Class forcing in its rightful setting

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Class forcing in set theory

Class forcing (forcing with a class partial order) is ubiquitous in set theory.

- Make GCH fail at every regular cardinal.
  - $\prod_{\alpha\in\text{Reg}}\mathrm{Add}(\alpha,\alpha^{++})$ with Easton support.
- Make $V=\mathrm{HOD}$ hold by coding all sets into the continuum function.
- Make all supercompact cardinals Laver indestructible.
- Add a global well-order (bijection $F:V\xrightarrow[\text{onto}]{1-1}\mathrm{Ord}$).
  - $\mathrm{Add}(\mathrm{Ord},1)$: conditions are partial functions $f:\alpha\to\mathrm{Ord}$ for $\alpha\in\mathrm{Ord}$ ordered by extension.
  - $<\mathrm{Ord}$-closed.
  - Does not add sets.
- Shoot a class club through a fat stationary class.
  - Conditions are closed sets of ordinals from the stationary class ordered by end-extension.
  - $<\mathrm{Ord}$-distributive.
  - Does not add sets.

The right setting for class forcing

Class forcing is fundamentally about classes.

- A generic filter for a class partial order is a subclass of the partial order meeting all dense subclasses.
- The properties of the partial order depend on which classes exist around it.

First-order set theory

- Classes are definable (with parameters) collections of sets.
- Classes are objects in the meta-theory.

Second-order set theory

- Classes are elements of the model.
- We can quantify over classes.
- We can study general properties of classes.
- The theory determines which classes exist.

We should not expect class forcing to be as nice as set forcing because classes even in the strongest second-order set theories do not behave as nicely as sets.

Second-order set theory

Second-order set theory has two sorts of objects: sets and classes.

Syntax: Two-sorted logic

- Separate variables and quantifiers for sets and classes
- Convention: upper-case letters for classes, lower-case letters for sets
- Notation:
  - Σ^0_n - first-order Σ_n-formula
  - Σ^1_n - n-alternations of class quantifiers followed by a first-order formula

Semantics: A model is a triple $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$.

- V consists of the sets.
- $\mathcal{C}$ consists of the classes.
- Every set is a class: $V \subseteq \mathcal{C}$.
- $C \subseteq V$ for every $C \in \mathcal{C}$.

Weakest second-order theories

Set axioms

- ZF^-: no choice, no powerset, collection instead of replacement
- ZFC^-: no powerset, collection instead of replacement, well-ordering principle instead of axiom of choice[1]

[1] Different formulations of the axiom of choice are not equivalent over ZF^-.

Weakest second-order set theories

Second-order theories: axioms for sets and classes

- GB
  - set axioms: ZF
  - Replacement: If F is a function and a is a set, then $F \restriction a$ is a set.
  - First-order comprehension: If $\varphi(x, A)$ is a first-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.
  - Every model of ZF with together with its definable collections is a model of GB.
- GBc: GB with ZF replaced by ZFC.
  - Every model of ZFC together with its definable collections is a model of GBc.
- GBC: Gödel-Bernays set theory
  - GBc
  - Global well-order: there is a bijection $F : V \xrightarrow[\text{onto}]{1-1} \mathrm{Ord}$.[2]
  - Every model of ZFC, with a definable global well-order, together with its definable collections is a model of GBC, e.g. L.
  - Every model of ZFC has a class forcing extension with the same sets satisfying GBC: force with $\mathrm{Add}(\mathrm{Ord}, 1)$.
  - Conservative over ZFC.
- GB^-, GBc^-, GBC^-: ZF^- instead of ZF.

[2] Different formulations of global choice are not equivalent over GBc^-.

Hyperclasses

Definition: In a model of second-order set theory, a hyperclass is a definable (with parameters) collection of classes.

- Analogue of classes in second-order.
- Third-order part of the model.

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$. Given a set a and a class A, the a-th slice of A is the class

$$\{x \mid \langle a, x\rangle \in A\},$$

where $\langle\, ,\, \rangle$ is the Gödel pairing function.

Definition: A hyperclass, given by a formula $\varphi(X, A)$, is coded by a class C if for every class B, $\varphi(B, A)$ holds if and only if there is a set b such that $C_b = B$.

"A coded hyperclass has class-many classes."

Transfinite recursion on classes along meta-ordinals

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: A meta-ordinal is a well-order $(\Gamma, \leq) \in \mathcal{C}$.

- Examples: Ord, $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \cdot \omega$.
- Notation: For $a \in \Gamma$, $\Gamma \upharpoonright a$ is the restriction of the well-order to $\leq$-predecessors of a.
- Meta-ordinals may not have unique representations.

Question: Does GBC prove that any two meta-ordinals are comparable?

Definition: Suppose $\Gamma \in \mathcal{C}$ is a meta-ordinal. A solution along Γ to a first-order recursion rule $\varphi(x, b, F)$ is a class S such that for every $b \in \Gamma$, $S_b = \{x \mid \varphi(x, b, S \upharpoonright b)\}$.

- $S_0 = \{x \mid \varphi(x, 0, \emptyset)\}$
- $S_1 = \{x \mid \varphi(x, 1, \langle S_0\rangle)\}$
- $S_2 = \{x \mid \varphi(x, 2, \langle S_0, S_1\rangle)\}$
- etc.

ETR: Elementary transfinite recursion

ETR: Every first-order recursion on classes along a meta-ordinal has a solution.

ETR_Γ: ETR restricted to a well-order Γ.

- $\mathrm{ETR}_{\mathrm{Ord}\cdot\omega}$, $\mathrm{ETR}_{\mathrm{Ord}}$, ETR_ω.

Theorem: (Fujimoto) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

- ETR_ω is equivalent to the assertion that for every class A, there is a truth predicate for the structure $\langle V, \in A\rangle$ (with a predicate for A).
  - Tarskian truth is given by a recursion of length ω each of whose levels is a class.
- ETR_Γ is equivalent to the existence, for every class A, of the iterated truth predicate of length Γ for the structure $\langle V, \in, A\rangle$.

Corollary: ETR is equivalent to the assertion that for every meta-ordinal Γ and every class A, there is an iterated truth predicate of length Γ for $\langle V, \in, A\rangle$.

Corollary: $\mathrm{GBC} + \mathrm{ETR}_\omega$ implies $\mathrm{Con}(\mathrm{ZFC})$.

Theorem: (Williams) If $\Gamma > \omega^\omega$ is a meta-ordinal, then $\mathrm{GBC} + \mathrm{ETR}_{\Gamma\cdot\omega}$ implies $\mathrm{Con}(\mathrm{GBC} + \mathrm{ETR}_\Gamma)$.

Theorem: $\mathrm{GBC} + \mathrm{ETR}$ implies that any two meta-ordinals are comparable.

A comprehension hierarchy to Kelley-Morse set theory

Σ^1_n-comprehension CA: Every Σ^1_n-formula defines a class.

Theorem: $\mathrm{GBC}+\Sigma^1_1\text{-}\mathrm{CA}$ implies ETR.

The theories $\mathrm{GBC}+\Sigma^1_{\mathrm{n}}\text{-}\mathrm{CA}$ form a hierarchy of strength culminating in Kelley-Morse set theory KM.

Kelley-Morse set theory KM

- GBC
- Every second-order formula defines a class.

Class choice principles

Choice Scheme CC: Given a second-order formula $\varphi(x, X, A)$, if for every set x, there is a class X witnessing $\varphi(x, X, A)$, then there is a class collecting witnesses for every x:

$$\forall x\, \exists X\, \varphi(x, X, A) \rightarrow \exists Y\, \forall x\, \varphi(x, Y_x, A).$$

Σ^1_n-CC: CC restricted to Σ^1_n-formulas.

Theorem: (G., Hamkins) Σ^0_1-CC can fail in a model of KM.

α**-Dependent Choice Scheme** DC_α: $(\omega \leq \alpha \leq \mathrm{Ord})$ Every second-order definable $<\alpha$-closed tree $T(X, A)$ has a branch of height α.

Σ^1_n-DC_α: DC_α restricted to Σ^1_n-definable trees.

Theorem: (Marek, Mostowski, Ratajczyk) If $\mathscr{V} \models \mathrm{GBC} + \Sigma^1_n\text{-CA}$, then its second-order constructible universe $\mathscr{L} \models \mathrm{GBC} + \Sigma^1_n$-Comprehension$+\Sigma^1_n$-$\mathrm{CC}$$+\Sigma^1_n$-$\mathrm{DC}_{\mathrm{Ord}}$.

- If $\mathscr{V} \models \mathrm{KM}$, then its second-order constructible universe $\mathscr{L} \models \mathrm{KM}{+}\mathrm{CC}{+}\mathrm{DC}_{\mathrm{Ord}}$.
    - Given a meta-ordinal Γ, we can build a meta-constructible universe L_Γ by a recursion of length Γ.
    - A meta-ordinal Γ is constructible if $\Gamma \in L_{\mathrm{Ord}^+}$.
    - **Theorem**: (Tharp) Constructible meta-ordinals have unique representations.
    - A class $A \in \mathcal{C}$ is constructible if there is a constructible meta-ordinal Γ such that $A \in L_{\mathrm{Ord}^+}$.
    - The second-order constructible universe is $\mathscr{L} = \langle L, \in, \mathcal{L} \rangle$, where $\mathcal{L}$ consists of the constructible classes.
- The theories KM and $\mathrm{KM}{+}\mathrm{CC}{+}\mathrm{DC}_{\mathrm{Ord}}$ are equiconsistent.

Applications of class choice

Theorem: Over GBC, Σ^1_n-CC implies Δ^1_n-CA. Consequently, $\mathrm{GBC}+\mathrm{CC}$ is equivalent to $\mathrm{KM}+\mathrm{CC}$.

Theorem: (G., Hamkins) Over GBC, Łoś' Lemma for second-order ultrapowers is equivalent to CC for set-many choices.

Theorem: (G., Hamkins) Over GBC, normal form for second-order formulas is equivalent to CC.

Theorem: (G., Hamkins, Karagila) Fodor's Lemma for class clubs

- can fail in KM,
- holds in $\mathrm{GBC}+\Sigma^0_2$-CC.

Aside

Theorem: (Enayat) $\mathrm{GBC}+\Sigma^1_1$-CC is conservative over ZFC.

- Every countable model of ZFC can be elementarily extended to countable recursively saturated model of ZFC.
- Every countable recursively saturated model of ZFC can be expanded to a model of $\mathrm{GBC}+\Sigma^1_1$-CC without adding sets.

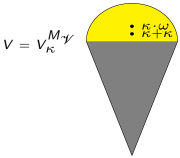
Moving to first-order with KM+CC

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM{+}CC}$.

- View each extensional well-founded class relation $R \in \mathcal{C}$ as coding a transitive set.
  - $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \cdot \omega$
  - $V \cup \{V\}$
- Define a membership relation E on the collection of all such relations R (modulo isomorphism).
- Let $\langle M_{\mathscr{V}}, \mathsf{E}\rangle$, the companion model of $\mathscr{V}$, be the resulting first-order structure.
  - $M_{\mathscr{V}}$ has the largest cardinal $\kappa \cong \mathrm{Ord}^{\mathscr{V}}$.
  - $V_\kappa^{M_{\mathscr{V}}} \cong V$.
  - $\mathcal{P}(V_\kappa)^{M_{\mathscr{V}}} \cong \mathcal{C}$.
  - $\langle M_{\mathscr{V}}, \mathsf{E}\rangle \models \mathrm{ZFC}^-_{\mathrm{I}}$.

$\mathrm{ZFC}^-_{\mathrm{I}}$

- ZFC^-
- There is a largest cardinal κ.
- κ is inaccessible: κ is regular and for all $\alpha < \kappa$, 2^α exists and $2^\alpha < \kappa$.
  - V_κ exists.
  - $V_\kappa \models \mathrm{ZFC}$.



Bi-interpretability

Suppose $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with a largest cardinal κ.

- $V = V^M_\kappa$
- $\mathcal{C} = \{X \in M \mid X \subseteq V^M_\kappa\}$
- $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM{+}CC}$
- $M_{\mathscr{V}} \cong M$ is the companion model of $\mathscr{V}$.

Theorem: (Marek) The theory $\mathrm{KM{+}CC}$ is bi-interpretable with the theory $\mathrm{ZFC}^-_{\mathrm{I}}$.

Class forcing interpretation

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing.

Definition:

- A class $\mathbb{P}$-name is a collection of pairs $\langle \sigma, p\rangle$ such that $p \in \mathbb{P}$ and $\sigma \in V^{\mathbb{P}}$.
- G is $\mathscr{V}$-generic for $\mathbb{P}$ if G meets every dense subclass $D \in \mathcal{C}$ of $\mathbb{P}$.
- The forcing extension $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G]\rangle$.

Definability of forcing relations

Suppose $\mathscr{V}=\langle V,\in,\mathcal{C}\rangle\models\mathrm{GBC}$ and $\mathbb{P}\in\mathcal{C}$ is a class forcing.

The Class Forcing Theorem for $\mathbb{P}$: There is a solution to the recursion defining the forcing relation for atomic formulas.

- $p\Vdash\sigma\in\tau$: there is a dense set of conditions $q\leq p$ for which there is $\langle\rho,r\rangle\in\tau$ with $q\leq r$ and $q\Vdash\sigma=\rho$.
- $p\Vdash\sigma=\tau$: $p\Vdash\sigma\subseteq\tau$ and $p\Vdash\tau\subseteq\sigma$.
- $p\Vdash\sigma\subseteq\tau$: whenever $\langle\rho,r\rangle\in\sigma$ and $q'\leq p,r$, there is $q\leq q'$ with $q\Vdash\rho\in\tau$.

The Class Forcing Theorem implies that forcing relations for all second-order formulas are definable.

- $p\Vdash\sigma\in\Gamma$: there are densely many $q\leq p$ for which there is $\langle\tau,r\rangle\in\Gamma$ with $q\leq r$ and $q\Vdash\sigma=\tau$.
- $p\Vdash\varphi\wedge\psi$: $p\Vdash\varphi$ and $p\Vdash\psi$.
- $p\Vdash\neg\varphi$: there is no $q\leq p$ with $q\Vdash\varphi$.
- $p\Vdash\forall x\,\varphi(x)$: $p\Vdash\varphi(\tau)$ for every $\mathbb{P}$-name τ.
- $p\Vdash\forall X\,\varphi(X)$: $p\Vdash\varphi(\Delta)$ for every class $\mathbb{P}$-name Δ.

Nice class forcing

Definition: A class forcing $\mathbb{P}$ has the Ord-cc if every antichain of $\mathbb{P}$ is a set.

Definition: (Friedman) A class forcing $\mathbb{P}$ is pretame if for every class sequence $\langle D_x \mid x \in a\rangle \in \mathcal{C}$ of dense classes of $\mathbb{P}$, indexed by elements of a set a, and condition $p \in \mathbb{P}$, there is a condition $q \leq p$ and a sequence $\langle d_x \mid x \in a\rangle$ of subsets of $\mathbb{P}$ such that each $d_x \subseteq D_x$ is pre-dense below q in $\mathbb{P}$.

- "Reduces dense classes to pre-dense sets."
- Condition to preserve replacement

Definition: (Friedman) A class forcing $\mathbb{P}$ is tame if it is pretame and for every $p \in \mathbb{P}$, there is $q \leq p$ and ordinal α such that whenever $\vec{D} = \{\langle D_0^x, D_1^x\rangle \mid x \in a\} \in \mathcal{C}$, for a set a, is a sequence of pre-dense partitions below q, then the class

$$\{r \in \mathbb{P} \mid \vec{D} \text{ is equivalent below } r \text{ to some partition } \vec{E} \in V_\alpha\}$$

is dense below q.

- Condition to preserve powerset
- Ord-cc forcings
- Progressively closed Ord-length products and iterations of set forcing
- $<\mathrm{Ord}$-distributive forcing

Preserving the theory

Class forcing need not preserve ZFC.

- $\mathrm{Coll}(\omega, \mathrm{Ord})$.
- (Friedman) Forcing $\mathbb{F}$ to code $\langle V, \in\rangle$ into a relation E on ω.
  - In the extension: $\langle V, \in\rangle \cong \langle\omega, E\rangle$.

Question: Are class forcing extensions always closed under complements?

Theorem: (Friedman) Pretameness is equivalent to preservation of GB^-. Tameness is equivalent to preservation of GB.

Theorem: Tame forcings preserve:

- GBc, GBC.
- (Antos) KM.
- (Antos, G., Friedman) $\mathrm{KM}+\mathrm{CC}$.
- (Antos, G., Friedman) $\mathrm{KM}+\mathrm{CC}+\mathrm{DC}_\alpha$ for every $\omega \leq \alpha \leq \mathrm{Ord}$.

Theorem: Tame forcings preserve $\mathrm{GBC} + \Sigma^1_\mathrm{n}\text{-CA}+\Sigma^1_n\text{-CC}$.

Question: Do tame forcings preserve $\mathrm{GBC} + \Sigma^1_\mathrm{n}\text{-CA}$?

- Without $\Sigma^1_n\text{-AC}$, the forcing relation for Σ^1_n-formulas may not be Σ^1_n-definable.

Class Forcing Theorem

Theorem: (Holy, Krapf, Lücke, Njegomir, Schlicht) The Class Forcing Theorem can fail in a model of GBC.

- The Class Forcing Theorem for Friedman's forcing $\mathbb{F}$ implies existence of truth predicate for $\langle V, \in \rangle$.

Theorem: (Stanley) In GB^-, the Class Forcing Theorem holds for all pretame forcings.

- In ZFC, pretame forcings have definable forcing relations.
- In ZFC^- pretame forcings have definable forcing relations.

Theorem: (G., Hamkins, Holy, Schlicht, Williams) Over GBC, the Class Forcing Theorem is equivalent to $\mathrm{ETR}_{\mathrm{Ord}}$.

- The recursion to define the forcing relation on atomic formulas has length Ord.
- The atomic forcing relation for Friedman's forcing $\mathbb{F}$ yields an iterated truth predicate of length Ord for $\langle V, \in \rangle$.
- Define a version of Friedman's forcing $\mathbb{F}_A$ for a class A.

Dense embeddings

Theorem: Suppose $\mathbb{P}$ and $\mathbb{Q}$ are set forcings and there is a dense embedding from $\mathbb{P}$ into $\mathbb{Q}$. Then $\mathbb{P}$ and $\mathbb{Q}$ have the same forcing extensions.

Definition:

- $\mathrm{Coll}(\omega, \mathrm{Ord})$: conditions $f : A \to \mathrm{Ord}$, where A is a finite subset of ω.
- $\mathrm{Coll}_*(\omega, \mathrm{Ord})$: conditions $f : n \to \mathrm{Ord}$ (no holes!).

Theorem (Holy, Krapf, Lücke, Njegomir, Schlicht)

- $\mathrm{Coll}_*(\omega, \mathrm{Ord})$ densely embeds into $\mathrm{Coll}(\omega, \mathrm{Ord})$.
- $\mathrm{Coll}(\omega, \mathrm{Ord})$ adds a bijection between every ordinal and ω.
- $\mathrm{Coll}_*(\omega, \mathrm{Ord})$
  - adds a class $f : \omega \xrightarrow[\text{onto}]{1-1} \mathrm{Ord}$,
  - does not add sets.

Theorem: (Holy, Krapf, Lücke, Njegomir, Schlicht) Assume GBC^-. If $\mathbb{P}$ and $\mathbb{Q}$ are pretame class forcings such that $\mathbb{P}$ densely embeds into $\mathbb{Q}$, then $\mathbb{P}$ and $\mathbb{Q}$ have the same forcing extensions.

Nice names

Definition: Suppose $\mathbb{P}$ is a forcing. A nice $\mathbb{P}$-name for a subset of ordinals has the form $\bigcup_{\xi<\alpha}\{\check{\xi}\} \times A_\xi$, where each $A_\xi \subseteq \mathbb{P}$ is an antichain.

Theorem: Suppose $\mathbb{P}$ is a set forcing. Then every subset of ordinals in a forcing extension by $\mathbb{P}$ has a nice $\mathbb{P}$-name.

Theorem: (Holy, Krapf, Schlicht) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$ and $G \subseteq \mathrm{Coll}(\omega, \mathrm{Ord})$ is $\mathscr{V}$-generic. In $\mathscr{V}[G]$, the set

$$A = \{n \in \omega \mid G(n) = 0\}$$

does not have a nice-$\mathrm{Coll}(\omega, \mathrm{Ord})$-name.

Theorem: (Holy, Krapf, Schlicht)

- Assume GBC^-. Every pretame class forcing $\mathbb{P}$ has nice $\mathbb{P}$-names for all subsets of ordinals.
- Assume $\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$. A class forcing $\mathbb{P}$ that has nice $\mathbb{P}$-names for all subsets of ordinals is pretame.
  - Uses existence of definable forcing relation.

Ground model definability

Theorem: (Laver, Woodin) The ground model V is uniformly definable from a parameter in any set-forcing extension.

Theorem: (Antos) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$. There is a class forcing $\mathbb{P}$ such that $\langle V, \in\rangle$ is not definable from parameters over $V[G]$ in any forcing extension $\mathscr{V}[G]$ by $\mathbb{P}$.

- $\mathbb{P}$ is the Easton support product $\prod_{\alpha\in\mathrm{Reg}} \mathrm{Add}(\alpha, 1)$.
- $\mathrm{Add}(\alpha, 1) \cong \mathrm{Add}(\alpha, 1) \times \mathrm{Add}(\alpha, 1)$.
- Use an automorphism.

Theorem: (G., Johnstone) It is consistent that $H_{\kappa^+}(\models \mathrm{ZFC}^-_\mathrm{I})$ is not definable, even with parameters, in its forcing extension $H_{\kappa^+}[G]$ by $\mathrm{Add}(\kappa, 1)$.

Corollary: There is a model $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}{+}\mathrm{CC}$ such that $\mathcal{C}$ is not a hyperclass of its forcing extensions $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G]\rangle$ by $\mathrm{Add}(\mathrm{Ord}, 1)$, even with a parameter from $\mathcal{C}[G]$.

Theorem: (Asperó) It is consistent (from large cardinals) that $H_{\kappa^+}(\models \mathrm{ZFC}^-_\mathrm{I})$ is not definable, even with parameters, in its forcing extension $H_{\kappa^+}[G]$ by $\mathrm{Add}(\omega, 1)$.

Corollary: Ground model definability can fail in $\mathrm{KM}{+}\mathrm{CC}$ even for set-forcing.

Boolean completions

Theorem: Every set forcing $\mathbb{P}$ densely embeds into a unique (modulo isomorphism) complete Boolean algebra.

- $U \subseteq \mathbb{P}$ is a cut if it is closed downwards: if $p \in U$ and $q \leq p$, then $q \in U$.
- For $p \in \mathbb{P}$, $U_p = \{q \in \mathbb{P} \mid q \leq p\}$ is a cut.
- A cut U is regular if for every $p \notin U$, there is $q \leq p$ such that $U_q \cap U = \emptyset$.
- The Boolean algebra $\mathbb{B}_\mathbb{P}$ consists of all regular cuts of $\mathbb{P}$.

Definition: A class Boolean algebra is:

- set-complete if it has suprema for all its subsets,
- class-complete if it has suprema for all its subclasses.

Theorem: (Holy, Krapf, Lücke, Njegomir, Schlicht) Assume GBC. A class forcing has:

- a Boolean set-completion if and only if the Class Forcing Theorem holds for it.
  - The forcing relation can be used to construct a Boolean set-completion.
  - Defining Boolean values of assertions in the forcing language does not require ETR.
- a Boolean class-completion if and only if it has the Ord-cc.
- a unique Boolean set-completion if and only if it has the Ord-cc.

A class forcing with a proper class antichain cannot have a Boolean class-completion!

Hyperclass Boolean completions

Definition: Suppose $\mathscr{V} \models \mathrm{GBC}$.

- Suppose $\mathbb{P} \in \mathcal{C}$ is a class forcing. The regular cuts of $\mathbb{P}$ form a hyperclass Boolean algebra $\mathbb{B}_{\mathbb{P}}$.
- A hyperclass Boolean algebra is:
  - class-complete if it has suprema for all its coded subhyperclasses.
  - hyperclass-complete if it has suprema for all its subhyperclasses.

Theorem: Suppose $\mathscr{V} \models \mathrm{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing. Then $\mathbb{B}_{\mathbb{P}}$ is class-complete.

Theorem: (Antos, Friedman, G.) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is not Ord-cc. If $\mathbb{B}_{\mathbb{P}}$ is hyperclass-complete, then $\mathscr{V} \models \mathrm{KM}$.

- Code instances of comprehension into suprema of antichains of $\mathbb{B}_{\mathbb{P}}$.

Theorem: (Antos, Friedman, G.) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing. Every hyperclass antichain of $\mathbb{B}_{\mathbb{P}}$ is coded.

Corollary: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM}+\mathrm{CC}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing. In the companion model $M_{\mathscr{V}}$, $\mathbb{B}_{\mathbb{P}}$ is an $\mathrm{Ord}^{M_{\mathscr{V}}}$-cc class-complete Boolean algebra.

The Class Intermediate Model Theorem

Intermediate Model Theorem: (Solovay)

- If $V \models \mathrm{ZFC}$ and $W \models \mathrm{ZFC}$ is an intermediate model between V and its set-forcing extension $V[G]$, then W is a set-forcing extension of V.
- If $V \models \mathrm{ZF}$ and $V[a] \models \mathrm{ZF}$, with $a \subseteq V$, is an intermediate model between V and its set-forcing extension $V[G]$, then $V[a]$ is a set-forcing extension of V.

Definition: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$. Then $\mathscr{W} = \langle W, \in, \mathcal{C}^*\rangle$ is a simple extension of $\mathscr{V}$ if $\mathcal{C}^*$ is generated by $\mathcal{C}$ together with a single new class.

- Forcing extensions are simple extensions.

Definition: Suppose T is a second-order set theory.

- The Intermediate Model Theorem holds for T if whenever $\mathscr{V} \models T$ and $\mathscr{W} \models T$ is an intermediate model between $\mathscr{V}$ and its class-forcing extension $\mathscr{V}[G] \models T$, then $\mathscr{W}$ is a class-forcing extension of $\mathscr{V}$.
- The simple Intermediate Model Theorem holds for T if whenever $\mathscr{V} \models T$ and $\mathscr{W} \models T$ is a simple extension of $\mathscr{V}$ between $\mathscr{V}$ and its class-forcing extension $\mathscr{V}[G] \models T$, then $\mathscr{W}$ is a class-forcing extension of $\mathscr{V}$.

The Class Intermediate Model Theorem: successes and failures

Theorem:

- (Friedman) The simple Intermediate Model Theorem for GBC fails.
- (Hamkins, Reitz) The simple Intermediate Model Theorem for GBC fails even for Ord-cc forcing.

Theroem: (Antos, Friedman, G.) If $M \models \mathrm{ZFC}^-$ and $M[a] \models \mathrm{ZFC}^-$, with $a \subseteq M$, is an intermediate model between M and its set-forcing extension $M[G]$, then $M[a]$ is a set-forcing extension of M.

- Use the Ord-cc class-complete Boolean completion $\mathbb{B}_{\mathbb{P}}$.

Corollary: (Antos, Friedman, G.) The simple Intermediate Model Theorem for $\mathrm{KM+CC}$ holds.

Theorem: (Antos, Friedman, G.) Every model $\mathscr{V} \models \mathrm{KM+CC}$ has a forcing extension $\mathscr{V}[G] \models \mathrm{KM} + \mathrm{CC}$ with a non-simple intermediate model. Therefore the Intermediate Model Theorem for $\mathrm{KM+CC}$ fails.

Question: Does the simple Intermediate Model Theorem for KM hold?

Hyperclass forcing

The theory of hyperclass forcing was developed by Antos and Friedman.

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM} + \mathrm{CC}$ and $\mathbb{P}$ is a hyperclass forcing of $\mathscr{V}$.

$G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic if it meets every dense subhyperclass of $\mathbb{P}$.

Move to the companion model $M_{\mathscr{V}}$, where $\mathbb{P}$ is a definable class forcing, and form the forcing extension $M_{\mathscr{V}}[G]$.

Suppose $M_{\mathscr{V}}[G] \models \mathrm{ZFC}^-_\mathrm{I}$ with κ as the largest cardinal.

- $\mathbb{P}$ is pretame.
- $\mathbb{P}$ preserves inaccessibility of κ.
- e.g, $\mathbb{P}$ is $\mathrm{Ord}^{M_{\mathscr{V}}}$-cc.

Define $\mathscr{V}[G] = \langle W, \in, \mathcal{C}^*\rangle$:

- $W = V_\kappa^{M_{\mathscr{V}}[G]}$
- $\mathcal{C}^* = \{C \subseteq V_\kappa^{M_{\mathscr{V}}[G]} \mid C \in M_{\mathscr{V}}[G]\}$

Useful hyperclass forcing

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM} + \mathrm{CC}$ and $M_{\mathscr{V}}$ is the companion model of $\mathscr{V}$.

Hyperclass Boolean completions

Let $\mathbb{P} \in \mathcal{C}$ be a class forcing.

In $M_{\mathscr{V}}$:

- $\mathbb{B}_{\mathbb{P}}$ is a definable $\mathrm{Ord}^{M_{\mathscr{V}}}$-cc class-complete Boolean algebra.
- $\mathbb{B}_{\mathbb{P}}$ is pretame.
- $\mathbb{P}$ and $\mathbb{B}_{\mathbb{P}}$ have the same forcing extensions.
- The hyperclass forcing extensions by $\mathbb{B}_{\mathbb{P}}$ are precisely forcing extensions by $\mathbb{P}$.

Hyperclass forcing and first-order set theory

- (Welch) Assuming large cardinals, models of the form $L[C]$ for a proper class club C of uncountable cardinals are characterized as hyperclass forcing extensions of a truncated iterate of a mouse with large cardinals.
- (Friedman, G., Müller) Assuming large cardinals, models $L[C_1, \ldots, C_n]$ for specially nested clubs C_i of uncountable cardinals as hyperclass forcing extensions of a truncated iterate of a mouse with stronger large cardinals.
- Uses class products of Prikry forcing with class supports.