A MITCHELL-LIKE ORDER FOR RAMSEY AND RAMSEY-LIKE CARDINALS

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Abstract. Smallish large cardinals κ are often characterized by the existence of a collection of filters on κ, each of which is an ultrafilter on the subsets of κ of some transitive ZFC^--model of size κ. We introduce a Mitchell-like order for Ramsey and Ramsey-like cardinals, ordering such collections of small filters. We show that the Mitchell-like order and the resulting notion of rank have all the desirable properties of the Mitchell order on normal measures on a measurable cardinal. The Mitchell-like order behaves robustly with respect to forcing constructions. We show that extensions with the cover and approximation properties cannot increase the rank of a Ramsey or Ramsey-like cardinal. We use the results about extensions with the cover and approximation properties together with recently developed techniques about soft killing of large-cardinal degrees by forcing to softly kill the ranks of Ramsey and Ramsey-like cardinals.

1. Introduction

Mitchell introduced the *Mitchell order* on normal measures on a measurable cardinal κ in [Mit74], where he defined that $U \lhd W$ for two normal measures U and W on κ whenever $U \in \mathrm{Ult}(V, W)$, the (transitive collapse of the) ultrapower of the universe V by W. After observing that $\lhd$ is well-founded, we can define the ordinal rank $o(U)$ of a normal measure and define $o(\kappa)$, the Mitchell rank of κ, to be the strict supremum of $o(U)$ over all normal measures U on κ. The Mitchell rank of κ tells us to what extent measurability is reflected below κ. Mitchell used the Mitchell order to study coherent sequences of normal measures, which allowed him to generalize Kunen's construction of $L[U]$ to canonical inner models with many measures (cf. [Mit74]). The Mitchell rank of a measurable cardinal has also proved instrumental in calibrating the consistency strength of set theoretic assertions. Gitik [Git93] showed, for instance, that the consistency strength of the GCH failing at a measurable cardinal is a measurable cardinal κ with $o(\kappa) = \kappa^{++}$. The notion of Mitchell order generalizes to extenders, where it has played a role in constructions of core models. In this generality, however,

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it loses some of its attractive properties, such as transitivity. Similarly, its well-foundedness becomes harder and harder to establish. Steel and Neeman showed that the Mitchell order on extenders is well-founded for all downward closed extenders of type below rank-to-rank, at which point it becomes ill-founded (cf. [Ste93] and [Nee04]).

In this article, we introduce a Mitchell-like order for Ramsey and Ramsey-like cardinals. Although we tend to associate smaller large cardinals κ with combinatorial definitions, many of them have characterizations in terms of elementary embeddings. The domains of these embeddings are transitive structures of size κ, possibly with additional properties, and the embeddings are ultrapower embeddings by mini-measures or mini-extenders that apply only to the κ-sized domain of the embedding. In most interesting cases, the mini-ultrafilter or extender is external to M, but we can still form the ultrapower by using functions on κ that are elements of M. A prototypical characterization of a smaller large cardinal κ states (see section 2 for all undefined terms) that every $A \subseteq \kappa$ is an element of a weak κ-model M (with additional requirements) for which there is an M-ultrafilter on κ (with additional requirements). The additional requirements on M and the M-ultrafilter are dictated by the large cardinal property. The simplest such characterization belongs to weakly compact cardinals, where the requirement on the M-ultrafilter is the least stringent, namely that the ultrapower of M is well-founded.

Given a large-cardinal property $\mathscr{P}$ with an embedding characterization as discussed above (such as weak compactness, Ramseyness, etc.), let us say that an M-ultrafilter is a *$\mathscr{P}$-measure* if it, together with M, witnesses $\mathscr{P}$ and that a $\mathscr{P}$-measure is *A-good* for some $A \subseteq \kappa$ if $A \in M$. To avoid having to specify which model M we associate to a given $\mathscr{P}$-measure U, we will always associate it with a fixed M_U (see Definition 2.8). Let us say that a collection $\mathcal{U}$ of $\mathscr{P}$- measures is a *witness* for $\mathscr{P}$ if for every $A \subseteq \kappa$, it contains some A-good $\mathscr{P}$-measure. So while a normal measure on κ witnesses the measurability of κ, a witness collection of $\mathscr{P}$-measures is precisely what witnesses $\mathscr{P}$ for one of these smaller large cardinals. This suggests that a reasonable Mitchell-like order should not be comparing the tiny $\mathscr{P}$-measures, but rather witness collections of $\mathscr{P}$-measures in a way that ensures that the corresponding rank $o_{\mathscr{P}}(\kappa)$ of κ measures the extent to which $\mathscr{P}$ is reflected below κ. We will call this order the M-order in honor of Mitchell.

Definition 1.1 (M-order)**.** Suppose that κ has a large-cardinal property $\mathscr{P}$ with a suitable embedding characterization. Given two witness collections $\mathcal{U}$ and $\mathcal{W}$ of $\mathscr{P}$-measures, we define that $\mathcal{U} \lhd \mathcal{W}$ if for every $W \in \mathcal{W}$ and $A \subseteq \kappa$ in the ultrapower N_W of M_W by W, there is an A-good $U \in \mathcal{U} \cap N_W$ such that N_W agrees that U is an A-good $\mathscr{P}$-measure on κ.

In other words, the elements of $\mathcal{U}$ should witness that κ retains the property $\mathscr{P}$ in the ultrapowers by the elements of $\mathcal{W}$. It is tempting to say that $\mathcal{U}$ itself should witness $\mathscr{P}$ in those ultrapowers, but note that $\mathcal{U}$ is too large to be an element of a weak κ-model.

Mitchell proved that Ramsey cardinals have an embedding characterization and Gitman used generalizations of it to define the Ramsey-like cardinals: *α-iterable*, *strongly Ramsey*, and *super Ramsey cardinals* (cf. [Mit79] and [Git11]). We will show that the M-order and the corresponding notion of M-rank for Ramsey and Ramsey-like cardinals share all the desirable features of the Mitchell order on normal measures on a measurable cardinal. For example, the order is transitive and well-founded. Note that since an ultrapower of a weak κ-model has size at most κ, the M-rank of a large cardinal κ of this type can be at most κ^+, in contrast with the upper bound of $(2^\kappa)^+$ in the case of the usual Mitchell rank for a measurable cardinal.

Theorem 1.2. *Suppose $\mathcal{U}$ is a witness collection of $\mathscr{P}$-measures, where $\mathscr{P}$ is Ramsey or Ramsey-like, such that $o_{\mathscr{P}}(\mathcal{U}) \geq \alpha$. Then:*

(1) *For every $U \in \mathcal{U}$, the ultrapower N_U of M_U by U satisfies $o_{\mathscr{P}}(\kappa) \geq \alpha$.*
(2) *There is a witness collection $\mathcal{W}$ with $o_{\mathscr{P}}(\mathcal{W}) = \alpha$ such that $N_W \models o_{\mathscr{P}}(\kappa) = \alpha$ for all $W \in \mathcal{W}$.*

We should not expect an analogue of Theorem 1.2 (1) with equality because we are now dealing with collections of measures instead of a single measure and so Theorem 1.2 (2) is the best possible result.

Theorem 1.3. *Any strongly Ramsey cardinal κ has the maximum Ramsey M-rank $o_{\mathrm{Ram}}(\kappa) = \kappa^+$, any super Ramsey cardinal κ has the maximum strongly Ramsey M-rank $o_{\mathrm{stRam}}(\kappa) = \kappa^+$, and any measurable cardinal κ has the maximum super Ramsey M-rank $o_{\mathrm{supRam}}(\kappa) = \kappa^+$.*

We will show that the new Mitchell order behaves robustly with respect to forcing constructions. We prove that extensions with cover and approximation properties cannot create new Ramsey or Ramsey-like cardinals or increase their M-rank. Hamkins [Ham03] showed that most large cardinals cannot be created in extensions with the cover and approximation properties

and we provide several modifications of his techniques to the embeddings characterizing Ramsey and Ramsey-like cardinals. This result is of independent interest since it was not previously known whether Ramsey cardinals can be created in extensions with the cover and approximation properties.

Theorem 1.4. *If $V \subseteq V'$ has the δ-cover and δ-approximation properties for some regular cardinal $\delta < \kappa$ of V', then $o^V_{\mathscr{P}}(\kappa) \geq o^{V'}_{\mathscr{P}}(\kappa)$, where $\mathscr{P}$ is strongly or super Ramsey, and if we additionally assume that $V^\omega \subseteq V$ in V', then $o^V_{\mathrm{Ram}}(\kappa) \geq o^{V'}_{\mathrm{Ram}}(\kappa)$.*

Using the results about extensions with the approximation and cover properties together with new techniques recently developed in Carmody's dissertation [Car15] about softly killing degrees of large cardinals with forcing, we show how to softly kill the M-rank of a Ramsey or Ramsey-like cardinal by forcing.

Theorem 1.5. *If κ has $o_{\mathscr{P}}(\kappa) = \alpha$, where $\mathscr{P}$ is Ramsey or Ramsey-like and $\beta < \alpha$, then there is a cofinality-preserving forcing extension in which $o_{\mathscr{P}}(\kappa) = \beta$.*

Although the general framework of the M-order we have sketched here applies to many smallish large cardinals, we focus in this paper on its application to Ramsey, strongly Ramsey and super Ramsey cardinals. Other instances of it definitely warrant further research.

2. Preliminaries

2.1. Ramsey and Ramsey-like cardinals.

Definition 2.1. Let κ be a cardinal. A transitive model M of ZFC^- of size κ and having $\kappa \in M$ is called a *weak κ-model*. If, in addition, we also have $M^{<\kappa} \subseteq M$, we call M a *κ-model*.

Definition 2.2. Suppose M is a transitive model of ZFC^- and κ is a cardinal of M. An *M-ultrafilter* is a family $U \subseteq \mathcal{P}(\kappa)^M$ such that the structure $\langle M, \in, U\rangle$, with a predicate for U, satisfies that U is a normal ultrafilter on κ.[1]

An M-ultrafilter U is *δ-intersecting*, for some cardinal δ, if the intersection of fewer than δ many sets from U is always nonempty.[2]

[1]The notation $\mathcal{P}(\kappa)^M$ is meant to denote $\mathcal{P}(\kappa) \cap M$, whether or not this is actually an element of M.

[2]In the literature such M-ultrafilters are often called δ-complete which we find confusing because δ-complete ultrafilters are supposed to have the property that the intersection of fewer than δ-many sets in the ultrafilter is itself in the ultrafilter. But in the situation of M-ultrafilters, the intersection may not even be an element of M.

We will always assume that an ultrafilter on a cardinal κ is uniform. It follows from this assumption, and normality, that an M-ultrafilter is κ-complete for sequences in M.

As discussed in the introduction, many large cardinals κ below a measurable cardinal have the prototypical characterization, where for every $A \subseteq \kappa$, there is a weak κ-model M, with some additional properties, containing A for which there is an M-ultrafilter on κ, with some additional properties, where the additional properties are what distinguishes the different large cardinal properties. We can form the ultrapower of a model M by an M-ultrafilter using functions on κ that are elements of M. It is not difficult to verify that the Łoś Theorem continues to hold for these ultrapowers (see, for example, [Kan03] section 19). However, since an M-ultrafilter is only κ-complete for sequences from M and M might be missing countable sequences, its ultrapower need not be well-founded. Standard arguments show that a sufficient condition for well-foundedness is that the M-ultrafilter is ω_1-intersecting, but as we will discuss later, in contrast with the case of countably complete ultrafilters, this condition is not necessary.

Many set theoretic constructions use iterated ultrapowers by a measure on a measurable cardinal. If U is an ultrafilter on some set, then the ultrapower (of V) construction with it can be iterated along the ordinals by taking the ultrapower by the image of the previous stage's ultrafilter at successor stages and direct limits at limit stages. Gaifman showed that if an internal ultrafilter is countably complete, which is equivalent to having a well-founded ultrapower, then all its iterated ultrapowers are well-founded (cf. [Gai74]).

With M-ultrafilters, we run into immediate difficulties when trying to carry out the iterated ultrapower construction. To be able to define the successor stage ultrafilters, we must require that the M-ultrafilter is at least partially internal to M, a property that is captured by the notion of weak amenability.

Definition 2.3. An M-ultrafilter U on κ is *weakly amenable* if for every $X \in M$ of size at most κ in M, the intersection $X \cap U$ is an element of M.[3]

Although weak amenability allows us to define all the iterated ultrapowers, it does not have any bearing on their well-foundedness. Kunen showed that being ω_1-intersecting is sufficient for well-foundedness (cf. [Kun70]),

[3]The property is a weakening of the usual definition of amenability because we restrict to X of size at most κ in M.

but is not necessary. Unlike measures on κ, where either all the iterated ultrapowers are well-founded or none of them are, we will see below that it is consistent to have M-ultrafilters with exactly α-many well-founded iterated ultrapowers for any countable ordinal α.[4]

Recall that κ is weakly compact if and only if $\kappa^{<\kappa} = \kappa$ and every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is an M-ultrafilter on κ with a well-founded ultrapower. This characterization can be strengthened in a number of significant ways while still yielding the same large cardinal notion. For instance, we can assume that M is a κ-model that is elementary in H_{κ^+} and hence reflects V to a certain extent. In fact, we can assume that *every* weak κ-model M has an M-ultrafilter with a well-founded ultrapower.

Definition 2.4. A cardinal κ is *Ramsey* if $\kappa \to (\kappa)_2^{<\omega}$; that is, if for any function $F\colon [\kappa]^{<\omega} \to 2$ there is a set $H \subseteq \kappa$ of size κ such that $F \restriction [H]^n$ is constant for each n.

For our purposes, a description of Ramsey cardinals in terms of elementary embeddings (or certain filters) will be more useful. Mitchell found exactly such a characterization in [Mit79].

Theorem 2.5 (Mitchell)**.** *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is a weakly amenable ω_1-intersecting M-ultrafilter on κ.*

Surprisingly, we cannot strengthen the characterization of Ramsey cardinals in the same fashion as we did with weakly compact cardinals above, without increasing the required consistency strength.

Definition 2.6. A cardinal κ is called *ω-closed Ramsey* if it is Ramsey and the model M in the characterization above may be taken to be closed under countable sequences.

The cardinal κ is *strongly Ramsey* if M may be taken to be closed under $<\kappa$-sequences (i.e. be a κ-model).

The cardinal κ is *super Ramsey* if M may be taken to be a κ-model which is furthermore an elementary submodel of H_{κ^+}.

The strongly and super Ramsey cardinals were introduced by Gitman, and she showed that these notions fit neatly above Ramsey cardinals. In

[4]It follows from Gaifman's arguments in [Gai74] for ultrapowers by a measure that an M-ultrafilter with ω_1-many well-founded iterated ultrapowers, must have all of its iterated ultrapowers well-founded.

particular, strongly Ramsey cardinals are stationary limits of Ramsey cardinals[5], and super Ramsey cardinals are stationary limits of strongly Ramsey cardinals (but weaker than a measurable cardinal). Strongly Ramsey cardinals can also be viewed as quite strong because they are limits of the completely Ramsey cardinals defined by Feng in [Fen90]. Surprisingly, assuming that *every* weak κ-model M has a weakly amenable ω_1-intersecting M-ultrafilter turns out to be inconsistent. For details, see [Git11].

The requirement that the M-ultrafilters are weakly amenable already takes us well beyond weak compactness. If every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is a weakly amenable M-ultrafilter with a well-founded ultrapower, then κ is a stationary limit of completely ineffable cardinals, which sit atop a hierarchy of ineffability. The following is a very useful characterization of weak amenability.

Fact 2.7. *If U is an M-ultrafilter on κ and $j\colon M \to N$ is the ultrapower by U, then U is weakly amenable if and only if $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$. Moreover, if $j : M \to N$ is any embedding with critical point κ and $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$, then the M-ultrafilter U derived from j in the usual way, is weakly amenable.*

We can stratify weakly amenable M-ultrafilters by degrees of iterability. Let us say that an M-ultrafilter is *α-iterable* if it has α-many well-founded iterated ultrapowers and that it is *iterable* if it is ω_1-iterable. Gitman defined that a cardinal κ is α-iterable (for $1 \leq \alpha \leq \omega_1$) if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is a weakly amenable α-iterable M-ultrafilter. Gitman and Welch [GW11] showed that the α-iterable cardinals form a hierarchy of strength, and Sharpe and Welch [SW11] showed that an ω_1-Erdős cardinal is a limit of ω_1-iterable cardinals.

As promised, we now describe how to associate to a given filter a distinguished family of sets that will serve as the domain for the ultrapower.

Definition 2.8. If U is a filter on some cardinal κ, let M_U be the collection of those sets such that their Mostowski code or the complement of that code is in U.

Suppose that M is a weak κ-model and U is an M-ultrafilter. Consider the submodel $\overline{M} = H^M_{\kappa^+}$ consisting of all sets that have hereditary size at most κ in M. Clearly $\overline{M}$ is itself a weak κ-model and if M was a κ-model then $\overline{M}$ is as well. Also, U is an $\overline{M}$-ultrafilter and it retains all other relevant properties with respect to $\overline{M}$ that it had with respect to M, such as being

[5]Note that a weakly amenable M-ultrafilter for a κ-model M is automatically ω_1-intersecting.

weakly amenable, α-iterable, or ω_1-intersecting. Now observe that $\overline{M} = M_U$ consists of precisely the sets whose Mostowski codes are measured by U. Thus, if U is an M-ultrafilter for a weak κ-model M, then M_U is a model of ZFC^- with the largest cardinal κ, with respect to which U retains all the relevant properties it originally had for the model M, and which can be recoved from U in any model of a sufficient fragment of set theory. If the ultrapower of M_U by U is well-founded, we will denote its Mostowski collapse by N_U. Note that if U is weakly amenable (with a well-founded ultrapower), then $M_U = H^{N_U}_{\kappa^+}$ is an element of N_U. In our arguments, for technical reasons which will become apparent later on, we will only consider M_U-ultrafilters U for which $V_\kappa \in M_U$, so that the ultrapower N_U thinks that $V_{j(\kappa)}$ exists.

It turns out that if U is an iterable (or, in particular, ω_1-intersecting) M_U-ultrafilter, then U also codes a weak κ-model M^*_U of full ZFC so that U is also an iterable M^*_U-ultrafilter. Specifically, we can take $M^*_U = V^{N_U}_{j(\kappa)}$, where $j : M_U \to N_U$ is the ultrapower map of M_U by U. The ultrapower map of M^*_U by U is $j^* : M^*_U \to N^*_U$, the restriction of the ultrapower of N_U by U. If M_U was a κ-model, then so is M^*_U. The embedding j^* has several useful properties, such as $M_U = H^{M^*_U}_{\kappa^+}$, that $M^*_U = V^{N^*_U}_{j^*(\kappa)}$ is in N^*_U, and $M^*_U \prec N^*_U$. The same construction cannot be carried out with a partially iterable M_U-ultrafilter because the iterability of U decreases when you pass to the model M^*_U. Indeed, assuming that there are α-iterable ultrafilters for models of ZFC produces a stronger notion than an α-iterable cardinal.

For forcing constructions with Ramsey cardinals, which we discuss below, we will need to make some additional assumptions about the weak κ-model M.

Definition 2.9. A weak κ-model M is *ω-special* if it is the union of an elementary chain of (not necessarily transitive) substructures

$$\kappa \in M_0 \prec M_1 \prec \cdots \prec M_n \prec \cdots$$

for $n < \omega$ such that each $M_n \in M$ and $|M_n|^M = \kappa$.

The ultrapower of an ω-special weak κ-model M by a weakly amenable M-ultrafilter on κ is ω-special as witnessed by the sequence $\langle X_n \mid n < \omega\rangle$, where

$$X_n = \{j(f)(\kappa) \mid f : \kappa \to M_n, f \in M_n\},$$

and if $M = M_U$, then M^*_U is ω-special as well (see Lemmas 2.7 and 2.9 of [CG15]).

Lemma 2.10. *Suppose κ is Ramsey. Then every $A \subseteq \kappa$ is contained in an ω-special weak κ-model M for which there is a weakly amenable ω_1-intersecting M-ultrafilter on κ.*

Proof. Fix $A \subseteq \kappa$ and choose some weak κ-model $\overline{M}$ containing A for which there is a weakly amenable ω_1-intersecting $\overline{M}$-ultrafilter $\overline{U}$ on κ. Let $\overline{N}$ be the ultrapower of $\overline{M}$ by $\overline{U}$. We can assume that κ is the largest cardinal of $\overline{M}$, and therefore $\overline{M} = H^{\overline{N}}_{\kappa^+}$ is an element of $\overline{N}$. Working in $\overline{N}$, let M_0 be any transitive elementary submodel of H_{κ^+} of size κ with $A \in M_0$. Since $\overline{N}$ and $\overline{M}$ have the same subsets of κ, M_0 and hence $U_0 = M_0 \cap \overline{U}$ are in $\overline{M}$. So working in $\overline{N}$, we can choose a transitive $M_1 \prec H_{\kappa^+}$ of size κ with $M_0, U_0 \in M_1$. Continuing in this fashion, we obtain a sequence $\langle (M_n, U_n) \mid n < \omega \rangle$. Let $M = \bigcup_{n<\omega} M_n$ and $U = \bigcup_{n<\omega} U_n$. The model M is ω-special as witnessed by the sequence $\langle M_n \mid n < \omega \rangle$ (the M_n are even transitive) and $U = M \cap \overline{U}$ is ω_1-intersecting and weakly amenable by construction. $\square$

Since M^*_U is ω-special whenever M_U is, it follows that if κ is Ramsey, then every $A \subseteq \kappa$ is contained in an ω-special weak κ-model $M \models \mathrm{ZFC}$ for which there is a weakly amenable ω_1-intersecting M-ultrafilter on κ.

2.2. **Forcing constructions.** Suppose $\mathbb{P} \subseteq V_\kappa$ is a poset and we would like to verify that κ is Ramsey in a forcing extension $V[G]$ by $\mathbb{P}$. Since every $A \subseteq \kappa$ in $V[G]$ has a $\mathbb{P}$-name $\dot{A}$ of hereditary size at most κ in V, $\dot{A}$ together with $\mathbb{P}$ can be put into a weak κ-model M, which ensures that A is in the weak κ-model $M[G]$. Thus, it suffices to show that every ultrapower $j : M \to N$ of a weak κ-model M by a weakly amenable ω_1-intersecting M-ultrafilter can be lifted to $j : M[G] \to N[j(G)]$ so that the lift is the ultrapower by a weakly amenable ω_1-intersecting $M[G]$-ultrafilter in $V[G]$. The *lifting criterion* states that j lifts to $j : M[G] \to N[H]$ with $H = j(G)$ if and only if $j \text{ "} G \subseteq H$. In this setting, when constructing a generic filter, we usually work with a κ-model M and a poset $\mathbb{P}$ that is $<\kappa$-closed in M. This suffices for the existence of an M-generic filter for $\mathbb{P}$. Instead of this approach, which does not apply to weak κ-models, we will use the following *diagonalization criterion*, introduced in [GJ].

Lemma 2.11 (Diagonalization criterion, [GJ])**.** *If M is an ω-special weak κ-model and $\mathbb{P}$ is a $\leq\kappa$-distributive poset in M, then there is an M-generic filter for $\mathbb{P}$.*

The lift of an ultrapower embedding is always an ultrapower embedding by the ultrafilter W obtained from the lift and we will usually use a direct

argument to verify that $M[G]$ and $N[j(G)]$ have the same subsets of κ (which demonstrates weak amenability). To show that W is ω_1-intersecting, we have the following lemma.

Lemma 2.12 (Gitman, Johnstone [GJ])**.** *Suppose that M is a weak κ-model and $j : M \to N$ is the ultrapower by an ω_1-intersecting M-ultrafilter on κ. Suppose further that $\mathbb{P} \in M$ is a countably closed forcing notion and $G \subseteq \mathbb{P}$ is M-generic. If the ultrapower map j lifts to $j : M[G] \to N[j(G)]$, then the lift j is the ultrapower map by an ω_1-intersecting $M[G]$-ultrafilter in $V[G]$.*

3. The M-order for Ramsey and Ramsey-like cardinals

Let us start by recalling and making more precise the definitions we made in the introduction.

Definition 3.1. Let κ be a cardinal. Say that a filter U on κ of size κ is a *small universe measure* if M_U is a weak κ-model such that $V_\kappa \in M_U$ and U is an M_U-ultrafilter.

We will write N_U for the Mostowski collapse of the ultrapower of M_U by U, provided that the ultrapower is well-founded.

Definition 3.2. Let κ be a cardinal and U a small universe measure on κ. We say that U is a *Ramsey measure* if it is weakly amenable (to M_U) and ω_1-intersecting; a *strong Ramsey measure* if it is a Ramsey measure and M_U is a κ-model; and a *super Ramsey measure* if it is a strong Ramsey measure and $M_U \prec H_{\kappa^+}$.

We will carry out all the arguments below for Ramsey cardinals since they are the most complicated, pointing out at the end that analogous or simpler arguments work for strongly Ramsey and super Ramsey cardinals. The interested reader can note along the way where the arguments adapt to other smallish large cardinals, which we do not discuss here.

A first attempt at defining a Mitchell-like order for Ramsey cardinals might be to consider ordering Ramsey measures on a fixed Ramsey κ analogously to the Mitchell order on normal measures on a measurable cardinal.

Definition 3.3. Given two Ramsey measures U and W on a cardinal κ, define that $U \lhd W$ if $U \in N_W$.

Lemma 3.4. *The relation $\lhd$ on Ramsey measures on a cardinal κ is transitive and well-founded.*

Proof. Transitivity is straightforward. To see that the relation is also well-founded, notice that $U \lhd W$ implies $j_U(\kappa) < j_W(\kappa)$, where j_U and j_W are the ultrapower maps with respect to U and W. This is so because if $U \in N_W$ then also $j_U, N_U \in N_W$ and $N_W \models |U| = |N_U|$. Furthermore, $|U|^{N_W} \leq (2^\kappa)^{N_W} < j_W(\kappa)$, because $j_W(\kappa)$ is inaccessible in N_W. Since $j_U(\kappa)$ is an element of N_U, a transitive substructure of N_W, we must have $j_U(\kappa) \leq |N_U|^{N_W} < j_W(\kappa)$. The well-foundedness of $\lhd$ now follows since an infinite decreasing chain of Ramsey measures would yield an infinite decreasing chain of ordinals. □

We should also point out that no Ramsey measure U can have more than κ predecessors in the order $\lhd$, since the ultrapower N_U has cardinality κ.

The order $\lhd$ on Ramsey measures is an interesting object in its own right, but it is not useful for defining degrees of Ramsey cardinals with the intention to capture the extent to which Ramseyness is reflected below κ because if κ is Ramsey then there are already Ramsey measures of all possible ranks $\alpha < \kappa^+$.

Lemma 3.5. *If κ is Ramsey and $\alpha < \kappa^+$, then there is a Ramsey measure on κ of rank at least α in the $\lhd$-order.*

Proof. Suppose by way of induction that for all $\beta < \alpha$, there is a Ramsey measure U_β on κ whose rank in the $\lhd$-order is at least β. Let U be some Ramsey measure on κ such that $\{U_\beta \mid \beta < \alpha\} \subseteq M_U$, which is possible since this set has hereditary size κ. Clearly the rank of U in $\lhd$ is at least α. □

The same analysis holds for strongly Ramsey and super Ramsey measures. It is to be expected that ordering the small universe measures is not the right analogue of the Mitchell order for smaller large cardinals because such a cardinal is characterized by the existence of many such measures and not just one. This brings us back to the definition of the M-order on collections of small universe measures, which we restate here again in full generality before going to back concrete arguments for Ramsey cardinals. In the following let $\mathscr{P}$ be a suitable large cardinal property, such as Ramsey, strongly Ramsey, etc., which is characterized by the existence of small universe measures.

Definition 3.6. A $\mathscr{P}$-measure U on κ is *A-good* for some $A \subseteq \kappa$ if $A \in M_U$. A collection of $\mathscr{P}$-measures on κ is a *witness collection* if it contains at least one A-good $\mathscr{P}$-measure for each $A \subseteq \kappa$.

In other words, κ has the property $\mathscr{P}$ exactly when there is a witness collection of $\mathscr{P}$-measures on κ.

Definition 3.7. (M-order) Suppose that κ is a large cardinal with property $\mathscr{P}$ having a suitable embedding characterization. Given two witness collections $\mathcal{U}$ and $\mathcal{W}$ of $\mathscr{P}$-measures, we define that $\mathcal{U} \lhd \mathcal{W}$ whenever, given $W \in \mathcal{W}$ and $A \subseteq \kappa$ in the ultrapower N_W of M_W by W, there is an A-good $U \in \mathcal{U} \cap N_W$ such that N_W agrees that U is an A-good $\mathscr{P}$-measure on κ.

Lemma 3.8. *The M-order on witness collections of Ramsey measures on a cardinal κ is transitive and well-founded.*

Proof. First, we show transitivity. Suppose that $\mathcal{U}$, $\mathcal{W}$, $\mathcal{Z}$ are witness collections of Ramsey measures on κ such that $\mathcal{U} \lhd \mathcal{W}$ and $\mathcal{W} \lhd \mathcal{Z}$. To see that $\mathcal{U} \lhd \mathcal{Z}$, take $Z \in \mathcal{Z}$ and $A \in N_Z$. Then, since $\mathcal{W} \lhd \mathcal{Z}$, there is $W \in N_Z \cap \mathcal{W}$ with $A \in N_W$, but then, since $\mathcal{U} \lhd \mathcal{W}$, there is $U \in N_W \cap \mathcal{U} \subseteq N_Z \cap \mathcal{U}$ with $A \in N_U$. Thus, $\mathcal{U} \lhd \mathcal{Z}$.

Next, suppose towards a contradiction that $\lhd$ is ill-founded for witness collections of Ramsey measures on κ and fix a descending sequence

$$\mathcal{U}_0 \rhd \mathcal{U}_1 \rhd \cdots \rhd \mathcal{U}_n \rhd \cdots$$

of witness collections. Let U_0 be any element of $\mathcal{U}_0$. Since $\mathcal{U}_1 \lhd \mathcal{U}_0$, then N_{U_0} has some element U_1 of $\mathcal{U}_1$, and so $U_1 \lhd U_0$ in the ordering on Ramsey measures. Continuing in the same manner, we obtain a descending sequence

$$U_0 \rhd U_1 \rhd \cdots \rhd U_n \rhd \cdots$$

in the $\lhd$-order on Ramsey measures, which is impossible by Lemma 3.4. $\square$

The lemma implies that the M-order on witness collections of Ramsey measures gives rise to a ranking function.

Definition 3.9. Let $\mathcal{U}$ be a witness collection of Ramsey measures on κ. We define $o_{\mathrm{Ram}}(\mathcal{U})$ to be its rank in the M-order and let

$$o_{\mathrm{Ram}}(\kappa) = \{o_{\mathrm{Ram}}(\mathcal{U}) \mid \mathcal{U} \text{ is a witness collection of Ramsey measures on } \kappa\}.$$

We define ranks for strongly and super Ramsey cardinals in similar fashion.

The defining property of the classical Mitchell order is that a normal measure U on κ has rank α if and only if $\mathrm{Ult}(V, U)$ satisfies $o(\kappa) = \alpha$. The analogous result for the M-order on witness collections of Ramsey measures on κ will be that $o_{\mathrm{Ram}}(\mathcal{U}) \geq \alpha$ if and only if $N_U \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$ for every $U \in \mathcal{U}$. It is not feasible to obtain equality because a witness collection $\mathcal{U}$ of rank α can easily have elements U with $N_U \models o_{\mathrm{Ram}}(\kappa) > \alpha$. Still, we will be able to show that "well-behaved" collections always exist: if $\alpha < o_{\mathrm{Ram}}(\kappa)$, then there is some witness collection $\mathcal{W}$ with $o_{\mathrm{Ram}}(\mathcal{W}) = \alpha$ such that $N_W \models o_{\mathrm{Ram}}(\kappa) = \alpha$ for every $W \in \mathcal{W}$.

A subtle issue arises when trying to prove these results in the case of Ramsey cardinals (but not for strongly Ramsey or super Ramsey cardinals): our notion of Ramsey measure is not upward absolute for weak κ-models. Specifically, a weak κ-model might believe that a small universe measure is a Ramsey measure, but a larger weak κ-model could realize that the measure is in fact not ω_1-intersecting. This problem does not arise with strongly and super Ramsey cardinals because, clearly, κ-models will be correct about filters being ω_1-intersecting. To prove the desired result we will temporarily use a stronger notion of Ramsey measure, which will be absolute for transitive ZFC^--models, and show that the two notions give the same M-rank.

Definition 3.10. Let U be a small universe measure on a regular uncountable cardinal κ. We say that U is a *certified Ramsey measure* if it is weakly amenable and there is some unbounded $I \subseteq \kappa$ such that $X \in M_U$ is in U if and only if X contains a tail of I. In this case we say that I *certifies* U.

Clearly, every certified Ramsey measure is a Ramsey measure because it is even κ-intersecting (every sequence of $<\kappa$-many sets in U has a non-empty intersection) and certified Ramsey measures have the advantage of being absolute between transitive models of set theory. It turns out that certified Ramsey measures always exist on Ramsey cardinals.

Fact 3.11. *If κ is a Ramsey cardinal then there is, for each $A \subseteq \kappa$, an A-good certified Ramsey measure on κ.*

For a proof of this fact see [Dod82], or [Git07] for a more detailed exposition. Briefly, the proof uses the notion of a *good set of indiscernibles* for a structure $L_\kappa[A]$ with $A \subseteq \kappa$ and shows that if κ is Ramsey, then for every $A \subseteq \kappa$, there is a good set I of indiscernibles for $L_\kappa[A]$ of size κ. The indiscernibles in I are then used to construct a weak κ-model M (with the largest cardinal κ) and a weakly amenable ω_1-intersecting M-ultrafilter that is certified by I.

Definition 3.12. Given a witness collection $\mathcal{U}$ of certified Ramsey measures on κ, let $o^*_{\mathrm{Ram}}(\mathcal{U})$ denote the rank of $\mathcal{U}$ in the M-order restricted to witness collections of certified Ramsey measures, and let $o^*_{\mathrm{Ram}}(\kappa)$ be the strict supremum of the ranks of all such $\mathcal{U}$.

For inductive arguments about the M-rank of Ramsey or Ramsey-like cardinals, we will often need to know that the results hold not just in V, but more generally in transitive set models of ZFC^- that know enough about the cardinal.

Definition 3.13. Let $\mathcal{M}$ be a transitive set model of ZFC^- and κ a cardinal in $\mathcal{M}$. We say that $\mathcal{M}$ is *practical* for κ if $V_{\kappa+3}^{\mathcal{M}}$ exists.

Working with a practical model ensures that the model can put together all the relevant witness collections on κ in order to rank them. Especially relevant in the following arguments will be the fact that, given a Ramsey measure U on κ, the ultrapower N_U is practical for κ and thus has a sensible notion of M-rank for κ. We require that practical models be sets so that we may quantify over them, and the following results will talk about the properties of the M-order in practical models. This suffices to describe these properties even in class-sized models (such as V), since they are reflected in structures of the form H_{λ^+} for some cardinal λ much bigger than κ, and these structures are practical for κ.

Lemma 3.14. *If α is an ordinal and $\mathcal{M}$ is practical for κ, then, in $\mathcal{M}$, a witness collection $\mathcal{U}$ of certified Ramsey measures on κ has $o^*_{\mathrm{Ram}}(\mathcal{U}) \geq \alpha$ if and only if $N_U \models o^*_{\mathrm{Ram}}(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$.*

Proof. We will argue by induction on α. The case $\alpha = 0$ is trivial. So suppose that the statement is true for all $0 \leq \beta < \alpha$. Fix an $\mathcal{M}$ practical for κ and work in $\mathcal{M}$.

In one direction, fix a witness collection $\mathcal{U}$ of certified measures on κ with $o^*_{\mathrm{Ram}}(\mathcal{U}) \geq \alpha$. Let $U \in \mathcal{U}$. We must show that N_U has witness collections of certified Ramsey measures of all ranks $\beta < \alpha$. Fix $\beta < \alpha$. Since $o^*_{\mathrm{Ram}}(\mathcal{U}) \geq \alpha$, there must be some witness collection $\mathcal{W} \lhd \mathcal{U}$ of certified Ramsey measures with $o^*_{\mathrm{Ram}}(\mathcal{W}) = \beta$. By the inductive hypothesis applied to $\mathcal{M}$ we have $N_W \models o^*_{\mathrm{Ram}}(\kappa) \geq \beta$ for all $W \in \mathcal{W}$. Therefore, since $\mathcal{W} \lhd \mathcal{U}$, there is for every $A \subseteq \kappa$ in N_U some A-good certified Ramsey measure $W \in N_U$ with $N_W \models o^*_{\mathrm{Ram}}(\kappa) \geq \beta$ and so N_U, by collecting these together, has a witness collection $\overline{\mathcal{W}}$ of certified Ramsey measures such that $N_W \models o^*_{\mathrm{Ram}}(\kappa) \geq \beta$ holds for all $W \in \overline{\mathcal{W}}$. But then, by applying our inductive hypothesis to N_U, we have that $o^*_{\mathrm{Ram}}(\overline{\mathcal{W}}) \geq \beta$ in N_U. This completes the proof in one direction.

In the other direction, suppose that $\mathcal{U}$ is a witness collection of certified Ramsey measures such that $N_U \models o^*_{\mathrm{Ram}}(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$. We must show that for all $\beta < \alpha$, there is a witness collection $\mathcal{W}$ of certified Ramsey measures with $\mathcal{W} \lhd \mathcal{U}$ and $o^*_{\mathrm{Ram}}(\mathcal{W}) \geq \beta$. Fix $\beta < \alpha$. For each $U \in \mathcal{U}$, we can fix some $\mathcal{W}_U$, which N_U thinks is a witness collection of certified Ramsey measures of rank at least β, and let $\mathcal{W} = \bigcup_{U \in \mathcal{U}} \mathcal{W}_U$. Since all our measures are certified, $\mathcal{W}$ is a witness collection of certified Ramsey measures and we

have arranged that $\mathcal{W} \lhd \mathcal{U}$. If $W \in \mathcal{W}$, then $W \in \mathcal{W}_U$ for some $U \in \mathcal{U}$ and therefore, since $N_U \models o^*_{\mathrm{Ram}}(\mathcal{W}_U) \geq \beta$, we get $N_W \models o^*_{\mathrm{Ram}}(\kappa) \geq \beta$ by applying the inductive hypothesis to N_U. Thus, by the inductive hypothesis, $o^*_{\mathrm{Ram}}(\mathcal{W}) \geq \beta$, which establishes this direction. □

Note that the fact that the Ramsey measures we are working with are certified only came into play in the second part of the proof. Essentially, being a Ramsey measure is downward absolute. On the other hand, we would have run into trouble in the second part if we had built the collection $\mathcal{W}$ using only ordinary Ramsey measures, since N_U and $\mathcal{M}$ might disagree on whether a given filter is ω_1-intersecting. This observation will be important when we revisit this proof in Theorem 3.15.

The desired result, which is the same lemma for witness collections of arbitrary Ramsey measures, will follow once we establish that $o^*_{\mathrm{Ram}}(\kappa) = o_{\mathrm{Ram}}(\kappa)$. First, we have to review a few basic facts which we will use now and in later sections.

Observe that if U is a Ramsey measure, then the intersection of any countably many sets in U has size κ because if the intersection was bounded by $\alpha < \kappa$, we could add $\kappa \setminus \alpha$ to the sets being intersected (recall that all tails of κ are in U by assumption) and violate the ω_1-intersecting property. Another useful fact is that for every ordinal $\kappa \leq \alpha < \kappa^+$, if E is a well-ordering of κ in order-type α, then there is a single function $g^E : \kappa \to \kappa$ such that whenever U is an M-ultrafilter with a well-founded ultrapower and $E \in M$, then $g^E \in M$ and $[g^E]_U = \alpha$ in the ultrapower. We call such g^E a *representing function* for α and can define it by simply letting $g^E(\xi)$ be the order-type of $E \restriction \xi \times \xi$.

Theorem 3.15. *Let α be an ordinal and $\mathcal{M}$ practical for κ. Then the following hold in $\mathcal{M}$:*

(1) *a witness collection $\mathcal{U}$ of Ramsey measures on κ has $o_{\mathrm{Ram}}(\mathcal{U}) \geq \alpha$ if and only if $N_U \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$;*
(2) *if there is a witness collection $\mathcal{U}$ of Ramsey measures on κ with $o_{\mathrm{Ram}}(\mathcal{U}) \geq \alpha$, then there is also a witness collection $\mathcal{U}^*$ of certified Ramsey measures with $o_{\mathrm{Ram}}(\mathcal{U}^*) \geq \alpha$.*

Note that part (2) of the lemma talks about the simple M-rank and not the M-rank restricted to witness collections of certified measures.

Proof. We will prove both parts of the statement simultaneously by induction on α. For the base case $\alpha = 0$ part (1) is trivial and part (2) follows by

Fact 3.11. So suppose inductively that the statement holds for all $0 \leq \beta < \alpha$. Fix an $\mathcal{M}$ practical for κ and work in $\mathcal{M}$.

Let us first show that part (1) holds for α by mirroring the proof of Lemma 3.14. The forward direction goes through exactly as in the proof of Lemma 3.14, since, as we noted after that proof, the fact that the Ramsey measures were certified played no part in this particular argument. For the converse, suppose that $\mathcal{U}$ is a witness collection of Ramsey measures such that $N_U \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$. Fix a $\beta < \alpha$. For each $U \in \mathcal{U}$ we can fix $\mathcal{W}_U$ which N_U thinks is a collection of Ramsey measures of rank at least β and let $\mathcal{W} = \bigcup_{U \in \mathcal{U}} \mathcal{W}_U$. We would like to say that $\mathcal{W}$ is a witness collection of Ramsey measures, but this need not be the case if we are working with arbitrary (noncertified) Ramsey measures. Instead, we apply part (2) of the induction hypothesis to each N_U to replace each $\mathcal{W}_U$ with a witness collection $\mathcal{W}_U^*$ of *certified* Ramsey measures satisfying $o_{\mathrm{Ram}}(\mathcal{W}_U^*) \geq \beta$ in N_U. If we now let $\mathcal{W}^* = \bigcup_{U \in \mathcal{U}} \mathcal{W}_U^*$, this actually is a witness collection of Ramsey measures and, again, $\mathcal{W}^* \lhd \mathcal{U}$. The rest of the argument proceeds as before: if $W \in \mathcal{W}^*$, then $W \in \mathcal{W}_U^*$ for some $U \in \mathcal{U}$ and therefore, since $N_U \models o_{\mathrm{Ram}}(\mathcal{W}_U^*) \geq \beta$, we get $N_W \models o_{\mathrm{Ram}}(\kappa) \geq \beta$ by applying part (1) of the induction hypothesis to N_U. Thus, by part (1) of the induction hypothesis again, $o_{\mathrm{Ram}}(\mathcal{W}^*) \geq \beta$. Altogether, this shows that $o_{\mathrm{Ram}}(\mathcal{U}) \geq \alpha$.

Now we move on to show that part (2) holds for α. Suppose that there is a witness collection $\mathcal{U}$ of Ramsey measures on κ with $o_{\mathrm{Ram}}(\mathcal{U}) \geq \alpha$. We need to show that there is a witness collection $\mathcal{U}^*$ of certified Ramsey measures on κ with $o_{\mathrm{Ram}}(\mathcal{U}^*) \geq \alpha$. By what we just argued it follows that $N_U \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$ for all $U \in \mathcal{U}$. The next step is to replace each U with some U^*, where U^* is certified and N_{U^*} also satisfies that $o_{\mathrm{Ram}}(\kappa) \geq \alpha$. For this, we need to look more closely at how a good set I of indiscernibles for $L_\kappa[A]$ is constructed, as mentioned in the discussion after Fact 3.11.

For every $A \subseteq \kappa$, there is an associated club C_A in κ and a regressive function $f_A : [C_A]^{<\omega} \to \kappa$ such that any homogeneous set for f_A is a good set of indiscernibles for $L_\kappa[A]$. The club C_A and function f_A are defined simply enough from A that any transitive model of ZFC^- containing A also contains C_A and f_A (for details, see [Git07], chapter 2). Given an A-good Ramsey measure U, we will find a homogeneous set I of size κ for f_A by showing that for each $n < \omega$, the restriction $f_n : [C_A]^n \to \kappa$ of f_A has a homogeneous set in U and using the ω_1-intersecting property of U. Since U is weakly amenable, we can define the finite product M_U-ultrafilters U^n for $n < \omega$ (where

$U^1 = U$) and, since all iterated ultrapowers of U are well-founded and the ultrapower by U^n is isomorphic to the nth-iterated ultrapower of U, it follows that all ultrapower maps $j_{U^n} : M_U \to N_{U^n}$ are embeddings into transitive models. Standard facts about products of normal ultrafilters also tell us that a set $B \subseteq \kappa^n$ is in U^n if and only if $\langle \kappa, j_U(\kappa), j_{U^2}(\kappa), \ldots, j_{U^{n-1}}(\kappa)\rangle \in j_{U^n}(B)$. Now fix $n < \omega$ and consider f_n. The set $[C_A]^n$ is in U^n, since every club in M_U is in U, so $j_{U^n}(f_n)$ is defined at $\langle \kappa, j_U(\kappa), j_{U^2}(\kappa), \ldots, j_{U^{n-1}}(\kappa)\rangle$. Let $j_{U^n}(f_n)(\kappa, j_U(\kappa), j_{U^2}(\kappa), \ldots, j_{U^{n-1}}(\kappa)) = \xi$, where we must have $\xi < \kappa$ since $j_{U^n}(f_n)$ is regressive by elementarity. It follows that the set

$$X_n = \{\langle \xi_1, \xi_2, \ldots, \xi_n\rangle \in [C]^n \mid f_n(\xi_1, \xi_2, \ldots, \xi_n) = \xi\}$$

is in U^n. Because U is M_U-normal, it follows from the definition of the product ultrafilter U^n that there is a set $\overline{X}_n \in U$ such that every sequence $\langle \xi_1, \xi_2, \ldots, \xi_n\rangle \in [C]^n$ with $\xi_i \in \overline{X}_n$ is in X_n. Clearly each $\overline{X}_n$ is homogeneous for f_n and so we can intersect all the $\overline{X}_n$ to obtain a homogeneous set I of size κ for f_A. Note that we can further refine I by adding some other sets in U to the intersection.

Now fix some $A \subseteq \kappa$ and find an $\{A, E\}$-good Ramsey measure U in $\mathcal{U}$, where E is some well-order of κ of order-type α, so that we have the representing function g^E in M_U (if $\alpha < \kappa$ we can use a constant function instead of g^E and omit E from the following discussion). Since $N_U \models o_{\text{Ram}}(\kappa) \geq \alpha$, the set

$$X = \{\xi < \kappa \mid o_{\text{Ram}}(\xi) \geq g^E(\xi)\}$$

is in U by Łoś's theorem. This is crucial to the ensuing construction. Let $A^* \subseteq \kappa$ code the triple $\{A, E, V_\kappa\}$. Now we consider the regressive function $f_{A^*} : [C_{A*}]^{<\omega} \to \kappa$ and construct a good set I_{A^*} of indiscernibles for $L_\kappa[A^*]$ by intersecting the sets $\overline{X}_n$, homogeneous for f_n, as described above, together with this new set X. This ensures that $I_{A^*} \subseteq X$. Using I_{A^*}, we construct a certified Ramsey measure U^* with $A, E, V_\kappa \in M_{U^*}$, which is certified by I_{A^*}. Note that X is an element of M_{U^*} because it is definable over V_κ from E and so it must be the case that $X \in U^*$, because $I_{A^*} \subseteq X$ and U^* is certified by I_{A^*}. But since $E \in M_{U^*}$, it follows that $[g^E] = \alpha$ in the ultrapower N_{U^*} and so $N_{U^*} \models o_{\text{Ram}}(\kappa) \geq \alpha$. Thus, we have succeeded in finding for every $A \subseteq \kappa$ an A-good certified Ramsey measure U^* such that $N_{U^*} \models o_{\text{Ram}}(\kappa) \geq \alpha$. Let $\mathcal{U}^*$ be the witness collection consisting of these U^*. By part (1) above, $o_{\text{Ram}}(U^*) \geq \alpha$, which completes the argument. $\square$

The proofs of analogous results for witness collections of strong Ramsey or super Ramsey measures are easier, in that we do not even need to introduce certified measures. A κ-model is always correct about a set being a strong Ramsey measure and a κ-model that is elementary in H_{κ^+} is always correct about a set being a super Ramsey measure: if $M \prec H_{\kappa^+}$ is a κ-model and U is a weakly amenable M-ultrafilter, then M is the H_{κ^+} of the ultrapower N and therefore if $\overline{M} \prec H^N_{\kappa^+}$, then $\overline{M} \prec H_{\kappa^+}$.

A more direct approach to defining the rank of a Ramsey (or Ramsey-like) cardinal, without introducing the order on the witness collections, would be as follows. Define that the Ramsey rank of κ is 0 if κ is not Ramsey, that the Ramsey rank of κ is ≥ 1 if κ is Ramsey, and now inductively that the Ramsey rank of κ is $\geq \alpha$ if for every $A \subseteq \kappa$ and $\beta < \alpha$, there is an A-good Ramsey measure U on κ such that the Ramsey rank of κ in N_U is $\geq \beta$. Finally, define that the rank of κ is exactly α if it is $\geq \alpha$, but it is not $\geq \alpha + 1$. As a corollary of Theorem 3.15, we get that the M-rank is precisely the Ramsey rank we just described.

Corollary 3.16. *If α is an ordinal and $\mathcal{M}$ is practical for κ, then, in $\mathcal{M}$, we have $o_{\mathrm{Ram}}(\kappa) \geq \alpha$ if and only if for every $A \subseteq \kappa$ and every $\beta < \alpha$, there is an A-good Ramsey measure W on κ with $N_W \models o_{\mathrm{Ram}}(\kappa) \geq \beta$. The same result holds for strongly Ramsey and super Ramsey measures.*

Corollary 3.16 allows us to calculate $o_{\mathrm{Ram}}(\kappa)$ inside H_{κ^+} and confirms the intuition that objects in H_{κ^+} should suffice to compute the M-rank of a Ramsey or Ramsey-like cardinal. An important advantage of this alternative description of the M-rank is that it is meaningful even in models of set theory which are not practical for κ, e.g. in κ-models where $\mathcal{P}(\kappa)$ does not exist, or H_{κ^+} itself. Such models might contain many Ramsey measures on κ but cannot collect them into a witness collection. Consequently, the M-rank, as originally defined, of κ in such a model would be 0, but computing it in this alternative way can give nontrivial values.

Next, as promised, we show that there are always "well-behaved" witness collections of Ramsey (strongly Ramsey, super Ramsey) measures.

Theorem 3.17. *If α is an ordinal and $\mathcal{M}$ is practical for κ, then, in $\mathcal{M}$, whenever $o_{\mathrm{Ram}}(\kappa) > \alpha$, there is a witness collection $\mathcal{U}$ of Ramsey measures on κ with $o_{\mathrm{Ram}}(\mathcal{U}) = \alpha$ such that $N_U \models o_{\mathrm{Ram}}(\kappa) = \alpha$ for all $U \in \mathcal{U}$. The same result holds for strongly Ramsey and super Ramsey measures.*

Proof. As usual, we prove the result for Ramsey measures. Fix an $\mathcal{M}$ practical for κ and work in $\mathcal{M}$. Suppose that $\mathcal{W}$ is a witness collection of Ramsey

measures on κ with $o_{\mathrm{Ram}}(\mathcal{W}) = \alpha$. If for every $A \subseteq \kappa$, there is some A-good Ramsey measure U such that $N_U \models o_{\mathrm{Ram}}(\kappa) = \alpha$, then we can let $\mathcal{U}$ be the witness collection of such Ramsey measures, one for every A, and by Theorem 3.15, we would have $o_{\mathrm{Ram}}(\mathcal{U}) = \alpha$. Thus, we can suppose towards a contradiction that there is some $A \subseteq \kappa$ such that for every A-good Ramsey measure U, if $N_U \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$, then $N_U \models o_{\mathrm{Ram}}(\kappa) > \alpha$. It follows from this assumption and Theorem 3.15 that there is for every $B \subseteq \kappa$ an $\{A, B\}$-good Ramsey measure $W \in \mathcal{W}$ with $N_W \models o_{\mathrm{Ram}}(\kappa) > \alpha$. Let $\mathcal{W}_0$ be a witness collection consisting of one such Ramsey measure for every B. Thus, if $W \in \mathcal{W}_0$, then N_W has what it thinks is a witness collection of certified Ramsey measures of rank $\geq \alpha$ by Theorem 3.15. So for each $B \subseteq \kappa$ in N_W, the model N_W has a B-good certified measure $\overline{W}$ with $N_{\overline{W}} \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$. But then by our assumption about A, N_W must also have, for any $B \subseteq \kappa$ in N_W, an $\{A, B\}$-good certified measure $\overline{W}$ with the property that $N_{\overline{W}} \models o_{\mathrm{Ram}}(\kappa) > \alpha$. Let $\mathcal{W}_1$ be the witness collection formed by putting together all such certified measures from all N_W for $W \in \mathcal{W}_0$. By construction $\mathcal{W}_1 \lhd \mathcal{W}_0$. But $\mathcal{W}_1$ has the same property as $\mathcal{W}_0$, namely that for every $W \in \mathcal{W}_1$, $N_W \models o_{\mathrm{Ram}}(\kappa) > \alpha$. Thus, we can repeat the process to construct $\mathcal{W}_2 \lhd \mathcal{W}_1$ with the same property and in this way obtain a descending infinite sequence in the M-order, which is impossible. □

The theorem allows us to obtain the following sharpened version of Corollary 3.16.

Corollary 3.18. *If α is an ordinal and $\mathcal{M}$ is practical for κ, then, in $\mathcal{M}$, we have $o_{\mathrm{Ram}}(\kappa) \geq \alpha$ if and only if for every $A \subseteq \kappa$ and every $\beta < \alpha$, there is an A-good Ramsey measure W on κ with $N_W \models o_{\mathrm{Ram}}(\kappa) = \beta$. The same result holds for strongly Ramsey and super Ramsey measures.*

We end the discussion of the basic properties of the M-order on witness collections by showing that strongly Ramsey cardinals have the maximum Ramsey rank, super Ramsey cardinals have the maximum strongly Ramsey rank and measurable cardinals have the maximum super Ramsey rank.

Theorem 3.19.

(1) *If κ is strongly Ramsey, then $o_{\mathrm{Ram}}(\kappa) = \kappa^+$.*
(2) *If κ is super Ramsey, then $o_{\mathrm{stRam}}(\kappa) = \kappa^+$.*
(3) *If κ is measurable, then $o_{\mathrm{supRam}}(\kappa) = \kappa^+$.*

Proof. Let us introduce an intermediate large cardinal property between Ramsey and super Ramsey cardinals by removing the κ-model assumption

from the definition of super Ramsey cardinals. Call a cardinal *weakly super Ramsey* if for every $A \subseteq \kappa$, there is a weak κ-model $M \prec H_{\kappa^+}$ containing A for which there is a weakly amenable ω_1-intersecting M-ultrafilter on κ. We will argue that a weakly super Ramsey κ must have maximum Ramsey rank. Suppose not, meaning that $o_{\mathrm{Ram}}(\kappa) = \alpha < \kappa^+$. By Corollary 3.18 there is in H_{κ^+}, for every $\beta < \alpha$ and $A \subseteq \kappa$, an A-good Ramsey measure W with $N_W \models o_{\mathrm{Ram}}(\kappa) = \beta$. Fix $A \subseteq \kappa$ and let $M \prec H_{\kappa^+}$ be a weak κ-model containing A and α for which there is a weakly amenable ω_1-intersecting M-ultrafilter on κ. Let N be the ultrapower of M by U. If $\beta < \alpha$, then, by elementarity, M satisfies that for every $B \subseteq \kappa$, there is a Ramsey measure W on κ with $N_W \models o_{\mathrm{Ram}}(\kappa) = \beta$, and so N must satisfy this as well. But then Corollary 3.18 implies that $N \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$. Thus, we have shown that for every $A \subseteq \kappa$, there is an A-good Ramsey measure U on κ with $N_U \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$, which means that $o_{\mathrm{Ram}}(\kappa) > \alpha$, contradicting our assumption.

Let κ be strongly Ramsey. We will show that for every $\alpha < \kappa^+$, there is a witness collection of Ramsey measures on κ of rank α. Fix $A \subseteq \kappa$ and let U be an $\{A, \alpha\}$-good strong Ramsey measure on κ. We will now argue that κ is weakly super Ramsey in N_U. Fix $B \subseteq \kappa$ in N_U. Using the construction from the proof of Lemma 2.10, we obtain a sequence $\langle (M_n, W_n) \mid n < \omega \rangle$ such that $M = \bigcup_{n<\omega} M_n$ is elementary in M_U, we have $B \in M$, and $W = \bigcup_{n<\omega} W_n$ is a weakly amenable ω_1-intersecting M-ultrafilter on κ. Since N_U is a κ-model both M and W are in N_U. Thus, we have verified that κ is weakly super Ramsey in N_U, and so it follows that $N_U \models o_{\mathrm{Ram}}(\kappa) = \kappa^+$, so in particular $N_U \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$. But this means that for every $A \subseteq \kappa$, there is an A-good Ramsey measure U with $N_U \models o_{\mathrm{Ram}}(\kappa) \geq \alpha$, from which it follows that there is a witness collection of Ramsey measures on κ of rank at least α, as required.

To show that super Ramsey cardinals have maximum strong Ramsey rank, we just mimic the argument that weakly super Ramsey cardinals have maximum Ramsey rank. To show that measurable cardinals have maximum super Ramsey rank, we use that measurable cardinals are super Ramsey and repeat the same argument. □

Note that we did not need that κ is strongly Ramsey in the argument that $o_{\mathrm{Ram}}(\kappa) = \kappa^+$, but merely that κ is ω-closed Ramsey ($M^\omega \subseteq M$), which gives a lower bound on the strength of having maximum Ramsey rank. In fact, the proof shows that ω-closed Ramsey cardinals are stationary limits of Ramsey cardinals of maximal Ramsey rank.

4. Extensions with cover and approximation properties cannot increase Ramsey or Ramsey-like rank

In [Ham03], Hamkins developed general techniques to show that if $V \subseteq V'$ has the δ-cover and δ-approximation properties for some regular cardinal δ of V', then, for most large cardinal properties, V' cannot have new large cardinals of that type above δ. The techniques cannot be applied directly to Ramsey or Ramsey-like cardinals because, for the smaller large cardinals, the arguments require embeddings to exist for all transitive models of size κ (as in the case of weakly compact cardinals), and in particular for all κ-models, which we know is not the case for Ramsey or Ramsey-like cardinals. Nevertheless, we will be able to adapt the machinery used in the proofs of theorems in [Ham03] to our situation. We will show that if $V \subseteq V'$ has the δ-cover and δ-approximation properties (for some regular δ of V') and $\kappa > \delta$ has Ramsey (or Ramsey-like) rank α in V', then it had at least rank α in V. The significance of the result lies in applying it to forcing extensions to show that no new Ramsey or Ramsey-like cardinals of any rank were created. Although it is easy to show that Ramsey cardinals cannot be created by small forcing, it was not previously known whether the result generalized to all extensions with the cover and approximation properties. We begin by recalling the definition of the cover and approximation properties and their connection to forcing extensions.

Definition 4.1 (Hamkins [Ham03])**.** Suppose $V \subseteq V'$ are transitive (set or class) models of (some fragment of) ZFC and δ is a cardinal in V'.

(1) The pair $V \subseteq V'$ satisfies the *δ-cover property* if for every $X \in V'$ with $X \subseteq V$ and $|X|^{V'} < \delta$, there is $Y \in V$ with $X \subseteq Y$ and $|Y|^V < \delta$.
(2) The pair $V \subseteq V'$ satisfies the *δ-approximation property* if whenever $X \in V'$ with $X \subseteq V$ and $X \cap x \in V$ for every x of size less than δ in V, then $X \in V$.

If $\mathbb{P}$ is a forcing notion of size at most δ, then the pair $V \subseteq V[G]$, where G is V-generic for $\mathbb{P}$, has the δ^+-cover and δ^+-approximation properties. We say that a poset $\mathbb{P}$ has a *closure point* at a cardinal δ if $\mathbb{P}$ factors as $\mathbb{R} * \dot{\mathbb{Q}}$, where $\mathbb{R}$ is nontrivial[6] of size at most δ and $\Vdash_{\mathbb{R}} \dot{\mathbb{Q}}$ is strategically $\leq \delta$-closed. We then have:

[6]A poset is *nontrivial* if it necessarily adds a new set.

Theorem 4.2 (Hamkins)**.** *If $\mathbb{P}$ is a forcing notion with a closure point at δ, then the pair $V \subseteq V[G]$ satisfies the δ^+-cover and δ^+-approximation properties for any forcing extension $V[G]$ by $\mathbb{P}$.*

Thus, we will be able to show that a large class of forcing notions, namely those with a closure point less than or equal to the first inaccessible cardinal (or in fact much higher), cannot create new Ramsey or Ramsey-like cardinals of any rank.

Some hypotheses about the extensions are necessary to ensure that no new Ramsey or Ramsey-like cardinals are created. In particular, it is consistent that κ is not Ramsey-like, but becomes Ramsey-like in a $<\kappa$-distributive κ-cc forcing extension. Kunen had showed that the two-step forcing iteration $\mathbb{Q}*\dot{T}$, first adding a homogeneous κ-Souslin tree T followed by adding a branch to T, has a dense subset isomorphic to $\mathrm{Add}(\kappa, 1)$, the forcing to add a Cohen subset to κ with bounded conditions [Kun78]. He used this to show that weakly compact cardinals are not downward absolute by destroying and then resurrecting a weakly compact cardinal in a forcing extension. The same construciton can be carried out for Ramsey and Ramsey-like cardinals. So suppose that κ is Ramsey or Ramsey-like and we force with the iteration $\mathbb{P}_\kappa * \mathbb{Q} * \dot{T}$, where $\mathbb{P}_\kappa$ is a κ-length Easton-support iteration forcing with $\mathrm{Add}(\alpha, 1)$ whenever α is a cardinal in $V^{\mathbb{P}_\alpha}$. Let $V[G][T][b] \subseteq \mathbb{P}_\kappa * \mathbb{Q} * \dot{T}$ be V-generic. By results in [GJ], κ remains Ramsey or Ramsey-like in $V[G][T][b]$ because the final two steps are forcing equivalent to $\mathrm{Add}(\kappa, 1)$. But κ is not even weakly compact in $V[G][T]$ because this model has a κ-Souslin tree.

4.1. **Strongly Ramsey and super Ramsey cardinals.** Most of the work in showing that strong Ramsey rank cannot increase in extensions with the cover and approximation properties goes into showing that strongly Ramsey cardinals cannot be created in such extensions. Once again, to carry out the inductive arguments, we will need the statements to be formulated in terms of practical models, with the hypothesis that $\mathcal{M}$ and $\mathcal{M}'$ are practical for κ and that the pair $\mathcal{M} \subseteq \mathcal{M}'$ has the δ-cover and δ-approximation properties and the same ordinals. The lemma below, combined with the discussion following Definition 3.13, will then allow us to deduce the corresponding results for extensions of class-sized models.

Lemma 4.3. *Suppose $V \subseteq V'$ are transitive (set or class) models of a sufficient fragment of* ZFC *(*ZFC^- *is more than enough), and assume that the pair $V \subseteq V'$ satisfies the δ-approximation and δ-cover properties and has the same ordinals. Let $\lambda \geq \delta$ be a regular cardinal in V'. Then the pair*

$H_\lambda^V \subseteq H_\lambda^{V'}$ also satisfies the δ-cover and δ-approximation properties and has the same ordinals.

Proof. Since λ is a cardinal in both V and V', the ordinals in H_λ^V and $H_\lambda^{V'}$ are exactly those below λ.

If $X \subseteq H_\lambda^V$ is a set of size less than δ in $H_\lambda^{V'}$, then, by the δ-covering property of $V \subseteq V'$ there is a Y of size less than δ in V with $X \subseteq Y$. Let $\overline{Y} = Y \cap H_\lambda^V$. Then $\overline{Y}$ is a set of size less than δ each of whose elements has transitive size less than λ, and so, by the regularity of λ, we get $\overline{Y} \in H_\lambda^V$, showing that $H_\lambda^V \subseteq H_\lambda^{V'}$ satisfies the δ-cover property.

Finally, suppose that $X \subseteq H_\lambda^V$ is in $H_\lambda^{V'}$ and $x \cap X \in H_\lambda^V$ for all $x \in H_\lambda^V$ of size less than δ in V. It follows from the δ-approximation property of $V \subseteq V'$ that $X \in V$. Since X has size less than λ in V', it must also have size less than λ in V, showing that $X \in H_\lambda^V$ and that $H_\lambda^V \subseteq H_\lambda^{V'}$ satisfies the δ-approximation property. □

Theorem 4.4. *Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are practical for κ, they have the same ordinals, and that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ-cover and δ-approximation properties for some regular cardinal δ of $\mathcal{M}'$. If $\kappa > \delta$ is strongly Ramsey in $\mathcal{M}'$, then κ was already strongly Ramsey in $\mathcal{M}$. The same result holds for super Ramsey cardinals.*

The proof adapts techniques developed in [Ham03] to the embeddings characterizing strongly Ramsey cardinals. We will note in the course of the argument where the constructions occurred in [Ham03].

Proof. First, suppose that $\kappa > \delta$ is strongly Ramsey in $\mathcal{M}'$. Fix $A \subseteq \kappa$ in $\mathcal{M}$. We need to show that $\mathcal{M}$ has a κ-model M containing A and a weakly amenable M-ultrafilter U on κ. In $\mathcal{M}'$, let W be an $\{A, V_\kappa^{\mathcal{M}}\}$-good strong Ramsey measure and let $j : M_W \to N_W$ be the ultrapower by W. Note that both $V_\kappa^{\mathcal{M}'}$ and $V_\kappa^{\mathcal{M}}$ are ZFC-models since κ is inaccessible in $\mathcal{M}'$ and therefore also in $\mathcal{M}$.

Claim 4.4.1. *The pair $V_\kappa^{\mathcal{M}} \subseteq V_\kappa^{\mathcal{M}'}$ has the δ-cover and δ-approximation properties.*

Proof. This is just Lemma 4.3. □

Thus, by elementarity, N_W satisfies that the pair $N = j(V_\kappa^{\mathcal{M}}) \subseteq j(V_\kappa^{\mathcal{M}'}) = N'$ has the δ-cover and δ-approximation properties, and it is correct about this. Observe that $N' = V_{j(\kappa)}^{N_W}$, meaning that it is a κ-model in $\mathcal{M}'$ since N_W is a κ-model in $\mathcal{M}'$ (it is not difficult to see that the ultrapower of any

κ-model M by an M-ultrafilter on κ is always a κ-model). The next several claims have the aim to conclude that N and $W \cap N$ are in $\mathcal{M}$.

Claim 4.4.2. $V_\kappa^N = V_\kappa^{\mathcal{M}}$.

Proof. If $X \in V_\kappa^{\mathcal{M}}$, then $X = j(X) \in j(V_\kappa^{\mathcal{M}}) = N$. So $V_\kappa^{\mathcal{M}} \subseteq N$. Conversely, if $X \in V_\kappa^N$, then note first that $X \in M_W = \mathrm{dom}(j)$. This is because $X \in V_\kappa^{N_W} = V_\kappa^{M_W}$. But then $j(X) = X \in N = j(V_\kappa^{\mathcal{M}})$, and so $X \in V_\kappa^{\mathcal{M}}$. □

Claim 4.4.3. *If $X \subseteq \mathrm{ORD}^N$ is a set of size less than δ in $\mathcal{M}'$, then there is $Y \in \mathcal{M} \cap N$ of size at most δ in N' such that $X \subseteq Y$.*

Proof. This construction mimics Lemma 3.2 in [Ham03]. Let $X_0 = X$, and observe that $X_0 \in N'$ since N' is a κ-model in $\mathcal{M}'$. So, by the δ-cover property of $N \subseteq N'$, there is $X_1 \subseteq \mathrm{ORD}^N$ of size less than δ in N such that $X_0 \subseteq X_1$. Then, by the δ-cover property of $\mathcal{M} \subseteq \mathcal{M}'$, there is $X_2 \subseteq \mathrm{ORD}^N$ of size less than δ in $\mathcal{M}$ such that $X_1 \subseteq X_2$ (this uses that $\mathcal{M}$ and $\mathcal{M}'$ have the same ordinals). The set X_2 is in the κ-model N', and so, again, there is X_3 of size less than δ in N such that $X_2 \subseteq X_3$. Continue bouncing between N and $\mathcal{M}$ in this way. To get through limit stages, observe that if $\gamma < \delta$ and $\langle X_\xi \mid \xi < \gamma \rangle$ is a sequence of sets of size less than δ in $\mathcal{M}'$, then $X_\gamma = \bigcup_{\xi<\gamma} X_\gamma$ has size less than δ in $\mathcal{M}'$ by the regularity of δ. Thus, after δ-many steps, we end up with an increasing sequence $\langle X_\xi \mid \xi < \delta \rangle$ such that cofinally many elements of it are in N and cofinally many are in $\mathcal{M}$. Let $Y = \bigcup_{\xi<\delta} X_\xi$. By closure, $Y \in N'$ and Y has size at most δ there. To see that $Y \in N$ we use the δ-approximation property of $N \subseteq N'$. Specifically, let $y \in N$ have size less than δ. Then there is some $\xi < \delta$ such that $Y \cap y = X_\xi \cap y$ and we may furthermore choose ξ so that $X_\xi \in N$. So then clearly $Y \cap y \in N$ and we obtain $Y \in N$ by the δ-approximation property. A similar argument, using the δ-approximation property of $\mathcal{M} \subseteq \mathcal{M}'$, shows that also $Y \in \mathcal{M}'$. □

Claim 4.4.4. *$\mathcal{M}$ and N have the same subsets of ORD^N of size less than δ in $\mathcal{M}'$.*

Proof. This argument mimics Lemma 3.3 in [Ham03]. Suppose that $X \subseteq \mathrm{ORD}^N$ has size less than δ in $\mathcal{M}'$. By Claim 4.4.3, there is a set Y of size at most δ in N' such that $X \subseteq Y$ and $Y \in N \cap \mathcal{M}$. Let $Y = \{y_\alpha \mid \alpha < \gamma\}$ be the enumeration of Y arising from its order-type and note that $\gamma < \delta^{+\mathcal{M}'} < \kappa$. Since the order-type of Y is absolute, the enumeration is in both N and $\mathcal{M}$. Let $\overline{X} = \{\alpha < \gamma \mid y_\alpha \in X\}$, which is a subset of γ. Now observe that X is in N or $\mathcal{M}$ if and only if $\overline{X}$ is there, and $\overline{X}$ is in $\mathcal{M}$ if and only if it is in N, by Claim 4.4.2, since it is a subset of γ. □

Claim 4.4.5. *$\mathcal{M} \cap N' = N$ and $N \in \mathcal{M}$.*

Proof. This argument mimics Lemma 3.4 in [Ham03]. First, we show that $N \subseteq \mathcal{M}$. It suffices to verify that all sets of ordinals in N are elements of $\mathcal{M}$. Suppose that $X \subseteq \mathrm{ORD}^N$ is in N. Fix a set $x \subseteq \mathrm{ORD}^N$ of size less than δ in $\mathcal{M}$. By Claim 4.4.4 we have $x \in N$ and so $X \cap x \in N$ as well. But $X \cap x$ is a set of ordinals in N of size less than δ in $\mathcal{M}'$, and therefore by Claim 4.4.4 again, $X \cap x \in \mathcal{M}$. So $X \in \mathcal{M}$ by the δ-approximation property of $\mathcal{M} \subseteq \mathcal{M}'$.

Next, we verify that $\mathcal{M} \cap N' \subseteq N$. Initially, we show that every set of ordinals in $\mathcal{M} \cap N'$ is in N. So suppose that X is a set of ordinals in $\mathcal{M} \cap N'$. Let x be a set of ordinals of size less than δ in N. Then $x \in \mathcal{M}$, by Claim 4.4.4, and so $x \cap X$ in $\mathcal{M}$. By Claim 4.4.4 again, $x \cap X \in N$. So, by the δ-approximation property of $N \subseteq N'$, we obtain $X \in N$. Now suppose that X is any set in $\mathcal{M} \cap N'$. By $\in$-induction, suppose that every element of X is in N. Since $X \in N'$ and N' is a model of ZFC, there must be some ordinal β in N' such that $X \in V_\beta^{N'}$ and thus $X \subseteq V_\beta^N = Y$. Enumerate $Y = \{y_\alpha \mid \alpha < \gamma\}$ in N, and note that since $N \subseteq \mathcal{M}$, the enumeration exists in $\mathcal{M}$ as well. Let $\overline{Y} = \{\alpha < \gamma \mid y_\alpha \in X\}$. The set $\overline{Y}$ is in $\mathcal{M}$ and also in N'. So by what we already argued for sets of ordinals, $\overline{Y}$ is in N, and hence so is X. This completes the argument that $N = \mathcal{M} \cap N'$.

We will use the δ-approximation property of $\mathcal{M} \subseteq \mathcal{M}'$ to argue that $N \in \mathcal{M}$. Fix a set x of size less than δ in $\mathcal{M}$. The intersection $N \cap x$ is in the κ-model N', therefore there is some β such that $N \cap x \in V_\beta^{N'}$. It now follows, since obviously $N \cap x \subseteq N$, that $N \cap x \subseteq V_\beta^N$. Hence $N \cap x = V_\beta^N \cap x$ is in $\mathcal{M}$ since both x and V_β^N are there. □

In what follows let $\overline{U} = N \cap W$.

Claim 4.4.6. *$\overline{U} \in \mathcal{M}$.*

Proof. We will use the δ-approximation property of $\mathcal{M} \subseteq \mathcal{M}'$. This construction mimics Theorem 10 in [Ham03]. Suppose that x is a set of size less than δ in $\mathcal{M}$. We can assume that $x \subseteq P(\kappa)^N$ and also that whenever some $B \subseteq \kappa$ is in x, then so is the complement of B in κ. Since W is an M_W-ultrafilter and M_W is a κ-model in $\mathcal{M}'$, it follows that W is κ-intersecting in $\mathcal{M}'$. So, working in $\mathcal{M}'$, we consider the intersection of all $B \in W \cap x$, which is non-empty, and hence must contain some element β. We will argue that, for $B \in x$, we have $\beta \in B$ precisely when $B \in W$.

By definition of β, if $B \in W \cap x$, then $\beta \in B$. If $B \notin W$, then its complement B^c is in $x \cap W$, and so $\beta \in B^c$, which means that $\beta \notin B$. Thus, $\overline{U} \cap x = W \cap x = \{B \in x \mid \beta \in B\}$, which is clearly in $\mathcal{M}$. $\square$

Claim 4.4.7. *N is closed under $<\kappa$-sequences in $\mathcal{M}$, we have $A \in N$, and $\overline{U}$ is a weakly amenable ω_1-intersecting N-ultrafilter.*

Proof. Since $A \in N' = V_{j(\kappa)}^{N_W}$ and also $A \in \mathcal{M}$, we get $A \in N$ by Claim 4.4.5. If $\vec{x} = \langle x_\xi \mid \xi < \gamma \rangle$, for some $\gamma < \kappa$, is a sequence of elements of N in $\mathcal{M}$, then $\vec{x}$ is in both $\mathcal{M}$ and N', and so $\vec{x} \in N$, again by Claim 4.4.5.

It is clear that $\overline{U}$ is an ω_1-intersecting N-ultrafilter. It remains to show that $\overline{U}$ is weakly amenable to N. Consider $X \cap \overline{U}$, where X is a set of size κ in N. The set $X \cap \overline{U}$ is in $\mathcal{M}$ and also in N' by the weak amenability of U. Hence $X \cap \overline{U} \in N$ by Claim 4.4.5. $\square$

We return to the proof of the theorem. We now have, in $\mathcal{M}$, a model N, closed under $<\kappa$-sequences, with $A \in N$ and a weakly amenable ω_1-intersecting N-ultrafilter. The only obstacle to seeing that κ is strongly Ramsey in $\mathcal{M}$ is that N might be too large in cardinality. But this is easily fixed by building an elementary substructure M of $H_{\kappa^+}^N$, containing A, in κ-many steps so that M is a κ-model and $U = \overline{U} \cap M$ is a weakly amenable M-ultrafilter.

To prove the same result for super Ramsey cardinals, we start with $M_W \prec H_{\kappa^+}^{\mathcal{M}'}$. We will be done if we can argue that $H_{\kappa^+}^N \prec H_{\kappa^+}^{\mathcal{M}}$. First, observe that we can use the δ-approximation property of $\mathcal{M} \subseteq \mathcal{M}'$ to define $\mathcal{P}(\kappa)^{\mathcal{M}}$ in $H_{\kappa^+}^{\mathcal{M}'}$, using the parameter $a = V_\kappa^{\mathcal{M}}$, as the collection of all $X \subseteq \kappa$ such that for all $x \in a$, $x \cap X \in a$. Let us see that the same formula defines $\mathcal{P}(\kappa)^N$ in $H_{\kappa^+}^{N'}$. Recall that $a = V_\kappa^{\mathcal{M}} = V_\kappa^N \in N$ by Claim 4.4.2. If $X \subseteq \kappa$ is in N, then $x \cap X \in a$ for every $x \in a$, and if $X \subseteq \kappa$ is in N' and $x \cap X \in a$ for all $x \in a$, then $X \in N$ by the δ-approximation property of $N \subseteq N'$. Thus, using the usual Mostowski coding, there is a formula $\psi(x, a)$ which defines $H_{\kappa^+}^{\mathcal{M}}$ in $H_{\kappa^+}^{\mathcal{M}'}$ and the same $\psi(x, a)$ also defines $H_{\kappa^+}^N$ in $H_{\kappa^+}^{N'}$. Now suppose that $H_{\kappa^+}^{\mathcal{M}} \models \exists x\, \varphi(x, b)$ for some $b \in H_{\kappa^+}^N$. So $H_{\kappa^+}^{\mathcal{M}'}$ satisfies that $\exists x\, \varphi(x, b)$ holds in the collection defined by $\psi(x, a)$. Since $H_{\kappa^+}^{N'} = M_W \prec H_{\kappa^+}^{\mathcal{M}'}$, it satisfies the same statement, which gives that $H_{\kappa^+}^N \models \exists x\, \varphi(x, b)$. $\square$

Next, we show that extensions with cover and approximation properties cannot increase the rank of a strongly Ramsey or super Ramsey cardinal.

Theorem 4.5. *Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are practical for κ, they have the same ordinals, and that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ-cover and δ-approximation*

properties for some regular cardinal δ of $\mathcal{M}'$. If $\kappa > \delta$, then $o_{\mathrm{stRam}}(\kappa)^{\mathcal{M}'} \leq o_{\mathrm{stRam}}(\kappa)^{\mathcal{M}}$. The same result holds for super Ramsey cardinals.

Proof. We will argue by induction on α that if $\mathcal{M}$ and $\mathcal{M}'$ are as in the hypothesis and $o_{\mathrm{stRam}}(\kappa) = \alpha$ in $\mathcal{M}'$, then $o_{\mathrm{stRam}}(\kappa) \geq \alpha$ in $\mathcal{M}$. So suppose inductively that the statement holds for all $\beta < \alpha$. Fix some pair $\mathcal{M}$ and $\mathcal{M}'$ as in the hypothesis and suppose that $o_{\mathrm{stRam}}(\kappa) = \alpha$ in $\mathcal{M}'$. By Corollary 3.16 we have to show that for every $A \subseteq \kappa$ in $\mathcal{M}$ and every $\beta < \alpha$, $\mathcal{M}$ has an A-good strong Ramsey measure U with $N_U \models o_{\mathrm{stRam}}(\kappa) \geq \beta$. So fix $\beta < \alpha$ and $A \subseteq \kappa$ in $\mathcal{M}$. In $\mathcal{M}'$ there is a strong Ramsey measure W such that M_W contains A and $V_\kappa^{\mathcal{M}}$ and $N_W \models o_{\mathrm{stRam}}(\kappa) = \beta$. In what follows we use the notation from the proof of Theorem 4.4. Construct N and N'. Clearly $N' \models o_{\mathrm{stRam}}(\kappa) = \beta$, since $N' = V_{j(\kappa)}^{N_W}$. Since $N \subseteq N'$ has the δ-cover and δ-approximation properties, we may apply the induction hypothesis to this pair and conclude that $N \models o_{\mathrm{stRam}}(\kappa) \geq \beta$. Corollary 3.16 now implies that $H_{\kappa^+}^N$ satisfies that for every $B \subseteq \kappa$ and $\xi < \beta$ there is a strong Ramsey measure U_B with $N_{U_B} \models o_{\mathrm{stRam}}(\kappa) \geq \xi$ and so $M \prec H_{\kappa^+}^N$ must satisfy this statement as well. It follows that the ultrapower of M by U must, by weak amenability, satisfy the same statement, meaning that $o_{\mathrm{stRam}}(\kappa) \geq \beta$ there, which is precisely what we set out to establish. □

4.2. **Ramsey cardinals.** The arguments presented in the previous section do not generalize directly to Ramsey cardinals because we can no longer work with κ-models, whose properties were used crucially in several places in the proof of Theorem 4.4 to pass between $\mathcal{M}'$ and N'. Nevertheless, we can modify the proof to work for Ramsey cardinals with the extra assumption that $\mathcal{M}'$ doesn't have new countable sequences of elements of $\mathcal{M}$.

Theorem 4.6. *Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are practical for κ, they have the same ordinals, that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ-cover and δ-approximation properties for some regular cardinal δ of $\mathcal{M}'$, and $\mathcal{M}^\omega \subseteq \mathcal{M}$ in $\mathcal{M}'$. If $\kappa > \delta$ is Ramsey in $\mathcal{M}'$, then κ was already Ramsey in $\mathcal{M}$.*

Proof. Fix some $A \subseteq \kappa$ in $\mathcal{M}$. In $\mathcal{M}'$, let W be an $\{A, V_\kappa^{\mathcal{M}}\}$-good Ramsey measure and let $j : M_W \to N_W$ be the ultrapower by W. We proceed at first as in the proof of Theorem 4.4. Let $N = j(V_\kappa^{\mathcal{M}})$ and $N' = j(V_\kappa^{\mathcal{M}'})$. As before, the pair $N \subseteq N'$ has the δ-cover and δ-approximation properties, but N' is no longer a κ-model. Also, as before, $V_\kappa^N = V_\kappa^{\mathcal{M}}$.

Claim 4.6.1. *$\mathcal{P}^N(\kappa) = \mathcal{M} \cap \mathcal{P}^{N'}(\kappa)$ and hence $H_{\kappa^+}^N \subseteq \mathcal{M}$.*

Proof. First, we show that $\mathcal{P}^N(\kappa) \subseteq \mathcal{M}$ using the δ-approximation property of $\mathcal{M} \subseteq \mathcal{M}'$. So suppose that $B \subseteq \kappa$ is in N. Fix a set $x \subseteq \kappa$ of size less than δ in $\mathcal{M}$, and note that $x \in V_\kappa^{\mathcal{M}} \subseteq N$. Thus, $x \cap B \in V_\kappa^N \subseteq \mathcal{M}$ and the δ-approximation property gives $B \in \mathcal{M}$.

Next, we show that subsets of κ in the intersection of $\mathcal{M}$ and N' must be in N using the δ-approximation property of $N \subseteq N'$. So suppose that $B \subseteq \kappa$ and $B \in \mathcal{M} \cap N'$. Let $x \subseteq \kappa$ be a set of size less than δ in N, meaning that $x \in V_\beta^{\mathcal{M}}$ for some $\beta < \kappa$. So $x \cap B$ is also in $V_\beta^{\mathcal{M}}$, and hence is in N.

Finally, any $X \in H^N_{\kappa^+}$ is coded by a subset of κ via Mostowski coding, and this coding can be undone in $\mathcal{M}$. □

We cannot prove that N or even $H^N_{\kappa^+}$ is an element of $\mathcal{M}$. So instead we will find a weak κ-model $M \prec H^N_{\kappa^+}$ for which $U = M \cap W$ is a weakly amenable M-ultrafilter so that both M and U are in $\mathcal{M}$.

First, we argue that $\overline{W} = W \cap N$ is weakly amenable to N. Let S be a subset of $\mathcal{P}(\kappa)$ of size κ in N. By weak amenability $\overline{S} = S \cap W$ is in N'. Now we will use the δ-approximation property of $N \subseteq N'$ to get $\overline{S}$ into N. Let $x \subseteq S$ be a set of size less than δ in N. We can assume that whenever $B \subseteq \kappa$ is in x, then so is the complement of B in κ. Since W is an N_W-ultrafilter and $\overline{S} \cap x$ is in N_W, it follows that there is some β that is an element of every $B \in \overline{S} \cap x$. Thus, the sets in $\overline{S} \cap x$ are precisely the sets in x having β as an element, and so $\overline{S} \cap x$ is in N.

Now, working in $\mathcal{M}'$, we build $M \prec H^N_{\kappa^+}$ as in the proof of Lemma 2.10 from the sequence $\langle (M_n, U_n) \mid n < \omega \rangle$, so that $M = \bigcup_{n<\omega} M_n$ and $U = \bigcup_{n<\omega} U_n \subseteq W$ is a weakly amenable ω_1-intersecting M-ultrafilter. Since each M_n and U_n are in $H^N_{\kappa^+} \subseteq \mathcal{M}$, it follows by our closure assumption that M and U are in $\mathcal{M}$. □

Theorem 4.7. *Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are practical for κ, they have the same ordinals, that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ-cover and δ-approximation properties for some regular cardinal δ of $\mathcal{M}'$, and $\mathcal{M}^\omega \subseteq \mathcal{M}$ in $\mathcal{M}'$. If $\kappa > \delta$, then $o_{\mathrm{Ram}}(\kappa)^{\mathcal{M}'} \leq o_{\mathrm{Ram}}(\kappa)^{\mathcal{M}}$.*

The proof is identical to Theorem 4.5.

Question 4.8. Can we remove the assumption that $\mathcal{M}^\omega \subseteq \mathcal{M}$ in $\mathcal{M}'$ from Theorem 4.7?

4.3. **α-iterable cardinals.** For completeness, we give a proof that extensions with the cover and approximation properties cannot create new α-iterable cardinals provided that the extension has no new countable sequences from the old model.

Theorem 4.9. *Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are practical for κ, they have the same ordinals, that $\mathcal{M} \subseteq \mathcal{M}'$ has the δ-cover and δ-approximation properties for some regular cardinal δ of $\mathcal{M}'$, and $\mathcal{M}^\omega \subseteq \mathcal{M}$ in $\mathcal{M}'$. If $\kappa > \delta$ is α-iterable in $\mathcal{M}'$, then it was already α-iterable in $\mathcal{M}$.*

Proof. Fix $A \subseteq \kappa$ in $\mathcal{M}$. Working in $\mathcal{M}'$, find an $\{A, V_\kappa^{\mathcal{M}}\}$-good α-iterable measure W and let $j : M_W \to N_W$ be the ultrapower by W. We follow the proof of Theorem 4.6 exactly by considering the pair $N \subseteq N'$. We can show that $\mathcal{P}^N(\kappa) = \mathcal{M} \cap \mathcal{P}^{N'}(\kappa)$ and that $\overline{W} = W \cap N$ is weakly amenable to N exactly as there. In $\mathcal{M}'$ we can then build the sequence $\langle (M_n, U_n) \mid n < \omega \rangle$ of elementary submodels of $H^N_{\kappa^+}$ and filters, and this sequence must be in $\mathcal{M}$ by our closure assumption. Thus, $M = \bigcup_{n<\omega} M_n$ and $U = \bigcup_{n<\omega} U_n$ are in $\mathcal{M}$. Finally, Lemma 3.8 of [GW11] implies that U is α-iterable. □

Question 4.10. Can we remove the assumption that $\mathcal{M}^\omega \subseteq \mathcal{M}$ in $\mathcal{M}'$ from Theorem 4.9?

5. Killing the M-rank softly

We can use forcing to softly kill the rank of a Ramsey or Ramsey-like cardinal, meaning that, if κ has rank α and $\beta < \alpha$, then there is a cofinality preserving forcing extension in which κ has rank β. Let us consider the case of Ramsey cardinals. We will obtain the desired forcing extension by carefully adding a club of ordinals δ with $o_{\mathrm{Ram}}(\delta) < g^E(\delta)$ (where g^E is a representing function for β), while preserving $o_{\mathrm{Ram}}(\kappa) \geq \beta$. The result will follow because no weak κ-model containing such a club can have its ultrapower satisfy $o_{\mathrm{Ram}}(\kappa) = \beta$.

Recall that, if U is a Ramsey or Ramsey-like measure with the ultrapower map $j\colon M_U \to N_U$, then $M_U^* = V^{N_U}_{j(\kappa)}$.

Lemma 5.1. *Suppose $o_{\mathrm{Ram}}(\kappa) = \alpha > 0$. Then for every $A \subseteq \kappa$ and $\beta < \alpha$, there is an A-good Ramsey measure U such that M_U is ω-special and $N_U \models o_{\mathrm{Ram}}(\kappa) = \beta$, and hence M_U^* is ω-special and $M_U^* \models o_{\mathrm{Ram}}(\kappa) = \beta$.*

Proof. Fix $A \subseteq \kappa$ and let $\overline{U}$ be any A-good Ramsey measure satisfying $N_{\overline{U}} \models o_{\mathrm{Ram}}(\kappa) = \beta$. Following the proof of Lemma 2.10, we construct, in ω-many steps, an A-good Ramsey measure U such that $M_U \prec M_{\overline{U}}$ is ω-special and $\beta \in M_U$. By Corollary 3.16, the model $M_{\overline{U}}$ has, for every $B \subseteq \kappa$ in $M_{\overline{U}}$ and $\xi < \beta$, a B-good Ramsey measure W with $N_W \models o_{\mathrm{Ram}}(\kappa) \geq \xi$, and β is the largest ordinal for which this is true. Thus, by elementarity, M_U has, for every $B \subseteq \kappa$ in M_U and $\xi < \beta$, a B-good Ramsey measure W

with $N_W \models o_{\mathrm{Ram}}(\kappa) \geq \xi$, and β is still the largest ordinal for which this is true. It follows that $N_U \models o_{\mathrm{Ram}}(\kappa) = \beta$.

Recall from Section 2 that, whenever M_U is ω-special, M_U^* is as well. Since $M_U^* = V_{j(\kappa)}^{N_U}$, where j is the ultrapower map by U, it satisfies $o_{\mathrm{Ram}}(\kappa) = \beta$. □

Theorem 5.2. *If $o_{\mathrm{Ram}}(\kappa) = \alpha$ and $\beta < \alpha$ is any ordinal, then there is a cofinality preserving forcing extension in which $o_{\mathrm{Ram}}(\kappa) = \beta$. The same result holds for strongly Ramsey and super Ramsey cardinals.*

Proof. Suppose $o_{\mathrm{Ram}}(\kappa) = \alpha$ and fix $\beta < \alpha$. Since $\beta < \kappa^+$, we can fix some well-ordering E of κ in order-type β and let $g^E : \kappa \to \kappa$ be a representing function for β (see the discussion preceding Theorem 3.15; if $\beta < \kappa$ we can let β be represented by a constant function and omit E and g^E from the following argument).

Let $\mathbb{P}_\kappa$ be the κ-length Easton support iteration, forcing at each inaccessible γ with $\mathbb{Q}_\gamma$ to shoot a club, by closed initial segments, through the set of cardinals $\delta < \gamma$ with $o_{\mathrm{Ram}}(\delta) < g^E(\delta)$, and using trivial forcing everywhere else. It is easy to see that each $\mathbb{Q}_\gamma$ is $<\gamma$-strategically closed. Fixing $\xi < \gamma$, the strategy to ensure that the union of a ξ-sequence of conditions in $\mathbb{Q}_\gamma$ with the supremum added on is itself a condition in $\mathbb{Q}_\gamma$ is to make sure that the supremum gets above ξ. This ensures that the supremum is not inaccessible and so trivially has the property $o_{\mathrm{Ram}}(\delta) < g^E(\delta)$. The forcing we shall use to achieve our goal is $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$. This poset preserves all cardinals and cofinalities, since each $\dot{\mathbb{Q}}_\gamma$ is $<\gamma$-strategically closed in $V^{\mathbb{P}_\gamma}$. Let $G * g \subseteq \mathbb{P}$ be V-generic.

The iteration $\mathbb{P}_\kappa$ has size κ and the κ-chain condition (cf. [Cum10]) and elements of $\dot{\mathbb{Q}}_\kappa$ are names for bounded subsets of κ. Since each such name can be associated with a bounded subset of κ by a nice-name argument, we can assume that $\mathbb{P} \subseteq V_\kappa$. This means in particular that every $A \subseteq \kappa$ in $V[G][g]$ has a $\mathbb{P}$-name $\dot{A}$ in H_{κ^+} and so A is an element of every model $M[G][g]$ where M is a weak κ-model in V containing $\mathbb{P}$ and $\dot{A}$. The following claim will show that the M-rank of κ in $V[G][g]$ is still at least β.

Claim 5.2.1. *If $\mathcal{M} \models \mathrm{ZFC}$ is a weak κ-model such that $V_\kappa, \beta, E \in \mathcal{M}$ and $o_{\mathrm{Ram}}(\kappa)^{\mathcal{M}} < \beta$, then $o_{\mathrm{Ram}}(\kappa)^{\mathcal{M}[G][g]} \geq o_{\mathrm{Ram}}(\kappa)^{\mathcal{M}}$.*

Proof. Note that $\mathbb{P} \in \mathcal{M}$ since it is definable from V_κ and E. We shall argue by induction on $\xi < \beta$ that if $\mathcal{M}$ is as in the hypothesis and $o_{\mathrm{Ram}}(\kappa)^{\mathcal{M}} = \xi$, then $o_{\mathrm{Ram}}(\kappa)^{\mathcal{M}[G][g]} \geq \xi$. So suppose inductively that the statement holds for all $\eta < \xi$. Fix some $\mathcal{M}$ as in the hypothesis and suppose that $o_{\mathrm{Ram}}(\kappa)^{\mathcal{M}} = \xi$.

For $A \subseteq \kappa$ in $\mathcal{M}[G][g]$ and $\eta < \xi$, we need to produce an A-good Ramsey measure W such that $N_W \models o_{\mathrm{Ram}}(\kappa) \geq \eta$.

Let $A \in \mathcal{M}[G][g]$ be a subset of κ and choose a $\mathbb{P}$-name $\dot{A} \in \mathcal{M}$ for it. Fix $\eta < \xi$. We work in $\mathcal{M}$. By Lemma 5.1 we can find an $\{\dot{A}, \beta, E\}$-good Ramsey measure U such that M_U is ω-special and $N_U \models o_{\mathrm{Ram}}(\kappa) = \eta$. Let $M = M_U^*$ and $h : M \to N$ be the ultrapower by U. Observe that $M \models o_{\mathrm{Ram}}(\kappa) = \eta$, and so the inductive assumption applied to M gives that $M[G][g] \models o_{\mathrm{Ram}}(\kappa) \geq \eta$. We shall lift h to $M[G][g]$ and argue that the ultrafilter derived from the lift of h to $M[G][g]$ is the desired W.

First, we lift h to $M[G]$. To do this we need to find an N-generic filter for $h(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{\mathrm{tail}}$ containing $h"G = G$. We will use the filter $G*g$ for the $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ part of $h(\mathbb{P}_\kappa)$. Note that $\mathbb{P}_{\mathrm{tail}} = (\dot{\mathbb{P}}_{\mathrm{tail}})_{G*g}$ is $\leq\kappa$-strategically closed and hence $\leq\kappa$-distributive in $N[G][g]$. Thus, by Lemma 2.11, $\mathcal{M}[G][g]$ has an $N[G][g]$-generic G_{tail} for $\mathbb{P}_{\mathrm{tail}}$, and so we can lift h to $h : M[G] \to N[h(G)]$, where $h(G) = G * g * G_{\mathrm{tail}}$.

Next, we lift h fully to $M[G][g]$ by finding an $N[h(G)]$-generic filter for $h(\mathbb{Q}_\kappa)$, where $\mathbb{Q}_\kappa = (\dot{\mathbb{Q}}_\kappa)_G$, containing $h"g$. Let $C = \bigcup g$ and $\overline{C} = C \cup \{\kappa\}$, which is in $N[h(G)]$ by our choice of $h(G)$. Note that $\overline{C}$ is a closed bounded subset of $h(\kappa)$. If we can show that $\overline{C}$ is a condition in $h(\mathbb{Q}_\kappa)$, then we can use it as a master condition for the lift and use Lemma 2.11 to find an $N[h(G)]$-generic filter g^* for $h(\mathbb{Q}_\kappa)$ containing $\overline{C}$. The only reason it might not be the case that $\overline{C}$ is an element of $\mathbb{Q}_\kappa$ is that $o_{\mathrm{Ram}}(\delta)^{N[h(G)]} \geq g^E(\delta)$ for some $\delta \in C$ or $o_{\mathrm{Ram}}(\kappa)^{N[h(G)]} \geq h(g^E)(\kappa) = \beta$.

The first option cannot occur, since we would get $o_{\mathrm{Ram}}(\delta)^{M[G]} \geq g^E(\delta)$ by elementarity otherwise. This contradicts the construction of C which is a club of ordinals δ satisfying $o_{\mathrm{Ram}}(\delta)^{M[G]} < g^E(\delta)$.

To see that the second option also cannot occur, observe that the forcing $h(\mathbb{P}_\kappa)$ has a closure point at the first inaccessible cardinal δ_0: the first non-trivial forcing happens at stage δ_0 and has size δ_0 and each $\dot{\mathbb{Q}}_\delta$ for $\delta > \delta_0$ is $\leq\delta_0$-strategically closed in $V^{\mathbb{P}_\delta}$, from which it will follow that the remainder of the iteration is $\leq\delta_0$-strategically closed. By Theorem 4.2 the pair $N \subseteq N[h(G)]$ has the δ_0^+-cover and δ_0^+-approximation properties. Also $h(\mathbb{P}_\kappa)$ is clearly countably closed, meaning that $N^\omega \subseteq N$ in $N[h(G)]$. Following our assumptions, $N \models o_{\mathrm{Ram}}(\kappa) = \eta$ and hence Theorem 4.7 yields that $N[h(G)] \models o_{\mathrm{Ram}}(\kappa) \leq \eta < \beta$. This completes the argument that $\overline{C}$ is a condition in $h(\mathbb{Q}_\kappa)$, allowing us to lift h to $h : M[G][g] \to N[h(G)][g^*]$, where g^* is obtained using Lemma 2.11 below the master condition $\overline{C}$.

Recall that the lift of an ultrapower embedding is always an ultrapower embedding by the filter generated from the lift. In our case, the lift $h : M[G][g] \to N[h(G)][g^*]$ is the ultrapower by the $M[G][g]$-ultrafilter $W = \{X \in M[G][g] \mid X \subseteq \kappa, \kappa \in h(X)\}$. Next, we argue that W is weakly amenable and ω_1-intersecting. By Lemma 2.12, since $\mathbb{P}$ is countably closed, W is ω_1-intersecting. To conclude that W is weakly amenable, we verify that $M[G][g]$ and $N[h(G)][g^*]$ have the same subsets of κ. Suppose B is a subset of κ in $N[h(G)][g^*]$. Since $\mathbb{P}_{\text{tail}} * h(\dot{\mathbb{Q}}_\kappa)$ is $\leq \kappa$-distributive, $B \in N[G][g]$ and so B has a $\mathbb{P}$-name $\dot{B} \in N$, which we can take to be an element of $H^N_{\kappa^+} = M$. So finally, $B \in M[G][g]$.

Recall that $M[G][g] \models o_{\text{Ram}}(\kappa) \geq \eta$, from which it follows that we also have $N_W \models o_{\text{Ram}}(\kappa) \geq \eta$. This finishes the inductive argument and allows us to conclude that $\mathcal{M}[G][g] \models o_{\text{Ram}}(\kappa) \geq \xi$, which finishes the proof of the claim. □

To see that $o_{\text{Ram}}(\kappa)^{V[G][g]} \leq \beta$, recall that $C = \bigcup g$ is a club in κ, consisting of cardinals δ with $o_{\text{Ram}}(\delta)^{V[G]} < g^E(\delta)$. But since $\mathbb{Q}_\kappa$ is $<\kappa$-distributive, it also follows that $o_{\text{Ram}}(\delta)^{V[G][g]} < g^E(\delta)$ for all $\delta \in C$. This means that in $V[G][g]$ there cannot be a Ramsey measure U with $C, E \in M_U$ and $N_U \models o_{\text{Ram}}(\kappa) \geq j(g^E)(\kappa) = \beta$.

Exactly the same argument would work to get the result for strongly and super Ramsey cardinals, except that we would rely on Theorem 4.5 instead of Theorem 4.7 in the proof of the main claim. □

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