Jensen's forcing at an inaccessible

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Jensen's forcing and applications

Jensen's forcing $\mathbb{P}^J$ is a subposet of the Sacks forcing constructed in L using $\diamondsuit$.

- Perfect trees ordered by $\subseteq$.
- Has the ccc.
- Adds a unique Π^1_2-definable generic real over L.

Theorem: (Jensen) It is consistent that there is a Π^1_2-definable non-constructible real.

Theorem: (Lyubetsky, Kanovei) Suppose $\mathbb{P}^{J^{<\omega}}$ is the finite-support product of $\mathbb{P}^J$ of length ω and $G \subseteq \mathbb{P}^{J^{<\omega}}$ is L-generic. Then in $L[G]$, a real r is L-generic for $\mathbb{P}^J$ if and only if it is added by the n-th slice of G for some $n < \omega$.

Theorem: (Lyubetsky, Kanovei) There is a countable OD set of reals without an OD real.

Theorem: (Friedman, G., Kanovei) There is a model of second-order arithmetic Z_2 with the choice scheme in which the Π^1_2-dependent choice scheme fails.

The model is the reals of a symmetric submodel of a forcing extension by a tree iteration of $\mathbb{P}^J$.

Perfect posets

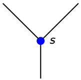
An infinite tree $T \subseteq 2^{<\omega}$ is perfect if every node of T has a splitting node above it.

Proposition: If T and S are perfect trees such that $T \cap S$ contains a perfect tree, then there is a maximal such perfect tree denoted $T \wedge S$.

A subposet $\mathbb{P}$ of Sacks forcing is perfect if:

- $(2^{<\omega})_s \in \mathbb{P}$ for every $s \in 2^{<\omega}$.

  For every $T, S \in \mathbb{P}$:
- $T \cup S \in \mathbb{P}$ (closed under unions),
- $T \wedge S \in \mathbb{P}$ (closed under meets).



Proposition: Suppose that $\mathbb{P}$ is a perfect poset and $G \subseteq \mathbb{P}$ is V-generic. Let $r = \bigcap_{T \in G} T$. Then in $V[G]$:

- r is a cofinal branch through every tree in G.
- If $r \in [T]$ for some $T \in \mathbb{P}$, then $T \in G$.

Smallest perfect poset $\mathbb{P}_{\text{min}} = \{(2^{<\omega})_s \mid s \in 2^{<\omega}\}$.

The fusion poset $\mathbb{Q}(\mathbb{P})$

Suppose that $\mathbb{P}$ is a perfect poset.

$\mathbb{Q}(\mathbb{P})$: elements are pairs (T, n) with $T \in \mathbb{P}$ and $n < \omega$ ordered by $(T, n) \leq (S, m)$ if $n \geq m$ and $T \cap 2^m = S \cap 2^m$.

Fusion arguments with trees from $\mathbb{P}$ can be expressed by meeting dense sets of $\mathbb{Q}(\mathbb{P})$.

Proposition: Suppose that $G \subseteq \mathbb{Q}(\mathbb{P})$ is V-generic. Then in $V[G]$,

$$\mathscr{T} = \bigcup_{(T,n)\in G} T \cap 2^n$$

is a perfect tree and $\mathscr{T} \subseteq T$ for every condition $(T, n) \in G$.

$\mathbb{Q}(\mathbb{P})^{<\omega}$: finite support ω-length product of the $\mathbb{Q}(\mathbb{P})$.

Growing perfect posets with generic perfect trees

Set-up

- $\mathbb{P}$ is a perfect poset
- $\mathbb{Q}(\mathbb{P})$ is a fusion poset for $\mathbb{P}$
- $G \subseteq \mathbb{Q}(\mathbb{P})^{<\omega}$ is V-generic
- $\mathscr{T}_n$ is the generic perfect tree added by the n-th slice of G

In $V[G]$

$\mathbb{P}^*$: close $\{\mathscr{T}_n \mid n < \omega\} \cup \mathbb{P}$ under meets and unions.

Proposition:

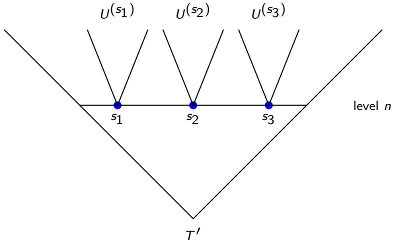
- $\{(\mathscr{T}_n)_s \mid n < \omega, s \in \mathscr{T}_n\}$ is dense in $\mathbb{P}^*$.
- $\{\mathscr{T}_n \mid n < \omega\}$ is a maximal antichain of $\mathbb{P}^*$.
- Every maximal antichain of $\mathbb{P}$ from V remains maximal in $\mathbb{P}^*$.

An argument template for $\mathbb{P}^*$

Proof sketch: Fix a maximal antichain $\mathcal{A}$ of $\mathbb{P}$ from V.

Suffices to show that for a level n, for every $s \in 2^n$, $\mathscr{T}_s \subseteq A$ for some $A \in \mathcal{A}$.

- Fix a condition $(T, n) \in \mathbb{Q}(\mathbb{P})$.
- For every s on level n of T, choose $A^{(s)} \in \mathcal{A}$ compatible with T_s.
- Let $U^{(s)} \subseteq T_s, A^{(s)}$ in $\mathbb{P}$.
- Let T' be obtained by replacing T_s with $U^{(s)}$ (closure under unions).
- Conditions of the form (T', n) are dense in $\mathbb{Q}(\mathbb{P})$.
- $\mathscr{T}_s \subseteq U^{(s)} \subseteq A^{(s)}$ for every s on level n of $\mathcal{T}$.



Jensen's forcing: $\mathbb{P}^J$

Work in L and fix a canonical $\diamondsuit$-sequence $\vec{D} = \{D_\xi \mid \xi < \omega_1\}$.

A model M is suitable if M is countable, $M = L_\alpha$ for some α, and $M \models \mathrm{ZFC}^- + \mathrm{P}(\omega)$ exists.

Observations:

- The Mostowski collapse M of any countable $X \prec L_{\omega_2}$ is suitable.
- If M is suitable and $\delta = (\omega_1)^M$, then $\langle D_\xi \mid \xi < \delta\rangle \in M$.

$\mathbb{P}^J$: union of a chain $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \cdots \subseteq \mathbb{P}_\xi \subseteq \cdots$ of length ω_1 of perfect posets.

$\mathbb{P}_0 = \mathbb{P}_{\text{min}}$

$\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$ at limits λ.

Suppose $\mathbb{P}_\xi$ has been defined.

If D_ξ codes a suitable model M_ξ such that $\mathbb{P}_\xi \in M_\xi$ and $(\omega_1)^{M_\xi} = \xi$:

- Let G_ξ be the L-least M_ξ-generic filter for $\mathbb{Q}(\mathbb{P}_\xi)^{<\omega}$.
- $\mathbb{P}_{\xi+1} = \mathbb{P}^*_\xi$ as constructed in $M_\xi[G_\xi]$.

Otherwise, $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi$.

Sealing Lemma: Every maximal antichain of $\mathbb{P}_\xi$ from M_ξ remains maximal in $\mathbb{P}^J$.

Perfect κ-trees

Suppose that κ is an inaccessible cardinal.

A perfect κ-tree is a tree $T \subseteq 2^{<\kappa}$ such that:

- T has size κ (T is a κ-tree).
- Every node of T has a splitting node above it (T is splitting).
- For every limit $\lambda < \kappa$ if $s \in 2^\lambda$ and $s \restriction \xi \in T$ for every $\xi < \lambda$, then $s \in T$ (T is closed).
- For every limit $\lambda < \kappa$ if $s \in 2^\lambda$ and for cofinally many $\xi < \lambda$, $s \restriction \xi$ splits, then s splits (the splitting nodes of T are closed).

Proposition: Suppose that $\{T_\xi \mid \xi < \beta\}$, for $\beta < \kappa$, is a $\subseteq$-decreasing sequence of perfect κ-trees. Then $T = \bigcap_{\xi<\beta} T_\xi$ is a perfect κ-tree.

Badly behaved perfect κ-trees

Proposition: There are perfect κ-trees whose intersection does not contain a maximal perfect κ-tree.

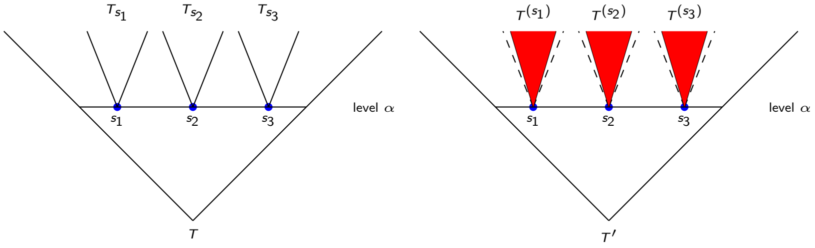
Proposition: There are ω-many perfect κ-trees whose union is not a perfect κ-tree.

κ-perfect posets

Suppose that κ is an inaccessible cardinal.

A collection $\mathbb{P}$ of perfect κ-trees ordered by $\subseteq$ is a κ-perfect poset if:

- $2^{<\kappa} \in \mathbb{P}$.
- If $T \in \mathbb{P}$ and $t \in T$, then $T_t \in \mathbb{P}$.
- If $\{T_\xi \mid \xi < \beta\} \subseteq \mathbb{P}$, with $\beta < \kappa$, then $T = \bigcap_{\xi<\beta} T_\xi \in \mathbb{P}$ ($<\kappa$-closure property).
- Suppose $T \in \mathbb{P}$, $\alpha < \kappa$ is a successor, and $\{T^{(s)} \subseteq T_s \mid s \in T \cap 2^\alpha\} \subseteq \mathbb{P}$. Then $T' = \bigcup_{s \in 2^\alpha} T^{(s)} \in \mathbb{P}$ (weak union property).



κ-perfect posets (continued)

Proposition: Suppose $\mathbb{P}$ is a κ-perfect poset and $G \subseteq \mathbb{P}$ is V-generic. Let $A = \bigcap_{T \in G} T$. Then in $V[G]$:

- A is cofinal branch through every tree in G.
- If $A \in [T]$ for some $T \in \mathbb{P}$, then $T \in G$.

Smallest κ-perfect poset $\mathbb{P}_{\text{min}}$: close $\{(2^{<\kappa})_s \mid s \in 2^{<\kappa}\}$ under $<\kappa$-intersection property and weak union property.

$\mathbb{P}_0 = \{(2^{<\kappa})_s \mid s \in 2^{<\kappa}\}$

$\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$ for limits λ

Suppose $\mathbb{P}_\xi$ has been defined.

$\mathbb{P}'_{\xi+1}$ consists of all $T' = \bigcup_{s \in 2^\alpha} T^{(s)}$ for $T \in \mathbb{P}_\xi$, successor $\alpha < \kappa$, and $\{T^{(s)} \subseteq T_s \mid s \in T \cap 2^\alpha\} \subseteq \mathbb{P}_\xi$.

$\mathbb{P}_{\xi+1}$ consists of all $T = \bigcap_{\xi<\beta} T_\xi$, for $\beta < \kappa$, and $\subseteq$-decreasing $\{T_\xi \mid \xi < \beta\} \subseteq \mathbb{P}'_{\xi+1}$.

Clean Levels Lemma: Every tree $T \in \mathbb{P}_{\text{min}}$ has a level α such that for every $t \in T \cap 2^\alpha$, $T_t = (2^{<\kappa})_t$.

The fusion poset $\mathbb{Q}(\mathbb{P})$

Suppose that $\mathbb{P}$ is a κ-perfect poset.

$\mathbb{Q}(\mathbb{P})$: elements are pairs (T, α), with $T \in \mathbb{P}$ and $\alpha < \kappa$ successor, ordered by $(T, \alpha) \leq (S, \beta)$ if $\alpha \geq \beta$ and $T \cap 2^\beta = S \cap 2^\beta$.

Proposition: The poset $\mathbb{Q}(\mathbb{P})$ is $<\kappa$-closed.

Proposition: Suppose $G \subseteq \mathbb{Q}(\mathbb{P})$ is V-generic. Then in $V[G]$,

$$\mathscr{T} = \bigcup_{(T,\alpha) \in G} T \cap 2^\alpha$$

is a perfect κ-tree and $\mathscr{T} \subseteq T$ for every condition $(T, \alpha) \in G$.

$\mathbb{Q}(\mathbb{P})^{<\kappa}$: bounded support κ-length product of the $\mathbb{Q}(\mathbb{P})$.

Growing κ-perfect posets with generic perfect κ-trees

Set-up

- $\mathbb{P}$ is a κ-perfect poset
- $\mathbb{Q}(\mathbb{P})$ is a fusion poset for $\mathbb{P}$
- $G \subseteq \mathbb{Q}(\mathbb{P})^{<\kappa}$ is V-generic
- $\mathscr{T}_\xi$ is the generic perfect κ-tree added by the ξ-th slice of G

In $V[G]$

$\mathbb{P}^*$: close $\{(\mathscr{T}_\xi)_t \mid \xi < \kappa, t \in \mathscr{T}_\xi\} \cup \mathbb{P}$ under $<\kappa$-intersection property and weak union property.

$\mathbb{P}_0 = \{(\mathscr{T}_\xi)_t \mid \xi < \kappa, t \in \mathscr{T}_\xi\} \cup \mathbb{P}$

$\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$ for limits λ

Suppose $\mathbb{P}_\xi$ has been defined.

$\mathbb{P}'_{\xi+1}$ consists of all $T' = \bigcup_{s \in T \cap 2^\alpha} T^{(s)}$ for $T \in \mathbb{P}_\xi$, $\alpha < \kappa$ successor, and $\{T^{(s)} \subseteq T_s \mid s \in T \cap 2^\alpha\} \subseteq \mathbb{P}_\xi$.

$\mathbb{P}_{\xi+1}$ consists of all $T = \bigcap_{\xi<\beta} T_\xi$ for $\beta < \kappa$ and $\subseteq$-decreasing $\{T_\xi \mid \xi < \beta\} \subseteq \mathbb{P}'_{\xi+1}$.

Growing κ-perfect posets with generic perfect κ-trees (continued)

Clean Levels Lemma: Every tree $T \in \mathbb{P}^*$ has a level α such that for every $t \in T \cap 2^\alpha$,

- $T_t = (\mathscr{T}_\xi)_t$ for some $\xi < \kappa$ or
- $T_t \in \mathbb{P}$.

Proposition:

- $\{(\mathscr{T}_\xi)_s \mid \xi < \kappa, s \in \mathscr{T}_\xi\}$ is dense in $\mathbb{P}^*$.
- $\{\mathscr{T}_\xi \mid \xi < \kappa\}$ is a maximal antichain of $\mathbb{P}^*$.
- Every maximal antichain from V remains maximal in $\mathbb{P}^*$.

Jensen's forcing at an inaccessible κ: $\mathbb{P}^{\kappa J}$

Work in L and fix a canonical $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$-sequence $\vec{D} = \{D_\xi \mid \xi \in \mathrm{Cof}(\kappa)\}$.

A model M is κ-suitable if:

- $|M| = \kappa$,
- $M^{<\kappa} \subseteq M$,
- $M = L_\alpha$ for some α,
- $M \models \mathrm{ZFC}^- + \mathrm{P}(\kappa)$ exists.

Observations:

- The Mostowski collapse M of any $X \prec L_{\kappa^{++}}$, with $X^{<\kappa} \subseteq X$ and $|X| = \kappa$, is κ-suitable.
- If M is κ-suitable and $\delta = (\kappa^+)^M$, then $\langle D_\xi \mid \xi < \delta\rangle \in M$.
- If M is κ-suitable and $\mathbb{P} \in M$ is $<\kappa$-closed, then there is an M-generic filter for $\mathbb{P}$.

$\mathbb{P}^{\kappa J}$: union of a chain $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \cdots \subseteq \mathbb{P}_\xi \subseteq \cdots$ of length κ^+ of κ-perfect posets.

Jensen's forcing at an inaccessible κ: $\mathbb{P}^{\kappa J}$ (continued)

$\mathbb{P}_0 = \mathbb{P}_{\text{min}}$

$\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$ at limits $\lambda \in \text{Cof}(\kappa)$.

Suppose $\mathbb{P}_\xi$ has been defined.

If $\xi \in \text{Cof}(\kappa)$ and D_ξ codes a κ-suitable model M_ξ such that $\mathbb{P}_\xi \in M_\xi$ and $(\kappa^+)^{M_\xi} = \xi$:

- Let G_ξ be the L-least M_ξ-generic filter for $\mathbb{Q}(\mathbb{P}_\xi)^{<\kappa}$.
- $\mathbb{P}_{\xi+1} = \mathbb{P}^*_\xi$ as constructed in $M_\xi[G_\xi]$.

Otherwise, $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi$.

Suppose λ is a limit of cofinality $<\kappa$.

$\mathbb{P}_\lambda$: close $\bigcup_{\xi<\lambda} \mathbb{P}_\xi$ under $<\kappa$-intersection property and weak union property.

Let $\mathscr{T}^{(\xi)}_\nu$ for $\xi < \kappa^+$ and $\nu < \kappa$ be the perfect κ-trees added in $M_\xi[G_\xi]$.

Clean Levels Lemma: Every tree $T \in \mathbb{P}^{\kappa J}$ has a level α such that for every $t \in 2^\alpha \cap T$,

- $T_t = (2^{<\kappa})_t$,
- $T_t = (\mathscr{T}^{(\xi)}_\nu)_t$ for some $\xi < \kappa^+$ and $\nu < \kappa$,
- $T_t = \bigcap_{\xi<\alpha} (\mathscr{T}^{(\mu_\xi)}_{\rho_\xi})_t$, with $\alpha < \kappa$, for some $\subseteq$-decreasing $\{(\mathscr{T}^{(\mu_\xi)}_{\rho_\xi})_t \mid \xi < \alpha\}$.

Sealing Lemma: Every maximal antichain of $\mathbb{P}_\xi$ from M_ξ remains maximal in $\mathbb{P}^{\kappa J}$.

Properties of $\mathbb{P}^{\kappa J}$

Theorem: The forcing $\mathbb{P}^{\kappa J}$

- is $<\kappa$-closed,
- has the κ^+-cc,
- adds a unique generic subset of κ over L.

Theorem: Suppose $\mathbb{P}^{\kappa J<\kappa}$ is the bounded support product of $\mathbb{P}^J$ of length κ and $G \subseteq \mathbb{P}^{\kappa J<\kappa}$ is L-generic. Then in $L[G]$, $A \subseteq \kappa$ is L-generic for $\mathbb{P}^{\kappa J}$ if and only if it is added by the α-th slice of G for some $\alpha < \kappa$.

Conjecture: (Work in progress) There is a model of Kelley-Morse set theory with the choice scheme in which the dependent choice scheme fails.

The model should be $V_{\kappa+1}$ of a symmetric submodel of a forcing extension by a tree iteration of $\mathbb{P}^{\kappa J}$.

Jensen's forcing outside of L

A version of Jensen's forcing can be constructed in any universe with $\diamondsuit$.

A version of Jensen's forcing at an inaccessible κ can be constructed in any universe with $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$.

- lose low complexity of generics
- keep uniqueness properties of generics