A primer on the set-theoretic multiverse

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Counting past infinity

What I assert and believe to have demonstrated in this and earlier works is that following the finite there is a transfinite (which one could also call the supra-finite), that is an unbounded ascending ladder of definite modes, which by their nature are not finite but infinite, but which just like the finite can be determined by well-defined and distinguishable numbers. – Georg Cantor

We use natural numbers to count through objects in order: (0th), 1st, 2nd, 3rd, etc.

In 1852, the mathematician Georg Cantor needed to iterate a mathematical operation past the natural numbers.

- Suppose X is a closed set of reals.
- Let $X' = \{x \in X \mid x \text{ is a limit point of } X\}$.
- Let $X^{(n)}$ be the result of iterating the $'$ operation n-many times.
- Past all the natural numbers, take the limit $\bigcap_n X^{(n)}$.
- Can we keep iterating?

Cantor extended the natural numbers to a (much) bigger counting system in which we can keep iterating!

Counting with natural numbers

Well-order

- Linear order
  - Reflexivity: $n \leq n$
  - Antisymmetry: if $n \leq m$ and $m \leq n$, then $n = m$
  - Transitivity: if $n \leq m$ and $m \leq k$, then $n \leq k$
  - Comparability: either $n \leq m$ or $m \leq n$
- Every subset has a least element.
  - Induction: Suppose that whenever a property $P(x)$ is true for every $n < m$, then it is also true for m. Then $P(x)$ is true for every n.
  - Justifies recursively defined operations.

Question: Can we extend the natural numbers while maintaining these key properties?

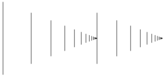
* Images credit: http://www.madore.org/ david/math/drawordinals.html

The transfinite: ordinals

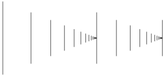
Add a new number ω above the natural numbers.

- The natural numbers ($n > 0$) are successor ordinals: n is an immediate successor of $n-1$.
- ω is a limit ordinal: it is not an immediate successor of anything.

Keep counting: $\omega+1, \omega+2, \omega+3, \ldots, \omega+n, \ldots$



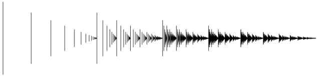
Next limit ordinals: $\omega+\omega=\omega\cdot 2, \omega\cdot 3, \ldots, \omega\cdot n, \ldots$



$\omega\cdot\omega=\omega^2, \omega^2+\omega, \omega^2+\omega\cdot 2, \ldots, \omega^2+\omega\cdot n, \ldots$

$\omega^2\cdot 2, \omega^2\cdot 2+\omega, \ldots, \omega^2\cdot 2+\omega\cdot n, \ldots, \omega^2\cdot 3, \ldots, \omega^3, \ldots$

$\omega^\omega, \omega^{\omega\cdot 2}, \omega^\omega, \omega^{\omega^2}, \omega^{\omega^\omega}, \ldots$



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Properties of ordinals

Let ORD denote the collection of all ordinals.

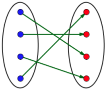
- The ordinals are well-ordered.
- Every ordinal is either 0, a successor or a limit.
- For every ordinal, there is a larger ordinal, its successor.
- We can iterate "forever" along the ordinals.

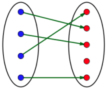
Measuring infinity

Definition: Suppose A and B are sets.

- A has the same size as B, $|A| = |B|$, if there is a bijection between them.
- A is smaller than or same size as B, $|A| \leq |B|$, if there is an injection from A to B.

Theorem: (Cantor-Bernstein) If there is an injection from A to B and also an injection from B to A, then there is a bijection between them. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Bijection: 

Injection: 

Examples

- The set of natural numbers has the same size as the set of integers and as the set of rational numbers.
- (Cantor) The set of natural numbers is smaller than the set of real numbers.
- Are there any sets whose size is strictly between the natural numbers and real numbers?

Theorem: (Cantor) The size of the powerset of a set X is larger than the size of X. For every infinity, there is a yet larger infinity!

The iterative conception of sets

A naive conception of set theory resulted in paradoxes such as the famous Russell's Paradox.

The iterative conception of sets was a way to carefully construct a universe of sets by iterating the powerset operation along the ordinals starting with (just!) the emptyset $\emptyset$.

The bottom up construction avoided the known paradoxes and was later formalized through the Zermelo Fraenkel ZFC axioms.

It is a convention to call universes of set theory V.

The V_α hierarchy

Let $\mathscr{P}(X)$ denote the powerset of X: the set of all subsets of X.

$V_0 = \emptyset$

$V_1 = \{\emptyset\}$

$V_2 = \{\emptyset, \{\emptyset\}\}$

$V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ (2^2 elements)

$\vdots$

$V_\omega = \bigcup_n V_n$

$V_{\omega+1} = \mathscr{P}(V_\omega)$

$V_{\omega+2} = \mathscr{P}(V_{\omega+1})$

$\vdots$

Natural numbers

- $0 = \emptyset$, $1 = \{0\} = \{\emptyset\}$, $2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, $3 = \{0, 1, 2\}, \ldots, n = \{0, 1, \ldots, n-1\}, \ldots$
- $n \in V_{n+1}$

Ordinals

- $\omega = \{0, 1, \ldots, n, \ldots\}$
- $\omega + 1 = \{0, 1, \ldots, n, \ldots, \omega\}$
- $\alpha = \{\xi \mid \xi < \alpha\}$
- $\alpha \in V_{\alpha+1}$

Reals

- Represent reals by subsets of natural numbers.
- Every real is in $V_{\omega+1}$.
- Every set of reals is in $V_{\omega+2}$.

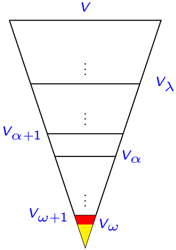
A universe of set theory

$V_0 = \emptyset$

$V_{\alpha+1} = \mathscr{P}(V_\alpha)$.

$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for a limit ordinal λ.

$V = \bigcup_{\alpha \in ORD} V_\alpha$.



Properties of the V_α-hierarchy

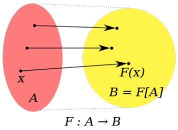
- Each V_α is transitive: if $a \in V_\alpha$ and $b \in a$, then $b \in V_\alpha$.
- If $\alpha < \beta$, then $V_\alpha \subseteq V_\beta$.

Everything we encounter in everyday mathematics is in some $V_{\omega+n}$.
Why do we study the rest of the set-theoretic universe?

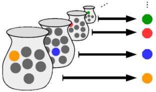
- Different universes of set theory have very different $V_{\omega+n}$!
- The properties of very large V_α affect the properties of $V_{\omega+n}$.

Zermelo-Fraenkel ZFC axioms

1. Axiom of Extensionality: If sets a and b have the same elements, then $a = b$.
2. Axiom of Pairing: For every set a and b, there is a set $\{a, b\}$.
3. Axiom of Union: For every set a, there is a set $b = \bigcup a$.
4. Axiom of Powerset: For every set a, there is a set $b = \mathscr{P}(a)$.
5. Axiom of Infinity: There exists an infinite set.
6. Axiom Schema of Separation: If $P(x)$ is a property, then for every set a, there is a set $b = \{x \in a \mid P(x) \text{ holds}\}$.
7. Axiom Scheme of Replacement: If $F(x) = y$ is a functional property and a is a set, then there is a set $b = \{F(x) \mid x \in a\}$.



8. Axiom of Regularity: Every non-empty set has an $\in$-minimal element. Equivalently there are no descending $\in$-sequences $\cdots \in a_n \in \cdots \in a_2 \in a_1 \in a_0$.
9. Axiom of Choice (AC): Every family of non-empty sets has a choice function.



The Axiom of Choice is necessary to select a set from an infinite number of socks, but not an infinite number of shoes. – Bertrand Russell

* Images credit: Wikipedia

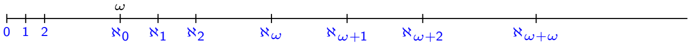
What ZFC knows and does not know

Theorem: $V = \bigcup_{\alpha \in ORD} V_\alpha$.

Theorem:

- (AC) Every set can be well-ordered.
- Every well-order is isomorphic to an ordinal.

Definition: A cardinal is an ordinal that is not bijective with any smaller ordinal.



Theorem: Every set a is bijective with a unique cardinal, which we call its cardinality $|a|$.

- A set a is countable if $|a| = \omega$, otherwise it is uncountable.
- $\aleph_1$ is the first uncountable ordinal.
- $\mathbb{R}$ (the set of reals) is uncountable.

Question: What is the cardinality of $\mathbb{R}$?

Continuum hypothesis (CH): $|\mathbb{R}| = \aleph_1$.

Pathological sets of reals

What's a set theorist? Someone who doesn't know what the real numbers are. – David Schrittesser (set theorist)

Every natural set of reals encountered in analysis has the following "regularity" properties.

- It is Lebesgue measurable.
- If uncountable, it has a perfect subset.
  - perfect set: nonempty, closed, and has no isolated points.
- It has the property of Baire.
  - A set with the property of Baire is "almost open".

Theorem: (AC)

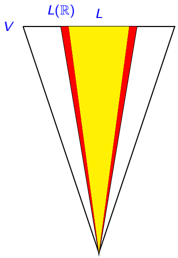
- There is a non-Lebesgue measurable set of reals.
- There is an uncountable set of reals without a perfect subset.
- There is a set of reals without the property of Baire.

Question: Is there a model of ZF (plus a small amount of choice) in which every set of reals is Lebesgue measurable, has the property of Baire, and if uncountable has a perfect subset?

Gödel's constructible universe L

Question: What happens if we construct the V_α-hierarchy by taking only subsets which we understand?

Suppose V is a universe of set theory.



The constructible hierarchy

$L_0 = \emptyset$

$L_{\alpha+1}$ is the set of all subsets of L_α given by some property $P(x)$.

$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ. $L = \bigcup_{\alpha\in\mathrm{ORD}} L_\alpha$.

Theorem: (Gödel) L satisfies $\mathrm{ZFC} + \mathrm{CH}$. The Continuum Hypothesis can hold in a universe of set theory.

L of the reals: $L(\mathbb{R})$

- $L_0(\mathbb{R}) = \mathbb{R}$
- $\vdots$

Theorem: $L(\mathbb{R})$ satisfies ZF plus a small amount of choice.

Question: Do the reals have some regularity properties in $L(\mathbb{R})$?

Cohen's Forcing

Definition: $(\mathbb{P}, \leq)$ is a partial order if it is reflexive, antisymmetric, and transitive.

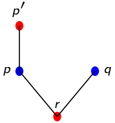
Intuition: If $p, q \in \mathbb{P}$ and $p \leq q$, then p has more information than q.

Examples:

- Finite binary sequences ordered by $s \leq t$ if s end-extends t.
- $\mathscr{P}(X)$ ordered by $A \leq B$ if $A \subseteq B$.
- Every linear order (but those are boring).

Definition:

- $D \subseteq \mathbb{P}$ is dense if for every $p \in \mathbb{P}$, there is $q \in D$ such that $q \leq p$.
  - Captures a behavior that cannot be ruled out by partial knowledge.
- $G \subseteq \mathbb{P}$ is a filter:
  - (upward closure) If $p \in G$, and $p \leq p'$, then $p' \in G$.
  - (compatibility) If $p, q \in G$, then there is $r \in G$ such that $r \leq p, q$.
- A filter $G \subseteq \mathbb{P}$ is generic if for every dense $D \subseteq \mathbb{P}$, $D \cap G \neq \emptyset$.
  - If a behavior cannot be ruled out by partial knowledge, then it occurs.



Theorem: A partial order $\mathbb{P} \in V$ cannot have a generic filter in V!

Cohen's Forcing: the big picture

A universe V together with an external generic filter G generate a larger universe: the forcing extension $V[G]$.

Analogy: Constructing the complex numbers from the reals.

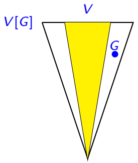
- $\mathbb{R}$ does not have $\sqrt{-1}$.
- $\mathbb{R}$ together with $\sqrt{-1}$ generate the complex numbers.

Cohen's forcing: the details

Fix a forcing notion: partial order $\mathbb{P} \in V$.

Define a collection $V^{\mathbb{P}}$ of names for elements of $V[G]$.

- Each element of $V[G]$ has a name $\tau \in V^{\mathbb{P}}$.
- An element of $V[G]$ can have more than one name.



Take a generic filter $G \notin V$ on $\mathbb{P}$.

The forcing extension $V[G] = \{\tau_G \mid \tau \in V^{\mathbb{P}}\}$ consists of the "interpretation" of all names in $V^{\mathbb{P}}$ by G.

- $V \subseteq V[G]$
- $G \in V[G]$

The forcing relation $p \Vdash P(\tau)$

- $p \in \mathbb{P}$, $\tau \in V^{\mathbb{P}}$
- $P(x)$ is a set-theoretic property

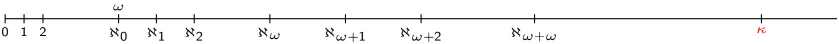
Whenever G is a generic filter and $p \in G$, then $P(\tau_G)$ holds in $V[G]$.

The Forcing Theorem: (Cohen) For every property $P(x)$, the relation $p \Vdash P(\tau)$ is expressible as a property of V.

We can talk about the forcing extension $V[G]$ inside V!

A universe in which the Continuum Hypothesis fails

Theorem: (Cohen) For ANY cardinal κ, there is a forcing extension $V[G]$ in which $|\mathbb{R}| \geq \kappa$.

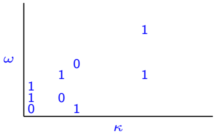


Continuum Hypothesis is independent of ZFC!

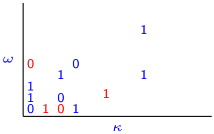
Partial order $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$

- Elements: finite partial functions $p : \omega \times \kappa \to \{0, 1\}$.
- Order: $q \leq p$ if q extends p.
- Generic filter $G : \omega \times \kappa \to \{0, 1\}$ with $G_\alpha \neq G_\beta$ for any $\alpha < \beta < \kappa$.

p :



q :

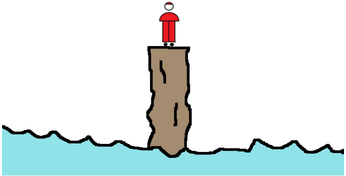


Inaccessible cardinals

The cardinal ω is inaccessible by smaller cardinals.

Suppose n is a natural number.

- $|\mathscr{P}(n)| = 2^n < \omega$.
- There is no cofinal function $f : n \to \omega$.



* Image credit: Vincenzo Dimonte

Definition: An uncountable cardinal κ is inaccessible if for every $\alpha < \kappa$:

- $|\mathscr{P}(\alpha)| < \kappa$.
- There is no cofinal function $f : \alpha \to \kappa$.

Question: Are there any inaccessible cardinals?

Theorem: It can't be that every ZFC universe has an inaccessible cardinal! **Why?**

Theorem: If κ is inaccessible, then V_κ is a ZFC universe!

Theorem (Gödel's Second Incompleteness Theorem) An axiom system extending ZFC cannot prove its own consistency.

An axiom system is consistent if a contradiction cannot be derived from it.

If every ZFC universe had an inaccessible cardinal, then ZFC would prove its own consistency.

A hierarchy of set-theoretic axioms

Definition: Suppose $\mathcal{T}$ and $\mathcal{S}$ are axiom systems.

- $\mathcal{T}$ and $\mathcal{S}$ are equiconsistent if consistency of $\mathcal{T}$ implies consistency of $\mathcal{S}$ and visa-versa.
- $\mathcal{T}$ is stronger than $\mathcal{S}$ if consistency of $\mathcal{T}$ implies consistency of $\mathcal{S}$ but not visa-versa.

$\mathrm{ZFC} + \mathrm{I}$: the axiom system ZFC together with an assertion that there is an inaccessible cardinal.

Examples

- $\mathrm{ZFC} + \mathrm{CH}$ and $\mathrm{ZFC} + \neg\mathrm{CH}$ are equiconsistent.
  - If there is a universe of $\mathrm{ZFC} + \mathrm{CH}$, then we can use forcing to construct a universe of $\mathrm{ZFC} + \neg\mathrm{CH}$ and visa versa.
- $\mathrm{ZFC} + \mathrm{I}$ is stronger than ZFC.
  - Suppose that consistency of ZFC implies consistency of $\mathrm{ZFC} + \mathrm{I}$.
  - Consider axiom *A*: There is a set universe of ZFC.
  - Every universe of $\mathrm{ZFC} + \mathrm{A}$ satisfies that ZFC is consistent, hence that $\mathrm{ZFC} + \mathrm{I}$ is consistent.
  - Every universe of $\mathrm{ZFC} + \mathrm{A}$ satisfies that $\mathrm{ZFC} + \mathrm{A}$ is consistent.
  - This violates Gödel's Second Incompleteness Theorem.

Theorem: (Solovay, Shelah) The theory

$$\mathrm{ZF} + (\text{some choice}) + \text{“}\mathbb{R} \text{ has regularity properties”}$$

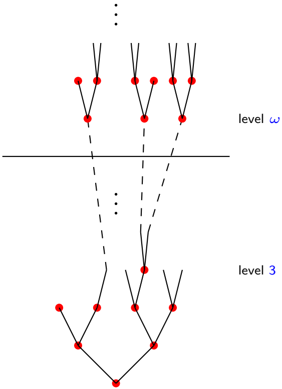
is equiconsistent with $\mathrm{ZFC} + \mathrm{I}$.

Weakly compact cardinals

Definition: A partial order T is a tree if for every $t \in T$, the set $\text{Pred}(t) = \{s \in T \mid s < t\}$ of predecessors of t in T is well-ordered (looks like an ordinal).

- Level α of T consists of all t such that $\text{Pred}(t)$ is isomorphic to α.
- The height of T is the largest ordinal β such that for all $\alpha < \beta$, T has level α.

Konig's Lemma: Every tree T of height ω all of whose levels are finite has a branch to the top.



Definition: An inaccessible cardinal κ is weakly compact if every tree of height κ all of whose levels have size less than κ has a branch to the top.

Theorem: If κ is weakly compact, then there are unboundedly many inaccessible cardinals below it. Therefore $\text{ZFC} +$ "Exists a weakly compact cardinal" is stronger than $\text{ZFC} + \text{I}$.

Filters, ultrafilters, and measures

Definition: A filter $\mathcal{F}$ on a set X is a collection of subsets of X satisfying:

- (closure under intersections) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
- (closure under superset) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.

Sets in a filter are "large".

Definition: Suppose $\mathcal{F}$ is a filter on X and κ is a cardinal.

- $\mathcal{F}$ is $<\kappa$-complete if it is closed under intersections of size less than κ.
  - We say that $\mathcal{F}$ is countably complete if it is $<\aleph_1$-complete.
- $\mathcal{F}$ is an ultrafilter if for every $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Examples

- The collection of sets of reals with Lebesgue measure 1 is a countably complete filter on $\mathbb{R}$.
- If X is a set and $a \in X$, then $\mathcal{F} = \{A \subseteq X \mid a \in A\}$ is an ultrafilter.
  - Such ultrafilters are trivial!
- Every filter is $<\omega$-complete.
- (AC) Every filter can be extended to an ultrafilter.
- Ultrafilters are measures with two values $\{0, 1\}$.

Ultrapowers of the universe: what ultrafilters are good for

Suppose $\mathcal{U}$ is an ultrafilter on a set X.

Suppose $f : X \to A$ and $g : X \to B$. Define:

- $f \sim g$ if and only if $\{x \in X \mid f(x) = g(x)\} \in \mathcal{U}$.
- $f \,\epsilon\, g$ if and only if $\{x \in X \mid f(x) \in g(x)\} \in \mathcal{U}$.
  - $\sim$ is an equivalence relation: reflexive, symmetric, transitive.
  - Let $[f]_{\mathcal{U}}$ be the equivalence class of f.
  - $[f]_{\mathcal{U}} \,\epsilon\, [g]_{\mathcal{U}}$ is well-defined.
  - For a set a, let $c_a : X \to \{a\}$ be the constant function with value a: $c_a(x) = a$.

Let W be the collection of all equivalence classes $[f]_{\mathcal{U}}$ with the membership relation ϵ.

Łoś Theorem: A property $P([f]_{\mathcal{U}})$ holds in W if and only if

$$\{x \in X \mid P(f(x)) \text{ holds in } V\} \in \mathcal{U}.$$

Corollary: There is an elementary embedding $h : V \to W$ defined by $h(a) = [c_a]_{\mathcal{U}}$: $P(a)$ holds in V if and only if $P([c_a]_{\mathcal{U}})$ holds in W.

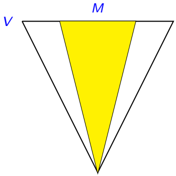
W is a universe of ZFC!

Special ultrapowers

Theorem: If $\mathcal{U}$ is a non-trivial countably complete ultrafilter, then W is isomorphic to a transitive sub-universe M of V. So there is an elementary embedding $j : V \to M$.

If $\mathcal{U}$ is not countably complete, then V sees that $\in$ is ill-founded: there is an infinite descending sequence

$$\cdots \in a_n \in \cdots \in a_2 \in a_1 \in a_0.$$



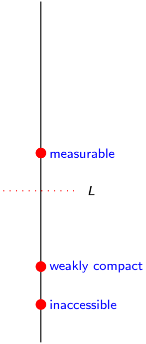
Measurable cardinals

Since every ultrafilter is $<\omega$-complete, there are many $<\omega$-complete ultrafilters on ω.

Definition: A cardinal κ is measurable if there is a $<\kappa$-complete ultrafilter on κ.

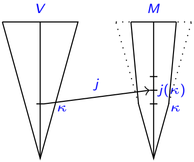
Theorem: If κ is measurable, then there are unboundedly many weakly compact cardinals below κ. Therefore
ZFC + "There exists a measurable cardinal" is stronger than
ZFC + "There exists a weakly compact cardinal".

Theorem: (Scott) There are no measurable cardinals in L.



Theorem: If κ is a measurable cardinal, then there is an elementary embedding $j : V \to M$ such that:

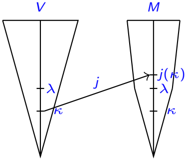
- $M \subseteq V$.
- Critical point $\mathrm{crit}(j) = \kappa$: $j(\alpha) = \alpha$ for every ordinal $\alpha < \kappa$, $j(\kappa) > \kappa$.
  - $j(x) = x$ for every $x \in V_\kappa$.
  - V and M agree up to $V_{\kappa+1}$.



Strong and supercompact cardinals

Question: Do there exist elementary embeddings $j : V \to M$ with "M close to V"?

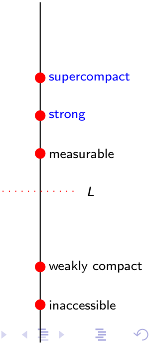
A cardinal κ is strong if for every $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $j(\kappa) > \lambda$.



- Characterized by existence of certain ultrafilters.
- For every $\lambda > \kappa$, there is $\alpha > \lambda$ and an elementary embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $V_\lambda \subseteq N$, and $j(\kappa) > \lambda$.

A cardinal κ is supercompact if for every $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $M^\lambda \subseteq M$ (every $f : \lambda \to M$ is in M), and $j(\kappa) > \lambda$.

- Characterized by existence of certain ultrafilters.
- For every $\lambda > \kappa$, there is $\alpha > \lambda$ and an elementary embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $N^\lambda \subseteq N$, and $j(\kappa) > \lambda$.



Theorem: (Woodin) Suppose there is a supercompact cardinal.

- The reals have regularity properties in $L(\mathbb{R})$.
- Forcing cannot change the properties of $L(\mathbb{R})$.

The set-theoretic multiverse and virtually large cardinals

There are universes of set-theory in which:

- CH holds,
- CH fails,
- every set is in L,
- there are various large cardinals,
- $L(\mathbb{R})$ has regularity properties,
- forcing cannot change the theory of the reals,
- etc.

We can use the multiverse view of set theory to introduce interesting new large cardinals.

Definition: A cardinal κ is virtually supercompact if in some forcing extension of V, for every $\lambda > \kappa$, there is $\alpha > \lambda$ and an elementary embedding $j : V_\alpha \to N$ with $\mathrm{crit}(j) = \kappa$, $N^\lambda \subseteq N$, and $j(\kappa) > \lambda$.

The template of virtual large cardinals applies to many large cardinals.

Theorem: (G., Schindler) Virtual large cardinals are stronger than weakly compact cardinals but much weaker than measurable cardinals. They can exist in L.

Theorem: (Schindler) The assertion that properties of $L(\mathbb{R})$ cannot be changed by proper forcing (an important class of forcing notions) is equiconsistent with a virtually supercompact cardinal.