A gentle introduction to class forcing

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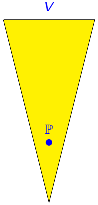
Logik Kolloquium
University of Konstanz

July 17, 2023

Let's review forcing: partial orders

Forcing is a technique developed by Paul Cohen in the 1960's for expanding a set-theoretic universe to a larger universe some desired properties.

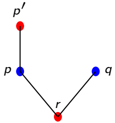


Set-up

- Universe $V \models \mathrm{ZFC}$
- Forcing $\mathbb{P} \in V$: partial order with largest element $\mathbb{1}$

Dense sets and generic filters

A set $D \subseteq \mathbb{P}$ is dense if for every $p \in \mathbb{P}$, there is $q \in D$ with $q \leq p$.



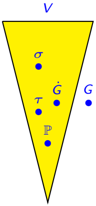
A set $G \subseteq \mathbb{P}$ is a filter:

- $\mathbb{1} \in G$.
- (upward closure) If $p \in G$ and $p' \geq p$, then $p' \in G$.
- (compability) If $p, q \in G$, then $r \in G$ such that $r \leq p, q$.

A filter $G \subseteq \mathbb{P}$ is V-generic if it meets every dense set $D \in V$ of $\mathbb{P}$: $D \cap G \neq \emptyset$.

The universe V has NO V-generic filters for $\mathbb{P}$.

Let's review forcing: building blocks



Set-up

- Universe $V \models \mathrm{ZFC}$
- Forcing $\mathbb{P} \in V$
- V-generic filter $G \subseteq \mathbb{P}$

$\mathbb{P}$-names

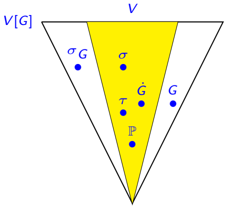
A $\mathbb{P}$-name σ is set of pairs $\langle \tau, p \rangle$, where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

$\mathbb{P}$-names are constructed by recursion on rank.

Special $\mathbb{P}$-names

- Given $a \in V$, $\check{a} = \{\langle \check{b}, \mathbb{1} \rangle \mid b \in a\}$.
- $\dot{G} = \{\langle \check{p}, p \rangle \mid p \in \mathbb{P}\}$.

Let's review forcing: forcing extension $V[G]$



Set-up

- Universe $V \models \mathrm{ZFC}$
- Forcing $\mathbb{P} \in V$
- Generic filter $G \subseteq \mathbb{P}$

From a $\mathbb{P}$-name to a set via G

$\sigma_G = \{\tau_G \mid \langle \tau, p\rangle \in \sigma \text{ and } p \in G\}$.

Constructed by recursion on rank.

The forcing extension $V[G] = \{\sigma_G \mid \sigma \in V \text{ is a } \mathbb{P}\text{-name}\}$.

- $V \subseteq V[G]$: $\check{a}_G = a$.
- $G \in V[G]$: $\dot{G}_G = G$.
- $V[G] \models \mathrm{ZFC}$

Let's review forcing: forcing theorem

The forcing relation "p forces $\varphi(\sigma)$"

$p \Vdash \varphi(\sigma)$: for every V-generic filter G if $p \in G$, then $V[G] \models \varphi(\sigma_G)$.

The forcing theorem

Fix a formula $\varphi(x)$.

1. The relation $p \Vdash \varphi(\sigma)$ is definable.
   rank recursion
   - $p \Vdash \sigma \in \tau$: there is a dense set of conditions $q \leq p$ for which there is $\langle \rho, r\rangle \in \tau$ with $q \leq r$ and $q \Vdash \sigma = \rho$.
   - $p \Vdash \sigma = \tau$: $p \Vdash \sigma \subseteq \tau$ and $p \Vdash \tau \subseteq \sigma$.
   - $p \Vdash \sigma \subseteq \tau$: whenever $\langle \rho, r\rangle \in \sigma$ and $q' \leq p, r$, there is $q \leq q'$ with $q \Vdash \rho \in \tau$.

   recursion on complexity of formulas
   - $p \Vdash \varphi \wedge \psi$: $p \Vdash \varphi$ and $p \Vdash \psi$.
   - $p \Vdash \neg\varphi$: there is no $q \leq p$ with $q \Vdash \varphi$.
   - $p \Vdash \forall x\, \varphi(x)$: $p \Vdash \varphi(\tau)$ for every $\mathbb{P}$-name τ.
2. If $V[G] \models \varphi(\sigma_G)$, then there is $p \in G$ such that $p \Vdash \varphi(\sigma)$.

Popular forcing notions: Cohen forcing

$\mathrm{Add}(\omega, 1)$: adds a new real

Conditions: binary sequences $p : D \to 2$ with $D \subseteq \omega$ finite.
Order: $q \leq p$ if q extends p.
$V[G]$: $r = \bigcup G$ is a new real

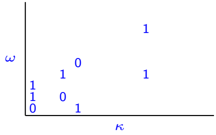
$p = \underset{0123456}{1\ 10\ \ 1}$

$q = \underset{01234567}{1110\ \ 11}$

Suppose κ is a cardinal.
$\mathrm{Add}(\omega, \kappa)$: adds (at least) κ-many reals

Conditions: functions $p : D \to 2$, where D is a finite subset of $\omega \times \kappa$.
Order: $q \leq p$ if q extends p.
$V[G]$:

- $\bigcup G$ gives κ-many new reals.
- $2^\omega \geq \kappa$



Suppose κ, δ are cardinals.
$\mathrm{Add}(\delta, \kappa)$: adds (at least) κ-many subsets of δ.

Popular forcing notions: collapse forcing

Suppose κ is a cardinal.

$\mathrm{Coll}(\omega, \kappa)$: adds a bijection between ω and κ

Conditions: injective functions $p : D \to \kappa$ with $D \subseteq \omega$ finite.

Order: $q \leq p$ if q extends p.

$V[G]$: $f = \bigcup G : \omega \to \kappa$ is a bijection

$\mathrm{Coll}_*(\omega, \kappa)$: adds a bijection between ω and κ

Conditions: injective functions $p : n \to \kappa$ with $n < \omega$.

Order: $q \leq p$ if q extends p.

$V[G]$: $f = \bigcup G : \omega \to \kappa$ is a bijection

Observation: $\mathrm{Coll}_*(\omega, \kappa)$ is a dense subset of $\mathrm{Coll}(\omega, \kappa)$.

Theorem: If a forcing $\mathbb{P}$ is a dense subset of a forcing $\mathbb{Q}$, then they have the same forcing extensions.

- If $V[G]$ is a forcing extension by $\mathbb{P}$, then there is $H \in V[G]$ such that $V[G] = V[H]$ and $H \subseteq \mathbb{Q}$ is a V-generic filter.
- If $V[H]$ is a forcing extension by $\mathbb{Q}$, then there is $G \in V[H]$ such that $V[G] = V[H]$ and $G \subseteq \mathbb{Q}$ is a V-generic filter.

Products and iterations of forcing notions

Products

Suppose $\{\mathbb{P}_\alpha \mid \alpha < \beta\}$ are forcing notions indexed by ordinals $\alpha < \beta$.

A product $\mathbb{P} = \prod_{\alpha<\beta} \mathbb{P}_\alpha$ is a natural forcing notion.

- Conditions: $\langle p_\alpha \mid \alpha < \beta \rangle$ with $p_\alpha \in \mathbb{P}_\alpha$.
- Common supports: finite, bounded, full.
- Example: $\mathrm{Add}(\omega, \kappa) = \prod_{\alpha<\kappa} \mathrm{Add}(\omega, 1)$ with finite support.
- Usage: adding several objects to a forcing extension.

Iterations

Suppose $\mathbb{P}$ is a forcing notion, $G \subseteq \mathbb{P}$ is V-generic, and $\mathbb{Q}$ is a forcing notion in $V[G]$.

V has a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$. Every element of $V[G]$ has a $\mathbb{P}$-name in V.

In V, we define a forcing notion $\mathbb{P} * \dot{\mathbb{Q}}$ such that forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ is the same as forcing with $\mathbb{P}$ followed by forcing with $\mathbb{Q}$.

- Conditions: $(p, \dot{q})$ with $p \in \mathbb{P}$ and $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$.
- Order: $(p, \dot{q}) \leq (r, \dot{s})$ if $p \leq r$ and $p \Vdash \dot{q} \leq \dot{s}$.
- Generalizes to ordinal length iterations with various supports.

Classes in set theory

A class is (first-order) definable with parameters collection of sets.

Examples

- V: collection of all sets
- Ord: collection of all ordinals
- Card: collection of all cardinals
- Reg: collection of all regular cardinals
  - A cardinal κ is regular if no $\alpha < \kappa$ can map cofinally into κ.
- Continuum function: $C : \mathrm{Card} \to \mathrm{Card}$ such that $C(\kappa) = 2^\kappa$.
- L: Gödel's constructible universe
- HOD: collection of all sets hereditarily definable from ordinal parameters
  - Takes work to prove that HOD is definable.
  - $\mathrm{HOD} \models \mathrm{ZFC}$
  - HOD is an important sub-universe of V.
- A global well-order function: bijection $W : \mathrm{Ord} \to V$.
  - Doesn't have to exist.

Forcing with class partial orders

We need to use class forcing if we want $V[G]$ to be globally different from V.

Examples

- Force $2^\kappa \neq \kappa^+$ at all regular cardinals

  $\prod_{\kappa\in\mathrm{Reg}} \mathrm{Add}(\kappa, \kappa^{++})$ with Easton support.
- Force $V[G] = \mathrm{HOD}$
  - We can code information about sets into the continuum function.
  - Suppose $r : \omega \to 2$ is a real.
  - Let $\{\kappa_n \mid n < \omega\}$ be ω-many "sufficiently spaced out" cardinals.
  - Code r into the continuum function by forcing $2^{\kappa_n} = \kappa_n^+$ if $r(n) = 0$ and $2^{\kappa_n} > \kappa_n^+$ otherwise.
- Force a global well-order.
  - $\mathrm{Add}(\mathrm{Ord}, 1)$: binary sequences $p : D \to 2$ with $D \subseteq \mathrm{Ord}$.
  - Doesn't add sets.
- Force that there is no global well-order.

  An Ord-length iteration where at every cardinal stage α we force with $\mathrm{Add}(\alpha, 1)$ (Easton support).

The right setting for class forcing

Class forcing is fundamentally about classes.

- A generic filter for a class partial order is a class.
- The properties of the partial order depend on which classes exist around it.

First-order set theory

- Sets are elements of the model.
- Classes are definable (with parameters) collections of sets.
- Classes are objects in the meta-theory.

Second-order set theory

- Classes are elements of the model.
- We can quantify over classes.
- We can study general properties of classes.
- The theory determines which classes exist.

Second-order set theory

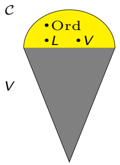
Second-order set theory has two sorts of objects: sets and classes.

Syntax: Two-sorted logic

- Separate variables and quantifiers for sets and classes
- Convention: upper-case letters for classes, lower-case letters for sets
- Notation:
  - Σ^0_n: first-order Σ_n-formula
  - Σ^1_n : n-alternations of class quantifiers followed by a first-order formula

Semantics: A model is a triple $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$.

- V consists of the sets.
- $\mathcal{C}$ consists of the classes.
- Every set is a class: $V \subseteq \mathcal{C}$.
- $C \subseteq V$ for every $C \in \mathcal{C}$ - C is determined by the sets $c \in C$.



Second-order axioms

Set axioms - ZFC

Class axioms

- class extensinality
- class replacement: every class function when restricted to a set is a set.
- global well-order: there exists a class global well-order function.
- first-order comprehension: every first-order formula defines a class.
- ETR: elementary transfinite recursion
- Σ^1_n-comprehension: every Σ^1_n-formula defines a class.

Second-order theories: GBC

GBc: Gödel-Bernays set theory without global well-order

- class extensionality, class replacement, first-order comprehension
- If $V \models \mathrm{ZFC}$ and $\mathcal{C}$ consists of the definable collections of V, then $\langle V, \in, \mathcal{C}\rangle \models \mathrm{GBc}$.

GBC: Gödel-Bernays set theory

- GBc, global well-order
- If $\mathcal{C}$ consists of the definable collections of L, then $\langle L, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.
- Every model $\langle V, \in, \mathcal{C}\rangle \models \mathrm{GBc}$ has a class forcing extension with the same sets satisfying GBC.
  - Force with $\mathrm{Add}(\mathrm{Ord}, 1)$.
- GBC is equiconsistent with ZFC
- GBC is conservative over ZFC - every assertion about sets provable in GBC is already provable in ZFC.

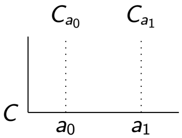
Second-order set theories: ETR

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBC}$.

Definition: A meta-ordinal is a well-order $(\Gamma, \leq) \in \mathcal{C}$.

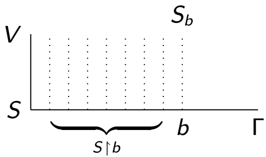
- Examples: Ord, $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \cdot \omega$.
- Notation: For $a \in \Gamma$, $\Gamma \restriction a$ is the restriction of the well-order to $\leq$-predecessors of a.

Definition: Suppose $A \in \mathcal{C}$ is a class. A sequence of classes $\langle C_a \mid a \in A \rangle$ is a single class C such that $C_a = \{x \mid \langle a, x \rangle \in C\}$.



Definition: Suppose $\Gamma \in \mathcal{C}$ is a meta-ordinal. A solution along Γ to a first-order recursion rule $\varphi(x, b, F)$ is a sequence of classes S such that for every $b \in \Gamma$, $S_b = \varphi(x, b, S \restriction b)$.

Elementary Transfinite Recursion ETR: For every meta-ordinal Γ, every first-order recursion rule $\varphi(x, b, F)$ has a solution along Γ.



Theorem: $\mathrm{GBC} + \mathrm{ETR}$ proves $\mathrm{Con}(\mathrm{GBC})$.

Second-order set theories: fragments of ETR

ETR_Γ: Elementary transfinite recursion for a fixed Γ.

- $\mathrm{ETR}_{\mathrm{Ord}\cdot\omega}$, $\mathrm{ETR}_{\mathrm{Ord}}$, ETR_ω

Theorem: (Williams) If $\Gamma \geq \omega^\omega$ is a (meta)-ordinal, then $\mathrm{GBC} + \mathrm{ETR}_{\Gamma\cdot\omega}$ implies $\mathrm{Con}(\mathrm{GBC} + \mathrm{ETR}_\Gamma)$.

Second-order set theories: the comprehension hierarchy to Kelley-Morse

The hierarchy

- GBC
- $\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$
- $\mathrm{GBC} + \mathrm{ETR}$
- $\mathrm{GBC} + \Sigma^1_1$-comprehension
- $\vdots$
- $\mathrm{GBC} + \Sigma^1_n$-comprehension
- $\vdots$
- KM Kelley Morse - $\mathrm{GBC} + \Sigma^1_n$-comprehension for every $n < \omega$

Finally class forcing

Set-up

- Universe $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBc}$
- Class forcing $\mathbb{P} \in \mathcal{C}$: partial order with largest element $\mathbb{1}$

Generic filters

A filter $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic if it meets every dense class $D \in \mathcal{C}$ of $\mathbb{P}$: $D \cap G \neq \emptyset$.

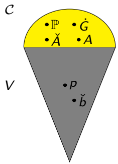
The universe $\mathscr{V}$ has NO $\mathscr{V}$-generic filters for $\mathbb{P}$.

Class $\mathbb{P}$-names

A class $\mathbb{P}$-name $\Gamma \in \mathcal{C}$ is class of pairs $\langle \tau, p\rangle$, where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

Special class $\mathbb{P}$-names

- Given $A \in \mathcal{C}$, $\check{A} = \{\langle \check{b}, \mathbb{1}\rangle \mid b \in A\}$.
- $\dot{G} = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$.



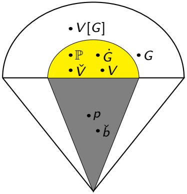
The class forcing extension

Set-up

- Universe $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBc}$
- Class forcing $\mathbb{P} \in \mathcal{C}$
- V-generic filter $G \subseteq \mathbb{P}$

The class forcing extension $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G] \rangle$

- $\mathcal{C}[G] = \{\Gamma_G \mid \Gamma \in \mathcal{C}$ is a class $\mathbb{P}$-name$\}$
- $\mathcal{C} \subseteq \mathcal{C}[G]$: $\check{A}_G = A$.
- $G \in \mathcal{C}[G]$: $\dot{G}_G = G$.
- $\mathscr{V}[G] \models \mathrm{GBc}$???



Quick aside on set theories without powerset

ZFC^-: ZFC without the powerset axiom

If T is any second-order set theory, then T^- is T with ZFC replaced by ZFC^-.

Preserving the axioms (or not)

Set-up

- Universe $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBc}$ (or GBC or KM)
- Class forcing $\mathbb{P} \in \mathcal{C}$
- V-generic filter $G \subseteq \mathbb{P}$
- $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G]\rangle$

Example: $\mathbb{P} = \mathrm{Add}(\omega, \mathrm{Ord})$

- Conditions: functions $p : D \to 2$, where $D \subseteq \omega \times \mathrm{Ord}$.
- $\mathbb{P}$ adds Ord-many reals.
- The powerset axiom fails in $V[G]$.
- $V[G] \not\models \mathrm{ZFC}$
- $\mathscr{V}[G] \models \mathrm{GBc}^-$ (or GBC^- or KM^-)

Preserving the axioms (or not): continued

Example: $\mathbb{P} = \mathrm{Coll}_*(\omega, \mathrm{Ord})$

- Conditions: injective functions $p : n \to \mathrm{Ord}$ with $n < \omega$.
- $\mathbb{P}$ does NOT add sets.
- $\mathbb{P}$ adds a class bijection $F : \omega \to \mathrm{Ord}$.
- The class replacement axiom fails in $\mathscr{V}[G]$.
- $V[G] = V \models \mathrm{ZFC}$.
- $\mathscr{V}[G] \not\models \mathrm{GBc}$.

Example: $\mathbb{P} = \mathrm{Coll}(\omega, \mathrm{Ord})$

- Conditions: injective functions $p : D \to \mathrm{Ord}$ with $D \subseteq \omega$ finite.
- For every cardinal κ, $\mathbb{P}$ adds a set bijection $f_\kappa : \omega \to \kappa$.
- $\mathbb{P}$ adds a class bijection $F : \omega \to \mathrm{Ord}$.
- $V[G] \not\models \mathrm{ZFC}$
- $V[G] \models \mathrm{ZFC}^-$.
- $\mathscr{V}[G] \not\models \mathrm{GBc}^-$.

Theorem: (Holy, Krapf, Lücke, Njegomir, Schlicht) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBc}$. There are class forcing $\mathbb{P}, \mathbb{Q} \in \mathcal{C}$ such that $\mathbb{P}$ is a dense subset of $\mathbb{Q}$, but they have different forcing extensions.

Nice class forcing: pretame and tame

Definition: (Friedman) A class forcing $\mathbb{P}$ is pretame if for every class sequence $\langle D_x \mid x \in a\rangle \in \mathcal{C}$ of dense classes of $\mathbb{P}$, indexed by elements of a set a, and condition $p \in \mathbb{P}$, there is a condition $q \leq p$ and a sequence $\langle d_x \mid x \in a\rangle$ of subsets of $\mathbb{P}$ such that each $d_x \subseteq D_x$ is pre-dense below q in $\mathbb{P}$.

Definition: (Friedman) A class forcing $\mathbb{P}$ is tame if it is pretame and for every $p \in \mathbb{P}$, there is $q \leq p$ and ordinal α such that whenever $\vec{D} = \{\langle D_0^x, D_1^x\rangle \mid x \in a\} \in \mathcal{C}$, for a set a, is a sequence of pre-dense partitions below q, then the class

$$\{r \in \mathbb{P} \mid \vec{D} \text{ is equivalent below } r \text{ to some partition } \vec{E} \in V_\alpha\}$$

is dense below q.

Theorem: (Stanley) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBc}$ (or GBC) and $\mathbb{P} \in \mathcal{C}$ is a class forcing notion.

- If $\mathbb{P}$ is pretame, then all forcing extensions $\mathscr{V}[G]$ of $\mathbb{P}$ satisfy GBc^- (or GBC^-).
- If all forcing extensions $\mathscr{V}[G]$ of $\mathbb{P}$ satisfy GBc^- (or GBC^-), then $\mathbb{P}$ is pretame.

Theorem: (Friedman) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBc}$ (or GBC) and $\mathbb{P} \in \mathcal{C}$ is a class forcing notion.

- If $\mathbb{P}$ is tame, then all forcing extensions $\mathscr{V}[G]$ satisfy GBc (or GBC).
- If all forcing extensions $\mathscr{V}[G]$ of $\mathbb{P}$ satisfy GBc (or GBC), then $\mathbb{P}$ is tame.

Theorem: (Antos) Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing notion.

- If $\mathbb{P}$ is pretame, then all forcing extensions $\mathscr{V}[G]$ of $\mathbb{P}$ satisfy KM^-.
- If $\mathbb{P}$ is tame, then all forcing extension $\mathscr{V}[G]$ of $\mathbb{P}$ satisfy KM.

Definability of the forcing relation (or not)

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBc}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing.

Definition: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBc}$ and $\mathbb{P} \in \mathcal{C}$ is a class forcing.
The Class Forcing Theorem holds for $\mathbb{P}$ if there is a solution to the rank recursion defining the forcing relation for atomic formulas.

- $p \Vdash \sigma \in \tau$: there is a dense set of conditions $q \leq p$ for which there is $\langle \rho, r\rangle \in \tau$ with $q \leq r$ and $q \Vdash \sigma = \rho$.
- $p \Vdash \sigma = \tau$: $p \Vdash \sigma \subseteq \tau$ and $p \Vdash \tau \subseteq \sigma$.
- $p \Vdash \sigma \subseteq \tau$: whenever $\langle \rho, r\rangle \in \sigma$ and $q' \leq p, r$, there is $q \leq q'$ with $q \Vdash \rho \in \tau$.

The Class Forcing Theorem for $\mathbb{P}$ implies that forcing relations for all second-order formulas are definable.

- $p \Vdash \sigma \in \Gamma$: there are densely many $q \leq p$ for which there is $\langle \tau, r\rangle \in \Gamma$ with $q \leq r$ and $q \Vdash \sigma = \tau$.
- $p \Vdash \varphi \wedge \psi$: $p \Vdash \varphi$ and $p \Vdash \psi$.
- $p \Vdash \neg\varphi$: there is no $q \leq p$ with $q \Vdash \varphi$.
- $p \Vdash \forall x\, \varphi(x)$: $p \Vdash \varphi(\tau)$ for every $\mathbb{P}$-name τ.
- $p \Vdash \forall X\, \varphi(X)$: $p \Vdash \varphi(\Delta)$ for every class $\mathbb{P}$-name Δ.

The Class Forcing Theorem fails, but pretame forcing is still nice

Theorem: (Holy, Krapf, Lücke, Njegomir, Schlicht) In a models of GBC, the Class Forcing Theorem can fail for some class forcing.

Theorem: (Stanley) In models of GBc, the Class Forcing Theorem holds for all pretame class forcing.

The Class Forcing Theorem is equivalent to $\mathrm{ETR}_{\mathrm{Ord}}$

Theorem: (G., Hamkins, Holy, Schlicht, Williams).

- In models of $\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$, the Class Forcing Theorem holds for all class forcing.
- If $\mathscr{V} \models \mathrm{GBC}$ and satisfies that the Class Forcing Theorem holds for all class forcing, then $\mathscr{V} \models \mathrm{ETR}_{\mathrm{Ord}}$.

Ground model definability

Theorem: (Laver, Woodin) The ground model V is uniformly definable with a parameter from V in any set-forcing extension $V[G]$.

Theorem: (Antos) Suppose $V \models \mathrm{ZFC}$. There is a class forcing $\mathbb{P}$ such that the ground model V is not definable in any forcing extension $V[G]$ by $\mathbb{P}$, even with a parameter from $V[G]$.

- $\mathbb{P}$ is the product $\prod_{\alpha \in \mathrm{Reg}} \mathrm{Add}(\alpha, 1)$ with Easton support.

Theorem: (G., Johnstone) There is a model $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM}$ such that the classes $\mathcal{C}$ of the ground model $\mathscr{V}$ are not definable in any forcing extension $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G] \rangle$ by $\mathrm{Add}(\mathrm{Ord}, 1)$, even with a parameter from $\mathcal{C}[G]$.

(Assuming existence of inaccessible cardinal.)

Theorem: (Asperó) There is a model $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM}$ such that the classes $\mathcal{C}$ of the ground model $\mathscr{V}$ are not definable in any forcing extension $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G] \rangle$ by $\mathrm{Add}(\omega, 1)$, even with a parameter from $\mathcal{C}[G]$.

(Assuming existence of very very large cardinals.)

The Class Intermediate Model Theorem

Intermediate Model Theorem: (Solovay) If $V \models \mathrm{ZFC}$ and $W \models \mathrm{ZFC}$ is an intermediate model between V and its set-forcing extension $V[G]$ ($V \subseteq W \subseteq V[G]$), then $W = V[H]$ is a set-forcing extension of V.

Definition: Suppose T is a second-order set theory.
The Intermediate Model Theorem holds for T if whenever $\mathscr{V} \models T$ and $\mathscr{W} \models T$ is an intermediate model between $\mathscr{V}$ and its class-forcing extension $\mathscr{V}[G] \models T$, then $\mathscr{W}$ is a class-forcing extension of $\mathscr{V}$.

Theorem: (Antos, Friedman, G.) The Intermediate Model Theorem fails for:

- GBC
- KM