The many universes of modern set theory

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Counting past infinity

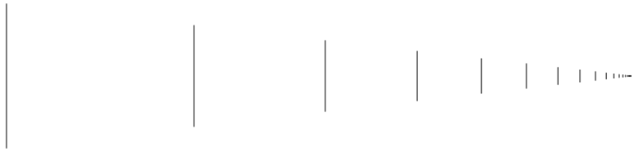
We use the natural numbers to count through mathematical objects, giving rise to recursive constructions.

In 1852, the mathematician Georg Cantor needed to iterate a mathematical operation past the natural numbers.

- Suppose X is a closed set of reals.
- Let $X' = \{x \in X \mid x \text{ is a limit point of } X\}$.
- Let $X^{(n)}$ be the result of iterating the $'$ operation n-many times.
- Past all the natural numbers, take the limit $\bigcap_n X^{(n)}$.
- Can we keep iterating?

Cantor extended the natural numbers to a (much) bigger counting system in which we can keep iterating!

Counting with natural numbers



Well-order

- Linear order
  - Reflexivity: $n \leq n$
  - Antisymmetry: if $n \leq m$ and $m \leq n$, then $n = m$
  - Transitivity: if $n \leq m$ and $m \leq k$, then $n \leq k$
  - Comparability: either $n \leq m$ or $m \leq n$
- Every subset has a least element.
  - Induction: Suppose that whenever a property $P(x)$ is true for every $n < m$, then it is also true for m. Then $P(x)$ is true for every n.
  - Justifies recursively defined operations.

Question: Can we extend the natural numbers while maintaining these key properties?

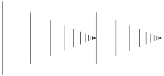
* Images credit: http://www.madore.org/ david/math/drawordinals.html

The transfinite: ordinals

Add a new number ω above the natural numbers.

- The natural numbers ($n > 0$) are successor ordinals: n is an immediate successor of $n - 1$.
- ω is a limit ordinal: it is not an immediate successor of anything.

Keep counting: $\omega + 1, \omega + 2, \omega + 3, \ldots, \omega + n, \ldots$

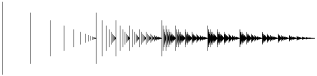


Next limit ordinals: $\omega + \omega = \omega \cdot 2, \omega \cdot 3, \ldots, \omega \cdot n, \ldots$

$\omega \cdot \omega = \omega^2, \omega^2 + \omega, \omega^2 + \omega \cdot 2, \ldots, \omega^2 + \omega \cdot n, \ldots$

$\omega^2 \cdot 2, \omega^2 \cdot 2 + \omega, \ldots, \omega^2 \cdot 2 + \omega \cdot n, \ldots, \omega^2 \cdot 3, \ldots, \omega^3, \ldots$

$\omega^\omega, \omega^{\omega \cdot 2}, \omega^\omega, \omega^{\omega^2}, \omega^{\omega^\omega}, \ldots$



* Images credit: http://www.madore.org/ david/math/drawordinals.html

The ordinals ORD

- The ordinals are well-ordered.
- Every ordinal is either 0, a successor or a limit.
- For every ordinal, there is a larger ordinal, its successor.

Question: Which mathematical structure contains the ordinals?

Introducing universes of set theory

The ordinals form the backbone of a universe of set theory $(V, \in)$.

- mathematical structure
- elements of V are sets
- $\in$ is the set membership relation.

Since all mathematical objects reduce down to sets, a universe of set theory absorbs all other mathematical structures.

A naive conception of a universe of set theory resulted in paradoxes such as the famous Russell's Paradox.

With the help of Cantor's ordinals, came the iterative conception of sets (Zermelo, von Neumann).

The universe of set theory is built up from the $\emptyset$ by iterating the powerset operation along the ordinals.

The bottom up construction avoids the known paradoxes and is formalized through the Zermelo-Fraenkel ZFC axioms.

The V_α hierarchy

Let $\mathscr{P}(X)$ denote the powerset of X: the set of all subsets of X.

$V_0 = \emptyset$

$V_1 = \{\emptyset\}$

$V_2 = \{\emptyset, \{\emptyset\}\}$

$V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ (2^2 elements)

$\vdots$

$V_\omega = \bigcup_n V_n$

$V_{\omega+1} = \mathscr{P}(V_\omega)$

$V_{\omega+2} = \mathscr{P}(V_{\omega+1})$

$\vdots$

Natural numbers

- $0 = \emptyset, 1 = \{0\} = \{\emptyset\}, 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, 3 = \{0, 1, 2\}, \ldots, n = \{0, 1, \ldots, n-1\}, \ldots$
- $n \in V_{n+1}$

Ordinals

- $\omega = \{0, 1, \ldots, n, \ldots\}$
- $\omega + 1 = \{0, 1, \ldots, n, \ldots, \omega\}$
- $\alpha = \{\xi \in \mathrm{ORD} \mid \xi < \alpha\}$
- $\alpha \in V_{\alpha+1}$

Reals

- Represent reals by subsets of natural numbers.
- Every real is in $V_{\omega+1}$.
- Every set of reals is in $V_{\omega+2}$.

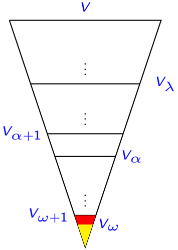
A universe of set theory

$V_0 = \emptyset$

$V_{\alpha+1} = \mathscr{P}(V_\alpha)$.

$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for a limit ordinal λ.

$V = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$.



Properties of the V_α-hierarchy

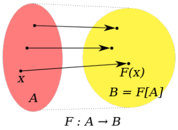
- Each V_α is transitive: if $a \in V_\alpha$ and $b \in a$, then $b \in V_\alpha$.
- If $\alpha < \beta$, then $V_\alpha \subseteq V_\beta$.

Everything we encounter in everyday mathematics is in some $V_{\omega+n}$.
Why do we study the rest of the set-theoretic universe?

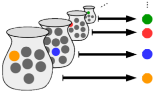
- Different universes of set theory have very different $V_{\omega+n}$!
- The properties of very large V_α affect the properties of $V_{\omega+n}$.

Zermelo-Fraenkel ZFC axioms

1. Axiom of Extensionality: If sets a and b have the same elements, then $a = b$.
2. Axiom of Pairing: For every set a and b, there is a set $\{a, b\}$.
3. Axiom of Union: For every set a, there is a set $b = \bigcup a$.
4. Axiom of Powerset: For every set a, there is a set $b = \mathscr{P}(a)$.
5. Axiom of Infinity: There exists an infinite set.
6. Axiom Schema of Separation: If $P(x)$ is a property, then for every set a, there is a set $b = \{x \in a \mid P(x) \text{ holds}\}$.
7. Axiom Schema of Replacement: If $F(x) = y$ is a functional property and a is a set, then there is a set $b = \{F(x) \mid x \in a\}$.



8. Axiom of Regularity: Every non-empty set has an $\in$-minimal element. Equivalently there are no descending $\in$-sequences $\cdots \in a_n \in \cdots \in a_2 \in a_1 \in a_0$.
9. Axiom of Choice (AC): Every family of non-empty sets has a choice function.



The Axiom of Choice is necessary to select a set from an infinite number of socks, but not an infinite number of shoes. – Bertrand Russell

* Images credit: Wikipedia

Consequences of ZFC

$V = \bigcup_{\alpha \in ORD} V_\alpha$.

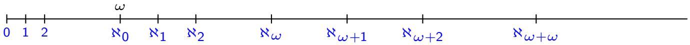
Every set is bijective with some ordinal α ($\alpha = \{\beta \in \mathrm{ORD} \mid \beta < \alpha\}$).

- Every set can be well-ordered.
- Equivalent to the Axiom of Choice over the axioms ZF.

(Cantor) $\mathscr{P}(a)$ is not bijective with a.

Definition: A cardinal is an ordinal that is not bijective with any smaller ordinal.

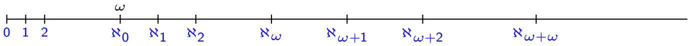
For every cardinal, there is a larger cardinal.



Every set a is bijective with a unique cardinal, which we call its cardinality $|a|$.

- A set a is countable if $|a| = \omega$, otherwise it is uncountable.
- $\aleph_1$ is the first uncountable ordinal.
- (Cantor) $\mathbb{R}$ (the set of reals) is uncountable.

The Continuum Hypothesis



Question: What is the cardinality of $\mathbb{R}$?

Continuum Hypothesis (CH): $|\mathbb{R}| = \aleph_1$.

Question: Is the Continuum Hypothesis true?

The ZFC axioms do not decide the Continuum Hypothesis.

- There are universes of set theory in which CH is true,
- and universes of set theory in which CH is false.

First Incompleteness Theorem: (Gödel) No reasonable axiomatization of sets can decide all properties of sets.

There will always be many universes of set theory!

Consequences of AC: pathological sets of reals

Every natural set of reals encountered in analysis has the following "regularity" properties.

- It is Lebesgue measurable.
- If uncountable, it has a perfect subset.
  - perfect set: nonempty, closed, and has no isolated points.
- It has the property of Baire.
  - A set with the property of Baire is "almost open".

But...

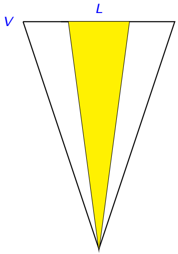
- There is a non-Lebesgue measurable set of reals.
- There is an uncountable set of reals without a perfect subset.
- There is a set of reals without the property of Baire.

Question: Is there a universe of set theory satisfying ZF (plus a little bit of choice) in which every set of reals is Lebesgue measurable, has the property of Baire, and if uncountable has a perfect subset?

Gödel's constructible universe L

Question: What happens if we construct the V_α-hierarchy by taking only subsets which we understand?

Suppose V is a universe of set theory.



The constructible hierarchy

$L_0 = \emptyset$

$L_{\alpha+1}$ is the set of all subsets of L_α given by some property $P(x)$.

$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ.

$L = \bigcup_{\alpha\in\mathrm{ORD}} L_\alpha$.

Theorem: (Gödel)

- L satisfies ZFC.
- The Continuum Hypothesis holds in L.
- Every transitive sub-universe of V contains L.

The universe $L(\mathbb{R})$

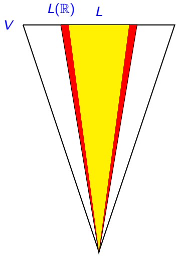
Suppose V is a universe of set theory.

The $L(\mathbb{R})$ hierarchy

$L_0(\mathbb{R}) = \mathbb{R}$

$L_{\alpha+1}(\mathbb{R})$ is the set of all subsets of $L_\alpha(\mathbb{R})$ given by some property $P(x)$.

$L_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ.
$L(\mathbb{R}) = \bigcup_{\alpha\in\mathrm{ORD}} L_\alpha(\mathbb{R})$.



$L(\mathbb{R})$ captures the set-theoretic theory of the reals.

Theorem: $L(\mathbb{R})$ satisfies ZF plus a little bit of choice.

Question: Do the reals have regularity properties in $L(\mathbb{R})$?

Cohen's Forcing

Definition: $(\mathbb{P}, \leq)$ is a partial order if it is reflexive, antisymmetric, and transitive.

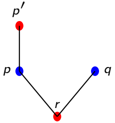
Intuition: If $p, q \in \mathbb{P}$ and $p \leq q$, then p has more information than q.

Examples:

- Finite binary sequences ordered by $s \leq t$ if s end-extends t.
- $\mathscr{P}(X)$ ordered by $A \leq B$ if $A \subseteq B$.
- Every linear order (but those are boring).

Definition:

- $D \subseteq \mathbb{P}$ is dense if for every $p \in \mathbb{P}$, there is $q \in D$ such that $q \leq p$.
  - Captures a behavior that cannot be ruled out by partial knowledge.
- $G \subseteq \mathbb{P}$ is a filter:
  - (upward closure) If $p \in G$, and $p \leq p'$, then $p' \in G$.
  - (compatibility) If $p, q \in G$, then there is $r \in G$ such that $r \leq p, q$.
- A filter $G \subseteq \mathbb{P}$ is generic if for every dense $D \subseteq \mathbb{P}$, $D \cap G \neq \emptyset$.
  - If a behavior cannot be ruled out by partial knowledge, then it occurs.



Theorem: A partial order $\mathbb{P} \in V$ cannot have a generic filter in V!

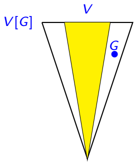
Cohen's Forcing: the big picture

A universe V together with an external generic filter G generate a larger universe: the forcing extension $V[G]$.

Analogy: Constructing the complex numbers from the reals.

- $\mathbb{R}$ does not have $\sqrt{-1}$.
- $\mathbb{R}$ together with $\sqrt{-1}$ generate the complex numbers.

Cohen's forcing: the details



Fix a forcing notion: partial order $\mathbb{P} \in V$.

Define a collection $V^{\mathbb{P}}$ of names for elements of $V[G]$.

- Each element of $V[G]$ has a name $\tau \in V^{\mathbb{P}}$.
- An element of $V[G]$ can have more than one name.

Take a generic filter $G \notin V$ on $\mathbb{P}$.

The forcing extension $V[G] = \{\tau_G \mid \tau \in V^{\mathbb{P}}\}$ consists of the "interpretation" of all names in $V^{\mathbb{P}}$ by G.

- $V \subseteq V[G]$
- $G \in V[G]$

The forcing relation $p \Vdash P(\tau)$

- $p \in \mathbb{P}$, $\tau \in V^{\mathbb{P}}$
- $P(x)$ is a set-theoretic property

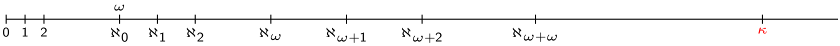
Whenever G is a generic filter and $p \in G$, then $P(\tau_G)$ holds in $V[G]$.

The Forcing Theorem: (Cohen) For every property $P(x)$, the relation $p \Vdash P(\tau)$ is expressible as a property of V.

We can talk about the forcing extension $V[G]$ inside V!

A forcing extension in which the Continuum Hypothesis fails

Theorem: (Cohen) For ANY cardinal κ, there is a forcing extension $V[G]$ in which $|\mathbb{R}| \geq \kappa$.



Continuum Hypothesis can fail badly in a universe of set theory!

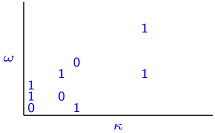
A forcing extension in which the Continuum Hypothesis fails (continued)

Partial order $\mathbb{P} = \mathrm{Add}(\omega, \kappa)$

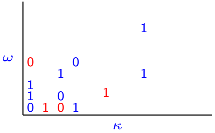
- Elements: finite partial functions $p : \kappa \times \omega \to \{0, 1\}$.
- Order: $q \leq p$ if q extends p.
- Generic filter G
  - $G : \kappa \times \omega \to \{0, 1\}$ is a total function (density)

    Let $G_\alpha : \omega \to \{0, 1\}$ be such that $G_\alpha(n) = G(\alpha, n)$.
  - $G_\alpha \neq G_\beta$ for $\alpha \neq \beta$ (density)

p :



q :



Inaccessible cardinals

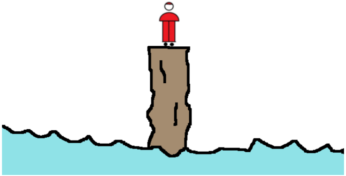
The cardinal ω is inaccessible by smaller cardinals.

Suppose n is a natural number.

- $|\mathscr{P}(n)| = 2^n < \omega$.
- There is no cofinal function $f : n \to \omega$.

Definition: An uncountable cardinal κ is inaccessible if for every $\alpha < \kappa$:

- $|\mathscr{P}(\alpha)| < \kappa$.
- There is no cofinal function $f : \alpha \to \kappa$.



* Image credit: Vincenzo Dimonte

Question: Are there any inaccessible cardinals?

Theorem: Every universe of set theory cannot have inaccessible cardinals.

Theorem: If κ is inaccessible, then V_κ is a universe of set theory satisfying ZFC!

Second Incompleteness Theorem: (Gödel) No reasonable axiom system can prove its own consistency.

An axiom system is consistent if a contradiction cannot be derived from it.

Large cardinal axioms

Axiom I: There is an inaccessible cardinal.

The axiom system $\mathrm{ZFC} + \mathrm{I}$ is stronger than ZFC.

Large cardinal axioms

- assert existence of very large infinite objects
- form a hierarchy of strong axiom systems

A hierarchy of axiom systems

Definition: Suppose $\mathcal{T}$ and $\mathcal{S}$ are axiom systems.

- $\mathcal{T}$ and $\mathcal{S}$ are equiconsistent if consistency of $\mathcal{T}$ implies consistency of $\mathcal{S}$ and visa-versa.
- $\mathcal{T}$ is stronger than $\mathcal{S}$ if consistency of $\mathcal{T}$ implies consistency of $\mathcal{S}$ but not visa-versa.

Examples

- $\mathrm{ZFC}+\mathrm{CH}$ and $\mathrm{ZFC}+\neg\mathrm{CH}$ are equiconsistent.
  - If there is a universe of $\mathrm{ZFC}+\mathrm{CH}$, then we can use forcing to construct a universe of $\mathrm{ZFC}+\neg\mathrm{CH}$ and visa versa.
- $\mathrm{ZFC}+\mathrm{I}$ is stronger than ZFC.

Theorem: (Solovay, Shelah) The theory

$$\mathrm{ZF}+\text{(some choice)}+\text{“}\mathbb{R}\text{ has regularity properties”}$$

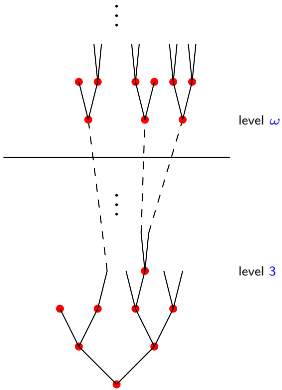
is equiconsistent with $\mathrm{ZFC}+\mathrm{I}$.

Weakly compact cardinals

Definition: A partial order T is a tree if for every $t \in T$, the set $\text{Pred}(t) = \{s \in T \mid s < t\}$ of predecessors of t in T is well-ordered.

- Level α of T consists of all t such that $\text{Pred}(t)$ is isomorphic to α.
- The height of T is the largest ordinal β such that for all $\alpha < \beta$, T has level α.

Konig's Lemma: Every tree T of height ω all of whose levels are finite has a cofinal branch.



Definition: An inaccessible cardinal κ is weakly compact if every tree of height κ all of whose levels have size less than κ has a cofinal branch.

Theorem: If κ is weakly compact, then V_κ is a universe of set theory with many inaccessible cardinals.

Weakly compact cardinals (continued)

Axiom WC: There is a weakly compact cardinal.

The axiom system $\mathrm{ZFC} + \mathrm{WC}$ is stronger than $\mathrm{ZFC} + \mathrm{I}$.

Filters, ultrafilters, and measures

Definition: A filter $\mathcal{F}$ on a set X is a collection of subsets of X satisfying:

- (closure under intersections) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
- (closure under superset) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.

Sets in a filter are "large".

Definition: Suppose $\mathcal{F}$ is a filter on X and κ is a cardinal.

- $\mathcal{F}$ is $<\kappa$-complete if it is closed under intersections of size less than κ.
  - We say that $\mathcal{F}$ is countably complete if it is $<\aleph_1$-complete.
- $\mathcal{F}$ is an ultrafilter if for every $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Examples

- The collection of sets of reals with Lebesgue measure 1 is a countably complete filter on $\mathbb{R}$.
- If X is a set and $a \in X$, then $\mathcal{F} = \{A \subseteq X \mid a \in A\}$ is an ultrafilter.
  - Such ultrafilters are trivial!
- Every filter is $<\omega$-complete.
- (AC) Every filter can be extended to an ultrafilter.
- Ultrafilters are measures with two values $\{0, 1\}$.

Ultrapowers of the universe: what ultrafilters are good for

Suppose $\mathcal{U}$ is an ultrafilter on a set X.

Suppose $f : X \to A$ and $g : X \to B$. Define:

- $f \sim g$ if and only if $\{x \in X \mid f(x) = g(x)\} \in \mathcal{U}$.
- $f \,\epsilon\, g$ if and only if $\{x \in X \mid f(x) \in g(x)\} \in \mathcal{U}$.
    - $\sim$ is an equivalence relation: reflexive, symmetric, transitive.
    - Let $[f]_{\mathcal{U}}$ be the equivalence class of f.
    - $[f]_{\mathcal{U}} \,\epsilon\, [g]_{\mathcal{U}}$ is well-defined.
    - For a set a, let $c_a : X \to \{a\}$ be the constant function with value a: $c_a(x) = a$.

Let W be the collection of all equivalence classes $[f]_{\mathcal{U}}$ with the membership relation ϵ.

Łoś Theorem: A property $P([f]_{\mathcal{U}})$ holds in W if and only if

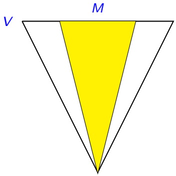
$$\{x \in X \mid P(f(x)) \text{ holds in } V\} \in \mathcal{U}.$$

Corollary: There is an elementary embedding $h : V \to W$ defined by $h(a) = [c_a]_{\mathcal{U}}$: $P(a)$ holds in V if and only if $P([c_a]_{\mathcal{U}})$ holds in W.

W is a universe of ZFC!

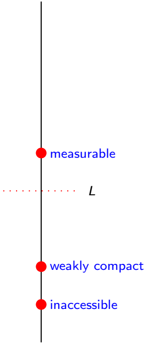
Special ultrapowers

Theorem: If $\mathcal{U}$ is a non-trivial countably complete ultrafilter, then W is isomorphic to a transitive sub-universe M of V. So there is an elementary embedding $j : V \to M$.



Measurable cardinals

Since every ultrafilter is $<\omega$-complete, there are many $<\omega$-complete ultrafilters on ω.



Definition: A cardinal κ is measurable if there is a $<\kappa$-complete ultrafilter on κ.

Theorem: If κ is measurable, then V_κ is a universe of ZFC with many weakly compact cardinals.

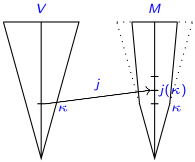
Axiom M: There is a measurable cardinal.

The axiom system $\mathrm{ZFC} + \mathrm{M}$ is stronger than $\mathrm{ZFC} + \mathrm{WC}$.

Theorem: (Scott) There are no measurable cardinals in L.

Theorem: If κ is a measurable cardinal, then there is an elementary embedding $j : V \to M$ such that:

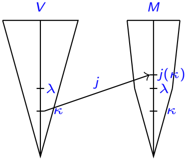
- $M \subseteq V$.
- Critical point $\mathrm{crit}(j) = \kappa$: $j(\alpha) = \alpha$ for every ordinal $\alpha < \kappa$, $j(\kappa) > \kappa$.
  - $j(x) = x$ for every $x \in V_\kappa$.
  - V and M agree up to $V_{\kappa+1}$.



Strong and supercompact cardinals

Question: Do there exist elementary embeddings $j : V \to M$ with "M close to V"?

A cardinal κ is strong if for every $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $j(\kappa) > \lambda$.



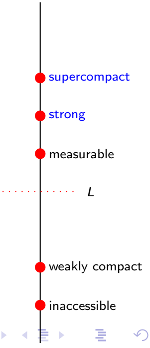
- Characterized by existence of certain ultrafilters.
- For every $\lambda > \kappa$, there is $\alpha > \lambda$ and an elementary embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $V_\lambda \subseteq N$, and $j(\kappa) > \lambda$.

A cardinal κ is supercompact if for every $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $M^\lambda \subseteq M$ (every $f : \lambda \to M$ is in M), and $j(\kappa) > \lambda$.

- Characterized by existence of certain ultrafilters.
- For every $\lambda > \kappa$, there is $\alpha > \lambda$ and an elementary embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $N^\lambda \subseteq N$, and $j(\kappa) > \lambda$.

Theorem: (Woodin) Suppose there is a supercompact cardinal.

- The reals have regularity properties in $L(\mathbb{R})$.
- Forcing cannot change the properties of $L(\mathbb{R})$.



The set-theoretic multiverse and virtually large cardinals

There are universes of set-theory in which:

- CH holds,
- CH fails,
- every set is in L,
- there are various large cardinals,
- $L(\mathbb{R})$ has regularity properties,
- forcing cannot change the theory of the reals,
- etc.

We can use the multiverse view of set theory to introduce interesting new large cardinals.

Definition: A cardinal κ is virtually supercompact if in some forcing extension of V, for every $\lambda > \kappa$, there is $\alpha > \lambda$ and an elementary embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $N^\lambda \subseteq N$, and $j(\kappa) > \lambda$.

The template of virtual large cardinals applies to many large cardinals.

Theorem: (G., Schindler) Virtual large cardinals are stronger than weakly compact cardinals but much weaker than measurable cardinals. They can exist in L.

Theorem: (Schindler) The assertion that properties of $L(\mathbb{R})$ cannot be changed by proper forcing (an important class of forcing notions) is equiconsistent with a virtually supercompact cardinal.