricted to a well-order \(\Gamma\).
- \(\text{ETR}_{\text{Ord} \cdot \omega}, \text{ETR}_{\text{Ord}}, \text{ETR}_\omega\).

**Theorem**: (Fujimoto) Suppose \(\mathcal{V} = \langle V, \in, C \rangle \models \text{GBC}\).
- **ETR\(_\omega\)** is equivalent to the assertion that for every class \(A\), there is a truth predicate for the structure \(\langle V, \in, A \rangle\) (with a predicate for \(A\)).
  - Tarskian truth is given by a recursion of length \(\omega\) each of whose levels is a class.
- **ETR\(_\Gamma\)** is equivalent to the existence, for every class \(A\), of the iterated truth predicate of length \(\Gamma\) for the structure \(\langle V, \in, A \rangle\).

**Corollary**: **ETR** is equivalent to the assertion that for every meta-ordinal \(\Gamma\) and every class \(A\), there is an iterated truth predicate of length \(\Gamma\) for \(\langle V, \in, A \rangle\).

**Corollary**: \(\text{GBC} + \text{ETR}_\omega\) implies \(\text{Con}(\text{ZFC})\).

**Theorem**: (Williams) If \(\Gamma > \omega^\omega\) is a meta-ordinal, then \(\text{GBC} + \text{ETR}_\Gamma \cdot \omega\) implies \(\text{Con}(\text{GBC} + \text{ETR}_\Gamma)\).

**Theorem**: \(\text{GBC} + \text{ETR}\) implies that any two meta-ordinals are comparable.
A comprehension hierarchy to Kelley-Morse set theory

\( \Sigma^1_n \)-comprehension CA: Every \( \Sigma^1_n \)-formula defines a class.

**Theorem:** GBC + \( \Sigma^1_1 \)-CA implies ETR.

The theories GBC + \( \Sigma^1_n \)-CA form a hierarchy of strength culminating in Kelley-Morse set theory KM.

**Kelley-Morse set theory KM**

- GBC
- Every second-order formula defines a class.
Class choice principles

**Choice Scheme CC**: Given a second-order formula \( \varphi(x, X, A) \), if for every set \( x \), there is a class \( X \) witnessing \( \varphi(x, X, A) \), then there is a class collecting witnesses for every \( x \):

\[
\forall x \exists X \varphi(x, X, A) \rightarrow \exists Y \forall x \varphi(x, Y, A).
\]

**\( \Sigma^1_n \)-CC**: CC restricted to \( \Sigma^1_n \)-formulas.

**Theorem**: (G., Hamkins) \( \Sigma^0_1 \)-CC can fail in a model of KM.

**\( \alpha \)-Dependent Choice Scheme DC\( \alpha \)**: (\( \omega \leq \alpha \leq \text{Ord} \)) Every second-order definable \( <\alpha \)-closed tree \( T(X, A) \) has a branch of height \( \alpha \).

**\( \Sigma^1_n \)-DC\( \alpha \)**: DC\( \alpha \) restricted to \( \Sigma^1_n \)-definable trees.

**Theorem**: (Marek, Mostowski, Ratajczyk) If \( \mathcal{V} \models \text{GBC} + \Sigma^1_n \)-CA, then its second-order constructible universe \( \mathcal{L} \models \text{GBC} + \Sigma^1_n \)-Comprehension + \( \Sigma^1_n \)-CC + \( \Sigma^1_n \)-DC\( \text{Ord} \).

- If \( \mathcal{V} \models \text{KM} \), then its second-order constructible universe \( \mathcal{L} \models \text{KM} + \text{CC} + \text{DC}_{\text{Ord}} \).
  - Given a meta-ordinal \( \Gamma \), we can build a meta-constructible universe \( L_{\Gamma} \) by a recursion of length \( \Gamma \).
  - A meta-ordinal \( \Gamma \) is **constructible** if \( \Gamma \in L_{\text{Ord}^+} \).
  - **Theorem**: (Tharp) Constructible meta-ordinals have unique representations.
  - A class \( A \in C \) is **constructible** if there is a constructible meta-ordinal \( \Gamma \) such that \( A \in L_{\text{Ord}^+} \).
  - The second-order constructible universe is \( \mathcal{L} = \langle L, \in, \mathcal{L} \rangle \), where \( \mathcal{L} \) consists of the constructible classes.

- The theories KM and KM+CC+DC\( \text{Ord} \) are equiconsistent.
Applications of class choice

**Theorem:** Over GBC, $\Sigma^1_n$-CC implies $\Delta^1_n$-CA. Consequently, GBC+CC is equivalent to KM+CC.

**Theorem:** (G., Hamkins) Over GBC, Łoś’ Lemma for second-order ultrapowers is equivalent to CC for set-many choices.

**Theorem:** (G., Hamkins) Over GBC, normal form for second-order formulas is equivalent to CC.

**Theorem:** (G., Hamkins, Karagila) Fodor’s Lemma for class clubs
- can fail in KM,
- holds in GBC+$\Sigma^0_2$-CC.

**Aside**

**Theorem:** (Enayat) GBC+$\Sigma^1_1$-CC is conservative over ZFC.
- Every countable model of ZFC can be elementarily extended to countable recursively saturated model of ZFC.
- Every countable recursively saturated model of ZFC can be expanded to a model of GBC+$\Sigma^1_1$-CC without adding sets.
Moving to first-order with KM+CC

Suppose $\mathcal{V} = \langle V, \in, C \rangle \models \text{KM}+\text{CC}$.

- View each extensional well-founded class relation $R \in C$ as coding a transitive set.
  - $\text{Ord} + \text{Ord}, \text{Ord} \cdot \omega$
  - $V \cup \{V\}$

- Define a membership relation $E$ on the collection of all such relations $R$ (modulo isomorphism).

- Let $\langle M_{\mathcal{V}}, E \rangle$, the companion model of $\mathcal{V}$, be the resulting first-order structure.
  - $M_{\mathcal{V}}$ has the largest cardinal $\kappa \cong \text{Ord}^{\mathcal{V}}$.
  - $V_{\kappa}^{M_{\mathcal{V}}} \cong V$.
  - $\mathcal{P}(V_{\kappa})^{M_{\mathcal{V}}} \cong C$.
  - $\langle M_{\mathcal{V}}, E \rangle \models \text{ZFC}^-_1$.

$\text{ZFC}^-_1$

- $\text{ZFC}^-$

- There is a largest cardinal $\kappa$.

- $\kappa$ is inaccessible: $\kappa$ is regular and for all $\alpha < \kappa$, $2^\alpha$ exists and $2^\alpha < \kappa$.
  - $V_{\kappa}$ exists.
  - $V_{\kappa} \models \text{ZFC}$.
Suppose $M \models \text{ZFC}_I^-$ with a largest cardinal $\kappa$.

- $V = V^M_\kappa$
- $C = \{X \in M \mid X \subseteq V^M_\kappa\}$
- $\mathcal{V} = \langle V, \in, C \rangle \models \text{KM+CC}$
- $M_\mathcal{V} \cong M$ is the companion model of $\mathcal{V}$.

Theorem: (Marek) The theory $\text{KM+CC}$ is bi-interpretable with the theory $\text{ZFC}_I^-$. 
Suppose $\mathcal{V} = \langle V, \in, C \rangle \models \text{GBC}$ and $P \in C$ is a class forcing.

**Definition:**
- A **class $P$-name** is a collection of pairs $\langle \sigma, p \rangle$ such that $p \in P$ and $\sigma \in V^P$.
- $G$ is $\mathcal{V}$-generic for $P$ if $G$ meets every dense subclass $D \in C$ of $P$.
- The forcing extension $\mathcal{V}[G] = \langle V[G], \in, C[G] \rangle$. 
Definability of forcing relations

Suppose $\mathcal{V} = \langle V, \in, C \rangle \models \text{GBC}$ and $\mathbb{P} \in C$ is a class forcing.

The Class Forcing Theorem for $\mathbb{P}$: There is a solution to the recursion defining the forcing relation for atomic formulas.

- $p \vdash \sigma \in \tau$: there is a dense set of conditions $q \leq p$ for which there is $\langle \rho, r \rangle \in \tau$ with $q \leq r$ and $q \vdash \sigma = \rho$.
- $p \vdash \sigma = \tau$: $p \vdash \sigma \subseteq \tau$ and $p \vdash \tau \subseteq \sigma$.
- $p \vdash \sigma \subseteq \tau$: whenever $\langle \rho, r \rangle \in \sigma$ and $q' \leq p, r$, there is $q \leq q'$ with $q \vdash \rho \in \tau$.

The Class Forcing Theorem implies that forcing relations for all second-order formulas are definable.

- $p \vdash \sigma \in \Gamma$: there are densely many $q \leq p$ for which there is $\langle \tau, r \rangle \in \Gamma$ with $q \leq r$ and $q \vdash \sigma = \tau$.
- $p \vdash \varphi \land \psi$: $p \vdash \varphi$ and $p \vdash \psi$.
- $p \vdash \neg \varphi$: there is no $q \leq p$ with $q \vdash \varphi$.
- $p \vdash \forall x \varphi(x)$: $p \vdash \varphi(\tau)$ for every $\mathbb{P}$-name $\tau$.
- $p \vdash \forall X \varphi(X)$: $p \vdash \varphi(\Delta)$ for every class $\mathbb{P}$-name $\Delta$. 
Nice class forcing

**Definition:** A class forcing $\mathbb{P}$ has the **Ord-cc** if every antichain of $\mathbb{P}$ is a set.

**Definition:** (Friedman) A class forcing $\mathbb{P}$ is **pretame** if for every class sequence $\langle D_x \mid x \in a \rangle \in \mathcal{C}$ of dense classes of $\mathbb{P}$, indexed by elements of a set $a$, and condition $p \in \mathbb{P}$, there is a condition $q \leq p$ and a sequence $\langle d_x \mid x \in a \rangle$ of subsets of $\mathbb{P}$ such that each $d_x \subseteq D_x$ is pre-dense below $q$ in $\mathbb{P}$.

- “Reduces dense classes to pre-dense sets.”
- **Condition to preserve replacement**

**Definition:** (Friedman) A class forcing $\mathbb{P}$ is **tame** if it is pretame and for every $p \in \mathbb{P}$, there is $q \leq p$ and ordinal $\alpha$ such that whenever $\vec{D} = \{\langle D_0^x, D_1^x \rangle \mid x \in a \} \in \mathcal{C}$, for a set $a$, is a sequence of pre-dense partitions below $q$, then the class

$$\{ r \in \mathbb{P} \mid \vec{D} \text{ is equivalent below } r \text{ to some partition } \vec{E} \in V_\alpha \}$$

is dense below $q$.

- **Condition to preserve powerset**
- **Ord-cc forcings**
- **Progressively closed Ord-length products and iterations** of set forcing
- **\langle Ord\text{-distributive forcing**
Preserving the theory

Class forcing need not preserve ZFC.

- $\text{Coll}(\omega, \text{Ord})$.
- (Friedman) Forcing $\mathbb{F}$ to code $\langle V, \in \rangle$ into a relation $E$ on $\omega$.
  - In the extension: $\langle V, \in \rangle \cong \langle \omega, E \rangle$.

Question: Are class forcing extensions always closed under complements?

Theorem: (Friedman) Pretameness is equivalent to preservation of $\text{GB}^-$. Tameness is equivalent to preservation of $\text{GB}$.

Theorem: Tame forcings preserve:

- $\text{GBc}$, $\text{GBC}$.
- (Antos) $\text{KM}$.
- (Antos, G., Friedman) $\text{KM}+\text{CC}$.
- (Antos, G., Friedman) $\text{KM}+\text{CC}+\text{DC}_\alpha$ for every $\omega \leq \alpha \leq \text{Ord}$.

Theorem: Tame forcings preserve $\text{GBC} + \Sigma^1_n\text{-CA} + \Sigma^1_n\text{-CC}$.

Question: Do tame forcings preserve $\text{GBC} + \Sigma^1_n\text{-CA}$?

- Without $\Sigma^1_n\text{-AC}$, the forcing relation for $\Sigma^1_n$-formulas may not be $\Sigma^1_n$-definable.
**Class Forcing Theorem**

**Theorem**: (Holy, Krapf, Lücke, Njegomir, Schlicht) The **Class Forcing Theorem** can fail in a model of GBC.

- The **Class Forcing Theorem** for Friedman’s forcing $\mathbb{F}$ implies existence of truth predicate for $\langle V, \in \rangle$.

**Theorem**: (Stanley) In $\text{GB}^-$, the **Class Forcing Theorem** holds for all pretame forcings.

- In $\text{ZFC}$, pretame forcings have definable forcing relations.
- In $\text{ZFC}^-$ pretame forcings have definable forcing relations.

**Theorem**: (G., Hamkins, Holy, Schlicht, Williams) Over GBC, the **Class Forcing Theorem** is equivalent to $\text{ETR}_{\text{Ord}}$.

- The recursion to define the forcing relation on atomic formulas has length $\text{Ord}$.
- The atomic forcing relation for Friedman’s forcing $\mathbb{F}$ yields an iterated truth predicate of length $\text{Ord}$ for $\langle V, \in \rangle$.
- Define a version of Friedman’s forcing $\mathbb{F}_A$ for a class $A$. 
Dense embeddings

**Theorem**: Suppose $\mathbb{P}$ and $\mathbb{Q}$ are set forcings and there is a dense embedding from $\mathbb{P}$ into $\mathbb{Q}$. Then $\mathbb{P}$ and $\mathbb{Q}$ have the same forcing extensions.

**Definition**:
- $\text{Coll}(\omega, \text{Ord})$: conditions $f : A \to \text{Ord}$, where $A$ is a finite subset of $\omega$.
- $\text{Coll}^*(\omega, \text{Ord})$: conditions $f : n \to \text{Ord}$ (no holes!).

**Theorem** (Holy, Krapf, Lücke, Njegomir, Schlicht)
- $\text{Coll}^*(\omega, \text{Ord})$ densely embeds into $\text{Coll}(\omega, \text{Ord})$.
- $\text{Coll}(\omega, \text{Ord})$ adds a bijection between every ordinal and $\omega$.
- $\text{Coll}^*(\omega, \text{Ord})$
  - adds a class $f : \omega \xrightarrow{1-1} \text{Ord}$,
  - does not add sets.

**Theorem**: (Holy, Krapf, Lücke, Njegomir, Schlicht) Assume $\text{GBC}^-$. If $\mathbb{P}$ and $\mathbb{Q}$ are pretame class forcings such that $\mathbb{P}$ densely embeds into $\mathbb{Q}$, then $\mathbb{P}$ and $\mathbb{Q}$ have the same forcing extensions.
Nice names

Definition: Suppose \( P \) is a forcing. A nice \( P \)-name for a subset of ordinals has the form \( \bigcup_{\xi<\alpha} \{\xi\} \times A_\xi \), where each \( A_\xi \subseteq P \) is an antichain.

Theorem: Suppose \( P \) is a set forcing. Then every subset of ordinals in a forcing extension by \( P \) has a nice \( P \)-name.

Theorem: (Holy, Krapf, Schlicht) Suppose \( \mathcal{V} = \langle V, \in, C \rangle \models \text{GBC} \) and \( G \subseteq \text{Coll}(\omega, \text{Ord}) \) is \( \mathcal{V} \)-generic. In \( \mathcal{V}[G] \), the set

\[
A = \{ n \in \omega \mid G(n) = 0 \}
\]

does not have a nice-\( \text{Coll}(\omega, \text{Ord}) \)-name.

Theorem: (Holy, Krapf, Schlicht)
- Assume \( \text{GBC}^- \). Every pretame class forcing \( P \) has nice \( P \)-names for all subsets of ordinals.
- Assume \( \text{GBC} + \text{ETR}_{\text{Ord}} \). A class forcing \( P \) that has nice \( P \)-names for all subsets of ordinals is pretame.
  - Uses existence of definable forcing relation.
Ground model definability

**Theorem:** (Laver, Woodin) The ground model $V$ is uniformly definable from a parameter in any set-forcing extension.

**Theorem:** (Antos) Suppose $\mathcal{V} = \langle V, \in, C \rangle \models \text{GBC}$. There is a class forcing $\mathbb{P}$ such that $\langle V, \in \rangle$ is not definable from parameters over $V[G]$ in any forcing extension $\mathcal{V}[G]$ by $\mathbb{P}$.

- $\mathbb{P}$ is the Easton support product $\prod_{\alpha \in \text{Reg}} \text{Add}(\alpha, 1)$.
- $\text{Add}(\alpha, 1) \cong \text{Add}(\alpha, 1) \times \text{Add}(\alpha, 1)$.
- Use an automorphism.

**Theorem:** (G., Johnstone) It is consistent that $H_{\kappa^+}(|= \text{ZFC}_1^-)$ is not definable, even with parameters, in its forcing extension $H_{\kappa^+}[G]$ by $\text{Add}(\kappa, 1)$.

**Corollary:** There is a model $\mathcal{V} = \langle V, \in, C \rangle \models \text{KM+CC}$ such that $C$ is not a hyperclass of its forcing extensions $\mathcal{V}[G] = \langle V[G], \in, C[G] \rangle$ by $\text{Add}(\text{Ord}, 1)$, even with a parameter from $C[G]$.

**Theorem:** (Asperó) It is consistent (from large cardinals) that $H_{\kappa^+}(|= \text{ZFC}_1^-)$ is not definable, even with parameters, in its forcing extension $H_{\kappa^+}[G]$ by $\text{Add}(\omega, 1)$.

**Corollary:** Ground model definability can fail in KM+CC even for set-forcing.
Boolean completions

**Theorem:** Every set forcing $\mathbb{P}$ densely embeds into a unique (modulo isomorphism) complete Boolean algebra.

- $U \subseteq \mathbb{P}$ is a cut if it is closed downwards: if $p \in U$ and $q \leq p$, then $q \in U$.
- For $p \in \mathbb{P}$, $U_p = \{ q \in \mathbb{P} \mid q \leq p \}$ is a cut.
- A cut $U$ is regular if for every $p \notin U$, there is $q \leq p$ such that $U_q \cap U = \emptyset$.
- The Boolean algebra $\mathbb{B}_\mathbb{P}$ consists of all regular cuts of $\mathbb{P}$.

**Definition:** A class Boolean algebra is:
- set-complete if it has suprema for all its subsets,
- class-complete if it has suprema for all its subclasses.

**Theorem:** (Holy, Krapf, Lücke, Njegomir, Schlicht) Assume GBC. A class forcing has:
- a Boolean set-completion if and only if the Class Forcing Theorem holds for it.
  - The forcing relation can be used to construct a Boolean set-completion.
  - Defining Boolean values of assertions in the forcing language does not require ETR.
- a Boolean class-completion if and only if it has the Ord-cc.
- a unique Boolean set-completion if and only if it has the Ord-cc.

A class forcing with a proper class antichain cannot have a Boolean class-completion!
Hyperclass Boolean completions

Definition: Suppose $\mathcal{V} \models \text{GBC}$.

- Suppose $\mathbb{P} \in \mathcal{C}$ is a class forcing. The regular cuts of $\mathbb{P}$ form a hyperclass Boolean algebra $\mathcal{B}_\mathbb{P}$.
- A hyperclass Boolean algebra is:
  - class-complete if it has suprema for all its coded subhyperclasses.
  - hyperclass-complete if it has suprema for all its subhyperclasses.

Theorem: Suppose $\mathcal{V} \models \text{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing. Then $\mathcal{B}_\mathbb{P}$ is class-complete.

Theorem: (Antos, Friedman, G.) Suppose $\mathcal{V} = \langle \mathcal{V}, \in, \mathcal{C} \rangle \models \text{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is not Ord-cc. If $\mathcal{B}_\mathbb{P}$ is hyperclass-complete, then $\mathcal{V} \models \text{KM}$.

- Code instances of comprehension into suprema of antichains of $\mathcal{B}_\mathbb{P}$.

Theorem: (Antos, Friedman, G.) Suppose $\mathcal{V} = \langle \mathcal{V}, \in, \mathcal{C} \rangle \models \text{KM}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing. Every hyperclass antichain of $\mathcal{B}_\mathbb{P}$ is coded.

Corollary: Suppose $\mathcal{V} = \langle \mathcal{V}, \in, \mathcal{C} \rangle \models \text{KM+CC}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing. In the companion model $M_{\mathcal{V}}$, $\mathcal{B}_\mathbb{P}$ is an $\text{Ord}^{M_{\mathcal{V}}}$-cc class-complete Boolean algebra.
The Class Intermediate Model Theorem

**Intermediate Model Theorem:** (Solovay)
- If $V \models \text{ZFC}$ and $W \models \text{ZFC}$ is an intermediate model between $V$ and its set-forcing extension $V[G]$, then $W$ is a set-forcing extension of $V$.
- If $V \models \text{ZF}$ and $V[a] \models \text{ZF}$, with $a \subseteq V$, is an intermediate model between $V$ and its set-forcing extension $V[G]$, then $V[a]$ is a set-forcing extension of $V$.

**Definition:** Suppose $\mathcal{V} = \langle V, \in, C \rangle \models \text{GBC}$. Then $\mathcal{W} = \langle W, \in, C^* \rangle$ is a simple extension of $\mathcal{V}$ if $C^*$ is generated by $C$ together with a single new class.

- Forcing extensions are simple extensions.

**Definition:** Suppose $T$ is a second-order set theory.
- The Intermediate Model Theorem holds for $T$ if whenever $\mathcal{V} \models T$ and $\mathcal{W} \models T$ is an intermediate model between $\mathcal{V}$ and its class-forcing extension $\mathcal{V}[G] \models T$, then $\mathcal{W}$ is a class-forcing extension of $\mathcal{V}$.
- The simple Intermediate Model Theorem holds for $T$ if whenever $\mathcal{V} \models T$ and $\mathcal{W} \models T$ is a simple extension of $\mathcal{V}$ between $\mathcal{V}$ and its class-forcing extension $\mathcal{V}[G] \models T$, then $\mathcal{W}$ is a class-forcing extension of $\mathcal{V}$. 
The Class Intermediate Model Theorem: successes and failures

**Theorem:**
- (Friedman) The simple Intermediate Model Theorem for GBC fails.
- (Hamkins, Reitz) The simple Intermediate Model Theorem for GBC fails even for Ord-cc forcing.

**Theorem:** (Antos, Friedman, G.) If $M \models \text{ZFC}^-$ and $M[a] \models \text{ZFC}^-$, with $a \subseteq M$, is an intermediate model between $M$ and its set-forcing extension $M[G]$, then $M[a]$ is a set-forcing extension of $M$.
  - Use the Ord-cc class-complete Boolean completion $\mathbb{B}_\mathbb{P}$.

**Corollary:** (Antos, Friedman, G.) The simple Intermediate Model Theorem for KM+CC holds.

**Theorem:** (Antos, Friedman, G.) Every model $\mathcal{V} \models \text{KM}+\text{CC}$ has a forcing extension $\mathcal{V}[G] \models \text{KM} + \text{CC}$ with a non-simple intermediate model. Therefore the Intermediate Model Theorem for KM+CC fails.

**Question:** Does the simple Intermediate Model Theorem for KM hold?
Hyperclass forcing

The theory of hyperclass forcing was developed by Antos and Friedman.

Suppose $\mathcal{V} = \langle V, \in, C \rangle \models KM + CC$ and $\mathbb{P}$ is a hyperclass forcing of $\mathcal{V}$.

$G \subseteq \mathbb{P}$ is $\mathcal{V}$-generic if it meets every dense subhyperclass of $\mathbb{P}$.

Move to the companion model $M_\mathcal{V}$, where $\mathbb{P}$ is a definable class forcing, and form the forcing extension $M_\mathcal{V}[G]$.

Suppose $M_\mathcal{V}[G] \models ZFC^-_I$ with $\kappa$ as the largest cardinal.

- $\mathbb{P}$ is pretame.
- $\mathbb{P}$ preserves inaccessibility of $\kappa$.
- e.g, $\mathbb{P}$ is $\text{Ord}^{M_\mathcal{V}}$-cc.

Define $\mathcal{V}[G] = \langle W, \in, C^* \rangle$:

- $W = V_{\kappa}^{M_\mathcal{V}[G]}$
- $C^* = \{ C \subseteq V_{\kappa}^{M_\mathcal{V}[G]} | C \in M_\mathcal{V}[G] \}$
Useful hyperclass forcing

Suppose $\mathcal{V} = \langle V, \in, C \rangle \models \text{KM} + \text{CC}$ and $M_{\mathcal{V}}$ is the companion model of $\mathcal{V}$.

Hyperclass Boolean completions

Let $P \in C$ be a class forcing.

In $M_{\mathcal{V}}$:
- $B_P$ is a definable $\text{Ord}^{M_{\mathcal{V}}}$-cc class-complete Boolean algebra.
- $B_P$ is pretame.
- $P$ and $B_P$ have the same forcing extensions.
- The hyperclass forcing extensions by $B_P$ are precisely forcing extensions by $P$.

Hyperclass forcing and first-order set theory

- (Welch) Assuming large cardinals, models of the form $L[C]$ for a proper class club $C$ of uncountable cardinals are characterized as hyperclass forcing extensions of a truncated iterate of a mouse with large cardinals.
- (Friedman, G., Müller) Assuming large cardinals, models $L[C_1, \ldots, C_n]$ for specially nested clubs $C_i$ of uncountable cardinals as hyperclass forcing extensions of a truncated iterate of a mouse with stronger large cardinals.
- Uses class products of Prikry forcing with class supports.