A COUNTABLE ORDINAL DEFINABLE SET OF REALS WITHOUT ANY ORDINAL DEFINABLE ELEMENTS

1. Introduction

In 2010 Garabed Gulbenkian asked the following question on MathOverflow.

Question 1.1. Is every countable OD-set of reals in HOD?

The question turned out to be open and it was solved 4 years later by Kanovei and Lyubetsky [KL].

Observation 1.2. *Every finite* OD*-set of reals is in* HOD*.*

Proof. Suppose S is an OD-set of reals of size some $n \in \omega$. A real $r \in S$ is defined as the m-th element of S in the lexicographical order for some $m < n$. □

The observation above makes a fundamental use of the lexicographical order on the reals. Indeed, it is consistent to have a two element OD-set (of sets of reals) neither of whose elements is OD. In a forcing extension of L by two mutually generic Sacks reals r and s, the set consisting of the L-degrees of r and s is definable but has no ordinal definable elements [GL87].

Observation 1.3. *It is consistent that there is an uncountable* OD*-set of reals without an* OD *element.*

Proof. Let $L[G]$ be a forcing extension of L by Cohen forcing. Let $S \in L[G]$ be the set of all non-constructible reals, which is definable without parameters. The set S cannot have any OD elements because any OD-real must be in L by the homogeneity of Cohen forcing.

Note that every real $r \in L$ induces an automorphism of the Cohen poset via bit-wise addition. This means that there are continuum many Cohen reals in $L[G]$, and hence S is uncountable. □

Theorem 1.4 (Kanovei-Lyubetsky)**.** *It is consistent that there is a countable* OD*-set of reals without any* OD*-elements.*

Kanovei and Lyubetsky used a poset constructed by Jensen in the 1970s to show that it is consistent to have a Δ^1_3 non-constructible real. By Shoenfield's Absoluteness, every Σ^1_2 or Π^1_2-real is constructible. Jensen constructed in L a ccc subposet $\mathbb{P}$ of Sacks forcing, using $\diamondsuit$ to seal maximal antichains, with the following properties [Jen70]. In any model of set theory, the set of all L-generic reals for $\mathbb{P}$ is Π^1_2-definable, a property which is also true of Cohen forcing. But unlike Cohen extensions of L which have uncountably many L-generic Cohen reals (see above), an L-generic extension by Jensen's forcing $\mathbb{P}$ adds a unique L-generic real, which is therefore Δ^1_3-definable. Kanovei and Lyubetsky extended the "uniqueness of the generic real" property of $\mathbb{P}$ to show that the ω-length finite-support product $\mathbb{P}^{<\omega}$ adds precisely the L-generic reals that appear on the coordinates of an L-generic filter for $\mathbb{P}^{<\omega}$.

Let $G \subseteq \mathbb{P}^{<\omega}$ be L-generic. In $L[G]$, the collection S of all $\mathbb{P}$-generic reals is precisely the ω-many reals coming from the coordinates of G. So S is countable. A standard automorphism argument for finite-support products shows that no element of S can be OD. Suppose that the n-th coordinate of G is ordinal definable by $\varphi(x,\alpha)$ in $L[G]$. Let $\dot{x}_n$ be the canonical name for the n-th coordinate of the generic filter for $\mathbb{P}^{<\omega}$, and let $p \in G$ force that $\dot{x}_n$ is defined by $\varphi(x,\check{\alpha})$. By density, there is $m > n$ such that $G(m) \restriction \mathrm{dom}(p(n)) = p(n)$. Let π be an automorphism of $\mathbb{P}^{<\omega}$ which switches coordinates n and m of the product. Observe that by our choice of m, $p \in H = \pi\, " \, G$. Since $L[G] = L[H]$, it must be the case that $\varphi(x,\alpha)$ defines $(\dot{x}_n)_H \neq (\dot{x}_n)_G$, which is impossible.

2. Perfect posets

Definition 2.1. A tree $T \subseteq {}^{<\omega}2$ is *perfect* if for every node $t \in T$, there is $s \supseteq t$ in T which splits in T, that is $s^\frown 0$ and $s^\frown 1$ are both in T.

Definition 2.2. We say that a collection $\mathbb{P}$ of perfect trees ordered by inclusion is a *perfect* poset if

- ${}^{<\omega}2 \in \mathbb{P}$,
- whenever $T \in \mathbb{P}$ and $s \in T$, then $T_s \in \mathbb{P}$,
- $\mathbb{P}$ is closed under finite unions.

So the smallest perfect poset is all finite unions of $({}^{<\omega}2)_s$ for $s \in {}^{<\omega}2$.

Lemma 2.3. *Suppose $\mathbb{P}$ is a perfect poset and $G \subseteq \mathbb{P}$ is V-generic. Then $\bigcap_{T\in G} T$ is a real r and $G = \{T \in \mathbb{P} \mid r \in [T]\}$.*

Proof. Suppose to the contrary that there are incompatible nodes $s, t \in \bigcap_{T\in G} T$. Fix some $T \in G$ and note that $s \in T$ by assumption. Consider the set

$$\mathcal{D} = \{S \leq T \mid s \notin S \text{ or } t \notin S\}.$$

Let's argue that $\mathcal{D}$ is dense below T. So fix $\bar{T} \leq T$. If either $s \notin \bar{T}$ or $t \notin \bar{T}$, then we are done. So assume $s, t \in \bar{T}$. In this case, $\bar{T}_s \leq \bar{T}$ and $t \notin \bar{T}_s$. So there is some $S \in \mathcal{D} \cap G$, but this is a contradiction. So r is indeed a real. Now suppose that $r \in [T]$. We will argue that $T \in G$. Let $S \in G$ force that $\dot{r} \in \check{T}$, where $\dot{r}$ is the canonical name for r. We claim that $S \leq T$, and hence $T \in G$. So suppose not. Then there is some $s \in S \setminus T$, which means that $S_s \cap T = \emptyset$. But then $S_s \leq S$ cannot force that $\dot{r} \in [T]$. □

To every perfect poset $\mathbb{P}$, we can associate the poset $\mathbb{Q}(\mathbb{P})$ whose elements are pairs (T, n) with $T \in \mathbb{P}$ and $n < \omega$ ordered so that $(S, m) \leq (T, n)$ whenever $S \subseteq T$, $m \geq n$, and $T \cap {}^n2 = S \cap {}^n2$. A generic filter $G \subseteq \mathbb{Q}(\mathbb{P})$ adds a perfect tree $\mathcal{T}$ which is the union of $T \cap {}^n2$ for every condition $(T, n) \in \mathbb{Q}(\mathbb{P})$.

We will show how to extend a perfect poset $\mathbb{P}$, using $\mathbb{Q}(\mathbb{P})$, to a larger perfect poset $\mathbb{P}^*$ with the property that for some fixed countable collection $\mathcal{C}$ of maximal antichains of $\mathbb{P}$, every $\mathcal{A} \in \mathcal{C}$ remains pre-dense in $\mathbb{P}^*$.

Suppose M is a countable transitive model of "ZFC$^-$ + $\mathcal{P}(\omega)$ exists" and $\mathbb{P} \in M$ is a perfect poset. Clearly $\mathbb{Q}(\mathbb{P}) \in M$. We will force with the ω-length finite-support product $\mathbb{Q}(\mathbb{P})^{<\omega}$ over M. So fix some M-generic $G \subseteq \mathbb{Q}(\mathbb{P})^{<\omega}$, and consider the extension $M[G]$. Let $\mathcal{T}_n$ be the perfect tree added by G on coordinate n. Let

$$\mathbb{U} = \{(\mathcal{T}_n)_s \mid n < \omega \text{ and } s \in \mathcal{T}_n\}.$$

Let $\mathbb{P}^*$ be the closure under finite unions of $\mathbb{P} \cup \mathbb{U}$. Clearly $\mathbb{P}^*$ is a perfect poset.

Lemma 2.4. *$\{\mathcal{T}_n \mid n < \omega\}$ is a maximal antichain in $\mathbb{P}^*$ and $\mathbb{U}$ is dense in $\mathbb{P}^*$.*

Proof. First, let's argue that the $\mathcal{T}_n$ are incompatible. Indeed, if $m \neq n$, then $\mathcal{T}_n \cap \mathcal{T}_m$ is finite. We will show that conditions $p \in \mathbb{Q}(\mathbb{P})$ with $p(i) = (T_i, k_i)$ such that $T_m \cap T_n$ is finite are dense in $\mathbb{Q}(\mathbb{P})$. Fix any $q \in \mathbb{Q}(\mathbb{P})$ with $q(i) = (S_i, k_i)$. By going to a stronger condition, we can assume without loss that $k_m = k_n = k$. For every node s on level k of T_n, let t_s be the first splitting node above it. If $t_s{}^\frown 0$ is not in T_m, we thin out $(T_n)_s$ to $(T_n)_{t_s{}^\frown 0}$. Similarly, if $t_s{}^\frown 1$ is not in T_m, we thin out accordingly. So finally, suppose that both $t_s{}^\frown 0$ and $t_s{}^\frown 1$ are in T_m, then we thin out $(T_n)_s$ to $(T_n)_{t_s{}^\frown 0}$ and $(T_m)_s$ to $(T_m)_{t_s{}^\frown 1}$. Now we strengthen q to p with $q(i) = p(i)$ for $i \neq n, m$, $q(n) = (\bar{T}_m, k)$, where $\bar{T}_m$ is the result of the appropriate thinning out of all subtrees $(T_n)_s$ and $q(m) = (\bar{T}_m, k)$, where $\bar{T}_m$ is the appropriate thinning out of substrees $(T_m)_s$. Note that p exists by closure under finite unions. Clearly p has the desired property.

Now observe that for any $T \in \mathbb{P}$, the set of conditions p in $\mathbb{Q}(\mathbb{P})^{<\omega}$ such that $(T, 0)$ appears on some coordinate of p is clearly dense because of finite-support. So there is some $\mathcal{T}_n$ below every $T \in \mathbb{P}$. It follows that $\{\mathcal{T}_n \mid n < \omega\}$ is a maximal antichain in $\mathbb{P}^*$ and $\mathbb{U}$ is dense in $\mathbb{P}^*$. □

Lemma 2.5. *Every maximal antichain $\mathcal{A} \in M$ of $\mathbb{P}$ remains pre-dense in $\mathbb{P}^*$. Indeed if $\mathcal{A}$ is a maximal antichain in $\mathbb{P}$, then every tree $\mathcal{T}_n$ has a level k such that for every node s on level k of $\mathcal{T}_n$, $(\mathcal{T}_n)_s$ is below some element of $\mathcal{A}$.*

Proof. Fix a maximal antichain $\mathcal{A} \in M$ of $\mathbb{P}$ and a tree $\mathcal{T}_n$. Consider the set $\mathcal{D}$ of all conditions $p \in \mathbb{Q}(\mathbb{P})^{<\omega}$ such that $p(n) = (T, k)$ and for every s on level k of T we have $T_s \leq A_s$ for some $A_s \in \mathcal{A}$. If $\mathcal{D}$ is dense in $\mathbb{Q}(\mathbb{P})^{<\omega}$, then we are done because it follows that for every s on level k of $\mathcal{T}_n$, $(\mathcal{T}_n)_s \leq T_s$ is below an element of $\mathcal{A}$. So let's argue that $\mathcal{D}$ is dense. So fix a condition $q \in \mathbb{Q}(\mathbb{P})^{<\omega}$ with $q(n) = (T', k)$. Fix a node s on level k of T'. Since $\mathcal{A}$ is maximal in $\mathbb{P}$, T'_s is compatible with some $R' \in \mathcal{A}$. Let $R_s \leq R', T'_s$. Repeat this for every node s on level k of T'. Now let T be the tree obtained by replacing each T'_s with R_s in T'. Let $r \leq q$ be defined so that $r(i) = q(i)$ for all $i \neq n$, and $r(n) = (T, k)$. Clearly $r \in \mathcal{D}$. Now fix some $(\mathcal{T}_n)_s \in \mathbb{U}$. Let k be the level from the hypothesis. If s sits on level k or below, then let s' be the node in $\mathcal{T}_n$ extending s on level k. Since $(\mathcal{T}_n)_{s'}$ is below $A_{s'} \in \mathcal{A}$, it follows that $(\mathcal{T}_n)_s$ is compatible with $A_{s'}$. If s sits above level k, then clearly $(\mathcal{T}_n)_s$ is below an element of $\mathcal{A}$. □

Note that an antichain $\mathcal{A} \in M$ of $\mathbb{P}$ may no longer be an antichain in $\mathbb{P}^*$ because incompatible elements in $\mathbb{P}$ can potentially become compatible in $\mathbb{P}^*$.

Now let's consider the ω-length finite-support product $\mathbb{P}^{<\omega}$ of the perfect poset $\mathbb{P}$ and the product $\mathbb{P}^{*<\omega}$ of its extension $\mathbb{P}^*$.

Lemma 2.6. *Every maximal antichain $\mathcal{A} \in M$ of $\mathbb{P}^{<\omega}$ remains pre-dense in $\mathbb{P}^{*<\omega}$.*

The proof is an easy generalization of the proof of Lemma 2.5.

We are now ready to construct the Jensen forcing $\mathbb{P}$ in L, which has the following properties:

- $\mathbb{P}$ is a perfect poset of size ω_1.
- $\mathbb{P}$ has the ccc.
- If $G \subseteq \mathbb{P}^{<\omega}$ is L-generic, then the L-generic reals for $\mathbb{P}$ in $L[G]$ are precisely the generic reals on the coordinates of G.

3. Jensen's poset

Before we begin, let's make the following definition. Let's call a countable transitive model $M \models$ "ZFC$^-$ + $\mathcal{P}(\omega)$" *suitable* if $M = L_\xi$ for some ordinal ξ. For instance, the collapse of any countable elementary submodel of L_{ω_2} will be suitable. Let's also fix a canonical $\diamondsuit$-sequence $\langle S_\alpha \mid \alpha < \omega_1\rangle$. Observe that a suitable model M correctly constructs $\langle S_\alpha \mid \alpha < \omega_1^M\rangle$.

We will construct the poset $\mathbb{P}$ in ω_1-many steps as the union of an increasing chain $\langle \mathbb{P}_\alpha \mid \alpha < \omega_1\rangle$ of countable perfect posets using $\diamondsuit$. Let $\mathbb{P}_0$ be the smallest perfect poset (or any countable perfect poset). At limit stages, we take unions. So suppose that $\mathbb{P}_\alpha$ has been defined. We let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha$, unless the following happens. Suppose S_α codes a well-founded, extensional binary relation $E \subseteq \alpha \times \alpha$ such that the collapse of E is a suitable model M_α with $\mathbb{P}_\alpha \in M_\alpha$ and $\alpha = \omega_1^{M_\alpha}$. In this case, we take the L-least M_α-generic filter $G \subseteq \mathbb{Q}(\mathbb{P}_\alpha)^{<\omega}$ and let $\mathbb{P}_{\alpha+1} = \mathbb{P}^*_\alpha$ (the closure under finite unions of $\mathbb{P}_\alpha \cup \mathbb{U}_\alpha$) as constructed in $M_\alpha[G]$.

By Proposition 2.6, every maximal antichain of $\mathbb{P}^{<\omega}_\alpha$ in M_α remains pre-dense in $\mathbb{P}^{<\omega}_{\alpha+1}$. Now let's argue that every maximal antichain of $\mathbb{P}^{<\omega}_\alpha$ in M_α remains pre-dense in the final poset $\mathbb{P}^{<\omega}$. It suffices to note that the models M_α form an increasing sequence. Indeed, if $\alpha < \beta$, then $M_\alpha \in M_\beta$ because $\beta = \omega_1^{M_\beta}$, and therefore, by suitability, M_β has S_α as an element and can collapse it to obtain M_α. This shows that every maximal antichain of $\mathbb{P}^{<\omega}_\alpha$ that exists in M_α is sealed.

Lemma 3.1. *The ω-length finite support product $\mathbb{P}^{<\omega}$ has the ccc.*

Proof. Fix a maximal antichain $\mathcal{A} \in \mathbb{P}^{<\omega}$. Let $N \prec L_{\omega_2}$ be a transitive elementary submodel of size ω_1 with $\mathcal{A} \in N$. We can decompose N as the union of a continuous elementary chain of countable structures

$$X_0 \prec X_1 \prec \cdots \prec X_\alpha \prec \cdots \prec N$$

with $\mathcal{A} \in X_0$. By the properties of $\diamondsuit$, there is some α such that $\alpha = \omega_1 \cap X_\alpha$, $\mathbb{P}_\alpha = \mathbb{P} \cap X_\alpha$, and S_α codes X_α. Let M_α be the transitive collapse of X_α. Then $\mathbb{P}_\alpha$ is the image of $\mathbb{P}$ under the collapse, $\bar{\mathcal{A}} = \mathcal{A} \cap X_\alpha$ is the image of $\mathcal{A}$, and α is the image of ω_1. So at stage α in the construction of $\mathbb{P}$, we chose a forcing extension $M_\alpha[G]$ of M_α by $\mathbb{Q}(\mathbb{P}_\alpha)^{<\omega}$ and let $\mathbb{P}_{\alpha+1} = \mathbb{P}^*_\alpha$ as constructed in $M_\alpha[G]$. Thus, by our observation above, $\bar{\mathcal{A}}$ remains pre-dense in $\mathbb{P}^{<\omega}$, and hence $\bar{\mathcal{A}} = \mathcal{A}$ is countable. □

Lemma 3.2. *A real r is L-generic for $\mathbb{P}$ and if and only if for every α, r is a branch through some $\mathcal{T}^\alpha_n \in \mathbb{U}_\alpha$.*

Proof. The condition is clearly necessary because $\{\mathcal{T}^\alpha_n \mid n < \omega\}$ remains pre-dense in $\mathbb{P}$. So let's argue that every r which satisfies the hypothesis is L-generic for $\mathbb{P}$. Fix a maximal antichain $\mathcal{A}$ of $\mathbb{P}$. Then $\mathcal{A}$ appears in some M_α in the construction of $\mathbb{P}$. Let r be a branch through $\mathcal{T}^\alpha_n \in \mathbb{U}_\alpha$. By Lemma 2.5, there is a level k in $\mathcal{T}^\alpha_n$ such that for every $s \in \mathcal{T}^\alpha_n$ on level k, $(\mathcal{T}^\alpha_n)_s \leq A_s \in \mathcal{A}$. Since $r \in [\mathcal{T}^\alpha_n]$, it follows that there is s on level k of $\mathcal{T}^\alpha_n$ such that $r \in [A_s]$. □

It follows that the collection of all L-generic reals for $\mathbb{P}$ is definable because everything in the construction of $\mathbb{P}$ is definable. In fact, it is not difficult to see that the collection of all L-generic generic reals for $\mathbb{P}$ is (light-face) Π^1_2.

It remains to argue that the product has the uniqueness of generic reals property. We will start with the following definitions.

4. Uniqueness property for products of Jensen reals

Given any poset $\mathbb{Q}$, we can think of a $\mathbb{Q}$-name σ for a real as a collection

$$\{C_n \mid n < \omega\}$$

of maximal antichains such that every $C_n = C_{n0} \sqcup C_{n1}$ and $\sigma_G(n) = i$ for a V-generic filter $G \subseteq \mathbb{Q}$ if and only if $G \cap C_{ni} \neq \emptyset$. Conversely, any collection $\{C_n \mid n < \omega\}$ of maximal antichains with this property gives a $\mathbb{Q}$-name for a real.

Example 4.1. Suppose $\mathbb{P}$ is a perfect poset. The $\mathbb{P}$-generic real $\dot{x}$ is given by the sequence $\{C_{ni}\}$, where each C_{ni} consists of the single condition

$$p_{ni} = \{s \in {}^{<\omega}2 \mid \mathrm{len}(s) > n \to s(n) = i\}.$$

Similarly, the canonical name $\dot{x}_k$ for the generic real on the k-th coordinate of $\mathbb{P}^{<\omega}$ is given by the sequence $\{C_{ni}\}$, where C_{ni} consists of the single condition p_{ni} with $\mathrm{dom}(p) = \{k\}$ and $p_{ni}(k) = \{s \in {}^{<\omega}2 \mid \mathrm{len}(s) > n \to s(n) = i\}$.

Definition 4.2. Suppose $\mathbb{Q}$ is a poset. Let σ and τ be two $\mathbb{Q}$-names for a real given by sequences $\{C_{ni}\}$ and $\{D_{ni}\}$ respectively. Given $p \in \mathbb{Q}$, we define that

- p *directly forces* $\sigma(n) = i$ if p is below some $q \in C_{ni}$,
- p *directly forces* $s \subseteq \sigma$, where $s \in {}^{<\omega}2$, if for all $n < \mathrm{len}(s)$, p directly forces $\sigma(n) = s(n)$,
- p *directly forces* $\sigma \neq \tau$ if there are incomparable strings $s, t \in {}^{<\omega}2$ such that p directly forces $s \subseteq \sigma$ and $t \subseteq \tau$,
- p *directly forces* $\sigma \notin [T]$, where T is a perfect tree, if there is $s \in {}^{<\omega}2 \setminus T$ such that p directly forces that $s \subseteq \sigma$.

We need the notion of direct forcing to go back and forth between what is forced by some $\mathbb{P}_\alpha$ and the final Jensen poset $\mathbb{P}$. Since each $\mathbb{P}_\alpha$ is just a subposet of $\mathbb{P}$ which might even be mistaken about compatibility, the forcing relations for the two posets have no reason to agree. But direct forcing is clearly absolute between the stages $\mathbb{P}_\alpha$ and the final poset $\mathbb{P}$.

It is not difficult to see that if a statement is forced, then it is dense that it is directly forced.

Proposition 4.3. *Let $\mathbb{Q}$ be a any perfect poset. Suppose σ is a $\mathbb{Q}^{<\omega}$-name for a real such that for some $k < \omega$, we have $\mathbb{1} \Vdash \sigma \neq \dot{x}_k$. Then the set*

$$\mathcal{D} = \{q \in \mathbb{Q}^{<\omega} \mid q \text{ directly forces } \sigma \neq \dot{x}_k\}$$

is dense.

Proof. Fix any condition $r \in \mathbb{Q}^{<\omega}$. Then there must be a condition $q \leq r$ and $s \neq t \in {}^{n}2$ such that q forces that $s \subseteq \dot{x}_\kappa$ and $t \subseteq \sigma$. We can assume without loss that $q(k) \subseteq ({}^{<\omega}2)_s$, so that q directly forces that $s \subseteq \dot{x}_\kappa$. Next, choose $q_0 \leq q$ such that q_0 is below some element of $C_{0t(0)}$ (this is possible since C_0 is a maximal antichain). Next, choose $q_1 \leq q_0$ such that q_1 is below some element of $C_{1t(1)}$. Continue in this manner, until we obtain q_n below some element of $C_{nt(n)}$, where $n = \mathrm{len}(t) - 1$. Clearly q_n directly forces that $\sigma \neq \dot{x}_k$. □

For the next crucial theorem, suppose that M is a countable suitable model, $\mathbb{P} \in M$ is a perfect poset, and $\mathbb{P}^*$ is constructed as above in a forcing extension $M[G]$ by $\mathbb{Q}(\mathbb{P})^{<\omega}$.

Theorem 4.4. *Suppose that σ is a $\mathbb{P}^{<\omega}$-name for a real such that each*

$$\mathcal{D}(k) = \{r \in \mathbb{P}^{<\omega} \mid r \text{ directly forces } \sigma \neq \dot{x}_\kappa\}$$

is dense in $\mathbb{P}^{<\omega}$. Then for every $\mathcal{T}_n \in \mathbb{U}$, conditions directly forcing that $\sigma \notin [\mathcal{T}_n]$ are dense in $\mathbb{P}^{*<\omega}$.

Proof. Fix a condition $p \in \mathbb{P}^{*<\omega}$ and $n \in \omega$. Since $\mathbb{U}^{<\omega}$ is dense in $\mathbb{P}^{*<\omega}$, we can assume without loss that $p \in \mathbb{U}^{<\omega}$. We need to find $q \leq p$ such that q directly forces $\sigma \notin [\mathcal{T}_n]$. We will illustrate the argument with a concrete, generalizable example. So suppose for concreteness that

$$n = 1 \text{ and } p = \langle (\mathcal{T}_3)_{t_3}, (\mathcal{T}_1)_{t_1}, (\mathcal{T}_7)_{t_7} \rangle.$$

So there must be a condition $\bar{p} \in G$ with $\bar{p}(i) = (T_i, m_i)$ such that for $i = 1, 3, 7$, there is $s_i \in {}^{m_i}2$ with $t_i \subseteq s_i$. Consider the set $\mathcal{D}$ of all conditions $\bar{q} \leq \bar{p}$ in $\mathbb{Q}(\mathbb{P})^{<\omega}$, with $\bar{q}(i) = (\bar{U}_i, k_i)$, for which there is an associated condition

$$a_{\bar{q}} = \langle \bar{W}_0, \bar{W}_1, \ldots, \bar{W}_j \rangle$$

in $\mathbb{P}^{<\omega}$ satisfying:

(1) $\bar{W}_0 = (\bar{U}_3)_{\bar{s}_3}$, $\bar{W}_1 = (\bar{U}_1)_{\bar{s}_1}$, $\bar{W}_2 = (\bar{U}_7)_{\bar{s}_7}$ for some nodes $\bar{s}_i \supseteq s_i$ on level k_i for $i = 1, 3, 7$.

(2) $a_{\bar{q}}$ directly forces $\sigma \notin [(\bar{U}_1)_t]$ for every node t on level k_1 of $\bar{U}_1$.

We will argue that $\mathcal{D}$ is dense below $\bar{p}$ in $\mathbb{Q}(\mathbb{P})^{<\omega}$. So fix some $r \leq \bar{p}$ in $\mathbb{Q}(\mathbb{P})^{<\omega}$ with $r(i) = (U_i, k_i)$. Let $W_0 = (U_3)_{\bar{s}_3}$, $W_1 = (U_1)_{\bar{s}_1}$, and $W_2 = (U_7)_{\bar{s}_7}$, where $\bar{s}_i$ are some nodes on level k_i above s_i for $i = 1, 3, 7$. Let W_i for $i \geq 2$ be the remaining subtrees $(U_1)_t$ for nodes $t \neq \bar{s}_1$ on level k_1 of U_1. Let

$$a' = \langle W_0, \ldots, W_{j'} \rangle,$$

where j' is the last index we end up with. By our assumption, we can find a condition

$$a = \langle \bar{W}_0, \ldots, \bar{W}_{j'}, \ldots, \bar{W}_j \rangle$$

below a' such that a directly forces $\sigma \neq \dot{x}_k$ for every $0 \leq k \leq j'$. So for every $0 \leq k \leq j'$, there are incompatible nodes s and t such that a directly forces $s \subseteq \sigma$ and $t \in \bar{W}_k$. Without loss of generality, we can restrict $\bar{W}_k$ to $(\bar{W}_k)_t$ so that a directly forces $\sigma \notin [\bar{W}_k]$. Let $\bar{U}_1$ be the tree obtained by replacing $(U_1)_{\bar{s}_1}$ with $\bar{W}_1$ and replacing the remaining $(U_1)_t$ with the corresponding $\bar{W}_i$ in U_1. Let $\bar{U}_3$ be the tree obtained by replacing $(U_3)_{\bar{s}_3}$ with $\bar{W}_0$ in U_3. Let $\bar{U}_7$ be the tree obtained by replacing $(U_7)_{\bar{s}_7}$ with $\bar{W}_2$ in U_7. Let $\bar{U}_i = U_i$ for the remaining i. Let $\bar{q}$ be such that $\bar{q}(i) = (\bar{U}_i, k_i)$ and let $a = a_{\bar{q}}$. Clearly $a_{\bar{q}}$ directly forces $\sigma \notin [(\bar{U}_1)_t]$ for every t on level k_1 of $\bar{U}_1$. Thus, we have verified that $\mathcal{D}$ is dense below $\bar{p}$.

So there must be some condition $\bar{q} \in G \cap \mathcal{D}$. It follows that $(\mathcal{T}_3)_{\bar{s}_3}$ is contained in $\bar{W}_0$, $(\mathcal{T}_7)_{\bar{s}_7}$ is contained in $\bar{W}_2$, and $(\mathcal{T}_1)_{\bar{s}_1}$ is contained in $\bar{W}_1$. So let

$$q = \langle (T_3)_{\bar{s}_3}, (T_1)_{\bar{s}_1}, (T_7)_{\bar{s}_7}, \bar{W}_3, \ldots, \bar{W}_j \rangle.$$

Clearly $q \leq p$, and also clearly $q \leq a_{\bar{q}}$. So q directly forces $\sigma \notin [\mathcal{T}_1]$ because $\mathcal{T}_1$ is contained in the union of the $(\bar{U}_1)_t$ for nodes t on level k_1. □

Theorem 4.5 (Kanovei-Lyubetsky)**.** *Suppose $H \subseteq \mathbb{P}^{<\omega}$ is L-generic. If $r \in L[H]$ is L-generic for $\mathbb{P}$, then $r = x_k$ for some $k < \omega$.*

Proof. Let's suppose to the contrary that this is not the case. Then there is a real $r \neq x_k$ for all $k < \omega$, which is L-generic for $\mathbb{P}$. Let σ be a $\mathbb{P}^{<\omega}$-name for r such that $\mathbb{1} \Vdash \dot{r} \neq x_k$ for all $k < \omega$. It follows that the sets

$$\mathcal{D}(k) = \{p \in \mathbb{P}^{<\omega} \mid p \text{ directly forces } \sigma \neq \dot{x}_\kappa\}$$

are dense in $\mathbb{P}^{<\omega}$.

Choose some transitive $N \prec L_{\omega_2}$ of size ω_1 with $\sigma \in N$. We can decompose N as the union of an elementary chain of countable substructures

$$X_0 \prec X_1 \prec \cdots \prec X_\alpha \prec \cdots \prec N$$

with $\sigma \in X_0$. By the properties of $\diamondsuit$, there is some α such that $\alpha = \omega_1 \cap X_\alpha$, $\mathbb{P}_\alpha = \mathbb{P} \cap X_\alpha$, and S_α codes X_α. Let M_α be the collapse of X_α. Then $\mathbb{P}_\alpha$ is the image of $\mathbb{P}$ under the collapse and α is the collapse of ω_1. Clearly the name σ is fixed by the collapse because all antichains of $\mathbb{P}^{<\omega}$ are countable. So at stage α of the construction of $\mathbb{P}$, we chose a forcing extension $M_\alpha[G]$ by $\mathbb{Q}(\mathbb{P}_\alpha)^{<\omega}$ and let $\mathbb{P}_{\alpha+1} = \mathbb{P}^*_\alpha$ as constructed in $M_\alpha[G]$. By elementarity, M_α satisfies that all sets

$$\mathcal{D}(k) = \{p \in \mathbb{P}_\alpha \mid p \text{ directly forces } \sigma \neq \dot{x}_k\}$$

are dense in $\mathbb{P}_\alpha$. Thus, by Theorem 4.4, for every $n < \omega$, $\mathbb{P}^{<\omega}_{\alpha+1}$ has a maximal antichain $\mathcal{A}_n$ consisting of conditions q which directly force $\sigma \notin [\mathcal{T}_n]$. Since H must meet every $\mathcal{A}_n$, it holds in $L[H]$ that $\sigma_H = r$ is not a branch through any $\mathcal{T}_n$. But this contradicts our assumption that r is L-generic for $\mathbb{P}$ because $\langle \mathcal{T}_n \mid n < \omega \rangle$ remains a maximal antichain of $\mathbb{P}$. □

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