The emerging zoo of second-order set theories

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Classes in first-order set theory

A (proper) class is a collection of sets that is "too big" to be a set.

Suppose $\langle V, \in \rangle \models \mathrm{ZFC}$ is a model of set theory.

Definition:

- A class is a first-order definable with parameters collection of sets.
- A proper class is a class that is not a set.

Proper classes:

- ORD - the collection of all ordinals
- L - Gödel's constructible universe
- HOD - the hereditarily ordinal definable sets
- An elementary embedding $j : V \to M$ (assuming large cardinals)
- The Easton partial order $\mathbb{P}$ - forces GCH to fail at every regular cardinal

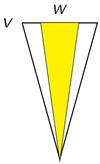
Natural questions:

- How to do we talk about non-definable collections of sets?
- How do we quantify over classes?

Why classes?

Inner models: transitive sub-models of $\mathrm{ZF(C)}$ containing all ordinals

- Inner model reflection principle (Barton, Caicedo, Fuchs, Hamkins, Reitz)
  Whenever a first-order formula $\varphi(a)$ holds in V, then it holds in some inner model $W \subsetneq V$.



- Inner model hypothesis (Friedman)
  Whenever a first-order sentence φ holds in some inner model of a universe $V^* \supseteq V$, then it already holds in some inner model of V.

Why classes?

Elementary embeddings

Reinhardt's Axiom: There is an elementary embedding $j : V \to V$.

- **Theorem**: (Kunen) If $V \models \mathrm{ZFC}$, there is no elementary embedding $j : V \to V$. Thus, Reinhardt's Axiom is inconsistent.
- If $V \models \mathrm{ZF}$, then there is no definable elementary embedding $j : V \to V$.
- Open question: If $V \models \mathrm{ZF}$, can there be an elementary $j : V \to V$?

Class partial orders

- Force global changes to the GCH pattern.
- Force $V = \mathrm{HOD}$ (code every set into the continuum pattern).
- Force to make the universe partially unchangeable by forcing.

Truth predicates (Coming up!)

Second-order set theory

There are two sorts of objects: sets and classes.

Syntax: Two-sorted logic

- Separate variables for sets and classes
- Separate quantifiers for sets and classes
- Convention: upper-case letters for classes, lower-case letters for sets
- Notation:
  - Σ^0_n - first-order Σ_n-formula
  - Σ^1_n - n-alternations of class quantifiers followed by a first-order formula

Semantics: A model is a triple $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$.

- V consists of the sets.
- $\mathcal{C}$ consists of the classes.
- Every set is a class: $V \subseteq \mathcal{C}$.
- $C \subseteq V$ for every $C \in \mathcal{C}$.

Alternatively, we can formalize second-order set theory in first-order logic:

- objects are classes,
- define that a set is a class that is an element of some class.

Gödel-Bernays set theory GBC

Axioms

- Sets: ZFC
- Classes:
  - Extensionality
  - Replacement: If F is a function and a is a set, then $F \upharpoonright a$ is a set.
  - Global well-order: There is a class well-order of sets.
  - Comprehension scheme for first-order formulas:

    If $\varphi(x, A)$ is a first-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Models

- Gödel's constructible universe L together with its definable classes is a model of GBC.
- Suppose $\langle V, \in \rangle \models \mathrm{ZFC}$ and V has a definable global well-order. Then V together with its definable classes is a model of GBC.
- **Theorem**: (Solovay) Every countable model of ZFC can be extended to a model of GBC with the same sets. (Force to add a global well-order class.)

Strength

- GBC is equiconsistent with ZFC.
- GBC has the same first-order consequences as ZFC.

Truth Predicates

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: A class $T \in \mathcal{C}$ is a truth predicate for $\langle V, \in\rangle$ if it satisfies Tarksi's truth conditions: For every $\ulcorner\varphi\urcorner \in V$ (φ possibly nonstandard),

- if φ is atomic, $V \models \varphi(\bar{a})$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$,
- $\ulcorner\neg\varphi(\bar{a})\urcorner \in T$ iff $\ulcorner\varphi(\bar{a})\urcorner \notin T$,
- $\ulcorner\varphi(\bar{a}) \wedge \psi(\bar{a})\urcorner \in T$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$ and $\ulcorner\psi(\bar{a})\urcorner \in T$,
- $\ulcorner\exists x\varphi(x, \bar{a})\urcorner \in T$ iff $\exists b\, \ulcorner\varphi(b, \bar{a})\urcorner \in T$.

Observation: If T is a truth predicate, then $V \models \varphi(\bar{a})$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$.

Theorem: (Tarski) A truth predicate is never definable over V.

Corollary: GBC cannot prove that there is a truth predicate.

Observation: If T is a truth predicate and $\ulcorner\varphi\urcorner \in \mathrm{ZFC}^{\mathrm{V}}$ (φ possibly nonstandard), then $\varphi \in T$.

Theorem: If there is a truth predicate $T \in \mathcal{C}$, then V is the union of an elementary chain of its rank initial segments V_α:

$$V_{\alpha_0} \prec V_{\alpha_1} \prec \cdots \prec V_{\alpha_\xi} \prec \cdots \prec V,$$

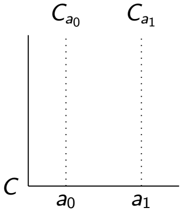
such that V thinks that each $V_{\alpha_\xi} \models \mathrm{ZFC}$.

Let $\langle V_\alpha, \in, T \cap V_\alpha\rangle \prec_{\Sigma_2} \langle V, \in, T\rangle$.

Iterated truth predicates

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBC}$.

Definition: Suppose $A \in \mathcal{C}$ is a class. A sequence of classes $\langle C_a \mid a \in A \rangle$ is a single class C such that $C_a = \{x \mid \langle a, x \rangle \in C\}$.



Definition: A sequence of classes $\vec{T} = \langle T_\beta \mid \beta < \alpha \rangle$ is an iterated truth predicate of length α if for every $\beta < \alpha$, T_β is a truth predicate for $\langle V, \in, \vec{T} \upharpoonright \beta \rangle$.

Given a class well-order Γ, similarly define the iterated truth predicate of length Γ.

- $\Gamma = \mathrm{ORD}$
- $\Gamma = \mathrm{ORD} + \mathrm{ORD}$
- $\Gamma = \mathrm{ORD} \cdot \omega$

$\mathrm{GBC} + \Sigma^1_1$-Comprehension

Axioms

- GBC
- Comprehension for Σ^1_1-formulas:
  If $\varphi(x, A) := \exists X \psi(x, X, A)$ with ψ first-order, then $\{x \mid \varphi(x, A)\}$ is a class.

Note: Σ^1_1-Comprehension is equivalent to Π^1_1-Comprehension.

Strength

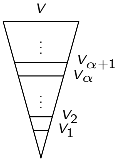
- **Theorem**: $\mathrm{GBC} + \Sigma^1_1$-Comprehension proves that there is a truth predicate.
  - Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC} + \Sigma^1_1$-Comprehension.
  - $\mathscr{V}$ satisfies the second-order assertion: $\exists\, \Sigma^0_0$-truth predicate and $\forall n \in \omega\, (\exists\, \Sigma^0_n$-truth predicate $\rightarrow \exists\, \Sigma^0_{n+1}$-truth predicate).
  - (induction) $\mathscr{V}$ satisfies: $\forall n \in \omega\, \exists\, \Sigma^0_n$-truth predicate.
  - (comprehension) $\mathscr{V}$ satisfies: There is a sequence of classes $\langle T_n \mid n < \omega\rangle$ such that for all $n < \omega$, T_n is a Σ^0_n-truth predicate.
- $\mathrm{GBC} + \Sigma^1_1$-Comprehension proves $\mathrm{Con}(\mathrm{ZFC})$, $\mathrm{Con}(\mathrm{Con}(\mathrm{ZFC}))$, etc.
- $\mathrm{GBC} + \Sigma^1_1$-Comprehension proves that for every class well-order Γ, there is an iterated truth predicate of length Γ.

The constructible universe

Theorem: (Definition by transfinite recursion.) Every recursive definition has a solution. Given a definable recursion rule $G : V \to V$, there is a definable solution $F : \mathrm{ORD} \to V$ such that $F(\alpha) = G(F \restriction \alpha)$.

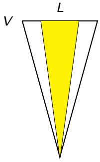
Definition: The von Neumann hierarchy.

- $V_0 = \emptyset$
- $V_{\alpha+1}$ consists of all subsets of V_α.
- $V_\lambda = \bigcup_{\alpha<\beta} V_\alpha$ for a limit λ.
- $V = \bigcup_{\alpha\in\mathrm{ORD}} V_\alpha$.



Definition: (Gödel) The constructible universe L.

- $L_0 = \emptyset$
- $L_{\alpha+1}$ consists of all definable subsets of L_α.
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ.
- $L = \bigcup_{\alpha\in\mathrm{ORD}} L_\alpha$.



Theorem: (Gödel) If $V \models \mathrm{ZF}$, then $L \models \mathrm{ZFC}$.

The second-order constructible universe

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC} + \Sigma^1_1$-Comprehension.

Definition: A meta-ordinal is a well-order $\Gamma \in \mathcal{C}$.

- $\mathrm{ORD} + \mathrm{ORD}$, $\mathrm{ORD} \cdot \omega$.

Theorem: $\mathrm{GBC} + \Sigma^1_1$-Comprehension proves that any two meta-ordinals are comparable.

Problem: Meta-ordinals don't need to have unique representations! Unless...

Given a meta-ordinal Γ, we can build the meta-constructible universe L_Γ up to Γ.

Definition: A meta-ordinal Γ is constructible if there is another meta-ordinal Δ such that L_Δ has a well-order of ORD isomorphic to Γ. ("$\Gamma \in L_{\mathrm{ORD}^+}$")

Theorem: (Tharp) Constructible meta-ordinals have unique representations.

Definition:

- A class $A \in \mathcal{C}$ is constructible if there is a constructible meta-ordinal Γ such that $A \in L_\Gamma$.
- The second-order constructible universe is $\mathscr{L} = \langle L, \in, \mathcal{L}\rangle$, where $\mathcal{L}$ consists of the constructible classes.

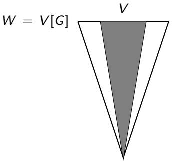
Theorem: $\mathscr{L} \models \mathrm{GBC} + \Sigma^1_1$-Comprehension. It also satisfies a version of the Axiom of Choice for classes. (Coming up!)

The Forcing Theorem

Suppose $\langle V, \in\rangle \models \mathrm{ZFC}$ and $\mathbb{P} \in V$ is a partial order.

The forcing construction: (Cohen) Build a model $W \models \mathrm{ZFC}$ extending V.

- Define a collection $V^{\mathbb{P}}$ of names for elements of W.
  - Each element of W has a name $\tau \in V^{\mathbb{P}}$.
  - An element of W can have more than one name.
- $G \notin V$ is a generic filter on $\mathbb{P}$: G meets every dense set $D \in V$ of $\mathbb{P}$.
- The forcing extension is $W = V[G] = \{\tau_G \mid \tau \in V^{\mathbb{P}}\}$.
  - $V[G] \models \mathrm{ZFC}$
  - $\mathrm{ORD}^V = \mathrm{ORD}^{V[G]}$



The forcing relation $p \Vdash \varphi(\tau)$

- $p \in \mathbb{P}$, $\tau \in V^{\mathbb{P}}$
- φ is a first-order formula

Whenever G is a generic filter and $p \in G$, then $V[G] \models \varphi(\tau_G)$.

The Forcing Theorem: (Cohen) For every first-order formula $\varphi(x)$, the relation $p \Vdash \varphi(\tau)$ is definable.

The definition of the forcing relation is given by a transfinite recursion.

The Class Forcing Theorem

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is a class partial order.

The forcing construction: Build a model $\mathscr{W} = \langle W, \in, \mathcal{C}^*\rangle$ extending $\mathscr{V}$.

- Define a collection $V^{\mathbb{P}}$ of names for elements of W.
- Define a collection $\mathcal{C}^{\mathbb{P}}$ of names for elements of $\mathcal{C}^*$.
- $G \notin \mathcal{C}$ is a generic filter on $\mathbb{P}$: G meets every dense class $D \in \mathcal{C}$ of $\mathbb{P}$.
- The forcing extension is $\mathscr{W} = \mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G]\rangle$:
  - $V[G] = \{\tau_G \mid \tau \in V^{\mathbb{P}}\}$
  - $\mathcal{C}[G] = \{\Gamma_G \mid \Gamma \in \mathcal{C}^{\mathbb{P}}\}$

Note: $\mathscr{V}[G]$ may not satisfy GBC.

The forcing relation $p \Vdash \varphi(\tau, \Gamma)$
Whenever G is a generic filter and $p \in G$, then $\mathscr{V}[G] \models \varphi(\tau_G, \Gamma_G)$.

Class Forcing Theorem: For every first-order formula $\varphi(x, X)$ and a name $\Gamma \in C^{\mathbb{P}}$, the relation $p \Vdash \varphi(\tau, \Gamma)$ is a class in $\mathcal{C}$.

Theorem: (Krapf, Njegomir, Holy, Lüke, Schlicht) The Class Forcing Theorem can fail in a model of GBC.
There is a class partial order such that the truth predicate is definable from its forcing relation for $\varphi(x, y) := $ "$x = y$".

Theorem: $\mathrm{GBC} + \Sigma^1_1$-Comprehension proves the class forcing theorem.

Games and determinacy

Let ω^ω be the topological space of ω-length sequences of natural numbers with the product topology (ω is given the discrete topology).

Fix $A \subseteq \omega^\omega$.

The Game $\mathcal{G}_A$

- Player I (Alice) and Player II (Bob) alternately play numbers for ω-many steps.

I	a_0	a_1	a_2	$\ldots$	a_n
II	b_0	b_1	b_2	$\ldots$	b_n

- Alice wins if $\vec{a} = \langle a_0, b_0, a_1, b_1 \ldots, a_n, b_n \ldots \rangle \in A$. Otherwise, Bob wins.
- The game is determined if one of the players has a winning strategy.

Theorem: (Gale, Stewart) If A is open, then the game $\mathcal{G}_A$ is determined.

- Suppose Alice does not have a winning strategy. There is no move a_0 that guarantees a win for Alice.
- There is a move b_0 for Bob such that Alice doesn't have a winning strategy in the game starting with $\langle a_0, b_0 \rangle$.
- There is no move a_1 that guarantees a win for Alice in the game starting with $\langle a_0, b_0 \rangle$.
- There is a move b_1 for Bob such that Alice doesn't have a winning strategy in the game starting with $\langle a_0, b_0, a_1, b_1 \rangle$.
- Bob wins because A open and Alice didn't win at any finite stage of the game.

Note: Everything here generalizes to spaces X^ω, where X is a set with the discrete topology.

Class games and determinacy

Let ORD^ω be the topological space of ω-length sequences of ordinals.
Fix a class $A \subseteq \mathrm{ORD}^\omega$.

The Game $\mathcal{G}_A$

- Player I (Alice) and Player II (Bob) alternately play ordinals for ω-many steps.

I	α_0		α_1		α_2		$\dots$		α_n	
II		β_0		β_1		β_2		$\dots$		β_n

- Alice wins if $\vec{\alpha} = \langle \alpha_0, \beta_0, \alpha_1, \beta_1 \dots, \alpha_n, \beta_n \dots \rangle \in A$. Otherwise, Bob wins.
- The game is determined if one of the players has a winning strategy.

Note: A strategy for a player in the game $\mathcal{G}_A$ is a class.

Open Class Determinacy: $\mathcal{G}_A$ is determined for every open $A \subseteq \mathrm{ORD}^\omega$.

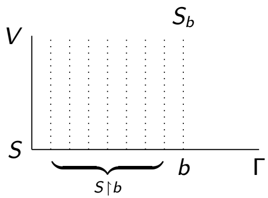
Clopen Class Determinacy: $\mathcal{G}_A$ is determined for every clopen $A \subseteq \mathrm{ORD}^\omega$.

Theorem: (G., Hamkins) $\mathrm{GBC}+\Sigma^1_1$-Comprehension implies Open Class Determinacy.

Elementary Transfinite Recursion ETR

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: Suppose $\Gamma \in \mathcal{C}$ is a meta-ordinal. A solution along Γ to a first-order recursion rule $\varphi(x, b, F)$ is a sequence of classes S such that for every $b \in \Gamma$, $S_b = \varphi(x, b, S \upharpoonright b)$.



Elementary Transfinite Recursion ETR: For every meta-ordinal Γ, every first-order recursion rule $\varphi(x, b, S)$ has a solution along Γ.

ETR_Γ: Elementary transfinite recursion for a fixed Γ.

- $\mathrm{ETR}_{\mathrm{ORD}\cdot\omega}$, $\mathrm{ETR}_{\mathrm{ORD}}$, ETR_ω

Theorem: (Williams) If $\Gamma \geq \omega^\omega$ is a (meta)-ordinal, then $\mathrm{GBC} + \mathrm{ETR}_{\Gamma\cdot\omega}$ implies $\mathrm{Con}(\mathrm{GBC} + \mathrm{ETR}_\Gamma)$.

Theorem: $\mathrm{GBC}+\Sigma^1_1$-Comprehension implies ETR.

Consequences of ETR

Theorem: $\mathrm{GBC} + \mathrm{ETR}_\omega$ proves that there exists a truth predicate.

Use Tarskian truth recursion on complexity of formulas.

Theorem: $\mathrm{GBC} + \mathrm{ETR}$ proves that for every meta-ordinal Γ, there is a constructible meta-universe L_Γ.

Theorem: (Fujimoto)

- Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$ and $\Gamma \geq \omega^\omega$ is a meta-ordinal. ETR_Γ is equivalent to the existence of an iterated truth predicate of length Γ.
- Over GBC, ETR is equivalent to the existence of an iterated truth predicate of length Γ for every meta-ordinal Γ.

Note: A single recursion, the iterated truth recursion, suffices to give all other recursions.

Theorem: (G., Hamkins, Holy, Schlicht, Williams) Over GBC, $\mathrm{ETR}_{\mathrm{ORD}}$ is equivalent to the Class Forcing Theorem.

Theorem: (G., Hamkins) Over GBC, ETR is equivalent to Clopen Class Determinacy.

Beyond ETR

Theorem: (Sato) Over GBC, Open Class Determinacy is stronger than ETR.

Question: Can forcing add meta-ordinals?

Theorem: (Hamkins, Woodin) Over GBC, Open Class Determinacy implies that forcing does not add meta-ordinals.

The petting zoo of second-order set theory

GBC + Σ^1_1-Comprehension

↓

GBC + Open Determinacy

↓

GBC + For every well-order Γ, there is an iterated truth predicate of length Γ

GBC + Clopen Class Determinacy

GBC + ETR

↓

GBC + There is an iterated truth predicate of length ORD

GBC + Class Forcing Theorem

GBC + $\mathrm{ETR}_{\mathrm{ORD}}$

↓

GBC + ETR_ω

↓

GBC + There exists a truth predicate

↓

GBC

Kelley-Morse set theory KM

Axioms

- GBC
- Full comprehension:
  If $\varphi(x, A)$ is a second-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Models

Suppose $\langle V, \in\rangle \models \mathrm{ZFC}$ and $\kappa \in V$ is an inaccessible cardinal.
$\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}$.

Theorem: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}$. Then its constructible universe $\mathscr{L} = \langle L, \in, \mathcal{L}\rangle \models \mathrm{KM}$. It also satisfies a version of the Axiom of Choice for classes. (Coming up!)

The Łoś Theorem for ultrapowers

Suppose $\langle V, \in\rangle \models \mathrm{ZFC}$.

- Suppose U is an ultrafilter on a cardinal κ.
- Define that functions $f : \kappa \to V$ and $g : \kappa \to V$ are equivalent when

$$\{\xi < \kappa \mid f(\xi) = g(\xi)\} \in U.$$

- Let M be the collection of the equivalence classes $[f]_U$.
- Define that $[f]_U \, \mathrm{E} \, [g]_U$ when $\{\xi < \kappa \mid f(\xi) \in g(\xi)\} \in U$.
- $\langle M, \mathrm{E}\rangle$ is the ultrapower of $\langle V, \in\rangle$ by U.
- Define $j : V \to M$ by $j(a) = [c_a]_U$ ($c_a(\xi) = a$ for all $\xi < \kappa$).

Łoś Theorem:

- $\langle M, \mathrm{E}\rangle \models \varphi([f]_U)$ if and only if $\{\xi < \kappa \mid \varphi(f(\xi))\} \in U$.
- The map j is an elementary embedding from $\langle V, \in\rangle$ to $\langle M, \mathrm{E}\rangle$.

- True for atomic formulas by definition.
- True for Boolean combinations by properties of ultrafilter.
- Suppose true for $\varphi(x, y)$.
- Suppose $\{\xi < \kappa \mid \exists x \varphi(x, f(\xi))\} \in U$.
- Choose a witness x_ξ for $f(\xi)$ and define $g(\xi) = x_\xi$ (uses Axiom of Choice)).

The Łoś Theorem for second-order ultrapowers

Suppose $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{KM}$.

- Suppose U is an ultrafilter on a cardinal κ.
- Define that functions $f : \kappa \to V$ and $g : \kappa \to V$ are equivalent when $\{\xi < \kappa \mid f(\xi) = g(\xi)\} \in U$.
- Let M be the collection of the equivalence classes $[f]_U$.
- Define that $[f]_U \, \mathrm{E} \, [g]_U$ when $\{\xi < \kappa \mid f(\xi) \in g(\xi)\} \in U$.
- Define that class sequences F and G are equivalent when $\{\xi < \kappa \mid F_\xi = G_\xi\} \in U$.
- Let $\mathcal{M}$ be the collection of the equivalence classes $[F]_U$.
- $[f]_U \, \mathrm{E} \, [F]_U$ when $\{\xi < \kappa \mid f(\xi) \in F(\xi)\} \in U$.
- $\langle M, \mathrm{E}, \mathcal{M}\rangle$ is the ultrapower of $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$ by U.
- Define $j : \langle V, \in, \mathcal{C}\rangle \to \langle M, \mathrm{E}, \mathcal{M}\rangle$ by $j(a) = [c_a]_U$ and $j(A) = [C_A]_U$.

The Łoś Theorem

Problem: For the class existential quantifier, we need

$$\mathscr{V} \models \forall \xi < \kappa \exists X \varphi(X, f(\xi)) \to \exists Y \forall \xi < \kappa \varphi(Y_\xi, f(\xi)).$$

Choice Scheme

The Choice Scheme is a version of the Axiom of Choice for classes.

Choice Scheme: Given a second-order formula $\varphi(x, X, A)$, if for every set x, there is a class X witnessing $\varphi(x, X, A)$, then there is a single class Y, collecting witnesses for every x:

$$\forall x \exists X \varphi(x, X, A) \rightarrow \exists Y \forall x \varphi(x, Y_x, A).$$

Choice Scheme over sets: Given a second-order formula $\varphi(x, X, A)$ and a set a:

$$\forall x \in a \exists X \varphi(x, X, A) \rightarrow \exists Y \forall x \in a \varphi(x, Y_x, A).$$

Theorem: (G., Johnstone, Hamkins) There is a model of KM in which the Choice Scheme fails for for ω-many choices for a first-order formula.

Theorem: (G., Johnstone, Hamkins) Over KM, the Łoś Theorem for second-order ultrapowers is equivalent to the Choice Scheme over sets.

The theory $\mathrm{KM}+\mathrm{AC}$

$\mathrm{KM}+\mathrm{AC}$: KM together with the Choice Scheme.

Models

- Suppose $\langle V,\in\rangle\models\mathrm{ZFC}$ and $\kappa\in V$ is an inaccessible cardinal. $\langle V_\kappa,\in,V_{\kappa+1}\rangle\models\mathrm{KM}+\mathrm{AC}$.
- Suppose $\mathscr{V}=\langle V,\in,\mathcal{C}\rangle\models\mathrm{KM}$. Then $\mathscr{L}=\langle L,\in,\mathcal{L}\rangle\models\mathrm{KM}+\mathrm{AC}$.

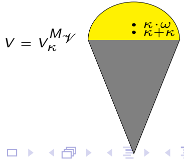
Theorem: The theories KM and $\mathrm{KM}+\mathrm{AC}$ are equiconsistent.

Moving back to first-order

Suppose $\mathscr{V}=\langle V,\in,\mathcal{C}\rangle\models\mathrm{KM}+\mathrm{AC}$.

- View each extensional well-founded class relation $R\in\mathcal{C}$ as coding a transitive set.
- Define a membership relation E on the collection of all such relations R (modulo isomorphism).
  - $\mathrm{ORD}+\mathrm{ORD}$, $\mathrm{ORD}\cdot\omega$.
  - $V\cup\{V\}$.
- Let $\langle M_{\mathscr{V}},\mathsf{E}\rangle$, the companion model of $\mathscr{V}$, be the resulting first-order structure.

- $M_{\mathscr{V}}$ has the largest cardinal $\kappa\cong\mathrm{ORD}^{\mathscr{V}}$.
- $V_\kappa^{M_{\mathscr{V}}}\cong V$.
- $\mathcal{P}(V_\kappa)^{M}_{\mathscr{V}}\cong\mathcal{C}$.
- $\langle M_{\mathscr{V}},\mathsf{E}\rangle\models\mathrm{ZFC}^-_{\mathrm{I}}$. (Next slide!)



The theory $\mathrm{ZFC}_{\mathrm{I}}^{-}$

Axioms

- ZFC without powerset (Collection scheme instead of Replacement scheme).
- There is the largest cardinal κ.
- κ is inaccessible.
  - κ is regular
  - for all $\alpha < \kappa$, 2^α exists and $2^\alpha < \kappa$.

Models

Suppose $\langle V, \in\rangle \models \mathrm{ZFC}$ and a cardinal $\kappa \in V$ is inaccessible.
Let H_{κ^+} be the collection of all sets of hereditary size at most κ.
$H_{\kappa^+} \models \mathrm{ZFC}_{\mathrm{I}}^{-}$.

Moving to second-order

Suppose $M \models \mathrm{ZFC}_{\mathrm{I}}^{-}$ with the largest cardinal κ.

- $V = V_\kappa^M$
- $\mathcal{C} = \{X \in M \mid X \subseteq V_\kappa^M\}$
- $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM} + \mathrm{AC}$
- $M_{\mathscr{V}} \cong M$ is the companion model of $\mathscr{V}$

Theorem: (Mostowski) The theory $\mathrm{KM} + \mathrm{AC}$ is bi-interpretable with the theory $\mathrm{ZFC}_{\mathrm{I}}^{-}$.

Reflection principles

Reflection Principle: Every formula is reflected by a transitive set:

For every first-order formula $\varphi(x)$, there is a transitive set M such that for all $a \in M$, $\varphi(a)$ holds if and only if $M \models \varphi(a)$.

Theorem: (Lévy) ZFC proves the reflection principle.
Every first-order formula is reflected by some V_α.

Class Reflection Principle: Every formula is reflected by a sequence of classes:

For every second-order formula $\varphi(X)$, there is a sequence of classes $S = \langle S_\xi \mid \xi \in \mathrm{ORD}\rangle$ such that for all $\xi \in \mathrm{ORD}$, $\varphi(S_\xi)$ if and only if $\langle V, \in, S\rangle \models \varphi(S_\xi)$.

Theorem: Suppose $\mathscr{V} \models \mathrm{KM} + \mathrm{AC}$ and $M_{\mathscr{V}} \models \mathrm{ZFC}^-_{\mathrm{I}}$ is its companion model.
Then $\mathscr{V}$ satisfies the Class Reflection Principle if and only if $M_{\mathscr{V}}$ satisfies the Reflection Principle.

Dependent Choice Scheme

The Dependent Choice Scheme is a version of the Axiom of Dependent Choice for classes.

Dependent Choice Scheme: Given a second-order formula $\varphi(X, Y, A)$, if for every class X, there is another class Y such that $\varphi(X, Y, A)$ holds, then we can make ω-many dependent choices according to φ:

$$\forall X \exists Y \varphi(X, Y, A) \to \exists Y \forall n \varphi(Y_n, Y_{n+1}, A).$$

Theorem: Over $\mathrm{KM} + \mathrm{AC}$, the Dependent Choice Scheme is equivalent to the Class Reflection Principle.

Open question: Does $\mathrm{KM} + \mathrm{AC}$ prove the Dependent Choice Scheme?

Conjecture: (Friedman, G.) $\mathrm{KM} + \mathrm{AC}$ does not prove the Dependent Choice Scheme.

The theory $\mathrm{KM} + \mathrm{AC} + \mathrm{DC}$

$\mathrm{KM} + \mathrm{AC} + \mathrm{DC}$: $\mathrm{KM} + \mathrm{AC}$ together with the Dependent Choice Scheme.

Models

- Suppose $\langle V, \in \rangle \models \mathrm{ZFC}$ and $\kappa \in V$ is an inaccessible cardinal.
  $\langle V_\kappa, \in, V_{\kappa+1} \rangle \models \mathrm{KM} + \mathrm{AC} + \mathrm{DC}$.
- Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM}$. Then $\mathscr{L} = \langle L, \in, \mathcal{L} \rangle \models \mathrm{KM} + \mathrm{AC} + \mathrm{DC}$.

Theorem: The theories KM and $\mathrm{KM} + \mathrm{AC} + \mathrm{DC}$ are equiconsistent.

Naturalist approach to forcing

Question: How do we make sense of the generic filter G being outside V?

Theorem: Every partial order $\mathbb{P}$ densely embeds into a complete Boolean algebra $\mathbb{B}$.

Elements of $\mathbb{B}$ are the regular cuts of $\mathbb{P}$.

Theorem: Densely embeddable partial orders produce the same forcing extensions.

Suppose a partial order $\mathbb{P}$ densely embeds into another partial order $\mathbb{Q}$.

- If G^* is a generic filter for $\mathbb{Q}$, then in a forcing extension $V[G^*]$, we can define a generic filter G for $\mathbb{P}$ such that $V[G^*] = V[G]$.
- If G is a generic filter for $\mathbb{P}$, then in a forcing extension $V[G]$, we can define a generic filter G^* for $\mathbb{Q}$ such that $V[G^*] = V[G]$.

Suppose $\mathbb{B}$ is a complete Boolean algebra and U is an ultrafilter on $\mathbb{B}$.

- Build the definable Boolean valued model $V^{\mathbb{B}}$.
- Use U to turn $V^{\mathbb{B}}$ into a definable model $(W, \mathrm{E}) \models \mathrm{ZFC}$.
- $W = \bar{V}[G]$ is a forcing extension of a submodel $\bar{V}$ by a complete Boolean algebra $\bar{\mathbb{B}}$.
- There is an elementary embedding $j : V \to \bar{V}$ such that $j(\mathbb{B}) = \bar{\mathbb{B}}$.

Given a complete Boolean algebra $\mathbb{B}$, we can build inside V a definable model $(\bar{V}, \mathrm{E})$:

- There is an elementary embedding $j : V \to \bar{V}$.
- V has a generic filter G for $j(\mathbb{B}) = \bar{\mathbb{B}}$.
- V can build the model $\bar{V}[G]$.

Naturalist approach to class forcing

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBC}$.

Definition: A class Boolean algebra $\mathbb{B}$ is class complete if every class antichain of $\mathbb{B}$ has a supremum.

Theorem: (Krapf, Njegomir, Holy, Lüke, Schlicht) A class Boolean algebra $\mathbb{B}$ with proper class antichains cannot be class complete.

Definition: A hyperclass is a (second-order) definable collection of classes.

Theorem: Every class partial order $\mathbb{P}$ can be embedded into a hyperclass class complete Boolean algebra $\mathbb{B}$.

Problem: How do we do the Boolean valued model construction with a hyperclass Boolean algebra?

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM} + \mathrm{AC}$ and $M_{\mathscr{V}} \models \mathrm{ZFC}^-_{\mathrm{I}}$ is its companion model.

- The hyperclass Boolean algebra $\mathbb{B}$ is a class of $M_{\mathscr{V}}$.
- In $M_{\mathscr{V}}$, we can build the Boolean valued model $M^{\mathbb{B}}_{\mathscr{V}}$.
- **Problem**: $M_{\mathscr{V}}$ may not have any ultrafilters on $\mathbb{B}$.
- **Solution**: $M_{\mathscr{V}}$ needs to have a definable global well-order. Equivalently $\mathscr{V}$ needs to have a hyperclass well-order of $\mathcal{C}$.

The strongest hypothesis

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM} + \mathrm{AC} + \mathrm{DC}$ and $M_{\mathscr{V}} \models \mathrm{ZFC}^-_{\mathrm{I}}$ is its companion model.

- $M_{\mathscr{V}}$ has a class partial order $\mathbb{P}$ such that:
  - $M_{\mathscr{V}}[G] \models \mathrm{ZFC}^-_{\mathrm{I}}$ has the form $L[A]$ for some $A \subseteq \kappa$,
  - $V_\kappa^{M_{\mathscr{V}}[G]} = V_\kappa^{M_{\mathscr{V}}}$.
- $M_{\mathscr{V}}[G]$ has a definable global well-order.
- Let $\bar{\mathscr{V}} = \langle \bar{V}, \in, \bar{\mathcal{C}} \rangle \models \mathrm{KM} + \mathrm{AC} + \mathrm{DC}$ such that $M_{\mathscr{V}}[G] = M_{\bar{\mathscr{V}}}$.
  - $\bar{V} = V$
  - $\mathcal{C} \subseteq \bar{\mathcal{C}}$
  - $\bar{V}$ has a canonical hyperclass well-order of classes.

The strongest hypothesis: $\mathrm{KM} + \mathrm{AC} + \mathrm{DC}$ and there exists a canonical hyperclass well-order of classes.

Zoo top tier

KM + There is a canonical hyperclass well-order of classes

$\downarrow$

KM + AC + Class Reflection Principle

KM + AC + DC

$\downarrow$

KM + AC

$\downarrow$

KM + Łoś Theorem for second-order ultrapowers

KM + Choice Scheme over sets

$\downarrow$

KM

Thank you!