Jensen's forcing at an inaccessible

Victoria Gitman

vgitman@nylogic.org
https://victoriagitman.github.io

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Jensen's forcing

Jensen's forcing is a subposet of Sacks forcing that is constructed using the guessing principle $\diamondsuit$.

- Elements are perfect trees ordered by the subtree relation: $T \leq S$ whenever $T \subseteq S$.
- Has the ccc.
- Adds a unique generic real.

Variables in the construction of Jensen's forcing allow for many forcings with the above properties. Jensen's construction of the forcing in L has the additional property:

- The generic real is a Π^1_2-definable singleton in the forcing extension.

Jensen's forcing and unique generics

Products and (carefully defined) iterations of Jensen's forcing also have "unique generics" properties.

Notation:

- $\mathbb{J}$: Jensen's forcing (or any similarly constructed forcing)
- $\mathbb{J}^{<\alpha}$: finite support α-length product of Jensen's forcing for $\alpha \geq \omega$ (ccc)
- $\mathbb{J}_n$: n-length iteration of Jensen's forcing for $n < \omega$ (ccc)

Theorem: (Lyubetsky, Kanovei) In a forcing extension by $\mathbb{J}^{<\alpha}$, the only generic reals for $\mathbb{J}$ are the α-many slices of the generic filter.

Theorem: (Abraham) In a forcing extension by $\mathbb{J}_n$, there is a unique generic n-length sequence of reals.

Applications of Jensen's forcing

Theorem: (Jensen) It is consistent that there is a Π^1_2-definable singleton non-constructible real.

- In a forcing extension $L[r]$ by $\mathbb{J}$, r is Π^1_2-definable.
- Every Σ^1_2-definable real is in L by Shoenfield's Absoluteness.

Theorem: (Lyubetsky, Kanovei) There is a countable ordinal definable set of reals without any definable members.

- The set of generic reals for $\mathbb{J}$ in a forcing extension by $\mathbb{J}^{<\omega}$.

Theorem: (Friedman, G., Kanovei) There is a model of second-order arithmetic Z_2 with the Choice Scheme in which Π^1_2-Dependent Choice Scheme fails.

The model is the reals of a symmetric submodel of a forcing extension by a tree iteration of Jensen's forcing.

Perfect posets

Definition: An infinite tree $T \subseteq 2^{<\omega}$ is perfect if every node of T has a splitting node above it.

Proposition: If T and S are perfect trees such that $T \cap S$ contains a perfect tree, then there is a maximal such perfect tree denoted $T \wedge S$.

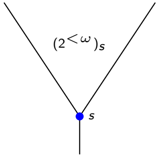
A subposet $\mathbb{P}$ of Sacks forcing is perfect if:

- $(2^{<\omega})_s \in \mathbb{P}$ for every $s \in 2^{<\omega}$.

  For every $T, S \in \mathbb{P}$:
- $T \cup S \in \mathbb{P}$ (closed under unions),

  Useful in constructions.
- $T \wedge S \in \mathbb{P}$ (closed under meets).

  If trees T and S are not compatible in $\mathbb{P}$, then they cannot become compatible in any larger perfect poset extending $\mathbb{P}$.



Proposition: Suppose that $\mathbb{P}$ is a perfect poset and $G \subseteq \mathbb{P}$ is V-generic. Let $r = \bigcap_{T \in G} T$. Then in $V[G]$:

- r is a cofinal branch through every tree in G.
- If $r \in [T]$ for some $T \in \mathbb{P}$, then $T \in G$.
- r and G are definable from each other.

Smallest perfect poset $\mathbb{P}_{\text{min}}$: close $\{(2^{<\omega})_s \mid s \in 2^{<\omega}\}$ under unions.

The fusion poset $\mathbb{Q}(\mathbb{P})$

Suppose that $\mathbb{P}$ is a perfect poset.

$\mathbb{Q}(\mathbb{P})$: elements are pairs (T, n) with $T \in \mathbb{P}$ and $n < \omega$ ordered by $(T, n) \leq (S, m)$ if $n \geq m$ and $T \cap 2^m = S \cap 2^m$.

Fusion arguments with trees from $\mathbb{P}$ can be expressed by meeting dense sets of $\mathbb{Q}(\mathbb{P})$.

Proposition: Suppose that $G \subseteq \mathbb{Q}(\mathbb{P})$ is V-generic. Then in $V[G]$,

- $\mathscr{T} = \bigcup_{(T,n)\in G} T \cap 2^n$ is a perfect tree,
- $\mathscr{T} \subseteq T$ for every condition $(T, n) \in G$.

Notation:
$\mathbb{Q}(\mathbb{P})^{<\omega}$: finite support ω-length product of the $\mathbb{Q}(\mathbb{P})$.

Growing perfect posets with generic perfect trees

Set-up

- $\mathbb{P}$ is a perfect poset
- $\mathbb{Q}(\mathbb{P})$ is a fusion poset for $\mathbb{P}$
- $G \subseteq \mathbb{Q}(\mathbb{P})^{<\omega}$ is V-generic
- $\mathscr{T}_n$ is the generic perfect tree added by the n-th slice of G

In $V[G]$

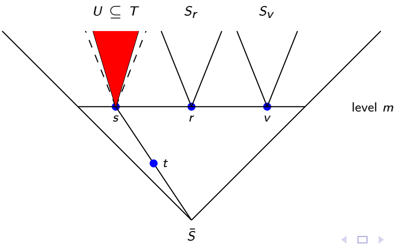
$\mathbb{P}^*$: close $\{\mathscr{T}_n \mid n < \omega\} \cup \mathbb{P}$ under meets and unions.

Properties of $\mathbb{P}^*$

Proposition: If $T \in \mathbb{P}$ and $\mathscr{T}_n \wedge T$ is a perfect tree, then for some s, $(\mathscr{T}_n)_s \subseteq T$.

Proof:

- Fix $t \in \mathscr{T}_n \wedge T$.
- Let $p \in G$, with $p(n) = (R, k)$, such that $p \Vdash t \in \dot{\mathscr{T}}_n \wedge T$.
- Since $\mathscr{T}_n \subseteq R$, $t \in R$.
- Fix $q \leq p$, with $q(n) = (S, m) \leq (R, k)$, such that $m > \text{lev}(t)$.
- There is $s \geq t$ on level m of S such that $U = S_s \wedge T$ is a perfect tree.
- Let $\bar{S}$ be S where we replace S_s with U.
- $\bar{S} \in \mathbb{P}$ by closure under unions.
- Let $\bar{q} \leq q$ such that $\bar{q}(n) = (\bar{S}, m)$ and $\bar{q}(i) = q(i)$ for all $i \neq n$.
- Conditions $\bar{q}$ are dense below p, so some $\bar{q} \in G$.
- $(\mathscr{T}_n)_s \subseteq T$. $\square$



Properties of $\mathbb{P}^*$ (continued)

Proposition:

(1) $\{(\mathscr{T}_n)_s \mid n < \omega, s \in \mathscr{T}_n\}$ is dense in $\mathbb{P}^*$.

(2) $\{\mathscr{T}_n \mid n < \omega\}$ is a maximal antichain of $\mathbb{P}^*$.

(3) Every maximal antichain of $\mathbb{P}$ from V remains maximal in $\mathbb{P}^*$.

Proof:

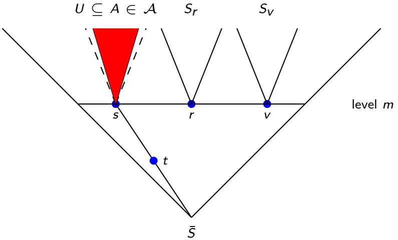
(1) By previous proposition.

(2) For $m \neq n$, $\mathscr{T}_m \cap \mathscr{T}_n$ is bounded.

Properties of $\mathbb{P}^*$ (continued)

(3) Fix a maximal antichain $\mathcal{A}$ of $\mathbb{P}$ from V. Suffices to show that every $(\mathscr{T}_n)_t$ is compatible with an element of $\mathcal{A}$.

- Fix $(\mathscr{T}_n)_t$.
- Let $p \in G$ such that $p \Vdash t \in \dot{\mathscr{T}}_n$.
- Fix $q \leq p$ such that $q(n) = (S, m)$ and $m > \text{lev}(t)$.
- Fix $s \geq t$ on level m of S.
- Choose $A \in \mathcal{A}$ such that A is compatible with S_s, let $U \subseteq A, S_s$.
- Let $\bar{S}$ be S where we replace S_s with U.
- Let $\bar{q} \leq q$ such that $\bar{q}(n) = (\bar{S}, m)$ and $\bar{q}(i) = q(i)$ for all $i \neq n$.
- Conditions $\bar{q}$ are dense below p, so some $\bar{q} \in G$.
- $(\mathscr{T}_n)_s \subseteq U \subseteq A \in \mathcal{A}$. $\square$



Suitable models

Work in L.

Let $\vec{D} = \{D_\xi \mid \xi < \omega_1\}$ be the canonical $\diamondsuit$-sequence.

Definition: A model M is suitable if

- $M = L_\alpha$ for some countable α
- $M \models \mathrm{ZFC}^- + \mathrm{P}(\omega)$ exists.

Observations:

- The Mostowski collapse M of any countable $X \prec L_{\omega_2}$ is suitable.
- If M is suitable and $\delta = (\omega_1)^M$, then $\langle D_\xi \mid \xi < \delta\rangle \in M$.

Jensen's forcing: $\mathbb{J}$

$\mathbb{J}$: union of a chain $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \cdots \subseteq \mathbb{P}_\xi \subseteq \cdots$ of length ω_1 of perfect posets.

$\mathbb{P}_0 = \mathbb{P}_{\text{min}}$

$\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$ at limits λ.

Suppose $\mathbb{P}_\xi$ has been defined.

If D_ξ codes a suitable model M_ξ such that $\mathbb{P}_\xi \in M_\xi$ and $(\omega_1)^{M_\xi} = \xi$:

- Let G_ξ be the L-least M_ξ-generic filter for $\mathbb{Q}(\mathbb{P}_\xi)^{<\omega}$.
- $\mathbb{P}_{\xi+1} = \mathbb{P}^*_\xi$ as constructed in $M_\xi[G_\xi]$.

Otherwise, $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi$.

Sealing Lemma: Every maximal antichain of $\mathbb{P}_\xi$ from M_ξ remains maximal in $\mathbb{J}$.

Notes:

Alternative choices of the $\diamondsuit$-sequence and the models M_ξ can yield a different Jensen's forcing.

If we don't work in L,

- lose low complexity of generics
- keep uniqueness properties of generics

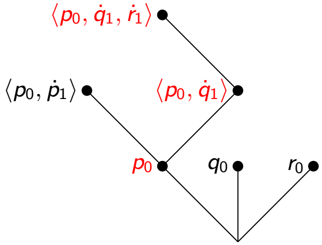
Finite iterations and tree iterations of Jensen's forcing

Finite iterations $\mathbb{J}_n$

- Jensen's forcings can be constructed in a forcing extension by $\mathbb{J}$ because ccc forcing of size ω_1 preserves $\diamondsuit$.
- In a forcing extension $V[r]$ by $\mathbb{J}$, use models $M_\xi[r]$, where M_ξ is given by $\diamondsuit$ in V, to construct a Jensen's forcing.

Tree iterations $\mathbb{P}(\mathbb{J}, \mathcal{T})$

- Fix a tree $\mathcal{T}$ of height ω.
- $\mathbb{P}(\mathbb{J}, \mathcal{T})$
  - Conditions: functions f_T from a finite subtree T of $\mathcal{T}$ into $\bigcup_{n<\omega} \mathbb{J}_n$ such that if $s \leq t$ in T, then $f(t) \upharpoonright \text{len}(s) = f(s)$.
  - Order: $f_S \leq f_T$ if $S \supseteq T$ and $f_S(t) \leq f_T(t)$ for every $t \in T$.
  - Generic filter: tree isomorphic to $\mathcal{T}$ whose nodes on level n are generic for $\mathbb{J}_n$.



Theorem: In a forcing extension $V[G]$ by $\mathbb{P}(\mathbb{J}, \mathcal{T})$, the only generics for $\mathbb{J}_n$ are those coming from the nodes of G on level n.

Generalizing Jensen's forcing to an inaccessible κ

Perfect κ-trees are not as nicely behaved as perfect trees because of limit levels.

- No meets.
- No unions.

The forcing should be $<\kappa$-closed.

- At limit stages, we have to close up unions under $<\kappa$-length sequences.
- Does this unseal maximal antichains?

Perfect κ-trees

Suppose that κ is an inaccessible cardinal.

A perfect κ-tree is a tree $T \subseteq 2^{<\kappa}$ such that:

- T has size κ (T is a κ-tree).
- Every node of T has a splitting node above it (T is splitting).
- For every limit $\lambda < \kappa$ if $s \in 2^\lambda$ and $s \restriction \xi \in T$ for every $\xi < \lambda$, then $s \in T$ (T is closed).
- For every limit $\lambda < \kappa$ if $s \in 2^\lambda$ and for cofinally many $\xi < \lambda$, $s \restriction \xi$ splits, then s splits (the splitting nodes of T are closed).

Proposition: Suppose that $\{T_\xi \mid \xi < \beta\}$, for $\beta < \kappa$, is a $\subseteq$-decreasing sequence of perfect κ-trees. Then $T = \bigcap_{\xi<\beta} T_\xi$ is a perfect κ-tree.

Badly behaved perfect κ-trees

Proposition: There are perfect κ-trees whose intersection does not contain a maximal perfect κ-tree.

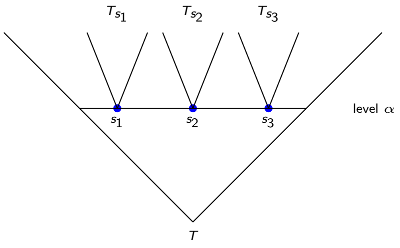
Proposition: There are ω-many perfect κ-trees whose union is not a perfect κ-tree.

κ-perfect posets

Suppose that κ is an inaccessible cardinal.

A collection $\mathbb{P}$ of perfect κ-trees ordered by $\subseteq$ is a κ-perfect poset if:

- $2^{<\kappa} \in \mathbb{P}$.
- If $T \in \mathbb{P}$ and $t \in T$, then $T_t \in \mathbb{P}$.
- If $\{T_\xi \mid \xi < \beta\} \subseteq \mathbb{P}$, with $\beta < \kappa$ is a decreasing sequence, then $T = \bigcap_{\xi<\beta} T_\xi \in \mathbb{P}$ ($<\kappa$-closure property).
- Suppose $T \in \mathbb{P}$, $\alpha < \kappa$ is a successor, and $\{T^{(s)} \subseteq T_s \mid s \in T \cap 2^\alpha\} \subseteq \mathbb{P}$. Then $T' = \bigcup_{s\in 2^\alpha} T^{(s)} \in \mathbb{P}$ (weak union property).



κ-perfect posets (continued)

Proposition: Suppose $\mathbb{P}$ is a κ-perfect poset and $G \subseteq \mathbb{P}$ is V-generic. Let $A = \bigcap_{T\in G} T$. Then in $V[G]$:

- A is cofinal branch through every tree in G.
- If $A \in [T]$ for some $T \in \mathbb{P}$, then $T \in G$.
- A and G are definable from each other.

Smallest κ-perfect poset $\mathbb{P}_{\text{min}}$: close $\{(2^{<\kappa})_s \mid s \in 2^{<\kappa}\}$ under $<\kappa$-intersection property and weak union property.

$\mathbb{P}_0 = \{(2^{<\kappa})_s \mid s \in 2^{<\kappa}\}$

$\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$ for limits λ

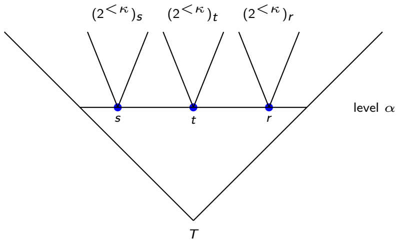
Suppose $\mathbb{P}_\xi$ has been defined.

$\mathbb{P}'_{\xi+1}$ consists of all $T' = \bigcup_{s\in 2^\alpha} T^{(s)}$ for $T \in \mathbb{P}_\xi$, successor $\alpha < \kappa$, and $\{T^{(s)} \subseteq T_s \mid s \in T \cap 2^\alpha\} \subseteq \mathbb{P}_\xi$.

$\mathbb{P}_{\xi+1}$ consists of all $T = \bigcap_{\xi<\beta} T_\xi$, for $\beta < \kappa$ and $\subseteq$-decreasing $\{T_\xi \mid \xi < \beta\} \subseteq \mathbb{P}'_{\xi+1}$.

κ-perfect posets (continued)

Clean Levels Lemma: Every tree $T \in \mathbb{P}_{\text{min}}$ has a level α such that for every $t \in T \cap 2^\alpha$, $T_t = (2^{<\kappa})_t$.



The fusion poset $\mathbb{Q}(\mathbb{P})$

Suppose that $\mathbb{P}$ is a κ-perfect poset.

$\mathbb{Q}(\mathbb{P})$: elements are pairs (T, α), with $T \in \mathbb{P}$ and $\alpha < \kappa$ successor, ordered by $(T, \alpha) \leq (S, \beta)$ if $\alpha \geq \beta$ and $T \cap 2^\beta = S \cap 2^\beta$.

Proposition: The poset $\mathbb{Q}(\mathbb{P})$ is $<\kappa$-closed.

Proposition: Suppose $G \subseteq \mathbb{Q}(\mathbb{P})$ is V-generic. Then in $V[G]$:

- $\mathscr{T} = \bigcup_{(T,\alpha)\in G} T \cap 2^\alpha$ is a perfect κ-tree.
- $\mathscr{T} \subseteq T$ for every condition $(T, \alpha) \in G$.

Notation:

$\mathbb{Q}(\mathbb{P})^{<\kappa}$: bounded support κ-length product of the $\mathbb{Q}(\mathbb{P})$.

Growing κ-perfect posets with generic perfect κ-trees

Set-up

- $\mathbb{P}$ is a κ-perfect poset
- $\mathbb{Q}(\mathbb{P})$ is a fusion poset for $\mathbb{P}$
- $G \subseteq \mathbb{Q}(\mathbb{P})^{<\kappa}$ is V-generic
- $\mathscr{T}_\xi$ is the generic perfect κ-tree added by the ξ-th slice of G

In $V[G]$

$\mathbb{P}^*$: close $\{(\mathscr{T}_\xi)_t \mid \xi < \kappa, t \in \mathscr{T}_\xi\} \cup \mathbb{P}$ under $<\kappa$-intersection property and weak union property.

$\mathbb{P}_0 = \{(\mathscr{T}_\xi)_t \mid \xi < \kappa, t \in \mathscr{T}_\xi\} \cup \mathbb{P}$

$\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$ for limits λ

Suppose $\mathbb{P}_\xi$ has been defined.

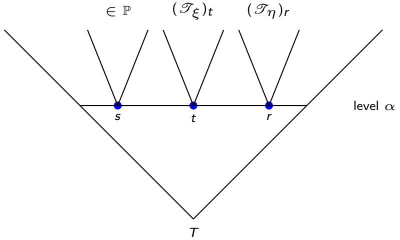
$\mathbb{P}'_{\xi+1}$ consists of all $T' = \bigcup_{s\in T\cap 2^\alpha} T^{(s)}$ for $T \in \mathbb{P}_\xi$, $\alpha < \kappa$ successor, and $\{T^{(s)} \subseteq T_s \mid s \in T \cap 2^\alpha\} \subseteq \mathbb{P}_\xi$.

$\mathbb{P}_{\xi+1}$ consists of all $T = \bigcap_{\xi<\beta} T_\xi$ for $\beta < \kappa$ and $\subseteq$-decreasing $\{T_\xi \mid \xi < \beta\} \subseteq \mathbb{P}'_{\xi+1}$.

Growing κ-perfect posets with generic perfect κ-trees (continued)

Clean Levels Lemma: Every tree $T \in \mathbb{P}^*$ has a level α such that for every $t \in T \cap 2^\alpha$,

- $T_t = (\mathscr{T}_\xi)_t$ for some $\xi < \kappa$ or
- $T_t \in \mathbb{P}$.



Proposition:

- $\{(\mathscr{T}_\xi)_s \mid \xi < \kappa, s \in \mathscr{T}_\xi\}$ is dense in $\mathbb{P}^*$.
- $\{\mathscr{T}_\xi \mid \xi < \kappa\}$ is a maximal antichain of $\mathbb{P}^*$.
- Every maximal antichain from V remains maximal in $\mathbb{P}^*$.

κ-suitable models

Work in L and fix a canonical $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$-sequence $\vec{D} = \langle D_\xi \mid \xi \in \mathrm{Cof}(\kappa)\rangle$.

A model M is κ-suitable if:

- $M = L_\alpha$ for some $|\alpha| = \kappa$
- $M^{<\kappa} \subseteq M$,
- $M \models \mathrm{ZFC}^- + \mathrm{P}(\kappa)$ exists.

Observations:

- The Mostowski collapse M of any $X \prec L_{\kappa^{++}}$, with $X^{<\kappa} \subseteq X$ and $|X| = \kappa$, is κ-suitable.
- If M is κ-suitable and $\delta = (\kappa^+)^M$, then $\langle D_\xi \mid \xi < \delta\rangle \in M$.
- If M is κ-suitable and $\mathbb{P} \in M$ is $<\kappa$-closed, then there is an M-generic filter for $\mathbb{P}$.
  - Diagonalize to meet all dense sets
  - Use closure to get through limit stages

Jensen's forcing at an inaccessible κ: $\mathbb{J}(\kappa)$

$\mathbb{J}(\kappa)$: union of a chain $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \cdots \subseteq \mathbb{P}_\xi \subseteq \cdots$ of length κ^+ of κ-perfect posets.

$\mathbb{P}_0 = \mathbb{P}_{\text{min}}$

Suppose $\mathbb{P}_\xi$ has been defined.

If $\xi \in \mathrm{Cof}(\kappa)$ and D_ξ codes a κ-suitable model M_ξ such that $\mathbb{P}_\xi \in M_\xi$ and $(\kappa^+)^{M_\xi} = \xi$:

- Let G_ξ be the L-least M_ξ-generic filter for $\mathbb{Q}(\mathbb{P}_\xi)^{<\kappa}$.
- $\mathbb{P}_{\xi+1} = \mathbb{P}^*_\xi$ as constructed in $M_\xi[G_\xi]$.

Otherwise, $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi$.

If $\mathrm{cf}(\lambda) = \kappa$:
$\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$.

If $\mathrm{cf}(\lambda) < \kappa$:
$\mathbb{P}_\lambda$: close $\bigcup_{\xi<\lambda} \mathbb{P}_\xi$ under $<\kappa$-intersection property and weak union property.

Let $\mathscr{T}^{(\xi)}_\nu$ for $\xi < \kappa^+$ and $\nu < \kappa$ be the perfect κ-trees added in $M_\xi[G_\xi]$.

Jensen's forcing at an inaccessible κ: $\mathbb{J}(\kappa)$ (continued)

Clean Levels Lemma: Every tree $T \in \mathbb{J}(\kappa)$ has a level α such that for every $t \in 2^\alpha \cap T$,

- $T_t = (2^{<\kappa})_t$,
- $T_t = (\mathscr{T}_\nu^{(\xi)})_t$ for some $\xi < \kappa^+$ and $\nu < \kappa$,
- $T_t = \bigcap_{\xi<\alpha}(\mathscr{T}_{\rho_\xi}^{(\mu_\xi)})_t$, with $\alpha < \kappa$, for some $\subseteq$-decreasing $\{(\mathscr{T}_{\rho_\xi}^{(\mu_\xi)})_t \mid \xi < \alpha\}$.

Sealing Lemma: Every maximal antichain of $\mathbb{P}_\xi$ from M_ξ remains maximal in $\mathbb{J}(\kappa)$.

Properties of $\mathbb{J}(\kappa)$

Theorem: The forcing $\mathbb{J}(\kappa)$

- is $<\kappa$-closed,
- has the κ^+-cc,
- adds a unique generic subset of κ.

Notation:

- $\mathbb{J}(\kappa)^{<\kappa}$: bounded support κ-length product of $\mathbb{J}(\kappa)$
- $\mathbb{J}(\kappa)_n$: finite iteration of length n of $\mathbb{J}(\kappa)$
- Given a tree $\mathcal{T}$ of height ω, $\mathbb{P}(\mathbb{J}(\kappa), \mathcal{T})$: tree iteration of $\mathbb{J}(\kappa)$ along $\mathcal{T}$

Theorem: In a forcing extension by $\mathbb{J}(\kappa)^{<\kappa}$, the only subsets of κ generic for $\mathbb{J}(\kappa)$ are the κ-many slices of the generic filter.

Theorem: In a forcing extension by $\mathbb{P}(\mathbb{J}(\kappa), \mathcal{T})$, the only generic filters for $\mathbb{J}_n$ are those coming from the nodes of the generic tree on level n.

Corollary: A forcing extension by $\mathbb{J}(\kappa)_n$ has a unique generic n-length sequence of subsets of κ.

Kelley-Morse and the choice principles for classes

Theorem: There is a model of Kelley-Morse set theory with the Choice Scheme in which the Dependent Choice Scheme fails.

The model is the $V_{\kappa+1}$ of a symmetric submodel of a forcing extension by a tree iteration of $\mathbb{J}(\kappa)$.