Notes to "Indestructible Strong Cardinals"

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Abstract

The goal of this talk is to show how to make a strong cardinal κ indestructible by all $\leq\kappa$-closed forcing. We show in Section 2 how to lift elementary embeddings that witness that κ is a θ-strong cardinal through $\leq\kappa$-distributive forcing. In Section 3 we contrast this result by showing why this does not mean that every strong cardinal κ is indestructible by $\leq\kappa$-closed forcing. In fact, we show how to make a strong cardinal destructible by arbitrarily highly closed forcing. This illustrates the significance of the main indestructibility theorem, which is proved in Section 4.

1 The filter $\langle j"G\rangle$ is a ran(j)-generic filter on $j(\mathbb{P})$

For this section, assume that $\mathbb{P}$ is any poset, and that $j : V \to M$ is an elementary embedding with $\mathrm{cp}(j) = \kappa$ and $M \subseteq V$. It is clear that $\mathrm{ran}(j)$ is an elementary substructure of M. Let $G \subseteq \mathbb{P}$ be V-generic. By the lifting criterion, we need an M-generic filter $H \subseteq j(\mathbb{P})$ such that $j"G \subseteq H$. So let's focus on the set $j"G$.

The set $j"G \subseteq j"\mathbb{P}$ is a V-generic filter on $j"\mathbb{P}$, as $j \restriction \mathbb{P}$ is an isomorphism. But how much genericity can we get for the poset $j(\mathbb{P})$? Since $j"\mathbb{P}$ is a subposet of $j(\mathbb{P})$, we see that $j"G$ is a directed subset of $j(\mathbb{P})$. We may thus close the set $j"G$ upwards in $j(\mathbb{P})$ and obtain the filter $\langle j"G\rangle \subseteq j(\mathbb{P})$.

The observation below shows that the filter $\langle j"G\rangle$ is a $\mathrm{ran}(j)$-generic filter on $j(\mathbb{P})$.

Observation 1. *For every set $D \in ran(j)$ such that D is a dense subset of $j(\mathbb{P})$, we have that $j"G \cap D \neq \emptyset$.*

Proof. Assume that $\mathbb{P}$ is any poset, and that $j : V \to M$ is an elementary embedding with $\mathrm{cp}(j) = \kappa$ and $M \subseteq V$. Let $D \in \mathrm{ran}(j)$ be a dense subset of $j(\mathbb{P})$. So $D = j(E)$ some dense subset $E \subseteq P$. Since $G \cap E \neq \emptyset$, it follows that $j"G \cap j"E \neq \emptyset$. As $j"E \subseteq j(E)$, the observation follows. □

Observation 2. *If $\mathbb{P}$ is a poset of size less than κ ($\mathbb{P}$ may or may not be a subset of V_κ), then j lifts uniquely to $j : V[G] \to M[j(G)]$, with $j(G) = \langle j"G\rangle$ necessarily.*

Proof. Since $\mathbb{P}$ is of size less than κ, we see that $j"\mathbb{P} = j(\mathbb{P})$, and consequently that $j"G$ is a fully V-generic filter on $j(\mathbb{P})$ by the remarks above. By the lifting criterion, we have no choice in choosing the M-generic filter $H \subseteq j(\mathbb{P})$: since $j"G$ is already V-generic and hence M-generic on $j(\mathbb{P})$, it follows that $H = j"G = \langle j"G\rangle$ necessarily and the embedding lifts uniquely to $j : V[G] \to M[j(G)]$ with $j(G) = j"G = \langle j"G\rangle$. □

Observation 2 has the following, important consequence.

Theorem 3 (Levy-Solovay, 1967, by a different proof)**.** *If κ is measurable and $\mathbb{P}$ is a poset of size less than κ, then κ remains measurable after forcing with $\mathbb{P}$.*

Proof. If κ is measurable, then there is an elementary embedding $j : V \to M$ with critical point κ. Suppose that $\mathbb{P}$ is a poset of size less than κ. By Observation 2 the embedding j lifts uniquely to $j : V[G] \to M[j(G)]$, with $j(G) = \langle j"G\rangle$. This lifted embedding witnesses that κ is measurable in $V[G]$, as desired. □

2 Lifting elementary embeddings through $\leq\kappa$-distributive forcing

Theorem 4. *Suppose that $\mathbb{P}$ is $\leq\kappa$-distributive and $j : V \to M$ is an ultrapower embedding by a normal measure μ on κ.*

1. *If $D \in M$ is a dense open subset of $j(\mathbb{P})$, then there is a $\bar{D} \in ran(j)$ such that $\bar{D}$ is a dense subset of $j(\mathbb{P})$ and $\bar{D} \subseteq D$.*
2. *The filter $\langle j"G\rangle \subseteq j(\mathbb{P})$ is an M-generic filter on $j(\mathbb{P})$.*
3. *The embedding j lifts uniquely to $j : V[G] \to M[j(G)]$, with $j(G) = \langle j"G\rangle$ necessarily.*

Proof. This proof presents a typical argument for the lifting techniques presented in this talk. Recall that as j is an ultrapower embedding by a measure μ, we have that the embedding j is generated by the unique seed $[\mathrm{id}]_\mu$. As μ is a normal measure on κ, we see that $[\mathrm{id}]_\mu = \kappa$ and consequently that every element of M has the form $j(f)(\kappa)$ for some function $f : \kappa \to V$ in V. In summary, we have that $j : V \to M$ is elementary with $\mathrm{cp}(j) = \kappa$ and $M = \{j(f)(\kappa) \mid f : \kappa \to V,\ f \in V\}$.

To prove assertion 1, fix a set $D \in M$ that is a dense open subset of $j(\mathbb{P})$. It follows that $D = j(f_0)(\kappa)$ for some function $f_0 : \kappa \to V$ in V. Clearly, D is definable from $j(f_0)$ and κ, but since κ is not an element of $\mathrm{ran}(j)$, it may be that $D \notin \mathrm{ran}(j)$. In order to find a the desired subset $\bar{D} \in \mathrm{ran}(j)$, we aim to avoid the dependency on κ. We thus define in M the set

$$\bar{D} = \bigcap\{j(f_0)(\alpha) : \alpha < j(\kappa) \text{ and } j(f_0)(\alpha) \text{ is a dense open subset of } j(\mathbb{P})\}$$

Note that this intersection is well-defined since for at least one ordinal $\alpha < j(\kappa)$, namely κ, the set $j(f_0)(\alpha)$ is really a dense open subset of $j(\mathbb{P})$. Since M thinks that $j(\mathbb{P})$ is $\leq j(\kappa)$-distributive, and the set $\bar{D}$ is defined in M, we see that $\bar{D}$ is a dense subset of $j(\mathbb{P})$. The set $\bar{D}$ is obviously a subset of D. The key point now is that $\bar{D}$ is definable from the elements $j(f_0), j(\kappa)$ and $j(\mathbb{P})$, each an element of $\operatorname{ran}(j)$. Since $\operatorname{ran}(j) \prec M$, it follows that $\bar{D} \in \operatorname{ran}(j)$, as desired.

Assertion 2 is immediate from assertion 1 since Observation 1 showed that $\langle j\text{"}G\rangle$ is always an $\operatorname{ran}(j)$-generic filter on $j(\mathbb{P})$. Assertion 3 is immediate from assertion 2 and the lifting criterion. □

Assertion 3 of Theorem 4 of course implies that κ remains measurable after forcing with $\mathbb{P}$. But this was already clear from the outset, as $\mathbb{P}$ doesn't add any subsets to κ and therefore the measure $\mu \in V$ remains a measure in $V[G]$.

Note that the argument of the proof of Theorem 4 in fact does not rely on the specifics of μ being a normal measure on κ. It is sufficient that j is an ultrapower embedding by some measure $\mu \subseteq \mathcal{P}(E)$ where E has size at most κ. It is thus a straightforward exercise to state and prove the corresponding theorem concerning these more general ultrapower embeddings.

Theorem 4 applies to ultrapower embeddings, and thus to embeddings where the target model M is generated by a single seed. Its proof avoids the dependency on that single seed by quantifying over a lot of seeds, namely over $j(\kappa)$ many possible such seeds. Using this idea, we shall see that the proof idea of Theorem 4 can be applied to certain θ-strongness embeddings of a cardinal κ (Theorem 5), or even more generally to embeddings whose target model is generated by a set S of seeds such that $S \subseteq j(E)$ for some set $E \in V$ of size at most κ in V (Theorem 6).

Recall that a cardinal κ is *strong* if it is θ-*strong* for every ordinal θ, meaning that there is an elementary embedding $j : V \to M$ such that $\operatorname{cp}(j) = \kappa$ with $j(\kappa) > \theta$ and $V_\theta \subseteq M$. We call such an embedding a θ-*strongness embedding* of κ in V. Results from elementary seed theory show that we may assume without loss of generality that the target model M of j is generated by the seed set V_θ, namely that $M = \{j(f)(s) \mid f : V_\kappa^{<\omega} \to V,\ f \in V,\ s \in V_\theta^{<\omega}\}$. Of course, as κ is a limit ordinal, it follows that V_κ is closed under finite sequences, so that $V_\kappa^{<\omega} \subseteq V_\kappa$. The same applies for V_θ, as long as θ is a limit ordinal. But even if θ is an infinite successor ordinal, we may use *flat pairing* (instead of the usual von Neumann pairing function) to see that finite sequences of V_θ may be viewed as elements of V_θ. Consequently, $V_\theta^{<\omega} \subseteq V_\theta$, and we may drop in the following discussions the exponent $<\omega$ and write the target model M simpler as $M = \{j(f)(s) \mid f : V_\kappa \to V,\ f \in V,\ s \in V_\theta\}$.

Theorem 5. *Suppose that $\mathbb{P}$ is $\leq\kappa$-distributive and $j : V \to M$ is a θ-strongness embedding of κ in V such that $M = \{j(f)(s) \mid f : V_\kappa \to V,\ f \in V,\ s \in V_\theta\}$.*

1. *If $D \in M$ is a dense open subset of $j(\mathbb{P})$, then there is a $\bar{D} \in \operatorname{ran}(j)$ such that $\bar{D}$ is a dense subset of $j(\mathbb{P})$ and $\bar{D} \subseteq D$.*

2. *The filter $\langle j\text{"}G\rangle \subseteq j(\mathbb{P})$ is an M-generic filter on $j(\mathbb{P})$.*

3. *The embedding j lifts uniquely to $j : V[G] \to M[j(G)]$, with $j(G) = \langle j\text{"}G \rangle$ necessarily.*

Proof. The proof is similar to the proof of Theorem 4, except that we now have a *set* of seeds, rather than a single seed only. To prove assertion 1, we again fix a set $D \in M$ that is a dense open subset of $j(\mathbb{P})$. It follows that $D = j(j_0)(s_0)$ for some function $f_0 : V_\kappa \to V$ in V and for some seed $s_0 \in V_\theta$. We do not want to take the intersection over all possible seeds $s \in V_\theta$, since V_θ need not be an element of $\text{ran}(j)$. Instead, we are even more generous and use the fact that $V_\theta^V \subseteq j(V_\kappa)$ as $\theta < j(\kappa)$ and $V_\theta^V \subseteq M$ to define in M the set

$$\bar{D} = \bigcap \{ j(f_0)(s) : s \in j(V_\kappa) \text{ and } j(f_0)(s) \text{ is a dense open subset of } j(\mathbb{P}) \}$$

Note that this intersection is well-defined, since for at least one seed $s \in j(V_\kappa)$, namely s_0, the set $j(f_0)(s)$ is really a dense open subset of $j(\mathbb{P})$. As $j(\mathbb{P})$ is $\leq j(\kappa)$-distributive in M, and $j(V_\kappa)$ has size $j(\kappa)$ is M, it follows that $\bar{D}$ is a dense open subset of $j(\mathbb{P})$. The set $\bar{D}$ is obviously a subset of D. As $\bar{D}$ is definable from $j(f_0), j(V_\kappa)$ and $j(\mathbb{P})$, we have that $\bar{D} \in \text{ran}(j)$, as desired.

Assertions 2 and 3 follow as before. □

The argument of the proof of Theorem 5 does not rely on the specifics that the target model M was generated by seeds $s \in V_\theta$. All that we used was that every element of M is expressible as $j(f)(s)$ for some function $f \in V$ and some seed $s \in j(E)$ where $j(E)$ has size at most $j(\kappa)$ in M. Suppose that $j : V \to M$ is an elementary embedding with $\text{cp}(j) = \kappa$. We say that the target model M is *generated by the seed set* S if there is some underlying set $E \in V$ with $S \subseteq j(E)$ such that $M = \{ j(f)(s) \mid f : E \to V,\ f \in V,\ s \in S \}$. Using this terminology we can state the following generalization of Theorem 5, which also generalizes Theorem 4.

Theorem 6. *Suppose that $\mathbb{P}$ is $\leq\kappa$-distributive and $j : V \to M$ is an elementary embedding of κ with $cp(j) = \kappa$ such that M is generated by the seed set S with $S \subseteq j(E)$ for some $E \in V$. Suppose furthermore that E has size at most κ in V.*

1. *If $D \in M$ is a dense open subset of $j(\mathbb{P})$, then there is a $\bar{D} \in ran(j)$ such that $\bar{D}$ is a dense subset of $j(\mathbb{P})$ and $\bar{D} \subseteq D$.*
2. *The filter $\langle j\text{"}G \rangle \subseteq j(\mathbb{P})$ is an M-generic filter on $j(\mathbb{P})$.*
3. *The embedding j lifts uniquely to $j : V[G] \to M[j(G)]$, with $j(G) = \langle j\text{"}G \rangle$ necessarily.*

Proof. Given the proof of Theorem 5, it is an easy exercise to prove this theorem. To prove assertion 1, we fix any dense open set $D \in M$, express D as $j(f_0)(s_0)$ for some function $f_0 : E \to V$ in V and some $s_0 \in S \subseteq j(E)$. We are again quite generous and quantify over all $s \in j(E)$ to build the set $\bar{D}$ in M. As E has size at most κ, and $\mathbb{P}$ is $\leq\kappa$-distributive, it follows by elementarity that $\bar{D}$ is a dense subset of $j(\mathbb{P})$, as desired. □

There is an important subtlety in Theorem 5 that is relevant for the next two sections, and that we now aim to discuss. The theorem shows that if $\mathbb{P}$ is a $\leq\kappa$-distributive poset and $j : V \to M$ is a θ-strongness embedding with $j(\kappa) > \theta$ and $M = \{j(f)(s) \mid f : V_\kappa \to V,\ f \in V,\ s \in V_\theta\}$, then j lifts uniquely to $j : V[G] \to M[j(G)]$, with $j(G) = \langle j"G\rangle$. Does this mean that the lifted embedding $j : V[G] \to M[j(G)]$ is a θ-strongness embedding *in* $V[G]$? In other words, do we have that $V_\theta^{V[G]} \subseteq M[j(G)]$?

If $\mathbb{P}$ is nontrivial forcing and $\mathbb{P} \in V_\theta$, then the answer is an emphatic No! Of course, we know that $V_\theta \subseteq M$, and thus $V_\theta[G] \subseteq M[G]$. Moveover, from $\mathbb{P} \in V_\theta$ it even follows $V_\theta^{V[G]} = V_\theta[G]$. We thus have $V_\theta^{V[G]} = V_\theta[G] \subseteq M[G]$, but is $M[G]$ a subclass of $M[j(G)]$? This question is equivalent to whether the generic filter G is an element of $M[j(G)]$. But, it is easy to see that G is *never* an element of $M[j(G)]$! For, if G would be an element of $M[j(G)]$, then it would there have size less than V_θ^V, and thus have size less than $j(\kappa)$ in $M[j(G)]$. But, since $j(\mathbb{P})$ is $\leq j(\kappa)$-distributive in M, this would mean that $G \in M$, an obvious contradiction as $M \subseteq V$ and G is a V-generic filter for the nontrivial forcing $\mathbb{P}$. Consequently, $G \notin M[j(G)]$ and thus $M[G] \not\subseteq M[j(G)]$. Moreover, since G has rank less than θ in $V[G]$, it follows that $V_\theta^{V[G]} \not\subseteq M[j(G)]$, and consequently that the lifted embedding $j : V[G] \to M[j(G)]$ is *not* a θ-strongness embedding in $V[G]$.

Thus, even though we saw that the θ-strongness embedding $j : V \to M$ does lift through $\leq\kappa$-distributive forcing, we now know that in general such a lift will not be a θ-strongness embedding in $V[G]$ anymore. But it might still be that κ is θ-strong in $V[G]$, and it is thus natural to ask the following question.

Question 1. *Assume that κ is a strong cardinal. Is κ necessarily indestructible by all $\leq\kappa$-distributive forcing?*

We shall show in the next section that the answer is again an emphatic No!

3 Making A Strong Cardinal destructible by highly closed forcing

Here is the main idea. As an example for highly closed forcing, let us consider the poset $\mathbb{Q} = \mathrm{Add}(\theta, 1)$ where θ is some very large regular cardinal θ, and conditions in $\mathbb{Q}$ have size less than θ. Let $G \subseteq \mathbb{Q}$ be V-generic. Let us assume that κ is strong in $V[G]$ and discuss some consequences. Fix thus any $(\theta+1)$-strongness embedding $j : V[G] \to \bar{M}$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \theta$. Since $V_{\theta+1}^{V[G]} \subseteq \bar{M}$, we see that $G \in \bar{M}$. As is always the case, if we let $M = \bigcup \mathrm{ran}(j \restriction V)$, then the restriction $j \restriction V : V \to M$ is elementary, and $\bar{M}$ becomes the forcing extension $M[j(G)]$ so that we have $j : V[G] \to M[j(G)]$.

We may now ask the following question. Is M a subclass of V? It is easy to see that the answer to this question is No! For, as $G \in \bar{M} = M[j(G)]$ and $\mathbb{Q}$ is (much more than) $<\kappa$-closed in V, we see that $j(\mathbb{Q})$ is $<j(\kappa)$-closed in M, and

forcing with $j(\mathbb{Q})$ could therefore not have added G. It follows that $G \in M$! But G is V-generic, so $G \notin V$ and consequently $M \nsubseteq V$.

I view of the next theorem, we remark that if $\mathbb{Q}$ would have been any forcing that had a *closure point* δ below κ, (or more generally forcing that had the δ cover and δ approximation property for some cardinal $\delta < \kappa$, see [Hamkins 2003]), then the answer to the previous question would have been Yes! Thus, assuming that κ is strong in $V[G]$, we would have concluded that $M \subseteq V$, but also that $M \nsubseteq V$. This obvious conflict is at the heart of the next theorem.

Theorem 7. *After small forcing, a strong cardinal κ is destructible by $Add(\theta, 1)$ for any regular $\theta \geq \kappa$. In fact, forcing with $Add(\theta, 1)$ destroys the $(\theta + 1)$-strongness of κ.*

Proof. Let $\mathbb{P}$ be a small poset relative to κ. So $|\mathbb{P}| < \kappa$. Without loss of generality, let $\mathbb{P} \in V_\kappa$. Let $g \subseteq \mathbb{P}$ be V-generic. Fix now any regular θ in $V[g]$ and let $\mathbb{Q} = \mathrm{Add}(\theta, 1)$. We will show that forcing with $\mathbb{Q}$ necessarily destroys the $(\theta + 1)$-strongness of κ.

Let $G \subseteq \mathbb{Q}$ be $V[g]$-generic. Suppose for contradiction that κ is $(\theta + 1)$-strong in $V[g * G]$. Then fix any $(\theta+1)$-strongness embedding $j : V[g*G] \to \bar{M}$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \theta$. As $V_{\theta+1}^{V[G]} \subseteq \bar{M}$, we see that $G \in \bar{M}$. If we let $M = \bigcup \mathrm{ran}(j \restriction V)$, then the restriction $j \restriction V : V \to M$ is elementary and $\bar{M}$ becomes the forcing extension $M[j(g) * j(G)]$ so that we have $j : V[g * G] \to M[j(g) * j(G)]$.

As $j(\mathbb{Q})$ is (much more than) $<j(\kappa)$-closed in M, and $G \in M[j(g)*j(G)]$, we see that forcing with $j(\mathbb{Q})$ could not have added G. It follows that $G \in M[j(g)]$. But $\mathbb{P} \in V_\kappa$ is small, so $j(g) = j\text{"}g = g$. So $G \in M[g]$. The crucial point now is as follows. Since $\mathbb{P} * \mathbb{Q}$ has a closure point below κ (namely the cardinal $|\mathbb{P}| < \kappa$), the fundamental theorem in [Hamkins 2003] implies that the restriction $j \restriction V : V \to M$ is a definable class in V, and consequently that $M \subseteq V$. This implies that $M[g] \subseteq V[g]$, and therefore that $G \in V[g]$. But this contradicts that $G \subseteq \mathbb{Q}$ is $V[g]$-generic, which completes the proof. □

Using the fact that measurability is equivalent to $(\kappa+1)$-strongness, we have the following.

Corollary 8. *After small forcing, a measurable cardinal κ is destructible by the forcing $Add(\kappa, 1)$.*

The proof of Theorem 7 did not really depend on $\mathbb{Q}$ actually being the particular forcing $\mathrm{Add}(\theta, 1)$. In fact, we get the following corollary which shows that no matter how highly closed a poset $\mathbb{Q}$ may be, it is possible that $\mathbb{Q}$ destroys the strongness of κ.

Corollary 9. *After small forcing, a strong cardinal κ is destructible by* any *$<\kappa$-closed nontrivial set forcing. In fact, if such $<\kappa$-closed forcing necessarily adds a subset to θ, then forcing with it destroys the $(\theta + 1)$-strongness of κ.*

Proof. We follow the proof of Theorem 7 closely. Again, we first fix a small poset $\mathbb{P} \in V_\kappa$ and let $g \subseteq \mathbb{P}$ be V-generic. Instead of using the poset $\mathrm{Add}(\theta, 1)$, we now fix the poset $\mathbb{Q}$ in $V[g]$ which is $<\kappa$-closed and nontrivial. It follows that there is a cardinal θ such that forcing with $\mathbb{Q}$ necessarily adds a new subset of θ. So, let $G \subseteq \mathbb{Q}$ be $V[g]$-generic, and let $A \subseteq \theta$ be a subset of θ added by $\mathbb{Q}$, so that $A \in V[g*G]$ but $A \notin V[g]$. Again, we suppose for contradiction that κ is $(\theta+1)$-strong in $V[g*G]$. Let thus $j : V[g*G] \to \bar{M}$ be a $(\theta+1)$-strongness embedding in $V[g*G]$. Note that $A \in \bar{M}$. Again, it follows that $\bar{M} = M[j(g) * j(G)]$ where $M = \bigcup \mathrm{ran}(j \restriction V)$. The set $A \subseteq \theta$ is in fact an element of $M[j(g)]$ by elementarity and the $<j(\kappa)$-closure of $j(\mathbb{Q})$. But, as before in the proof of Theorem 7, we have that $\mathbb{P} * \dot{\mathbb{Q}}$ has a closure point below κ, which implies that $M \subseteq V$. Consequently, $M[g] \subseteq V[g]$ and therefore $A \in M[j(g)] = M[g] \subseteq V[g]$, contradicting that A was added by the poset $\mathbb{Q}$. $\square$

We just showed that it is possible that a strong cardinal κ is destroyed by $<\kappa$-closed forcing, no matter how highly closed the forcing may be. In view of the next section, it is thus natural to ask the following.

Question 2. *Assume that κ is a strong cardinal. Can we make κ indestructible by all $\leq\kappa$-closed forcing?*

We shall show in Section 4 that the answer is (an emphatic) Yes! We shall use a Laver-like forcing iteration $\mathbb{P}$ of length κ, defined relative to a Laver function of a strong cardinal κ, such that forcing with $\mathbb{P}$ makes the strong cardinal κ indestructible by all $\leq\kappa$-closed forcing. Moreover, the Gitik-Shelah result presented by Victoria Gitman will generalize this result even further and make κ indestructible by all $\leq\kappa$-weakly-closed forcing with the Prikry property.

4 Making a strong cardinal indestructible by all $\leq\kappa$-closed forcing

In Sections 1 and 2, we used the $\mathrm{ran}(j)$-generic filter $\langle j\text{"}G\rangle \subseteq j(\mathbb{P})$ in order to lift the elementary embedding $j : V \to M$ to $j : V[G] \to M[j(G)]$. We used the $\leq\kappa$-distributivity of $\mathbb{P}$ to establish that for every dense open set $D \subseteq j(\mathbb{P})$ with $D \in M$, there is some dense set $\bar{D} \in \mathrm{ran}(j)$ of $j(\mathbb{P})$ such that $\bar{D} \subseteq D$. This meant that $\langle j\text{"}G\rangle$ was in fact a fully M-generic filter on $j(\mathbb{P})$, which allowed us to lift the embedding j. As we saw in Section 3, this method though is not sufficient to preserve the θ-strongness of the cardinal κ. The particular problem was that in general, the filter $G \subseteq \mathbb{P}$ is not an element of $M[j(G)]$.

To solve this problem and make a strong cardinal κ indestructible by $\leq\kappa$-closed forcing, we will use some preparatory forcing, namely an Easton support κ-iteration $\mathbb{P}$ defined relative to a Laver function $l : \kappa \to V_\kappa$. The Laver function will be able to anticipate any particular $\leq\kappa$-closed poset, so that if $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name of any $\leq\kappa$-closed poset, we may find a θ-strongness embedding $j : V \to M$ such that $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\mathrm{tail}}$. If $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is V-generic, it will be our goal to lift the embedding j in two steps in $V[G*g]$ to $j : V[G*g] \to M[j(G)*j(g)]$.

Since $j(G)$ will equal $G * g * G_{\text{tail}}$ for some $M[G * g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$, it will be clear from the construction that $G * g$ is an element of $M[j(G) * j(g)]$, and consequently that the lifted embedding will be a θ-strongness embedding in $V[G*g]$. The key problem though is: *How* can we manage to lift the embedding in two steps in $V[G * g]$? Here is a sketch of the idea that underlies the proof of Theorem 11.

In the first lifting step, we shall lift through the preparatory forcing $\mathbb{P}$. Since $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\text{tail}}$ and $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is fully V-generic, it suffices to find in $V[G * g]$ an $M[G * g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$. But we cannot apply the techniques from Sections 1 and 2 directly, since $\mathbb{P}_{\text{tail}}$ will not be fully $\leq j(\kappa)$-distributive, and $\langle j\text{"}G\rangle$ will thus not be sufficiently generic. Instead, we shall use a single seed, namely κ, to define an elementary substructure $X \prec M$ such that $\{\kappa, \mathbb{P}, \dot{\mathbb{Q}}, \dot{\mathbb{P}}_{\text{tail}}\} \subseteq X$ and $\operatorname{ran}(j) \subseteq X$. It will follow that $X^\kappa \subseteq X$ in V, and consequently that $X[G * g]^\kappa \subseteq X[G * g]$ in $V[G * g]$. By an additional $2^\kappa = \kappa^+$ assumption in V, it will follow that $X[G*g]$ contains only κ^+ many dense subsets of $\mathbb{P}_{\text{tail}}$. Since $\mathbb{P}_{\text{tail}}$ will be (much more than) $\leq\kappa$-closed in $X[G * g]$, we will be able to use diagonalization to build an $X[G * g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ in $V[G * g]$. Moreover, it will follow from the high distributivity of $\mathbb{P}_{\text{tail}}$ somewhat analogously as in Section 2, but now by using $X[G * g] \prec M[G * g]$ rather than the elementarity $\operatorname{ran}(j) \prec M$, that in fact G_{tail} is a fully $M[G * g]$-generic filter on $\mathbb{P}_{\text{tail}}$! This will be the key step that will allow us to let $j(G) = G * g * G_{\text{tail}}$ and lift the embedding in $V[G * g]$ to $j : V[G] \to M[j(G)]$. This will conclude the first lifting step.

The second lifting step will be considerably easier as we only have to lift the embedding $j : V[G] \to M[j(G)]$ through the $\leq\kappa$-closed forcing $\mathbb{Q}$. The $\leq\kappa$-distributivity of $\mathbb{Q}$ will allow us to use the technique of Section 2 directly and simply use the $M[j(G)]$-generic filter $\langle j''g\rangle \subseteq j(\mathbb{Q})$ to lift j in $V[G * g]$ to $j : V[G * g] \to M[j(G) * j(g)]$, with $j(g) = \langle j''g\rangle$. This concludes the discussion of the idea that underlies the proof of Theorem 11.

Strong cardinals admit a Laver function l, a kind of generalized $\Diamond$-sequence, which anticipates any object in the universe.

Theorem 10. *If κ is a strong cardinal, then there is a function $l \colon \kappa \to V_\kappa$ such that for any x and θ with $x \in H_{\theta^+}$ there is a θ-strongness embedding $j : V \to M$ with $j(l)(\kappa) = x$.*

We call a function $l \colon \kappa \to V_\kappa$ as in the theorem above, a *Laver function* for the strong cardinal κ.

Theorem 11. *If κ is strong and $2^\kappa = \kappa^+$, then there is a set forcing extension in which the strongness of κ becomes indestructible by any $\leq\kappa$-closed forcing.*

Proof. The definition of the iteration is very similar to the original Laver preparation of a supercompact cardinal, yet we will neither use a master condition argument nor will we use closure of the target model to lift the embedding. Let l be a Laver function for the strong cardinal κ, as in the theorem above. We use l to define a Easton support iteration $\mathbb{P}$ of length κ. If $\mathbb{P}_\gamma$ is defined for $\gamma < \kappa$, and $l(\gamma)$ happens to be a $\mathbb{P}_\gamma$-name for a poset that is $\leq\gamma$-closed in $V^{\mathbb{P}_\gamma}$, then we let the stage γ forcing $\mathbb{Q}_\gamma$ be this poset; otherwise, $\mathbb{Q}_\gamma$ is trivial forcing.

Suppose that $G \subseteq \mathbb{P}$ is V-generic and that $\mathbb{Q}$ is any $\leq\kappa$-closed poset in $V[G]$. It suffices to show that κ is strong in $V[G][g]$, where $g \subseteq \mathbb{Q}$ is $V[G]$-generic. Fix a name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$ which necessarily yields a $\leq\kappa$-closed poset. Fix any ordinal θ above κ with $\dot{\mathbb{Q}} \in H_{\theta^+}$. Since l is a Laver function, there is a θ-strongness embedding $j : V \to M$ such that $j(l)(\kappa) = \dot{\mathbb{Q}}$. Without loss of generality, we may assume that M is generated by the seed set V_θ, namely that $M = \{j(f)(s) \mid f : V_\kappa \to V,\ f \in V,\ s \in V_\theta\}$. Since $\mathbb{P}$ is defined relative to l and $M[G]$ agrees that $\dot{\mathbb{Q}}$ is a name for a $\leq\kappa$-closed poset, it follows that the stage κ forcing in $j(\mathbb{P})$ is precisely $\dot{\mathbb{Q}}$. The forcing factors therefore as $j(\mathbb{P}) = \mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{P}}_{\text{tail}}$. We may assume that $l``\gamma \subseteq V_\gamma$ for all $\gamma \in \text{dom}(l)$. We may also assume that the next element of the domain of $j(l)(\kappa)$ is beyond $\beth_\theta$, so that $\mathbb{P}_{\text{tail}}$ is $\leq\beth_\theta$-closed in $M[G][g]$. Lastly, we may also assume that $\theta = j(l')(\kappa)$ for some function $l' : \kappa \to \kappa$.

Step 1. In $V[G*g]$, lift the embedding $j : V \to M$ to $j : V[G] \to M[j(G)]$.

We shall build in $V[G*g]$ an M-generic filter $j(G) \subseteq j(\mathbb{P})$. Since $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\text{tail}}$, and $G * g \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is V-generic and hence M-generic, it suffices to find an $M[G*g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$. We cannot use diagonalization over $M[G*g]$, since we have no closure of $M[G*g]$ to rely on. Instead, we consider the structure $X = \{j(f)(\kappa) \mid f : \kappa \to V,\ f \in V\}$ and we will work with $X[G*g]$ in $V[G*g]$. As usual, we have that $X \prec M$ with $\text{ran}(j) \subseteq X$ and $\kappa \in X$. The structure X thus contains the elements $j(\mathbb{P}), j(l), \mathbb{P}, \dot{\mathbb{Q}}$, and $\dot{\mathbb{P}}_{\text{tail}}$. It also contains the ordinal θ and thus the set V_θ also. Since $\mathbb{P} \in X$, and $\mathbb{Q} \in X[G]$, it is a standard application of Tarski's criterion to see that $X[G] \prec M[G]$ and that $X[G*g] \prec M[G*g]$. The following claim is key and resembles the arguments of Sections 1 and 2. It is the reason for our definition of X and $X[G*g]$.

Claim 1. If $D \in M[G*g]$ is a dense open subset of $\mathbb{P}_{\text{tail}}$, then there is a $\bar{D} \in X[G*g]$ such that $\bar{D}$ is a dense subset of $\mathbb{P}_{\text{tail}}$ and $\bar{D} \subseteq D$.

Proof of Claim 1. Fix any $D \in M[G*g]$ which is a dense open subset of $\mathbb{P}_{\text{tail}}$. Since $j : V \to M$ is an extender embedding, we see that $D = j(f_0)(s_0)_{G*g}$ for some function $f_0 : V_\kappa \to V$ with $f_0 \in V$ and some seed $s_0 \in V_\theta^V$. Clearly, D is definable from $j(f_0), G*g$ and s_0, but it may be that neither s_0 nor D is an element of $X[G*g]$. In order to find the desired subset $\bar{D} \in X[G*g]$, we aim to avoid the dependency on s_0. We thus define in $M[G*g]$ the set

$$\bar{D} = \bigcap\{j(f_0)(s)_{G*g} \mid s \in V_\theta^V \text{ and } j(f_0)(s) \text{ is a dense open subset of } \mathbb{P}_{\text{tail}}\}.$$

Note that this intersection is well-defined since for at least one seed $s \in V_\theta^V \subseteq V_\theta^{M[G*g]}$, namely s_0, the set $j(f_0)(s)_{G*g}$ is really a dense open subset of $\mathbb{P}_{\text{tail}}$. The set $\bar{D}$ is thus an element of $M[G*g]$. Moreover, since $\mathbb{P}_{\text{tail}}$ is $\leq\beth_\theta$-distributive in $M[G*g]$, the set $\bar{D}$ is in fact a dense subset of $\mathbb{P}_{\text{tail}}$. The set $\bar{D}$ is obviously a subset of D. The key point now is that $\bar{D}$ is definable from the elements $j(f_0), G*g, V_\theta^V$ and $\mathbb{P}_{\text{tail}}$, each an element of $X[G*g]$. Since $X[G*g] \prec M[G*g]$, it follows that $\bar{D} \in X[G*g]$, which proves Claim 1.

To complete Step 1, it suffices to find in $V[G*g]$ an $X[G*g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$. By the claim this filter will then be fully $M[G*g]$-generic, and the embedding j will thus lift in $V[G*g]$ to $j : V[G] \to M[j(G)]$ where $j(G) = G*g*G_{\text{tail}}$. We shall use diagonalization over $X[G*g]$ to build G_{tail}. It is a standard argument to see that $X^\kappa \subseteq X$ in V. As $\mathbb{P} \subseteq X$ and $\mathbb{P}$ is κ-cc, it follows that $X[G]^\kappa \subseteq X[G]$ in $V[G]$. Since $\mathbb{Q}$ is $\leq\kappa$-distributive, it follows that $X[G*g]^\kappa \subseteq X[G*g]$ in $V[G*g]$. Since $X[G*g] \prec M[G*g]$, it follows that $\mathbb{P}_{\text{tail}}$ is (much more than) $\leq\kappa$-closed in $X[G*g]$. Lastly, we need to count in $V[G*g]$ the maximal antichains of $\mathbb{P}_{\text{tail}}$ that exist in $X[G*g]$. As usual, since $\mathbb{P}_{\text{tail}}$ has size $j(\kappa)$ and $\mathbb{P}*\dot{\mathbb{Q}}$ doesn't increase the size of $\mathcal{P}(j(\kappa))$, it suffices to count $\mathcal{P}(j(\kappa)) \cap X$ in the ground model V. Since every $A \in X$ with $A \subseteq j(\kappa)$ is represented by a function from κ to $\mathcal{P}(\kappa)$, we see that $\mathcal{P}(j(\kappa)) \cap X$ has size 2^κ in V. Consequently there are at most $(2^\kappa)^V$ many maximal antichains of $\mathbb{P}_{\text{tail}}$ that exist in $X[G*g]$. Since $(2^\kappa)^V = (\kappa^+)^V \leq (\kappa^+)^{V[G*g]}$, we see that we can enumerate in $V[G*g]$ all maximal antichains of $X[G*g]$ of $\mathbb{P}_{\text{tail}}$ as a κ^+-sequence. In view of the next theorem, note that this is the only place in the proof where one relies on the assumption that $2^\kappa = \kappa^+$ in V. We can now use diagonalization, applied to $X[G*g]$, to build a descending κ^+-sequence of conditions in $X[G*g] \cap \mathbb{P}_{\text{tail}}$ which meets every maximal antichain of $\mathbb{P}_{\text{tail}}$ that exists in $X[G*g]$. Let $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ be the filter generated by this descending sequence. Since G_{tail} is $X[G*g]$-generic by construction, it follows from the claim that in fact $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ is an $M[G*g]$-generic filter on $\mathbb{P}_{\text{tail}}$. If we let $j(G) = G*g*G_{\text{tail}}$, we see that the embedding j lifts in $V[G*g]$ to $j : V[G] \to M[j(G)]$. This completes Step 1.

Step 2. In $V[G*g]$, lift the embedding to $j : V[G*g] \to M[j(G)*j(g)]$.

This second lifting step is significantly easier. Since we already lifted the embedding to $j : V[G] \to M[j(G)]$, it makes sense to consider in $V[G*g]$ the set $j"g$. As always, this directed set generates a filter $\langle j"g \rangle \subseteq j(\mathbb{Q})$ which meets every dense subset $D \in \text{ran}(j)$ of $j(\mathbb{Q})$. In other words, the filter $\langle j"g \rangle$ is a $\text{ran}(j)$-generic filter on $j(\mathbb{Q})$. The following claim shows that in fact this filter is $M[j(G)]$-generic.

Claim 2. If $D \in M[j(G)]$ is a dense open subset of $j(\mathbb{Q})$, then there is a $\bar{D} \in \text{ran}(j)$ such that $\bar{D}$ is a dense subset of $j(\mathbb{Q})$ and $\bar{D} \subseteq D$.

Proof of Claim 2. Fix any $D \in M[j(G)]$ which is a dense open subset of $j(\mathbb{Q})$. Since $j : V \to M$ is an embedding where M is generated by the seed set V_θ, it follows that $D = j(f_0)(s_0)_{j(G)}$ for some function $f_0 : V_\kappa \to V$ with $f_0 \in V$ and some seed $s_0 \in V_\theta^V$. Clearly, D is definable from $j(f_0), j(G)$ and s_0, but it may be that s_0 is not an element of $\text{ran}(j)$. In order to find the desired subset $\bar{D} \in \text{ran}(j)$, we aim to avoid the dependency on s_0. We thus define in $M[j(G)]$ the set

$$\bar{D} = \bigcap \{ j(f_0)(s)_{j(G)} \mid s \in j(V_\kappa^V) \text{ and } j(f_0)(s) \text{ is a dense open subset of } j(\mathbb{Q}) \}.$$

Note that this intersection is well-defined since for at least one seed $s \in V_\theta^V \subseteq j(V_\kappa^V)$, namely s_0, the set $j(f_0)(s)_{j(G)}$ is really a dense open subset of $j(\mathbb{Q})$.

Since $\bar{D}$ is defined in $M[j(G]$ and $j(\mathbb{Q})$ is $\leq j(\kappa)$-distributive there, we see that the set $\bar{D} \in M[G*g]$ is a dense subset of $j(\mathbb{Q})$. The set $\bar{D}$ is obviously a subset of D. The key point now is that $\bar{D}$ is definable from the elements $j(f_0), j(G), j(V_\kappa^V)$ and $j(\mathbb{Q})$, each an element of $\mathrm{ran}(j)$. Since $\mathrm{ran}(j) \prec M[j(G)]$, it follows that $\bar{D} \in \mathrm{ran}(j)$. This proves Claim 2.

If we let $j(g) = \langle j\text{"}g \rangle$, we therefore see that the embedding j lifts in $V[G*g]$ to $j : V[G*g] \to M[j(G)*j(g)]$. This completes Step 2.

Since $\mathbb{P} * \dot{\mathbb{Q}}$ has rank less than θ and $V_\theta \subseteq M$, we see that $V_\theta^{V[G*g]} = V_\theta[G*g] \subseteq M[G*g]$. The filter $G*g$ is an initial segment of $j(G)$ by construction, and so $G*g$ is an element of $M[j(G)]$. It follows that $V_\theta^{V[G*g]} \subseteq M[G*g] \subseteq M[j(G)*j(g)]$, and consequently that $j : V[G*g] \to M[j(G)*j(g)]$ is a θ-strongness embedding in $V[G*g]$. This completes the proof of the theorem. $\square$

So, how can we get rid of the additional $2^\kappa = \kappa^+$ assumption? Here it goes.

Theorem 12. *If κ is strong, then there is a set forcing extension preserving the strongness of κ and in which $2^\kappa = \kappa^+$.*

Proof. We use the same Easton support iteration $\mathbb{P}$ as defined in Theorem 11. But, instead of forcing with an arbitrary poset $\mathbb{Q}$, we let $\mathbb{Q} = \mathrm{Add}(\kappa^+, 1)$, the canonical forcing to force $2^\kappa = \kappa^+$. Clearly, $\mathbb{Q}$ is $\leq\kappa$-closed, and we use a θ-strongness embedding $j : V \to M$ such that $j(l)(\kappa) = \dot{\mathbb{Q}}$. Let $G*g \subseteq \mathbb{P}*\dot{\mathbb{Q}}$ be any V-generic filter. I claim that $V[G*g]$ is the desired forcing extension. We clearly have $2^\kappa = \kappa^+$ in $V[G*g]$. In order to see that κ remains strong, we follow the argument of the previous proof exactly. In Step 1, when we counted the maximal antichains of $\mathbb{P}_{\mathrm{tail}}$ that existed in $X[G*g]$ we had used the assumption $(2^\kappa)^V = (\kappa+)^V$ to obtain the conclusion $(2^\kappa)^V \leq (\kappa+)^{V[G*g]}$. But, for the particular poset $\mathbb{Q} = \mathrm{Add}(\kappa^+, 1)$, we of course can obtain the same conclusion even without any assumption on the size of 2^κ in V. The remainder of the proof is identical. $\square$

Theorems 11 and 12 hence imply the goal indestructibility theorem of this talk.

Theorem 13. *If κ is strong, then there is a set forcing extension in which the strongness of κ becomes indestructible by any $\leq\kappa$-closed forcing.*