REFLECTION PRINCIPLES IN SET THEORY WITHOUT POWERSETS

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Abstract. The Reflection principle is the scheme of assertions that every formula is reflected by a transitive set. The Reflection principle follows from the axioms ZFC, but can fail in models of ZFC^-, set theory without powersets [FGK19]. We define that a *partial Reflection principle for* (a set or class) A is the scheme of assertions that whenever a formula $\varphi(a)$ with $a \in A$ holds, then it holds in some transitive set containing a. We show that the Reflection principle can fail in a model of ZFC^-, while the partial Reflection principle for V (all parameters are allowed) holds, separating the Reflection Principle from the strongest of the partial Reflection principles. We show that the partial Reflection principle for V can fail in a model of ZFC^-. Finally, we show the partial Reflection Principle for $\mathbb{R}$ holds in both of the two currently known models of ZFC^- in which the Reflection principle fails.

1. Introduction

The *Reflection principle* is the scheme of assertions that every formula is reflected by a transitive set, namely that for every formula $\varphi(x, a)$, with parameter a, there is a transitive set S containing a such that for all $s \in S$, $\varphi(s, a)$ holds if and only if it holds in S. The Reflection principle follows from ZFC, with the witnessing sets being elements of the V_α-hierarchy. Given a formula φ, the reflecting V_α is constructed as the union of a sequence $V_{\alpha_0} \subseteq V_{\alpha_1} \subseteq \cdots V_{\alpha_n} \subseteq \cdots$ of length ω, where each $V_{\alpha_{n+1}}$ is a closure of V_{α_n} under witnesses for all existential sub-formulas of φ, and so reflects φ by the Tarski-Vaught test. Since the argument uses only the existence of the V_α-hierarchy together with Replacement to verify that desired α exists, it goes through in ZF as well.

The definition of the V_α-hierarchy requires the existence of powersets, so it is natural to ask whether the Reflection principle continues to hold in set theories without powersets. Since some naturally equivalent versions of the ZFC axioms stop being equivalent once the Powerset axiom is removed, we end up with several versions of set theory without powersets. Without the Powerset axiom, the Replacement and Collection schemes are no longer equivalent and neither are the versions of the Axiom of Choice which we use interchangeably [Zar82]. For instance, ZFC without the Powerset axiom, with the Collection scheme instead of Replacement, although it has AC, does not imply that every set can be well-ordered. Let ZFC$-$ be the theory consisting of the axioms of ZFC (with Replacement) and the assertion that every set can be well-ordered. The theory ZFC$-$ exhibits many undesirable behaviors. It can have models in which ω_1 is a countable union of countably many sets, in which ω_1 exists, but every set of reals is countable, or where the Łoś theorem can fail for ultrapowers [GHJ16]. All these issues can be eliminated by instead taking the theory ZFC^-, where we replace the Replacement scheme with Collection, suggesting that this is the more natural version of set theory without powersets.

The most common set-theoretic structures encountered by set theorists in which the Powerset axiom fails are H_{κ^+}, the collection of all sets whose transitive closure has size at most κ, and these indeed satisfy ZFC$^-$.

The Reflection principle clearly implies Collection over the other axioms of ZFC$^-$. In ZFC$^-$, every set can be closed under witnesses for existential subformulas of a given formula, but after that it appears that some class version of dependent choice is required to iterate this construction ω-many times. Recall that the DC$_\omega$*-scheme* is the class version of dependent choice which asserts that we can make ω-many dependent choices along any definable relation without terminal nodes. More formally, the DC$_\omega$-scheme is a scheme of assertions for every formula $\varphi(x, y, a)$, with parameter a, that if for every x, there is y such that $\varphi(x, y, a)$ holds, then there is a sequence $\{b_n \mid n < \omega\}$ such that for every $n < \omega$, $\varphi(b_n, b_{n+1}, a)$ holds.[1] It is easy to see that ZFC$^-$ together with the DC$_\omega$-scheme imply the Reflection principle and indeed ZFC$^-$ together with the Reflection principle imply the DC$_\omega$-scheme because we can reflect the relation and the assertion that it has no terminal nodes to a transitive set and then use AC to construct the sequence of dependent choices. It is shown in [FGK19] that the theory ZFC$^-$ does not prove the DC$_\omega$-scheme, and so it does not prove the Reflection principle.

In this article, we consider a family of partial Reflection principles and examine their status in models of ZFC$^-$. Given a set or class A, let the *partial Reflection Principle for A* be the scheme of assertions for every formula $\varphi(a)$, with parameter $a \in A$, that if $\varphi(a)$ holds, then it holds in some transitive set. We will abbreviate the partial Reflection principle for V, where any set can be used as a parameter, as just the *partial Reflection principle* and at the other extreme, we will also call the partial Reflection principle for $\emptyset$, where no parameters are allowed, the *parameter-free Reflection principle*. Note that if we remove the requirement of transitivity from the parameter-free Reflection principle, then it is provable in ZFC$^-$ by standard proof-theoretic arguments. Freund considered some partial Reflection principles in [Fre20]. He showed that, over the axioms of ZFC$^-$ without Collection, the parameter-free Reflection principle is equivalent to the principle of induction along Δ_1-definable well-founded relations and that the partial Reflection principle for $\mathbb{R}$ is equivalent to the principle of induction along Δ_1-definable with real parameters well-founded relations [Fre20]. Partial reflection in the context of Zermelo set theory was studied by Lévy and Vaught, who showed that Zermelo set theory together with the partial Reflection principle do not imply Replacement [LV61]. More recently, Bokai Yao studied the partial Reflection principle in set theories with urelements [Yao24].

Models of ZFC$^-$ in which the DC$_\omega$-scheme fails, being the only candidates in which the partial Reflection principles can fail, are notoriously hard to construct. There are currently only two known models of ZFC$^-$ in which the DC$_\omega$-scheme fails. Both are constructed as submodels of forcing extensions by a tree iteration of Jensen's forcing. Jensen's forcing $\mathbb{J}$ is a subposet of Sacks forcing that is constructed in L using the $\diamondsuit$ principle. The poset $\mathbb{J}$ has the ccc and adds a unique generic real that is Π^1_2-definable as a singleton [Jen70]. In L, we can appropriately define finite iterations $\mathbb{J}_n$ of $\mathbb{J}$ of length n. These also have the ccc and add a unique n-length generic sequence of reals that is Π^1_2-definable (see [Abr84] and [FGK19]). Again

[1] Freund has observed that over the axioms ZFC$^-$ without Collection, the DC$_\omega$-scheme is equivalent to the principle of induction along definable well-founded relations (personal communication).

working in L, given a set or class tree T of height ω, a *tree iteration* $\mathbb{P}(\mathbb{J},T)$ *of* $\mathbb{J}$ *along* T is a poset whose elements are functions p from a finite subtree D_p of T to $\bigcup_{n<\omega}\mathbb{J}_n$ such that if t is a node on level n of D_p, then $p(t)\in\mathbb{J}_n$ and whenever $s\leq t$ in T, then $p(s)=p(t)\restriction\text{len}(s)$. The functions p are ordered so that $q\leq p$ whenever $D_p\subseteq D_q$ and for every $t\in D_p$, $q(t)\leq p(t)$. An L-generic $G\subseteq\mathbb{P}(\mathbb{J},T)$ adds a tree T^G isomorphic to T whose nodes on level n are generic n-length sequences for $\mathbb{J}_n$ and the sequences extend according to the tree order. For certain sufficiently homogeneous trees, such as $T=\omega^{<\omega}$ or $T=\omega_1^{<\omega}$, the forcing $\mathbb{P}(\mathbb{J},T)$ has the ccc and the uniqueness of generics property that the only L-generic filters for $\mathbb{J}_n$ in $L[G]$ are the nodes of T^G on level n. Also, the collection of all n-generic sequences for $\mathbb{J}_n$ over all $n<\omega$, namely the elements of T^G, is Π^1_2-definable [FGK19]. The class forcing $\mathbb{P}(\mathbb{J},\text{Ord}^{<\omega})$ along the class tree $\text{Ord}^{<\omega}$ has all the same properties as well [GM24]. Since the forcing $\mathbb{P}(\mathbb{J},\text{Ord}^{<\omega})$ has the ccc, it is easily seen to be pretame. It follows that the forcing relation for $\mathbb{P}(\mathbb{J},\text{Ord}^{<\omega})$ is definable and the forcing extension satisfies ZFC^- [Fri00]. The extension has all the same cardinals as L, but since we added class many reals, powerset of ω does not exist. The first model of ZFC^- in which the DC_ω-scheme fails is constructed in a forcing extension $L[g]$ by $\mathbb{P}(\mathbb{J},\omega_1^{<\omega})$ as the H_{ω_1} of an appropriately chosen symmetric submodel N satisfying $\text{ZF}+\text{AC}_\omega$ [FGK19]. We will refer to $H^N_{\omega_1}$ as M^g_{small}. The second model of ZFC^- in which the DC_ω-scheme fails is constructed as a submodel of a forcing extension $L[G]$ by $\mathbb{P}(\mathbb{J},\text{Ord}^{<\omega})$ [GM24]. Given a set subtree T of $\text{Ord}^{<\omega}$, let G_T be the restriction of G to $\mathbb{P}(\mathbb{J},T)$, which is easily seen to be L-generic for it (see Proposition 2.2). In $L[G]$, we let N be the union of $L[G_T]$, where T is a well-founded set subtree of $\text{Ord}^{<\omega}$. It is shown in [GM24] that N satisfies ZFC^-, but the DC_ω-scheme fails. We will refer to the model N as M^G_{large}.

In this article, we show:

Theorem 1.1. *Suppose* $g\subseteq\mathbb{P}(\mathbb{J},\omega_1^{<\omega})$ *and* $G\subseteq\mathbb{P}(\mathbb{J},\text{Ord}^{<\omega})$ *are* L*-generic.*

1. *The partial Reflection principle holds in* M^g_{small}.
2. *The partial Reflection principle for* $\{\omega_1\}$ *fails in* M^G_{large}. *Consequently, the partial Reflection principle fails in* M^G_{large}.
3. *The partial Reflection principle for* $\mathbb{R}$ *holds in* M^G_{large}. *Consequently, the parameter-free Reflection principle holds in* M^G_{large}.

Corollary 1.2. *Over* ZFC^-*, the Reflection principle is not equivalent to the partial Reflection principle.*

Corollary 1.3. *Over* ZFC^-*, the partial Reflection principle is not equivalent to the partial Reflection principle for* $\mathbb{R}$.

2. Partial reflection principles in models of ZFC^- without the DC_ω-scheme

Theorem 2.1. *Suppose* $g\subseteq\mathbb{P}(\mathbb{J},\omega_1^{<\omega})$ *is* L*-generic. The partial Reflection principle holds in the model* M^g_{small}.

Proof. Recall that M^g_{small} is the H_{ω_1} of a symmetric submodel $N\models\text{ZF}+\text{AC}_\omega$ of the forcing extension $L[g]$. So suppose that $M^g_{\text{small}}\models\varphi(r)$. Since every set in M^g_{small} is countable, we can assume without loss of generality that r is a real. Working in $L[g]$, we can construct a countable elementary submodel $\bar{M}\prec M^g_{\text{small}}$ with $r\in\bar{M}$.

The $\in$-relation on $\bar{M}$ can be coded by a subset of $\omega\times\omega$, and hence by some real a. Thus, $L[g]$ satisfies that there is a real a coding a well-founded model containing a real r_a that is isomorphic to r and satisfying $\varphi(r_a)$. Since this is a Σ^1_2-statement about r and $r\in N$, it must hold true in the symmetric submodel N as well by Shoenfield's absoluteness. Thus, N has some real b coding a well-founded model with a real r_b that is isomorphic to r and satisfying $\varphi(r_b)$. But now since $M^g_{\rm small}$ is the H_{ω_1} of N, we have $b\in M^g_{\rm small}$. Finally, since $M^g_{\rm small}$ satisfies ZFC$^-$, it can Mostowski collapse the relation on $\omega\times\omega$ coded by b to obtain a transitive model containing r and satisfying $\varphi(r)$. □

There are two standard approaches to constructing a class forcing extension of a first-order set-theoretic universe, in our case L, by a class forcing, in our case $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$. One approach, developed in [HKS18], is to force over the second-order model $(L,\in,\mathcal{C})\models$ GBC, where the classes $\mathcal{C}$ consist of the definable (with parameters) unary relations over L. The forcing language consists of the usual set $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$-names together with the class $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$-names consisting of pairs $\langle\sigma,p\rangle$ where σ is a set $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$-name and $p\in\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$. Suppose that $G\subseteq\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$ is $\langle L,\in,\mathcal{C}\rangle$-generic, meaning that it meets every dense sub-class of $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$ in $\mathcal{C}$. The forcing relation for $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$ is definable and the forcing extension is a second-order model $(L[G],\in,\mathcal{C}[G])\models$ GBC$^-$, the second-order analogue of ZFC$^-$, by the pretameness of $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$ [HKS18]. The classes $\mathcal{C}[G]$ of the forcing extension are obtained as interpretations of the class names. Note that every class in $\mathcal{C}[G]$ is definable from the class G over $L[G]$. Let $\mathcal{T}^G\in\mathcal{C}[G]$ be the class tree, isomorphic to ${\rm Ord}^{<\omega}$, of L-generic sequences for $\mathbb{J}_n$ added by G. Let $\Gamma\in\mathcal{C}[G]$ be the isomorphism between ${\rm Ord}^{<\omega}$ and $\mathcal{T}^G$ and let $\dot\Gamma$ be the canonical class $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$-name for Γ. Another approach, developed in [Fri00], is force over the first-order structure L, but to augment the forcing relation with a predicate $\dot G$ for the generic filter. The forcing extension, then consists of the first-order model $\langle L[G],\in,G\rangle$, where G is, as before, a filter meeting all the definable dense subclasses of $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$. The forcing relation is once again definable and the forcing extension satisfies ZFC$^-$ in the extended language [Fri00]. Despite the differences between the two approaches, we can clearly think of them as yielding the same forcing extension because the definable classes of the structure $\langle L[G],\in,G\rangle$ are precisely the classes $\mathcal{C}[G]$ obtained from the first approach.

Recall that given two partial orders $\mathbb{P}$ and $\mathbb{Q}$, an embedding $f:\mathbb{P}\to\mathbb{Q}$ is *complete* if it is an injection that preserves incompatibility and maximal antichains.

Proposition 2.2. *If T is a subtree of ${\rm Ord}^{<\omega}$, then $\mathbb{P}(\mathbb{J},T)$ completely embeds into $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$.*

Proof. Clearly, if two conditions are incompatible in $\mathbb{P}(\mathbb{J},T)$, then they stay incompatible in $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$. So it remains to check that every maximal antichain in $\mathbb{P}(\mathbb{J},T)$ remains maximal in $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$. Suppose that $\mathcal{A}$ is a maximal antichain of $\mathbb{P}(\mathbb{J},T)$. Fix $p\in\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$ and let $p\restriction T\in\mathbb{P}(\mathbb{J},T)$ be the restriction of p to nodes in T. Then there is $a\in\mathcal{A}$ that is compatible with $p\restriction T$ in $\mathbb{P}(\mathbb{J},T)$. We will argue that a is compatible with p in $\mathbb{P}(\mathbb{J},{\rm Ord}^{<\omega})$. Let $q\in\mathbb{P}(\mathbb{J},T)$ be such that $q\leq a,p\restriction T$. Define a condition q' with domain $D_{q'}=D_q\cup D_p$ as follows. If $s\in D_q$, then $q'(s)=q(s)$. If $s\in D_{q'}\setminus D_q$, then let i be largest such that $s\restriction i\in D_q$ and let $q'(s)$ be $q(s\restriction i)$ concatenated with the tail of $p(s)$ from i. It is easy to see that $q'\leq p,a$ in $\mathbb{P}(\mathbb{J},T)$. □

Thus, in particular, if T is a subtree of $\mathrm{Ord}^{<\omega}$, then G_T, the restriction of G to $\mathbb{P}(\mathbb{J},T)$ is L-generic. The following lemma is analogous to Theorem 3.1 in [Git24]. It is easy to see that any automorphism π of $\mathrm{Ord}^{<\omega}$ gives rise to a corresponding automorphism π^* of the forcing $\mathbb{P}(\mathbb{J},\mathrm{Ord}^{<\omega})$.

Lemma 2.3. *Suppose that $T \in L$ is a subtree of $\mathrm{Ord}^{<\omega}$. Then the only L-generic sequences for $\mathbb{J}_n$ in $L[G_T]$ are $\Gamma(s)$ for $s \in T$.*

Proof. Let $r \in L[G_T]$ be an n-length sequence of reals L-generic for $\mathbb{J}_n$. Then by the uniqueness of generic property of $\mathbb{P}(\mathbb{J},\mathrm{Ord}^{<\omega})$, $r = \Gamma(s)$ for some node s on level n of $\mathrm{Ord}^{<\omega}$. Suppose towards a contradiction that $s \notin T$. Let $i \geq 1$ be least such that $s \restriction i \notin T$. Let $\dot{r}$ be a $\mathbb{P}(\mathbb{J},T)$-name for r. Let $p \in G$ be such that $p \Vdash \dot{r} = \dot{\Gamma}(\check{s})$. Fix any condition $q \leq p$. Let $t \in \mathrm{Ord}^{<\omega}$ be a node on the same level as s such that $t \restriction i-1 = s \restriction i-1$ and $t \notin D_q \cup T \cup \{s\}$. Many such nodes t must exist since D_q and T are both sets. Let π be an automorphism of $\mathrm{Ord}^{<\omega}$ which maps $(\mathrm{Ord}^{<\omega})_s$ onto $(\mathrm{Ord}^{<\omega})_t$, while fixing everything outside these subtrees. In particular, π fixes T and $\pi(s) = t$. Let $D_{\bar{q}} = D_q \cup \pi\,``\,D_q$ and let $\bar{q}$ be the condition defined so that for $a \in D_q$, $\bar{q}(a) = \bar{q}(\pi(a)) = q(a)$. In other words, conditions on nodes in $D_q \cap (\mathrm{Ord}^{<\omega})_s$ are copied over to $(\mathrm{Ord}^{<\omega})_t$. This means, in particular, that $\pi^*(\bar{q}) = \bar{q}$. We have just argued that such conditions $\bar{q}$ are dense below p, and so some such $\bar{q} \in G$. Let $H = \pi^*\,``\,G$, which is also L-generic for $\mathbb{P}(\mathbb{J},\mathrm{Ord}^{<\omega})$, and note that $\pi^*(\bar{q}) = \bar{q} \in H$. Observe that $\dot{r}_H = \dot{r}_G = r$ since the name $\dot{r}$ only mentioned conditions with domain contained in T, and π fixes T. Since $p \geq \bar{q}$, $p \in H$. So it must be the case that $r = \dot{\Gamma}_H(s)$. But this is impossible because $\dot{\Gamma}_H(s) = \dot{\Gamma}_G(t)$ and, by genericity, $\Gamma(s) = \dot{\Gamma}_G(s) \neq \dot{\Gamma}_G(t) = \Gamma(t)$. □

Proposition 2.4. *Suppose that π is an automorphism of $\mathrm{Ord}^{<\omega}$. Then the model M^G_{large} constructed from G is the same as the model M^H_{large} constructed from $H = \pi^*\,``\,G$.*

Proof. Recall that the model M^G_{large} is the union of the models $L[G_T]$, where T is a well-founded subtree of $\mathrm{Ord}^{<\omega}$ and G_T is the restriction of G to $\mathbb{P}(\mathbb{J},T)$. The union over all well-founded subtrees $T \in L$ of $L[G_T]$ is clearly equal to the union of $L[G_{(\pi^{-1})^*\,``T}]$ because $\pi^{-1}$ is an automorphism of $\mathrm{Ord}^{<\omega}$. Next, observe that if $T \in L$ is a well-founded subtree of $\mathrm{Ord}^{<\omega}$, then $H_T = \pi^*\,``\,G_{\pi^{*-1}\,``T}$. Since $\mathbb{P}(\mathbb{J},T)$ is isomorphic to $\mathbb{P}(\mathbb{J},\pi^*\,``\,T)$ via π^*, it follows that $L[H_T] = L[G_{(\pi^{-1})^*\,``T}]$. □

Theorem 2.5. *The partial Reflection principle for $\{\omega_1\}$ fails in M^G_{large}.*

Proof. Consider the formula $\varphi(\omega_1)$ asserting that (1) L_{ω_1} exists, (2) for every $n < \omega$, the poset $\mathbb{J}_n$ exists, (3) there is a L-generic real for $\mathbb{J}$, and (4) every n-length L-generic sequence for $\mathbb{J}_n$ can be extended to an L-generic $n+1$-length sequence for $\mathbb{J}_{n+1}$. Note that if a transitive set A contains L_{ω_1}, then it can correctly identify the posets $\mathbb{J}_n$ and check for L-genericity because the posets $\mathbb{J}_n$ have the ccc. Because every node $s \in \mathrm{Ord}^{<\omega}$ is contained in some well-founded tree $T \in L$, all the L-generic sequences $\Gamma(s)$ are in M^G_{large}. Thus, in particular, M^G_{large} satisfies $\varphi(\omega_1)$. Suppose some transitive set $A \in M$ contains ω_1 and satisfies $\varphi(\omega_1)$. Then $A \in L[G_T]$ for some well-founded subtree $T \in L$. By Lemma 2.3, the L-generic n-length sequences for $\mathbb{J}_n$ contained in $L[G_T]$ are precisely the nodes $\Gamma(s)$ of $\mathcal{T}^G$ for $s \in T$. Since A is correct about an n-length sequence being L-generic for $\mathbb{J}_n$, we can apply choice in $L[G_T]$ to obtain an ω-sequence $\vec{s} = \{s_n \mid n < \omega\}$ such that each s_n is an n-length L-generic sequence for $\mathbb{J}_n$ and s_{n+1} extends s_n, but then the sequence $\vec{s}$ witnesses that the tree T is ill-founded. Thus, we have reached the desired contradiction showing that there cannot be such a set A in M^G_{large}. $\square$

Next, we would like to show that the partial Reflection principle for $\mathbb{R}$ holds in M^G_{large}. As a warm-up, we first show that the parameter-free Reflection principle holds.

Theorem 2.6. *The parameter-free Reflection principle holds in* M_{large}.

Proof. In this argument, we will view the forcing extension by $\mathbb{P}(\mathbb{J}, \text{Ord}^{<\omega})$ as the first-order structure $\langle L[G], \in G\rangle$ and the forcing relation as augmented by a predicate $\dot{G}$ for the generic filter. Let $M(x, G)$ be the formula with the predicate for G defining M^G_{large}. Suppose that M^G_{large} satisfies a sentence φ. Choose a condition $p \in \mathbb{P}(\mathbb{J}, \text{Ord}^{<\omega})$ forcing that the structure given by $M(x, \dot{G})$ satisfies φ. Choose a Σ_n-elementary substructure L_α of L, with $p \in L_\alpha$ and n large enough, so that it is correct enough about the properties of the forcing relation, and, in particular, satisfies that p forces that the structure given by $M(x, \dot{G})$ satisfies φ. Now let X be a countable elementary substructure of L_α with $p \in X$ and let $L_{\bar{\alpha}}$ be the collapse of X. Let $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, \text{Ord}^{<\omega})$ be $L_{\bar{\alpha}}$'s version of $\mathbb{P}(\mathbb{J}, \text{Ord}^{<\omega})$ and let p^* be the image of p under the collapse. Since $L_{\bar{\alpha}}$ is countable, we have in L, an $L_{\bar{\alpha}}$-generic filter g for $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, \text{Ord}^{<\omega})$ with $p^* \in g$. Since p^* is the image of p under the collapse, p^* forces, in the forcing relation for $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, \text{Ord}^{<\omega})$, that the structure given by $M(x, \dot{G})$ satisfies φ. Thus, $L_{\bar{\alpha}}[g]$ has a transitive class submodel M satisfying φ. Finally, since $M \in L \subseteq M^G_{\text{large}}$, M witnesses that there is a transitive set satisfying φ. $\square$

The above proof gives a heuristic argument that we cannot obtain a model in which the parameter-free Reflection principle fails as a submodel of a forcing extension, which is currently our only method of obtaining models of ZFC$^-$ in which Reflection fails.

Theorem 2.7. *The partial Reflection principle for* $\mathbb{R}$ *holds in* M^G_{large}.

Proof. We will once again view the forcing extension as the first-order structure $\langle L[G], \in, G\rangle$. Let $M(x, G)$ be the formula with the predicate for G defining M^G_{large}. Suppose that M^G_{large} satisfies $\varphi(r)$ for some real r. Let $T \in L$ be a well-founded subtree of $\text{Ord}^{<\omega}$ such that $r \in L[G_T]$ and let $\dot{r}$ be a nice $\mathbb{P}(\mathbb{J}, T)$-name such that $\dot{r}_{G_T} = r$. Since $\mathbb{P}(\mathbb{J}, \text{Ord}^{<\omega})$ has the ccc, the poset $\mathbb{P}(\mathbb{J}, T)$ does as well. So the name $\dot{r}$ mentions only countably many conditions. Therefore there is a countable subtree of T^* of T such that $\dot{r}$ is a $\mathbb{P}(\mathbb{J}, T^*)$-name. Thus, we can assume, by replacing T with T^*, that T is countable. By applying an automorphism to the tree $\text{Ord}^{<\omega}$, we can also assume without loss of generality such that T is a subtree of $\omega^{<\omega}$. By Proposition 2.4, for an automorphism π^* of $\mathbb{P}(\mathbb{J}, \text{Ord}^{<\omega})$ which arises from an automorphism π of the tree $\text{Ord}^{<\omega}$, we have $M^G_{\text{large}} = M^{\pi^{*}\text{"}G}_{\text{large}}$, meaning that the definition $M(x, \dot{G})$ is not affected by whether we use the generic G or $\pi\text{"}G$.

Choose a condition $p \in G$ forcing that the structure given by $M(x, \dot{G})$ satisfies $\varphi(\dot{r})$. Choose a Σ_n-elementary substructure L_α of L, with $\dot{r}, p, T \in L_\alpha$ and n large enough, so that it is correct enough about the properties of the forcing relation, and, in particular, satisfies that p forces that the structure given by $M(x, \dot{G})$ satisfies $\varphi(\dot{r})$. Now let X be a countable elementary substructure of L_α with $\dot{r}, p, T \in X$

and let $L_{\bar{\alpha}}$ be the collapse of X. Observe that an element of a poset $\mathbb{J}_n$ can be coded by a subset of ω: elements of $\mathbb{J}$ are perfect trees, and supposing inductively that every element of $\mathbb{J}_n$ can be coded by a subset of ω, an element of $\mathbb{J}_{n+1}$ is a pair $(p, \dot{q})$, where $\dot{q}$ can be assumed to be a nice $\mathbb{J}_n$-name for a perfect tree, which can be coded by a subset of ω since $\mathbb{J}_n$ has the ccc. For $n < \omega$, let $\mathbb{J}_n^{L_{\bar{\alpha}}}$ be the image of $\mathbb{J}_n$ under the collapse map, and observe that $\mathbb{J}_n^{L_{\bar{\alpha}}} = \mathbb{J}_n \cap L_{\bar{\alpha}}$ since elements of $\mathbb{J}_n$ can be coded by subsets of ω and are therefore fixed by the collapse map. Let $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, T)$ be the image of $\mathbb{P}(\mathbb{J}, T)$ under the collapse map, and observe that $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, T) = \mathbb{P}(\mathbb{J}, T) \cap L_{\bar{\alpha}}$. Because any two perfect trees have a unique greatest lower bound and all these greatest lower bounds are in $\mathbb{J}$ by the definition of $\mathbb{J}$ (see [FGK19]), whenever two perfect trees make it into a generic for $\mathbb{J}$, so does the greatest lower bound. From $\mathbb{J}$, we can easily extend this property all $\mathbb{J}_n$, and hence to the to the poset $\mathbb{P}(\mathbb{J}, T)$. This property combined with the ccc imply that the restriction G_T^* of G_T to $\mathbb{P}(\mathbb{J}, T) \cap L_{\bar{\alpha}}$ is $L_{\bar{\alpha}}$-generic.

Let $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, \text{Ord}^{<\omega}) = \mathbb{P}(\mathbb{J}, \bar{\alpha}^{<\omega}) \cap L_{\bar{\alpha}}$ be $L_{\bar{\alpha}}$'s version of $\mathbb{P}(\mathbb{J}, \text{Ord}^{<\omega})$. Let p^* be the image of p under the collapse map. Since $\dot{r}$ is a nice $\mathbb{P}(\mathbb{J}, T)$-name and elements of $\mathbb{P}(\mathbb{J}, T)$ mentioned in $\dot{r}$ are fixed by the collapse map, then so is $\dot{r}$. Thus, $\dot{r}_{G_T^*} = r$, and so $r \in L_{\bar{\alpha}}[G_T^*]$. Since $L \subseteq M^G_{\text{large}}$ and $G_T \in M^G_{\text{large}}$, it follows that $G_T^* \in M^G_{\text{large}}$.

Next, observe that the forcing $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, T)$ completely embeds into $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, \text{Ord}^{<\omega})$ by elementarity combined with Proposition 2.2. Thus, we can express $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, \text{Ord}^{<\omega})$ as the quotient forcing $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, T) * \mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, \text{Ord}^{<\omega})/\dot{G}$, where, as usual, $\dot{G}$ is a $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, T)$-name for the generic filter. Since $p \in G$, $p \cap T \in G_T$ and $p \cap T$ is fixed by the collapse map. Thus, $p \cap T \in G_T^*$.

Since $L_{\bar{\alpha}}[G_T^*]$ is a countable set in M^G_{large}, we can choose, in M^G_{large}, some $L_{\bar{\alpha}}[G_T^*]$-generic filter h for $\mathbb{P}^{L_{\bar{\alpha}}}(\mathbb{J}, \text{Ord}^{<\omega})/G_T^*$ with $p^* \in G_T^* * h$. This is possible since $p \cap T \in G_T^*$. Since, by elementarity, p^* forces that the structure given by $M(x, \dot{G})$ satisfies $\varphi(\dot{r})$ and $\dot{r}_{G_T^*} = r$, it follows that $L_{\bar{\alpha}}[G_T^*][h]$ has a transitive class model satisfying $\varphi(r)$. This model is a transitive set in M^G_{large}.

□

3. Questions

The two currently available models of ZFC$^-$ in which the Reflection principle fails both satisfy the partial Reflection principle for $\mathbb{R}$. So it remains to determine whether ZFC$^-$ implies the parameter-free Reflection principle and the partial Reflection principle for $\mathbb{R}$. As the proofs of Theorem 2.5 and Theorem 2.7 demonstrate, a model of ZFC$^-$ in which either of these principles fails is not going to be constructed as a submodel of a forcing extension.

Question 3.1. Does ZFC$^-$ imply the parameter-free Reflection principle?

Question 3.2. Does ZFC$^-$ imply the partial Reflection principle for $\mathbb{R}$?

A related question about Reflection involving the ill-behaved theory ZFC− was asked by Bokai Yao. As was noted earlier, ZFC− together with the Reflection principle (even without Replacement) imply Collection.

Question 3.3. (Bokai Yao) Does ZFC− together with the partial Reflection Principle imply Collection?

This question is once again difficult to answer with the available techniques because all known models of ZFC− are constructed according to the following general scheme (see [GHJ16]). We take an appropriately chosen product forcing $\mathbb{P}_\alpha$ of length some ordinal α and let G be V-generic for it. We construct the model of ZFC− as the union W of the forcing extensions $V[G_\xi]$ by the initial segments $\mathbb{P}_\xi$ of the product forcing. The model W has all the initial segment generics G_ξ for $\xi < \alpha$, but cannot collect them into a set because then it would have the full generic G_α. The model W satisfies that there is a V-generic filter for every product $\mathbb{P}_\xi$ with $\xi < \alpha$, but no sufficiently large transitive set can reflect this statement because it has to be an element of some $V[G_\xi]$. Note that we can ensure that the set is correct about V-genericity by including in it, via the use of a parameter, $P^V(\mathbb{P}_\alpha)$.

References

[Abr84] Uri Abraham. A minimal model for ¬CH: iteration of Jensen's reals. *Trans. Amer. Math. Soc.*, 281(2):657–674, 1984.

[FGK19] Sy-David Friedman, Victoria Gitman, and Vladimir Kanovei. A model of second-order arithmetic satisfying AC but not DC. *J. Math. Log.*, 19(1):1850013, 39, 2019.

[Fre20] Anton Freund. Set-theoretic reflection is equivalent to induction over well-founded classes. *Proc. Amer. Math. Soc.*, 148(10):4503–4515, 2020.

[Fri00] Sy D. Friedman. *Fine structure and class forcing*, volume 3 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2000.

[GHJ16] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone. What is the theory ZFC without power set? *MLQ Math. Log. Q.*, 62(4-5):391–406, 2016.

[Git24] Victoria Gitman. Parameter-free schemes in second-order arithmetic. *To appear in the Journal of Symbolic Logic*, 2024.

[GM24] Victoria Gitman and Richard Matthews. ZFC without power set II: reflection strikes back. *Fund. Math.*, 264(2):149–178, 2024.

[HKS18] Peter Holy, Regula Krapf, and Philipp Schlicht. Characterizations of pretameness and the Ord-cc. *Ann. Pure Appl. Logic*, 169(8):775–802, 2018.

[Jen70] Ronald Jensen. Definable sets of minimal degree. In *Mathematical logic and foundations of set theory (Proc. Internat. Colloq., Jerusalem, 1968)*, pages 122–128. North-Holland, Amsterdam, 1970.

[LV61] A. Lévy and R. Vaught. Principles of partial reflection in the set theories of Zermelo and Ackermann. *Pacific J. Math.*, 11:1045–1062, 1961.

[Yao24] Bokai Yao. Axiomatization and forcing in set theory with urelements. *To appear in the Journal of Symbolic Logic*, 2024.

[Zar82] Andrzej Zarach. Unions of ZF^--models which are themselves ZF^--models. In *Logic Colloquium '80 (Prague, 1980)*, volume 108 of *Stud. Logic Foundations Math.*, pages 315–342. North-Holland, Amsterdam, 1982.

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