Generic Vopěnka's Principle

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Vopěnka's Principle and imitations

Vopěnka's Principle: For every proper class $\mathcal{C}$ of first-order structures of the same type, there are $B \neq A$, both in $\mathcal{C}$, such that B elementarily embeds into A.

- Vopěnka's Principle is a second-order assertion.
- $\mathrm{VP}(\boldsymbol{\Sigma}_n)$: Vopěnka's Principle fragment for $\boldsymbol{\Sigma}_n$-definable classes.
- $\mathrm{VP}(\Sigma_n)$: Vopěnka's Principle fragment for Σ_n-definable without parameters classes.

First-order Vopěnka's Principle: Scheme asserting $\mathrm{VP}(\boldsymbol{\Sigma}_{\mathrm{n}})$ for every $n \in \omega$.

Theorem: (Hamkins) There are models of Gödel-Bernays set theory in which first-order Vopěnka's Principle holds, but Vopěnka's Principle fails. The two principles are equiconsistent.

Let κ be a cardinal.

$\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_{\mathrm{n}})$: (Bagaria) For every proper class $\mathcal{C}$ of first-order structures of the same type that is Σ_n-definable with parameters from H_κ, for every $A \in \mathcal{C}$, there is $B \in \mathcal{C} \cap H_\kappa$ such that B elementarily embeds into A.

- "The class $\mathcal{C}$ of structures reflects below κ".
- At first sight, $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_{\mathrm{n}})$ appears to be much stronger than $\mathrm{VP}(\Sigma_{\mathrm{n}})$.

Strength of Vopěnka-like principles

Relevant large cardinals

Let $C^{(n)}$ be the class of all δ such that $V_\delta \prec_{\Sigma_n} V$.

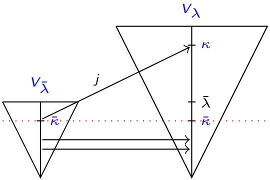
Definition:

- A cardinal κ is extendible if for every $\alpha > \kappa$, there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(k) > \alpha$.
- (Bagaria) A cardinal κ is $C^{(n)}$-extendible if for every $\alpha > \kappa$, there is an extendibility embedding $j : V_\alpha \to V_\beta$ with $j(\kappa) \in C^{(n)}$.

Lemma:

- Extendible cardinals are $C^{(1)}$-extendible.
- (Bagaria) A $C^{(n+1)}$-extendible cardinal is a limit of $C^{(n)}$-extendible cardinals.

Theorem: (Magidor) A cardinal κ is supercompact iff for every $\lambda > \kappa$, there is $\bar{\lambda} < \kappa$ such that there is $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$.



Strength of Vopěnka-like principles (continued)

Lemma: $\mathrm{VP}(\kappa, \mathbf{\Sigma}_1)$ holds for every uncountable κ.

Theorem: (Bagaria) (Lightface)

- The following are equivalent:
  1. There exists a supercompact cardinal.
  2. $\mathrm{VP}(\Sigma_2)$ holds.
  3. $\mathrm{VP}(\kappa, \mathbf{\Sigma}_2)$ holds for some κ.
- The following are equivalent for $n \geq 1$:
  1. There exists a $C^{(n)}$-extendible cardinal.
  2. $\mathrm{VP}(\Sigma_{n+2})$ holds.
  3. $\mathrm{VP}(\kappa, \mathbf{\Sigma}_{n+2})$ holds for some κ.

Theorem: (Bagaria) (Boldface)

- The following are equivalent:
  1. There exists a proper class of supercompact cardinals.
  2. $\mathrm{VP}(\mathbf{\Sigma}_2)$ holds.
  3. $\mathrm{VP}(\kappa, \mathbf{\Sigma}_2)$ holds for a proper class of κ.
- The following are equivalent for $n \geq 1$:
  1. There exists a proper class of $C^{(n)}$-extendible cardinals.
  2. $\mathrm{VP}(\mathbf{\Sigma}_{n+2})$ holds.
  3. $\mathrm{VP}(\kappa, \mathbf{\Sigma}_{n+2})$ holds for a proper class of κ.

Characterization of first-order Vopěnka's Principle

Corollary: First-order Vopěnka's Principle holds iff for every $n \in \omega$, there is a proper class of $C^{(n)}$-extendible cardinals.

Theorem: (Bagaria)

- The least κ such that $\mathrm{VP}(\kappa, \mathbf{\Sigma}_2)$ holds is the least supercompact cardinal.
- The least κ such that $\mathrm{VP}(\kappa, \mathbf{\Sigma}_{n+2})$ hold for $n \geq 1$ is the least $C^{(n)}$-extendible cardinal.

Generic Vopěnka's Principle and imitations

Generic Vopěnka's Principle: For every proper class $\mathcal{C}$ of first-order structures of the same type, there are $B \neq A$, both in $\mathcal{C}$, such that B elementarily embeds into A in some set-forcing extension.

- $\mathrm{gVP}(\boldsymbol{\Sigma}_n)$: Generic Vopěnka's Principle fragment for $\boldsymbol{\Sigma}_n$-definable classes.
- $\mathrm{gVP}(\Sigma_n)$: Generic Vopěnka's Principle fragment for Σ_n-definable without parameters classes.

First-order Generic Vopěnka's Principle: Scheme asserting $\mathrm{gVP}(\boldsymbol{\Sigma}_\mathrm{n})$ for every $n \in \omega$.

Let κ be a cardinal.

$\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_\mathrm{n})$: For every proper class $\mathcal{C}$ of first-order structures of the same type that is Σ_n-definable with parameters from H_κ, for every $A \in \mathcal{C}$, there is $B \in \mathcal{C} \cap H_\kappa$ such that B elementarily embeds into A in some set-forcing extension.

Absoluteness Lemma for embeddings on countable structures

Suppose B and A are (first-order) structures in the same language.

Lemma: (Absoluteness Lemma for embeddings on countable structures)
Suppose B is countable and B elementarily embeds into A. If W is a transitive (set or class) model of (a sufficiently large fragment of) ZFC such that

- $B, A \in W$,
- B is countable in W,

then B elementarily embeds into A in W.

Proof:

- Enumerate $B = \{b_n \mid n < \omega\}$ in W. Let $B \restriction n = \{b_i \mid i < n\}$.
- Let T be the tree of all partial finite isomorphisms

$$f : B \restriction n \to A$$

  ordered by extension.
- B elementarily embeds into A if and only if T has a cofinal branch.
- T is ill-founded in V, and hence in W. $\square$

Why do embeddings exist in a set-forcing extension?

Observation: In a $V^{\mathrm{Coll}(\omega,\mathbb{R})}$, there is an isomorphism $j : \mathbb{R}^V \to \mathbb{Q}$.

Proof: In $V^{\mathrm{Coll}(\omega,\mathbb{R})}$, $\mathbb{R}^V$ is a countable dense linear order without endpoints. $\square$

Observation: Suppose $0^\#$ exists. Then there is an elementary $j : L_{\omega_1^V} \to L_{\omega_2^V}$ in $L^{\mathrm{Coll}(\omega,\omega_1^V)}$.

Proof:

- V has an elementary $h : L_{\omega_1^V} \to L_{\omega_2^V}$.
- Suppose $G \subseteq \mathrm{Coll}(\omega, \omega_1^V)$ is V-generic.
- $h \in V[G]$ and $L[G] \subseteq V[G]$.
- $L[G]$ has some $j : L_{\omega_1^V} \to L_{\omega_2^V}$ (by Absoluteness Lemma). $\square$

When do embeddings exist in a set-forcing extension? Part I

Suppose B and A are (first-order) structures in the same language.

Theorem: The following are equivalent.

(1) B elementarily embeds into A in some set-forcing extension.

(2) B elementarily embeds into A in $V^{\mathrm{Coll}(\omega,B)}$.

Proof:
(2) $\Rightarrow$ (1): Trivial.
(1) $\Rightarrow$ (2): Suppose a set-forcing extension $V[G]$ has an elementary $j : B \to A$.

- Let $|B|^V = \delta$.
- Consider a further extension $V[G][H]$ by $\mathrm{Coll}(\omega, \delta)$.
- $j \in V[G][H]$ and B is countable in $V[G][H]$.
- $V[H] \subseteq V[G][H]$ has some elementary $j^* : B \to A$ (by Absoluteness Lemma). □

When do embeddings exist in a set-forcing extension? Part II

Suppose B and A are (first-order) structures in the same language.

Let $G(B,A)$ be an ω-length Ehrenfeucht-Fraïssé type game:

- Stage n: player I plays some $b_n \in B$ and player II plays some $a_n \in A$.
- Player II wins if for every $n \in \omega$ and formula $\varphi(x_0,\ldots,x_n)$,
$$B \models \varphi(b_0,\ldots,b_n) \leftrightarrow A \models \varphi(a_0,\ldots,a_n),$$
  and otherwise player I wins.
- If player II loses, she must do so in finitely many steps.
- $G(B,A)$ is closed, and hence determined by the Gale-Stewart Theorem.

Theorem: The following are equivalent.

(1) Player II has a winning strategy in $G(B,A)$.

(2) B elementarily embeds into A in $V^{\mathrm{Coll}(\omega,B)}$.

Proof:

(1) $\Rightarrow$ (2): A winning strategy for player II, remains winning in $V^{\mathrm{Coll}(\omega,B)}$ because no new finite sequences are added.

(2) $\Rightarrow$ (1): Fix $p \Vdash$ "$\tau : \check{B} \to \check{A}$ is an elementary embedding".

- To every finite $\vec{b}$ from B, associate $p_{\vec{b}} \Vdash \tau(\vec{b}) = \vec{a}$ below p so that: if $\vec{b}'$ extends $\vec{b}$, then $p_{\vec{b}'} \leq p_{\vec{b}}$.
- A winning strategy for player II: play $\vec{a}$ in response to $\vec{b}$. $\square$

Strength of Generic Vopěnka-like principles

Relevant large cardinals: virtual large cardinals

Suppose $\mathcal{A}$ is a very large cardinal property

- supercompact
- extendible, $C^{(n)}$-extendible
- rank-into-rank

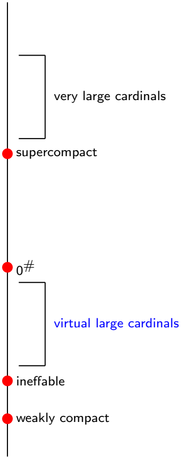
characterized by the existence of "suitable" set-sized embeddings.

(Certain closure requirements are not allowed.)

We say that a cardinal is virtually $\mathcal{A}$ if the embeddings of V-structures characterizing $\mathcal{A}$ exist in set-forcing extensions.

Virtual large cardinals are mini versions of their actual counterparts.

- Silver indiscernibles are virtual large cardinals.
- Virtual large cardinals are situated between ineffables and $0^\#$.
- Virtual large cardinals are downward absolute to L.



Remarkable cardinals

Definition: (Schindler) A cardinal κ is remarkable if for every $\lambda > \kappa$, there is $\bar{\lambda} < \kappa$ such that in some set-forcing extension there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$.

Remarkable cardinals are virtually supercompact!

Theorem The following assertions are equiconsistent with a remarkable cardinal.

- (Schindler) The theory of $L(\mathbb{R})$ cannot be changed by proper forcing.
- (Schindler) The weak proper forcing axiom wPFA.
- (Fuchs, Minden) The weak forcing axiom for subcomplete forcing wSCFA.

Lemma: If κ is remarkable, then for every $\lambda > \kappa$ and $a \in V_\lambda$, there is $\bar{\lambda} < \kappa$ such that in some set-forcing extension there is a remarkability embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $a \in \mathrm{ran}(j)$.

Corollary: If κ is remarkable, then for every $\lambda > \kappa$ and $\alpha < \kappa$, there is $\bar{\lambda} < \kappa$ such that in some set-forcing extension there is a remarkability embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $\mathrm{crit}(j) > \alpha$.

n-remarkable cardinals

Definition: (Bagaria, G., Schindler) A cardinal κ is n-remarkable, for $n \geq 1$, if for every $\lambda > \kappa$ in $C^{(n)}$, there is $\bar{\lambda} < \kappa$ also in $C^{(n)}$ such that in some set-forcing extension, there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$.

Lemma:

- A remarkable cardinal is 1-remarkable.
- An $n+1$-remarkable cardinal is a limit of n-remarkable cardinals.

Lemma: If κ is n-remarkable, then for every $\lambda > \kappa$ in $C^{(n)}$ and $a \in V_\lambda$, there is $\bar{\lambda} < \kappa$ also in $C^{(n)}$ such that in some set-forcing extension there is a remarkability embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $a \in \text{ran}(j)$.

Question: What are n-remarkable cardinals virtualizing?

More virtual large cardinals

Definition:

- A cardinal κ is virtually extendible if for every $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.
- A cardinal κ is virtually $C^{(n)}$-extendible if for every $\alpha > \kappa$, in a set-forcing extension, there is an extendibility embedding $j : V_\alpha \to V_\beta$ with $j(\kappa) \in C^{(n)}$.
- A cardinal κ is virtually rank-into-rank if for some $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V_\alpha \to V_\alpha$ with $\text{crit}(j) = \kappa$.

Observation: (Beyond Kunen's Inconsistency) A set-forcing extension can have an elementary $j : V_\alpha \to V_\alpha$ with $\alpha \gg \lambda$, the supremum of the critical sequence.

Proof: If κ is a Silver indiscernible and $\alpha \gg \kappa$ is uncountable in V, then there is an elementary $j : L_\alpha \to L_\alpha$ with $\text{crit}(j) = \kappa$. □

Lemma: If κ is virtually rank-into-rank, then V_κ is a model of proper class many virtually $C^{(n)}$-extendible cardinals for every $n \in \omega$.

Virtually $C^{(n)}$-extendible cardinals and n-remarkable cardinals

Theorem: (Bagaria, G., Schindler) A cardinal κ is $n+1$-remarkable iff it is virtually $C^{(n)}$-extendible.

We can generalize Magidor's characterization of supercompact cardinals to $C^{(n)}$-extendible cardinals and produce a unifying definition.

Theorem: A cardinal κ is $C^{(n)}$-extendible iff for every $\lambda > \kappa$ in $C^{(n+1)}$, there is a $\bar{\lambda} < \kappa$ also in $C^{(n+1)}$ such that there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$.

Question: How strong are the virtual large cardinals?

α-iterable cardinals

Definition: A weak κ-model (for a cardinal κ) is a transitive $M \models \mathrm{ZFC}^-$ of size κ and height above κ.

Suppose M is a weak κ-model.

Proposition: The following are equivalent.

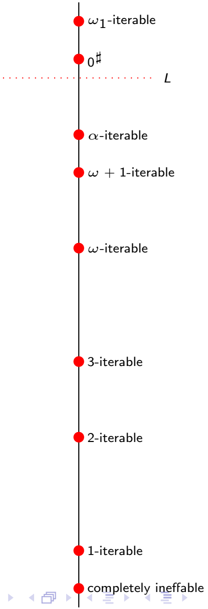
- There exists an elementary $j : M \to N$ with $\mathrm{crit}(j) = \kappa$.
- There exists an M-ultrafilter U with a well-founded ultrapower.
  - U is an M-ultrafilter if $\langle M, \in, U\rangle \models U$ is a normal ultrafilter.
  - $U = \{A \in M \mid \kappa \in j(A)\}$.

Definition: An M-ultrafilter U is weakly amenable if for every $X \in M$ with $|X|^M \leq \kappa$, $X \cap U \in M$.

- U is partially internal to M.
- Weak amenability is needed to iterate the ultrapower construction.

Definition: (G.) A cardinal κ is α-iterable ($1 \leq \alpha \leq \omega_1$) if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable M-ultrafilter on κ with α-many well-founded iterated ultrapowers.

α-iterable cardinals in the hierarchy



Theorem: (G., Welch, '08)

- A 1-iterable cardinal is a limit of completely ineffable cardinals.
- An α-iterable cardinal is a limit of β-iterable cardinals for every $\beta < \alpha$.

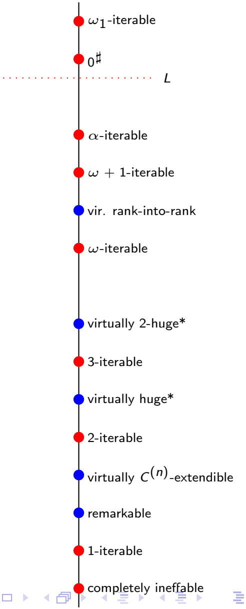
Virtual large cardinals in the hierarchy

Theorem: (G., Schindler)

- A remarkable cardinal is a 1-iterable limit of 1-iterable cardinals.
- If κ is 2-iterable, then V_κ is a model of proper class many virtually $C^{(n)}$-extendible cardinals for every $n \in \omega$.
- A virtually rank-into-rank cardinal is an ω-iterable limit of ω-iterable cardinals.
- An $\omega + 1$-iterable cardinal implies the consistency of a virtually rank-into-rank cardinal.

Question: What is between an n-iterable cardinal and an $n + 1$-iterable cardinal?

Definition: A cardinal κ is virtually n-huge* if for some $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$.



$\mathrm{gVP}(\kappa, \mathbf{\Sigma}_{\mathrm{n+1}})$ holds for n-remarkable κ

Theorem: Suppose κ is n-remarkable. Then $\mathrm{gVP}(\kappa, \mathbf{\Sigma}_{\mathrm{n+1}})$ holds.

Proof:

- Let $\mathcal{C}$ be a proper class of first-order structures defined by the formula $\exists y\, \varphi(x, y, a)$ with $\varphi(x, y, z) \in \Pi_n$ and $a \in H_\kappa$.
- Fix $A \in \mathcal{C}$ and let $A \in V_\lambda$ with $\lambda \in C^{(n+1)}$.
- There is $\bar{\lambda} \in C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega, V_{\bar{\lambda}})}$, there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$:
  - $\mathrm{rk}(a) < \mathrm{crit}(j)$,
  - $j(\mathrm{crit}(j)) = \kappa$,
  - $A = j(B)$.
- $j : B \to A$ is elementary.
- $V_\lambda \models \exists y\, \varphi(A, y, a) \to V_{\bar{\lambda}} \models \exists y\, \varphi(B, y, a)$.
- $B \in \mathcal{C}$. □

Corollary:

- If there a proper class of n-remarkable cardinals, then $\mathrm{gVP}(\kappa, \mathbf{\Sigma}_{\mathrm{n+1}})$ holds for a proper class of κ.
- If $0^\sharp$ exists, then Generic Vopěnka's Principle holds in L (Silver indiscernibles are n-remarkable).

$\mathrm{gVP}(\Sigma_{n+1})$ implies an *n*-remarkable or virtually rank-into-rank cardinal

Theorem: (Bagaria, G., Schindler) Suppose $\mathrm{gVP}(\Sigma_{n+1})$ holds. Then there is an *n*-remarkable cardinal or there is a virtually rank-into-rank cardinal.

Proof: Suppose there are no *n*-remarkable cardinals.

- Let $\mathcal{C}$ be the class of structures $\langle V_{\lambda+2}, \in, \alpha\rangle$ where $\lambda > \alpha$ is least in $C^{(n)}$ such that no $\delta \leq \alpha$ is *n*-remarkable up to λ.
- $\mathcal{C}$ is proper.
- $\mathcal{C}$ is Σ_{n+1}-definable without parameters.
- In $V^{\mathrm{Coll}(\omega, V_{\lambda+2})}$, there is an elementary

$$j : \langle V_{\lambda+2}, \in, \alpha\rangle \to \langle V_{\mu+2}, \in, \beta\rangle$$

  for some $\langle V_{\lambda+2}, \in, \alpha\rangle \neq \langle V_{\mu+2}, \in, \beta\rangle$.
- Since j cannot be the identity map, let $\mathrm{crit}(j) = \kappa$.
- $\alpha < \beta$ because either $\lambda = \mu$ or $\lambda < \mu$, in which case
  - no $\delta \leq \alpha$ is *n*-remarkable up to λ,
  - there is $\delta \leq \beta$ *n*-remarkable up to λ (by minimality of μ).
- It follows that $\kappa \leq \alpha$.

$\mathrm{gVP}(\Sigma_{n+1})$ implies there is an n-remarkable or virtual rank-into-rank cardinal (continued)

- If $j(\kappa) \geq \lambda$, then κ is n-remarkable up to λ.
  - Let $\eta < \lambda$ in $C^{(n)}$.
  - $j : V_\eta \to V_{j(\eta)}$ with $j(\mathrm{crit}(j)) = j(\kappa)$.
  - $V_{\mu+2}$ satisfies:

    There is $\bar{\eta} < j(\kappa)$ in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega, V_{\bar{\eta}})}$,
    there is $j^* : V_{\bar{\eta}} \to V_{j(\eta)}$ with $j^*(\mathrm{crit}(j^*)) = j(\kappa)$.

  - $V_{\lambda+2}$ satisfies:

    There is $\bar{\eta} < \kappa$ in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega, V_{\bar{\eta}})}$,
    there is $j^* : V_{\bar{\eta}} \to V_\eta$ with $j^*(\mathrm{crit}(j^*)) = \kappa$.

- $j(\kappa) < \lambda$ and $V_{\mu+2} \models \kappa$ is n-remarkable up to $j(\kappa)$.
- Fix $\gamma < \kappa$ that is n-remarkable up to κ.
- By elementarity, if $j^m(\kappa)$ exists, then γ is n-remarkable up to $j^m(\kappa)$.
- It follows that all $j^m(\kappa)$ exist and their supremum is below λ.
- Let ρ be largest such that γ is n-remarkable up to ρ.
- $j(\rho) = \rho$.
- $j : V_{\rho+2} \to V_{\rho+2}$. (We would be done here if Kunen's Incosistency held!)
- κ is virtually rank-into-rank in a very strong sense. □

Strength of Generic Vopěnka-like principles (continued)

Corollary: If $\mathrm{gVP}(\boldsymbol{\Sigma}_{n+1})$ holds, then there is a proper class of κ, where each κ is either remarkable or virtually rank-into-rank.

Theorem: (Bagaria, G., Schindler) (lightface) The following are equiconsistent.

(1) There is an *n*-remarkable cardinal.

(2) $\mathrm{gVP}(\Sigma_{n+1})$ holds.

(3) $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_{n+1})$ holds for some κ.

Proof:

- If κ is *n*-remarkable cardinal, then $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_{n+1})$ holds.
- $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_{n+1})$ holds for some κ implies $\mathrm{gVP}(\Sigma_{n+1})$.
- $\mathrm{gVP}(\Sigma_{n+1})$ implies that there is an *n*-remarkable cardinal or a there is a model with proper class many *n*-remarkable cardinals.

Theorem: (Bagaria, G., Schindler) (boldface) The following are equiconsistent.

(1) There is a proper class of *n*-remarkable cardinals.

(2) $\mathrm{gVP}(\boldsymbol{\Sigma}_{n+1})$ holds.

(3) $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_{n+1})$ holds for a proper class of κ.

Questions

- Does $\mathrm{gVP}(\Sigma_{\mathrm{n}+1})$ imply that there is an *n*-remarkable cardinal?
- Is the least κ such that $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_{\mathrm{n}+1})$ holds the least *n*-remarkable cardinal?
- What other natural set theoretic properties are equiconsistent with virtual large cardinals?

Thank you!