A model of second-order arithmetic satisfying AC but not DC

Victoria Gitman

vgitman@nylogic.org
http://victoriagitman.github.io

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Why second-order arithmetic

Models of second-order arithmetic have two types of objects:

- numbers
- sets of numbers (reals).

A second-order arithmetic axiom system postulates what kind of sets of numbers (reals) exist.

Given a theorem in analysis, we can ask what kind of reals need to exist in order to be able to prove it.

Second-order arithmetic provides an incredibly effective measuring stick for answering such questions.

(Most) classical results in analysis are provable within one of the main second-order arithmetic systems and indeed are equivalent to some such system (over a weak base system).

Second-order arithmetic

Syntax: Two-sorted logic

- Separate variables and quantifiers for numbers and sets of numbers.
- Convention: lower-case letters for numbers, upper-case letters for sets of numbers.
- Notation:
  - Σ_n^0 - first-order Σ_n-formula
  - Σ_n^1 - n-alternations of set quantifiers followed by a first-order formula.

Semantics: A model is $\mathscr{M} = \langle M, +, \times, <, 0, 1, \mathcal{S}\rangle$.

- M is the collection of numbers.
- $\mathcal{S}$ is the collection of sets of numbers: if $A \in \mathcal{S}$, then $A \subseteq M$.
- Example: the full standard model $\mathscr{M} = \langle \omega, +, \times, <, 0, 1, P(\omega)\rangle$.

Axiom systems

First-order axioms

- PA^-
- (Induction axiom) $\forall X\ ((0 \in X \wedge \forall n(n \in X \to n+1 \in X)) \to \forall n\, n \in X)$

Arithmetical comprehension ACA_0

- Comprehension scheme for first-order formulas: for all n, Σ^0_n-CA_0
  if $\varphi(n, A)$ is a first-order formula, then $\{n \mid \varphi(n, A)\}$ is a set.

⋆ Example: If $\langle M, +, \times, <, 0, 1\rangle \models \mathrm{PA}$ and $\mathcal{S}$ consists of definable subsets of M, then

$$\mathscr{M} = \langle M, +, \times, <, 0, 1, \mathcal{S}\rangle \models \mathrm{ACA}_0.$$

⋆ Every model of PA is naturally also a model of ACA_0.

⋆ ACA_0 is conservative over PA.

Elementary Transfinite Recursion ATR_0

- ACA_0
- Every first-order recursion on sets along a well-order has a solution.
  - A well-order is a linear order Γ whose every subset has a minimal element.
  - A solution to a recursion is a code of a function $F : \mathrm{dom}(\Gamma) \to \mathcal{S}$.

A code for F is $\bar{F} = \{\langle n, m\rangle \mid n \in \mathrm{dom}(\Gamma,)\ m \in F(n)\}$

⋆ Can iterate Turing jump.

⋆ Can build an internal constructible universe L.

Axiom systems (continued)

Σ^1_n-comprehension $\mathrm{\Sigma^1_n\text{-}CA_0}$

- Σ^1_n-comprehension: If $\varphi(n, A)$ is a Σ^1_n-formula, then $\{n \mid \varphi(n, A)\}$ is a set.

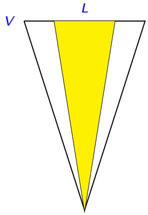
⋆ $\Sigma^1_1\text{-}\mathrm{CA}_0$ is stronger than ATR_0.

Full second-order arithmetic Z_2

Full second-order comprehension: for all n, Σ^1_n-comprehension.

Example: If $V \models \mathrm{ZF}$, then the full standard model $\mathscr{M}^V = \langle \omega, +, \times, <, 0, 1, P(\omega)\rangle \models \mathrm{Z}_2$.

Gödel's constructible universe L



Suppose $V \models \mathrm{ZF}$.

- $L_0 = \emptyset$
- $L_{\alpha+1}$ is the set of all subsets of L_α definable over L_α.
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ.
- $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$.

Suppose $\mathscr{M} = \langle M, +, \times, <, 0, 1, \mathcal{S}\rangle \models \mathrm{Z}_2$ and $\Gamma \in \mathcal{S}$ is a well-order.

- $\mathscr{M}$ can construct the L-hierarchy along Γ (uses ATR_0).
- There is a set coding a sequence of L_Δ for $\Delta \leq \Gamma$ obeying the definition of L.

A model of Z_2 has its own constructible universe $L^{\mathscr{M}}$!

Theorem: (Shoenfield Absoluteness) If φ is a Σ^1_2-assertion, then $\mathscr{M} \models \varphi$ iff $L^{\mathscr{M}} \models \varphi$.

In $L^{\mathscr{M}}$ interpret φ as an assertion about numbers and reals.

ω-models and β-models of second-order arithmetic

Definition: A model of second-order arithmetic is an ω-model if it has the standard first-order part: $\mathscr{M} = \langle \omega, +, \times, <, 0, 1, \mathcal{S} \rangle$.

Definition: A β-model of second-order arithmetic is an ω-model $\mathscr{M} = \langle \omega, +, \times, <, 0, 1, \mathcal{S} \rangle$ that is correct about well-foundedness:

for every relation $\Gamma \in \mathcal{S}$, $\mathscr{M} \models \Gamma$ is well-founded iff Γ is well-founded.

An ω-model of second-order arithmetic can be wrong about well-foundedness because it is missing a witnessing subset.

Example: If $V \models \mathrm{ZF}$, then $\mathscr{M}^V = \langle \omega, +, \times, <, 0, 1, P(\omega) \rangle$ is a β-model of Z_2.

Example: Suppose $\mathscr{M} = \langle \omega, +, \times, <, 0, 1, \mathcal{S} \rangle$ is a β-model of Z_2.

- $\mathscr{M}$ is correct about ordinals.
- $\mathscr{M}$ is correct about the constructible universe L up to its height.

Set choice principles

Choice Scheme

"If for every n, there is a set X witnessing $\varphi(n, X, A)$, then there is a single set Z collecting witnesses for every n."

Choice scheme Σ^1_n-AC: A scheme consisting of assertions for every Σ^1_n-formula $\varphi(n, X, A)$,

$$\forall n\, \exists X \varphi(n, X, a) \to \exists Z\, \forall n\, \varphi(n, Z_n, A),$$

where $Z_n = \{m \mid (n, m) \in Z\}$ is the n-th slice of Z.

Σ^1_∞-AC: for all n, Σ^1_n-AC.

Example: If $V \models \mathrm{ZF} + \mathrm{AC}_\omega$, then $\mathscr{M}^V = \langle \omega, +, \times, <, 0, 1, P(\omega)\rangle$ is a β-model of $\mathrm{Z}_2 + \Sigma^1_\infty$-$\mathrm{AC}$.

Dependent Choice Scheme

"Every relation on sets without terminal nodes has an infinite branch."

Dependent choice scheme Σ^1_n-DC: A scheme consisting of assertions for every Σ^1_n-formula $\varphi(X, Y, A)$,

$$\forall X\, \exists Y\, \varphi(X, Y, A) \to \exists Z\, \forall n\, \varphi(Z_n, Z_{n+1}, A).$$

Σ^1_∞-DC: for all n, Σ^1_n-DC.

Example: If $V \models \mathrm{ZF} + \mathrm{DC}$, then $\mathscr{M}^V = \langle \omega, +, \times, <, 0, 1, P(\omega)\rangle$ is a β-model of $\mathrm{Z}_2 + \Sigma^1_\infty$-$\mathrm{DC}$.

Choice principles in Z_2

Theorem: Z_2 proves Σ^1_2-AC.

Proof: Suppose $\mathscr{M} \models \mathrm{Z}_2$ and $\mathscr{M} \models \forall n \, \exists X \, \varphi(n, X)$, where φ is Σ^1_2.

- By Shoenfield Absoluteness, $L^{\mathscr{M}}$ has a witness for every Σ^1_2-assertion $\varphi(n, X)$.
- Choose the least $L^{\mathscr{M}}$-witness X and use comprehension to collect.

Theorem: (Mansfield, Simpson) Z_2 proves Σ^1_2-DC.

Strategy for constructing models with a failure of choice

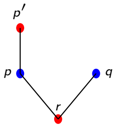
- Construct a forcing extension $V[G]$ having a submodel $N \models \mathrm{ZF}$ with a definable failure of choice.
- Let $\mathscr{M}^N = \langle \omega, +, \times, <, 0, 1, P(\omega)^N \rangle$.
- Necessarily produces a β-model.

Quick review of forcing

Suppose $V \models \mathrm{ZFC}$ and $\mathbb{P}$ is a forcing notion: partial order with largest element $\mathbb{1}$.

Dense sets and generic filters

$D \subseteq \mathbb{P}$ is dense if for every $p \in \mathbb{P}$, there is $q \in D$ with $q \leq p$.



$G \subseteq \mathbb{P}$ is a filter:

- (upward closure) If $p \in G$ and $p' \geq p$, then $p' \in G$.
- (compability) If $p, q \in G$, then $r \in G$ such that $r \leq p, q$.

Note: If $G \neq \emptyset$, then $\mathbb{1} \in G$.

A filter $G \subseteq \mathbb{P}$ is V-generic if it meets every dense set $D \in V$ of $\mathbb{P}$: $D \cap G \neq \emptyset$.

Theorem: V has no V-generic filters for $\mathbb{P}$.

The forcing extension $V[G]$ is constructed from V together with an external V-generic filter G.

Quick review of forcing (continued)

$\mathbb{P}$-**names**: names for elements of $V[G]$.

Defined recursively so that a $\mathbb{P}$-name σ consists of pairs $\langle \tau, p\rangle$: $p \in \mathbb{P}$ and τ is a $\mathbb{P}$-name.

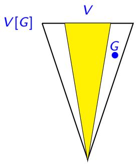
Special $\mathbb{P}$-names

- Given $a \in V$, $\check{a} = \{\langle \check{b}, \mathbb{1}\rangle \mid b \in a\}$.
- $\dot{G} = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$.

Forcing extension $V[G]$

Suppose $G \subseteq \mathbb{P}$ is V-generic and σ is a $\mathbb{P}$-name. The interpretation of σ by G: $\sigma_G = \{\tau_G \mid \langle \tau, p\rangle \in \sigma \text{ and } p \in G\}$.

Defined recursively.



The forcing extension $V[G] = \{\sigma_G \mid \sigma \text{ is a } \mathbb{P}\text{-name in } V\}$.

- $V \subseteq V[G]$: $\check{a}_G = a$.
- $G \in V[G]$: $\dot{G}_G = G$.
- $V[G] \models \mathrm{ZFC}$

Forcing relation $p \Vdash \varphi(\sigma)$

Whenever G is V-generic and $p \in G$, then $V[G] \models \varphi(\sigma_G)$.

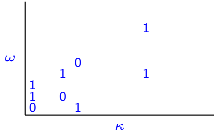
Theorem: (definability of the forcing relation) For a fixed first-order formula $\varphi(x)$, the relation $p \Vdash \varphi(\sigma)$ is definable.

Useful forcing notions

$\mathrm{Add}(\omega,\kappa)$ - Add κ-many subsets to ω

- Conditions: functions $p : D \to 2$, where D is a finite subset of $\omega \times \kappa$.
- Order: $p \leq q$ if p extends q.

⋆ If $G \subseteq \mathrm{Add}(\omega,\kappa)$ is V-generic, then in $V[G]$, $2^\omega \geq \kappa$.



$\mathrm{Add}(\omega_1,\kappa)$ - Add κ-many subsets to ω_1

- Conditions: functions $p : D \to 2$, where D is a countable subset of $\omega_1 \times \kappa$.
- Order: $p \leq q$ if p extends q.

$\mathrm{Coll}(\omega,\kappa)$ - Collapse κ to ω

- Conditions: functions $p : D \to \kappa$, where D is a finite subset of ω.
- Order: $p \leq q$ if p extends q.

⋆ If $G \subseteq \mathrm{Coll}(\omega,\kappa)$ is V-generic, then in $V[G]$, κ is a countable ordinal.

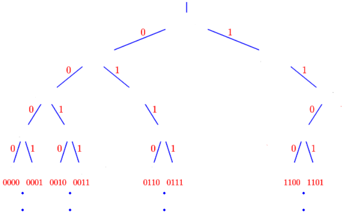
Useful forcing notions (continued)

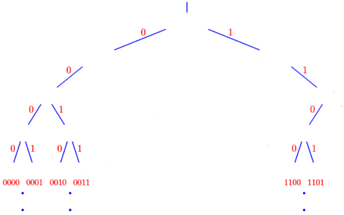
Sacks forcing $\mathbb{S}$ - Add a generic real

- Conditions: Perfect trees $T \subseteq 2^{<\omega}$: every node has a splitting node above it.
- Order: $T \leq S$ if T is a subtree of S.

⋆ If G is V-generic for $\mathbb{S}$, then there is a real $b \in V[G]$ such that $T \in G$ iff b is a branch of T.

⋆ The generic real b determines G.





Products and iterations of forcing notions

Products

Suppose $\mathbb{P}_\alpha$ for $\alpha < \beta$ are forcing notions.

A product $\mathbb{P} = \Pi_{\alpha<\beta}\mathbb{P}_\alpha$ is also a natural forcing notion.

- Conditions: $\langle p_\alpha \mid \alpha < \beta\rangle$ with $p_\alpha \in \mathbb{P}_\alpha$.
- Common supports: finite, bounded, full.
- Example: $\mathrm{Add}(\omega,\kappa) = \Pi_{\alpha<\kappa}\mathrm{Add}(\omega,1)$ with finite support.
- Usage: adding several objects to a forcing extension.

Iterations

Suppose $\mathbb{P}$ is a forcing notion, $G \subseteq \mathbb{P}$ is V-generic, and $\mathbb{Q}$ is a forcing notion in $V[G]$.

V has a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$. Every element of $V[G]$ has a $\mathbb{P}$-name in V.

In V, we can define a forcing notion $\mathbb{P} * \dot{\mathbb{Q}}$ such that forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ is the same as forcing with $\mathbb{P}$ followed by forcing with $\mathbb{Q}$.

- Conditions: $(p, \dot{q})$ with $p \in \mathbb{P}$ and $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$.
- Order: $(p, \dot{q}) \leq (r, \dot{s})$ if $p \leq r$ and $p \Vdash \dot{q} \leq \dot{s}$.
- n-step iterations are defined similarly (infinite iterations can be defined as well).

Example: $\mathbb{S} * \dot{\mathbb{S}}$, where $\dot{\mathbb{S}}$ is the name for the Sacks forcing of the forcing extension.

Sacks forcing of $V[G]$ is different from Sacks forcing of V because $V[G]$ has new perfect trees.

Symmetric submodels of forcing extensions

Set-up

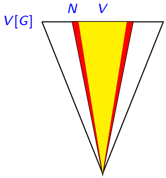
- $\mathbb{P}$ is a forcing notion.
- $\mathcal{G}$ is a group of automorphisms of $\mathbb{P}$.
- $\mathscr{F}$ is a normal filter of subgroups of $\mathcal{G}$.
  - (upward closure) If $H_1 \in \mathscr{F}$ and $H_2 \supseteq H_1$, then $H_2 \in \mathscr{F}$.
  - (closure under intersections) If $H_1, H_2 \in \mathscr{F}$, then $H_1 \cap H_2 \in \mathscr{F}$.
  - (normality) If $H \in \mathscr{F}$ and $\pi \in \mathcal{G}$, then $\pi H \pi^{-1} \in \mathscr{F}$.
- $G \subseteq \mathbb{P}$ is V-generic

Definition: If σ is a $\mathbb{P}$-name and $\pi \in \mathcal{G}$, then $\pi(\sigma) = \{\langle \pi(\tau), \pi(p)\rangle \mid \langle \tau, p\rangle \in \sigma\}$.

Proposition: For every $p \in \mathbb{P}$, $p \Vdash \varphi(\sigma)$ iff $\pi(p) \Vdash \varphi(\pi(\sigma))$.

Definition: Suppose σ is a $\mathbb{P}$-name.

- σ is symmetric if there is $H \in \mathscr{F}$ such that every $\pi \in H$ fixes σ: $\pi(\sigma) = \sigma$.
- σ is hereditarily symmetric if σ is symmetric and all $\mathbb{P}$-names occurring hereditarily in σ are also symmetric.
- HS be the collection of all hereditarily symmetric names.



$N = \{\sigma_G \mid \sigma \in \mathrm{HS}\}$ is a symmetric submodel of $V[G]$.

Theorem: $N \models \mathrm{ZF}$.

The Feferman-Lévy symmetric model

We work in the constructible universe L.

The forcing $\mathbb{P}$

- Finite-support product $\Pi_{n<\omega}\mathrm{Coll}(\omega, \omega_n)$. ω_n is n-th cardinal.
- Let $G \subseteq \mathbb{P}$ be L-generic and let $G_m = G \upharpoonright \Pi_{n<m}\mathrm{Coll}(\omega, \omega_n)$.

Automorphisms of $\mathbb{P}$

- $\mathcal{G}$ is the group of all product automorphisms $\Phi = \Pi_{n<\omega}\phi_n$, where ϕ_n is an automorphism of $\mathrm{Coll}(\omega, \omega_n)$.
- $\mathscr{F}$ is generated by subgroups $H_n = \{\Phi \mid \Phi(n) = \mathrm{Id} \text{ for } i < n\}$.

The symmetric submodel N

Theorem: The subsets of ordinals in N are precisely those added by initial stages of the product: $S \subseteq \mathrm{Ord}$ is in N iff $S \in V[G_m]$ for some $m < \omega$.

The following holds in N:

- Each ω_n^L is countable.
- ω_ω^L is the first uncountable cardinal. ω_ω is countable in $V[G]$.

Independence of Π^1_2-AC from Z_2

Theorem: (Feferman, Lévy) Π^1_2-AC can fail in a β-model of Z_2.

Proof: Let N be the Feferman-Lévy symmetric submodel.
Let $\mathscr{M}^N = \langle \omega, +, \times, <, 0, 1, P(\omega)^N \rangle \models Z_2$.

- Every L_{ω_n} is coded in $\mathscr{M}^N$, but L_{ω_ω} is not coded in $\mathscr{M}^N$.
- We cannot collect the (codes of) L_{ω_n}.
- The assertion
$$\forall n\, \exists X = L_{\omega_n} \to \exists Z\, \forall n\, Z_n = L_{\omega_n}$$
fails in $\mathscr{M}^N$.
- The assertion "X codes L_{ω_n}" is Π^1_2:
$$\underbrace{X \text{ codes } L_\alpha}_{\Pi^1_1} \wedge \forall Y (\underbrace{Y \text{ codes } L_\beta \text{ with } \beta > \alpha}_{\Pi^1_1} \longrightarrow \underbrace{L_\beta \text{ thinks } \alpha = \omega_n}_{\Pi^1_0}). \ \square$$

Independence of Π^1_2-DC from $\mathrm{Z}_2 + \Sigma^1_\infty$-$\mathrm{AC}$

Theorem (Friedman, G., Kanovei) Π^1_2-DC can fail in a β-model of $\mathrm{Z}_2 + \Sigma^1_\infty$-$\mathrm{AC}$.

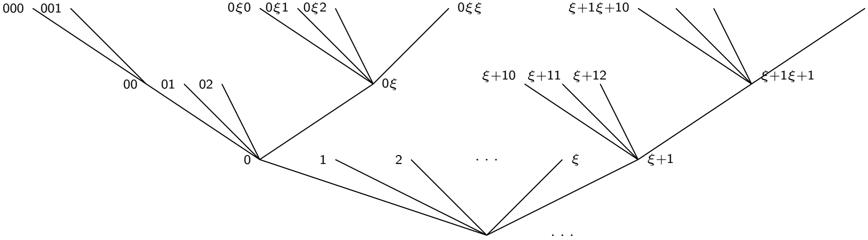
History

- Simpson claims proof in abstract in Notices of American Mathematical Society in 1973, but proof is lost.
- Kanovei publishes proof in Russian journal in 1979.
- We prove the theorem independently and ask Kanovei to join us on the paper when we learn about the 1979 result.

Strategy

- Construct a symmetric submodel N of some forcing extension $V[G]$ such that in N:
  - AC_ω holds,
  - DC fails for a Π^1_2-definable relation on the reals.
- Let $\mathscr{M}^N = \langle \omega, +, \times, <, 0, 1, P(\omega)^N \rangle$.

Classical symmetric submodel of $\mathrm{AC}_\omega + \neg\mathrm{DC}$ (Jensen)



The forcing $\mathbb{P}$ - "$\mathrm{Add}(\omega_1, \omega_1^{<\omega})$"

- Adds a tree isomorphic to $\omega_1^{<\omega}$ whose nodes are V-generic for $\mathrm{Add}(\omega_1, 1)$.
- Conditions: $p : D \to 2$, where D is a countable subset of $\omega_1^{<\omega} \times \omega_1$.
- Order: $p \leq q$ if p extends q.
- $\mathbb{P}$ is countably closed: every descending ω-sequence of conditions has a lower bound.

$$p \leq \cdots \leq p_n \leq p_{n-1} \leq \cdots \leq p_1 \leq p_0$$

- Let $G \subseteq \mathbb{P}$ be V-generic.

Classical symmetric submodel of $\mathrm{AC}_\omega + \neg\mathrm{DC}$ (Jensen)

Automorphisms of $\mathbb{P}$

- Every automorphism π of the tree $\omega_1^{<\omega}$ extends to an automorphism π^* of $\mathbb{P}$.
- $\mathcal{G}$ is the group of all such automorphisms π^*.
- A countable tree $T \subseteq \omega_1^{<\omega}$ is good if it has no infinite branch.
- Given a good tree T, let H_T be the group of all π^* with π point-wise fixing T.
- $\mathscr{F}$ is generated by all such subgroups H_T.

The symmetric submodel N

Suppose $\sigma \in \mathrm{HS}$ and T is a good tree such that $\pi^*(\sigma) = \sigma$ for all $\pi^* \in H_T$. Then we say that T witnesses that σ is symmetric.

If T is a good tree, let G_T be the restriction of G to nodes of T.

Theorem: $S \subseteq \mathrm{Ord}$ is in N iff $S \in V[G_T]$ for some good tree T.

Classical model of $\mathrm{AC}_\omega + \neg\mathrm{DC}$ (continued)

Preliminaries

- Let $\dot{\mathscr{T}}$ be the canonical $\mathbb{P}$-name for the tree of Cohen subsets of ω_1 added by $\mathbb{P}$.
- $\dot{\mathscr{T}}$ is hereditarily symmetric, and hence $\mathscr{T} = (\dot{\mathscr{T}})_G \in N$.

Lemma: DC fails in N.

Proof sketch:

- Suppose that $b \in N$ is an infinite branch through $\mathscr{T}$.
- Let $\sigma \in HS$ be a $\mathbb{P}$-name for b, witnessed by a good tree T.
- Use that eventually b lies outside of T to derive a contradiction. $\square$

Lemma: AC_ω holds in N.

Proof sketch:

- Let $F = \{F_n \mid n < \omega\} \in N$ be a family of non-empty sets.
- Let $\sigma \in \mathrm{HS}$ be a $\mathbb{P}$-name for F, witnessed by a good tree S.
- Build a descending sequence of conditions $p_0 \geq p_1 \geq \cdots \geq p_i \geq \cdots$ such that:
  - $p_i \Vdash \tau_i \in \sigma(i)$ for some $\tau_i \in \mathrm{HS}$, witnessed by a good tree T_i.
  - For $i < j$, $T_i \cap T_j = S$.
- Let $\tau \in \mathrm{HS}$ be a $\mathbb{P}$-name for the sequence of the τ_i, as witnessed by $T = \bigcup_{i<\omega} T_i$.
- Let $p \leq p_i$ for all $i < \omega$.
- $p \Vdash$ "τ is a choice function for σ". $\square$

Obstacle: $\mathscr{T}$ is not a tree of reals.

A variation on the classical model (Friedman, G.)

The forcing $\mathbb{P}$ - "$\mathrm{Add}(\omega, \omega_1^{<\omega})$"

- Adds a tree isomorphic to $\omega_1^{<\omega}$ whose nodes are V-generic for $\mathrm{Add}(\omega, 1)$.
- Conditions: $p : D \to 2$, where D is a finite subset of $\omega_1^{<\omega} \times \omega$.
- Order: $p \leq q$ if p extends q.
- $\mathbb{P}$ has the ccc: countable chain condition - every antichain is countable.

Automorphisms of $\mathbb{P}$

Same as before.

The symmetric model N

- DC fails in N.
- AC_ω holds in N (use ccc instead of countable closure).

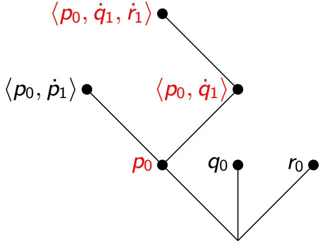
Obstacle: Why is $\mathscr{T}$ definable over $P^N(\omega)$?

- Domain
  - How do we pick out which generic reals for $\mathrm{Add}(\omega, 1)$ lie on the tree?
  - Forcing with $\mathrm{Add}(\omega, 1)$ adds 2^ω-many generic reals.
- Order
  - How do we know how the generic reals are ordered in $\mathcal{T}$?

A model of $\mathrm{ZF} + \mathrm{AC}_\omega + \neg\Pi^1_2\text{-}\mathrm{DC}$

The forcing $\mathbb{P}$

- Let $\langle \mathbb{P}_n \mid n < \omega\rangle$ be a sequence of forcing iterations such that:
  - $\mathbb{P}_n$ is an iteration of length n,
  - a generic filter for $\mathbb{P}_n$ is determined by an n-length sequence of reals,
  - for $m > n$, $\mathbb{P}_m \restriction n = \mathbb{P}_n$,
  - The collection of all generic n-length sequences of reals for $\mathbb{P}_n$ is Π^1_2-definable.
- Conditions: $p : D_p \to \bigcup_{n<\omega} \mathbb{P}_n$ such that:
  - D_p is a finite subtree of $\omega_1^{<\omega}$,
  - for all $s \in D_p$, $p(s) \in \mathbb{P}_{\mathrm{len}(s)}$,
  - for $s \subseteq t$ in D_p, $p(s) = p(t) \restriction \mathrm{len}(s)$.
- Order: $p \leq q$ if $D_p \supseteq D_q$ and for all $s \in D_q$, $p(s) \leq q(s)$.
- $\mathbb{P}$ is an "iteration along the tree $\omega_1^{<\omega}$".
- Suppose $G \subseteq \mathbb{P}$ is V-generic.
- An n-length sequence of reals in $V[G]$ is V-generic for $\mathbb{P}_n$ if and only if it comes from a node of the tree added by G.
- $\mathbb{P}$ has the ccc.



A model of $\mathrm{ZF} + \mathrm{AC}_\omega + \neg\Pi^1_2\text{-}\mathrm{DC}$

Automorphisms of $\mathbb{P}$

Same as before.

The symmetric model N

- DC fails in N.
- Using that $\mathbb{P}$ has the ccc, it follows that AC_ω holds in N.

The tree $\mathcal{T}$

- Domain: Π^1_2-definable.
- Order: extension.

Obstacle: Find $\langle \mathbb{P}_n \mid n < \omega \rangle$ with desired properties.

Jensen's forcing $\mathbb{J}$

Constructed in L using the $\diamondsuit$-principle.

Sub-forcing of Sacks forcing (conditions are perfect trees $T \subseteq 2^{<\omega}$).

Adds a unique generic real over L. $\mathrm{Add}(\omega, 1)$ adds 2^{ω}-many generic reals.

Has the ccc.

The forcing $\mathbb{J}$ is constructed as a chain of countable partial orders of length ω_1 using $\diamondsuit$ to seal antichains along the way.

Products of $\mathbb{J}$

The "uniqueness of generic reals" property of $\mathbb{J}$ extends to products.

Theorem: (Lyubetsky, Kanovei) If G is L-generic for the finite-support product $\Pi_{n<\omega}\mathbb{J}$, then the only L-generic reals for $\mathbb{J}$ in $L[G]$ are those on the coordinates of G.

The tree iteration of Jensen's forcing

Let $\mathbb{J}_n$ for $n < \omega$ be the n-length iterations of $\mathbb{J}$.

$\mathbb{J}_n$ adds a generic n-length sequence of reals.

Let $\mathbb{T}$ be the tree iteration using the sequence $\langle \mathbb{J}_n \mid n < \omega \rangle$.

The forcing $\mathbb{T}$ adds a tree $\mathcal{T}$ isomorphic to $\omega_1^{<\omega}$:
nodes on level n are L-generic sequences of reals for $\mathbb{J}_n$.

The "uniqueness of generic reals" property of $\mathbb{J}$ extends to tree iterations.

Main Theorem: (Friedman, G.) If G is L-generic for the tree iteration $\mathbb{T}$ along the tree $\omega_1^{<\omega}$, then the only L-generic sequences of reals for $\mathbb{J}^n$ are those on the nodes of the generic tree $\mathcal{T}$.

- The domain of $\mathcal{T}$ is Π^1_2-definable.
- The order is extension.

The main theorem in ZFC

Theorem: There is a β-model of $\mathrm{Z}_2 + \Sigma^1_\infty$-AC in which Π^1_2-DC fails.

Proof:

- Let $\mathbb{T} \in L$ be the forcing used in the proof of the Main Theorem.
- Let $G \subseteq \mathbb{T}$ be V-generic.
- $L[G] \subseteq V[G]$.
- $V[G]$ has a β-model $\mathscr{M} \models \mathrm{Z}_2 + \Sigma^1_\infty\text{-AC} + \neg\Pi^1_2\text{-DC}$.
- $\mathscr{M}$ has a countable β-sub-model with the same properties.
- The submodel is coded by a real.
- By Shoenfield Absoluteness, V has a β-model $\bar{\mathscr{M}} \models \mathrm{Z}_2 + \Sigma^1_\infty\text{-AC} + \neg\Pi^1_2\text{-DC}$.

The set theory of second-order arithmetic

Set theory without powerset ZFC^-

- ZFC without powerset
- Collection Scheme instead of the Replacement Scheme
- Well-ordering principle instead of the Axiom of Choice

  Well-ordering principle: every set can be well-ordered.

Let HC be the assertion that every set is countable.

Theorem: The following theories are bi-interpretable.

- $\mathrm{Z}_2 + \Sigma^1_\infty\text{-AC}$
- $\mathrm{ZFC}^- + \mathrm{HC}$

Proof:

- Suppose $\mathscr{M} = \langle M, +, \times, <, 0, 1, \mathcal{S}\rangle \models \mathrm{Z}_2 + \Sigma^1_\infty\text{-AC}$.
  - View each extensional well-founded relation $R \in \mathcal{S}$ as coding a transitive set.
  - Define a membership relation E on the collection of all such relations R (modulo isomorphism).
  - The resulting first-order structure $\mathcal{N}^{\mathscr{M}} = \langle N, \mathrm{E}\rangle \models \mathrm{ZFC}^- + \mathrm{HC}$.
- Suppose $\mathcal{N} = \langle N, \mathrm{E}\rangle \models \mathrm{ZFC}^- + \mathrm{HC}$.
  - The structure $\mathscr{M} = \langle \omega^N, +, \times, <, 0, 1, P(\omega)^{\mathcal{N}}\rangle \models \mathrm{Z}_2 + \Sigma^1_\infty\text{-AC}$.
  - $\mathcal{N}^{\mathscr{M}} \cong \mathcal{N}$. $\square$

What we lose without powerset

The theory ZFC^- lacks many nice properties of ZFC.

- (Zarach and G., Hamkins, Johnstone) The Replacement Scheme and Collection Scheme are not equivalent.
- (Zarach) Various formulations of the Axiom of Choice are not equivalent.
- (G., Johnstone) Ground model definability (a model is definable with parameters in its forcing extensions) can fail.

  Ground model definability: A model is definable with parameters in its forcing extensions.
- (Antos, Friedman, G.) The Intermediate Model Theorem can fail.

  Intermediate Model Theorem: Intermediate models between a model and its forcing extensions are forcing extensions.

DC-Scheme

DC-Scheme

"Every definable relation without terminal nodes has an infinite branch."

A scheme consisting of assertions for every formula $\varphi(x, y, a)$,

$$\forall x \exists y \, \varphi(x, y, a) \to \exists z \forall n < \omega \, \varphi(z(n), z(n+1), a).$$

Theorem: (G., Friedman, Kanovei) The DC-Scheme can fail over a model of ZFC^-.

Proof:

- Let $L[G]$ be the forcing extension from the Main Theorem and let N be the symmetric submodel.
- The DC-Scheme fails in $H^N_{\omega_1}$ (hereditarily countable sets). $\square$

Reflection Scheme

Reflection Scheme:

"Every formula with parameters is reflected by a transitive set."

A scheme consisting of assertions for every formula $\varphi(x)$ and set a:

> There is a transitive model M, with $a \in M$, which reflects V with respect to $\varphi(x)$.

Theorem: ZFC proves the Reflection Scheme.

Proof: Arbitrarily high rank initial segments V_α reflect $\varphi(x)$ $\square$.

Theorem: Over ZFC^-, the Reflection Scheme is equivalent to the assertion that there is a class forcing notion which forces Global Choice without adding sets.

Theorem: (G., Johnstone, Hamkins) Over ZFC^-, the Reflection Scheme is equivalent to the DC-Scheme.

Theorem: (G., Friedman, Kanovei) The Reflection Scheme can fail over a model of ZFC^-.

Global Choice: There exists a class well-ordering of the universe.

Weak Reflection Scheme

Weak Reflection Scheme (Freund)

"Reflection Scheme for sentences".

A scheme of assertions for every sentence φ,

If φ holds, then there is a transitive model in which φ holds.

Theorem: The Weak Reflection Scheme holds in the model $H_{\omega_1}^N$, where N is the symmetric submodel from the proof of the Main Theorem.

Proof:

- Fix a sentence φ such that $H_{\omega_1}^N \models \varphi$.
- The forcing extension $L[G]$ can construct a countable transitive model m such that $m \models \varphi$.
- A countable transitive model can be coded by a real.
- The existence of m is a Σ^1_2-assertion and therefore absolute to N by Shoenfield Absoluteness. $\square$

Questions

Question: (Freund) Does ZFC^- imply the Weak Reflection Scheme?

Because of Shoenfield Absoluteness, our strategy cannot answer this question.

Question: Can the Reflection Scheme fail in a model of ZFC^- with uncountable cardinals?

Thank you!