An overview of virtual large cardinals

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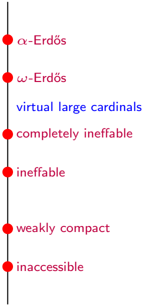
The large cardinal hierarchy in L

A cardinal κ is:

- inaccessible if it is a regular strong limit.
- weakly compact if every coloring $f : [\kappa]^2 \to 2$ has a homogeneous set of size κ.
- ineffable if every coloring $f : [\kappa]^2 \to 2$ has a stationary homogeneous set.
- completely ineffable if there is a non-empty collection $\mathcal{S}$ of stationary subsets of κ such that for every coloring $f : [A]^2 \to 2$ and $A \in \mathcal{S}$, there is $B \subseteq A$ in $\mathcal{S}$ homogeneous for f.
- ω-Erdős if every coloring $f : [\kappa]^{<\omega} \to 2$ has homogeneous set of order-type ω.
- α-Erdős ($\omega < \alpha < \omega_1$) if every coloring $f : [\kappa]^{<\omega} \to 2$ has homogeneous set of order-type α.



The large large cardinal hierarchy

measurable cardinal κ: there is an embedding $j : V \to M$ with $\text{crit}(j) = \kappa$.

- set embeddings: for every $\lambda > \kappa$, there is $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$

λ-strong cardinal κ: there is an embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $V_\lambda \subseteq M$.

- set embedding: $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$ and $V_\lambda \subseteq M_\lambda$
- can assume $j(\kappa) > \lambda$ (proof uses Kunen's inconsistency)

strong cardinal κ: λ-strong for every $\lambda > \kappa$.

λ-supercompact cardinal κ: there is an embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $M^\lambda \subseteq M$.

- equivalently $j \upharpoonright \lambda \in M$
- set embedding: $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$ and $M_\lambda^\lambda \subseteq M_\lambda$
- can assume $j(\kappa) > \lambda$

supercompact cardinal κ: λ-supercompact for every $\lambda > \kappa$.

The large large cardinal hierarchy (continued)

extendible cardinal κ: for every $\kappa < \lambda$, there is β_λ and an embedding $j_\lambda : V_\lambda \to V_{\beta_\lambda}$ with $\text{crit}(j_\lambda) = \kappa$.

$C^{(n)}$**-extendible cardinal** κ $(1 \leq n < \omega)$: for every $\kappa < \lambda \in C^{(n)}$, there is $\beta_\lambda \in C^{(n)}$ and an embedding $j_\lambda : V_\lambda \to V_{\beta_\lambda}$ with $\text{crit}(j_\lambda) = \kappa$.

- $C^{(n)} = \{\alpha \in \text{Ord} \mid V_\alpha \prec_{\Sigma_n} V\}$
- extendible cardinals are $C^{(1)}$-extendible
- can assume $j_\lambda(\kappa) > \lambda$

Vopěnka's Principle: Every proper class of first-order structures in the same language has two structures which elementarily embed.

- (Bagaria) for every $1 \leq n < \omega$ there is a proper class of $C^{(n)}$-extendible cardinals
- (Bagaria) for every $1 \leq n < \omega$ there is a $C^{(n)}$-extendible cardinal
- can assume structures are $\langle V_\lambda, \in, R\rangle$, where R is a unary predicate
- can assume language is finite

The largest large cardinal hierarchy (too large?)

rank-into-rank cardinal κ: there is an embedding $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$.

- λ is limit or $\lambda = \bar{\lambda} + 1$
- Kunen's inconsistency: there is no non-trivial embedding $j : V_{\lambda+2} \to V_{\lambda+2}$

Berkeley cardinal δ: for every transitive set M, with $\delta \subseteq M$, and $\gamma < \delta$, there is an embedding $j : M \to M$ with $\gamma < \text{crit}(j) < \delta$.

- inconsistent with ZFC
- consistent with ZF?

club Berkeley cardinal δ: for every transitive set M, with $\delta \subseteq M$, and club $C \subseteq \delta$, there is an embedding $j : M \to M$ with $\text{crit}(j) \in C$.

Large cardinal embeddings in a forcing extension

Question: What happens if we ask that embeddings characterizing a given large cardinal exist in a forcing extension of V?

Versions of measurability

In a forcing extension $V[G]$:

- there is an embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $M \subseteq V$.
  - (Usuba) κ is measurable!
- generically measurable: there is an embedding $j : V \to M \subseteq V[G]$ with $\text{crit}(j) = \kappa$.
  - equiconsistent with a measurable cardinal
  - κ can be a small cardinal like ω_1
- generically setwise measurable: for every $\lambda > \kappa$, there is an embedding $j_\lambda : V_\lambda \to M_\lambda \in V[G]$ with $\text{crit}(j_\lambda) = \kappa$.
  - (Nielsen) equiconsistent with a virtually measurable cardinal!
- virtually measurable: for every $\lambda > \kappa$, there is an embedding $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$ and $M_\lambda \subseteq V$.
  - equivalently $M_\lambda \in V$
  - κ is completely ineffable and more
  - consistent with L

Large cardinal embeddings in a forcing extension (continued)

Versions of strongness

For every $\lambda > \kappa$, in a forcing extension $V[G]$:

- generically strong: there is an embedding $j : V \to M \subseteq V[G]$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subseteq M$.
  - at least measurable in consistency strength
- generically setwise strong: there is an embedding $j_\lambda : V_\lambda \to M_\lambda \in V[G]$ with $\text{crit}(j_\lambda) = \kappa$, $j(\kappa_\lambda) > \lambda$, and $V_\lambda = V_\lambda^{M_\lambda}$.
  - (Dimopolous, G., Nielsen) equivalent to a virtually strong cardinal!
- virtually strong: there is an embedding $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$, $j_\lambda(\kappa) > \lambda$, $M_\lambda \subseteq V$, and $V_\lambda \subseteq M_\lambda$.
- virtually* strong: there is an embedding $j_\lambda : V_\lambda \to M_\lambda \in V[G]$ with $\text{crit}(j_\lambda) = \kappa$, $j_\lambda(\kappa) > \lambda$, and $V_\lambda = V_\lambda^{M_\lambda}$, but M_λ need not be well-founded.
  - defined by Wilson
  - (G.) weaker than a virtually strong cardinal

Large cardinal embeddings in a forcing extension (continued)

Versions of supercompactness

For every $\lambda > \kappa$, in a forcing extension $V[G]$:

- generically supercompact: there is an embedding $j : V \to M \subseteq V[G]$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $j \upharpoonright \lambda \in M$.
- virtually supercompact: there is an embedding $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$, $j_\lambda(\kappa) > \lambda$, $M_\lambda \subseteq V$, and $M_\lambda^\lambda \subseteq M_\lambda$ in V.
  - equivalently $M_\lambda \in V$
  - consistent with L
- generically setwise supercompact: there is an embedding $j_\lambda : V_\lambda \to M_\lambda \in V[G]$ with $\text{crit}(j_\lambda) = \kappa$, $j_\lambda(\kappa) > \lambda$, and $M_\lambda^\lambda \subseteq M_\lambda$ in $V[G]$.
  - defined by Schlicht and Nielsen
  - (Usuba) equiconsistent with a virtually extendible cardinal

Large cardinal embeddings in a forcing extension (continued)

Versions of extendibility

For every $\lambda > \kappa$, in a forcing extension $V[G]$:

- virtually extendible: there is an embedding $j : V_\lambda \to V_{\beta_\lambda}$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.
    - consistent with L
- generically extendible: there is an embedding $j : V_\lambda \to V_\beta^{V[G]}$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.
    - recently defined by Ikegami and Vänäänen
    - (Ikegami, Vänäänen) strong compactness cardinal for second-order Boolean-valued logic
    - (Usuba) equiconsistent with a virtually extendible cardinal

Virtual versus generic large cardinals

Virtual

- set embeddings
- the target M is in V
- the target M has closure in V
- completely ineffable and more
- consistent with L

Generic

- class or set embeddings
- the target M may not be a subset of V
- the target M has closure in $V[G]$
- could be a small cardinal like ω_1
- usually high consistency strength

Virtual embeddings

There is a virtual embedding between first-order structures M and N if they elementarily embed in a forcing extension.

Proposition: There is a virtual isomorphism between the reals $\mathbb{R}$ and the rationals $\mathbb{Q}$.

Proof:

- Force with $\mathrm{Coll}(\omega, \mathbb{R})$ to make $\mathbb{R}$ countable in the forcing extension $V[G]$.
- In $V[G]$, $\mathbb{R}^V$ is a countable dense linear order without endpoints. $\square$

Absoluteness lemma for countable embeddings

Lemma: (Silver) Suppose M and N are first-order structures such that

- M is countable,
- there is an embedding $j : M \to N$.

Suppose W is a transitive (set or class) model of (a large enough fragment of) ZFC such that

- $M, N \in W$,
- M is countable in W.

Then for any finite $\bar{a} \subseteq M$, W has an embedding $j^* : M \to N$ agreeing with j on $\bar{a}$, and (where applicable) $\mathrm{crit}(j) = \mathrm{crit}(j^*)$.

Proof:

- Enumerate $M = \{a_n \mid n < \omega\}$ in W. Let $M \upharpoonright n = \{a_i \mid i < n\}$.
- Let T be the tree of all partial finite isomorphisms

$$f : M \upharpoonright n \to N,$$

  satisfying the requirements, ordered by extension.
- M embeds into N if and only if T has a cofinal branch.
- T is ill-founded in V, and hence in W. □

Lemma: Suppose M and N are first-order structures and some set-forcing extension has an embedding $j : M \to N$. Then for every finite $\bar{a} \subseteq M$, $V^{\mathrm{Coll}(\omega,M)}$ has an embedding $j^* : M \to N$ agreeing with j on $\bar{a}$ and (where applicable) $\mathrm{crit}(j) = \mathrm{crit}(j^*)$.

Proof: Suppose a set-forcing extension $V[G]$ has an elementary $j : M \to N$.

- Let $|M|^V = \delta$.
- Consider a further extension $V[G][H]$ by $\mathrm{Coll}(\omega, \delta)$.
- $j \in V[G][H]$ and M is countable in $V[G][H]$.
- $V[H] \subseteq V[G][H]$ has the embedding $j^* : M \to N$ (by Absoluteness lemma). □

Virtually Berkeley cardinals

virtually Berkeley cardinal δ: for every transitive set M, with $\delta \subseteq M$, and $\gamma < \delta$, there is a virtual embedding $j : M \to M$ with $\gamma < \text{crit}(j) < \delta$.

virtually club Berkeley cardinal δ: for every transitive set M, with $\delta \subseteq M$, and club $C \subseteq \delta$, there is a virtual embedding $j : M \to M$ with $\text{crit}(j) \in C$.

Theorem: (Wilson)

- Virtually club Berkeley cardinals are precisely the ω-Erdös cardinals.
- The least ω-Erdős cardinal is the least virtually Berkeley cardinal.

Corollary:

- Virtually club Berkeley cardinals are consistent with L.
- There is NO virtual Kunen's inconsistency!

Virtually rank-into-rank cardinals

virtually rank-into-rank cardinal κ: there is a virtual embedding $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$.

- λ is not restricted by Kunen's inconsistency.

Theorem: (G., Schindler) The least ω-Erdős cardinal is a limit of virtually rank-into-rank cardinals.

Virtually $C^{(n)}$-extendible cardinals

virtually $C^{(n)}$-extendible cardinal κ: for every $\kappa < \lambda \in C^{(n)}$, there is a virtual embedding $j_\lambda : V_\lambda \to V_{\beta_\lambda}$ with $\text{crit}(j_\lambda) = \kappa$, $j_\lambda(\kappa) > \lambda$, and $\beta_\lambda \in C^{(n)}$.

weakly virtually $C^{(n)}$-extendible cardinal κ: for every $\kappa < \lambda \in C^{(n)}$, there is a virtual embedding $j_\lambda : V_\lambda \to V_{\beta_\lambda}$ with $\text{crit}(j_\lambda) = \kappa$ and $\beta_\lambda \in C^{(n)}$.

Theorem: (G., Schindler) If κ is virtually rank-into-rank, then V_κ is a model of proper class many virtually $C^{(n)}$-extendible cardinals.

Theorem: (G.) If there is a weakly virtually extendible cardinal which is not virtually extendible, then there is a virtually rank-into-rank cardinal.

Corollary: If there are NO virtually rank-into-rank cardinals, then a cardinal is weakly virtually $C^{(n)}$-extendible if and only if it is virtually $C^{(n)}$-extendible.

Corollary: A weakly virtually $C^{(n)}$-extendible cardinal is equiconsistent with a virtually $C^{(n)}$-extendible cardinal.

Question: If there is a weakly virtually extendible cardinal which is not virtually extendible, is there a virtually Berkeley cardinal?

Virtual Vopěnka's Principle

Virtual Vopěnka's Principle: Every proper class of first-order structures in the same language has two structures which virtually elementarily embed.

Theorem: (G., Hamkins) Virtual Vopenka's Principle holds if and only if for every $n < \omega$, there is a proper class of weakly virtually $C^{(n)}$-extendible cardinals.

Theorem: (G., Hamkins) It is consistent that Virtual Vopěnka's Principle holds, but there are no virtually supercompact cardinals.

Virtual Vopěnka's Principle for finite languages

Virtual Vopěnka's Principle for finite languages: Every proper class of first-order structures in the same finite language has two structures which virtually elementarily embed.

Theorem: (Dimopolous, G., Nielsen) Virtual Vopěnka's Principle for finite languages holds if and only if for every $n < \omega$, there is a weakly virtually $C^{(n)}$-extendible cardinal.

Theorem: (G., Nielsen) It is consistent that the Virtual Vopěnka's Principle fails and the virtual Vopěnka's Principle for finite languages holds.

Theorem: (Nielsen) If for every $n < \omega$, there is a weakly virtually $C^{(n)}$-extendible cardinal, but for some $n < \omega$, there is no virtually $C^{(n)}$-extendible cardinal, then there is a virtually Berkeley cardinal.

Corollary: If there are NO virtually Berkeley cardinals, then the following are equivalent:

- virtual Vopěnka's Principle
- virtual Vopěnka's Principle for finite languages
- for every $n < \omega$ there is a virtually $C^{(n)}$-extendible cardinal

Virtually supercompact cardinals

virtually supercompact cardinal κ: for every $\lambda > \kappa$, there is a virtual embedding $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$, $j_\lambda(\kappa) > \lambda$, and $M_\lambda^\lambda \subseteq M_\lambda$.

Theorem: (G., Schindler) A cardinal κ is virtually supercompact if and only if it is remarkable.

Theorem: (G., Schindler) A virtually extendible cardinal is a limit of virtually supercompact cardinals.

setwise generically supercompact cardinal κ: for every $\lambda > \kappa$, in a forcing extension $V[G]$, there is an embedding $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$ and $M_\lambda^\lambda \subseteq M_\lambda$ in $V[G]$.

Theorem: (Usuba) The following are equiconsistent.

- virtually extendible cardinal
- (ω_1 or ω_2 is a) generically setwise supercompact cardinal
  - $\kappa > \omega_2$ is generically setwise supercompact implies $0^\#$.
- generically extendible cardinal

Virtually strong cardinals

virtually strong cardinal κ: for every $\lambda > \kappa$, there is a virtual embedding $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subseteq M$.

weakly virtually strong cardinal κ: for every $\lambda > \kappa$, there is a virtual $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$ and $V_\lambda \subseteq M$.

Theorem: (Nielsen) If a cardinal is weakly virtually strong cardinal, but not not virtually strong, then it is virtually rank-into-rank.

Corollary: Weakly virtually strong cardinals are equiconsistent with virtually strong cardinals.

Theorem: (G., Schindler) A cardinal is virtually supercompact if and only if it is virtually strong.

generically setwise strong cardinal κ: for every $\lambda > \kappa$, in a forcing extension $V[G]$, there is an embedding $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda = V_\lambda^M$.

Theorem: (G., Dimopolous, Nielsen) A cardinal is virtually strong if and only if it is generically setwise strong.

Virtually measurable cardinals

virtually measurable cardinal κ: for every $\lambda > \kappa$, there is a virtual embedding $j : V_\lambda \to M$ with $\text{crit}(j) = \kappa$.

generically setwise measurable cardinal κ: for every $\lambda > \kappa$, in a forcing extension $V[G]$, there is an embedding $j : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$.

Theorem: (Nielsen) Generically setwise measurable cardinals are equiconsistent with virtually supercompact cardinals.

Proof: A generically setwise measurable cardinal κ is weakly virtually strong in L. $\square$

Theorem: (G.) It is consistent that there is a generically setwise measurable cardinal which is not weakly virtually strong.

Virtually* strong cardinals

virtually* strong cardinal κ: (Wilson) for every $\lambda > \kappa$, in a forcing extension $V[G]$, there is an embedding $j_\lambda : V_\lambda \to M_\lambda$ with $\text{crit}(j_\lambda) = \kappa$ and $V_\lambda = V_\lambda^{M_\lambda}$, but M_λ need not be well-founded.

Theorem: (G.)

- A cardinal κ is $\kappa + 1$-virtually* strong if and only if it is completely ineffable.
- virtually* strong cardinals are weaker than virtually measurable cardinals.

Weak Vopěnka's Principle: Technical weakening of Vopěnka's Principle.

Theorem: (Wilson) Weak Vopěnka's Principle holds if and only if for every $n < \omega$, there is a $C^{(n)}$-strong cardinal.

Theorem: (Wilson) Virtual Weak Vopenka's Principle holds if and only if for every $n < \omega$, there is a weakly virtually* $C^{(n)}$-strong cardinal.

Applications

The model $L(\mathbb{R})$

- start the L-construction with $\mathbb{R}$ instead of $\emptyset$
- satisfies ZF
- assuming large cardinals, satisfies the Axiom of Determinacy.

Even though forcing easily changes the theory of V, it is consistent (from large cardinals) that the theory of $L(\mathbb{R})$ cannot be changed by forcing.

Theorem: (Woodin) If there is a supercompact cardinal, then there is a model in which theory of $L(\mathbb{R})$ cannot be changed by forcing.

Theorem: (Schindler) The assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing is equiconsistent with a remarkable (virtually supercompact) cardinal.

Applications (continued)

A set of reals is universally Baire if for every continuous function from a compact Hausdorff space to the reals, its preimage has the Baire property.

- include Σ^1_1-sets and Π^1_1-sets
- Lebesgue measurable
- Baire property
- assuming large cardinals, perfect set property

Theorem: (Schindler, Wilson) The assertion that every universally Baire set has the perfect set property is equiconsistent with a virtually Shelah for supercompactness cardinal.

The virtual large cardinal hierarchy: consistency strength

virtually Berkeley
ω-Erdős = virtually club Berkeley

virtually rank-into-rank

virtual Vopěnka's Principle
virtual Vopěnka's Principle for finite languages

virtually $C^{(n)}$-extendible
weakly virtually $C^{(n)}$-extendible

generically extendible
generically setwise supercompact
virtually extendible

virtually supercompact
weakly virtually supercompact
virtually strong
weakly virtually strong
generically setwise strong
virtually measurable
generically setwise measurable

virtually* strong

completely ineffable