

ZFC WITHOUT POWER SET II: REFLECTION STRIKES BACK

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ABSTRACT. The theory ZFC implies the scheme that for every cardinal δ we can make δ many dependent choices over any definable relation without terminal nodes. Friedman, the first author, and Kanovei constructed a model of ZFC^- (ZFC without power set) with largest cardinal ω in which this principle fails for ω many choices. In this article we study failures of dependent choice principles over ZFC^- by considering the notion of big proper classes. A proper class is said to be big if it surjects onto every non-zero ordinal.

We shall see that if one assumes the scheme of dependent choices of any arbitrary set length then every proper class is indeed big. However, by building on work of Zarach, we provide a general framework for separating dependent choice schemes of various lengths by producing models of ZFC^- with proper classes that are not big. Using a similar idea, we then extend the earlier result by producing a model of ZFC^- in which there are unboundedly many cardinals but the scheme of dependent choices of length ω still fails.

Finally, the second author has proven that a model of ZFC^- cannot have a non-trivial, cofinal, elementary self-embedding for which the von-Neumann hierarchy exists up to its critical point. We answer a related question posed by the second author by showing that the existence of such an embedding need not imply the existence of any non-trivial fragment of the von-Neumann hierarchy. In particular, that in such a situation $\mathcal{P}(\omega)$ can be a proper class.

1. INTRODUCTION

Many natural set-theoretic structures satisfy all the axioms of ZFC excluding the power set axiom. These include the structures H_{κ^+} (the collection of all sets whose transitive closure has size at most κ for a cardinal κ), forcing extensions of models of ZFC by pretame (but not tame) class forcing, and first-order structures bi-interpretable with models of the strong second-order set theory Kelley-Morse together with the choice scheme. The set theory that these structures satisfy is the theory ZFC^- , whose axioms consist of the axioms of ZFC with the collection scheme in place of the replacement scheme and with the well-ordering principle (the assertion that every set can be well-ordered) in place of the axiom of choice (the assertion that every non-empty family of sets has a choice function). The reason for the particular choice of axioms comprising ZFC^- is that without the existence of power sets we lose certain equivalences between set theoretic assertions that we tend to take for granted.

Definition 1.1.

- Let ZF^- be the theory ZF with the power set axiom removed. That is, ZF^- consists of the axioms: extensionality, empty set, pairing, unions, infinity, the foundation scheme, the separation scheme and the replacement scheme.
- Let ZFC^- denote the theory ZF^- plus the well-ordering principle.
- Let $\text{ZF}(\text{C})^-$ denote the theory $\text{ZF}(\text{C})^-$ plus the collection scheme.

Szczepaniak showed that the axiom of choice is not equivalent to the well-ordering principle over ZF^- (see [Zar82]) and therefore we choose to take the stronger principle when formulating the theory ZFC^- . Zarach showed that the theory ZFC^- does not imply the collection scheme [Zar96]. The first author et al. showed in [GHJ16] that the theory ZFC^- has many other undesirable behaviors: there are models of ZFC^- in which ω_1 is singular, in which every set of reals is countable but ω_1 exists, and in which the Łoś Theorem fails for (class) ultrapowers.

Although the theory ZFC^- avoids these pathological behaviors, there is a number of useful properties of models of full ZFC that fail or are not known to hold in models of ZFC^- , mostly as a consequence of the absence in these models of a hierarchy akin to the von Neumann hierarchy. It is known that ground model definability, the assertion that the model is definable in its set forcing extensions, can fail in models of ZFC^- [GJ14]. The intermediate model theorem, the assertion that any intermediate model between the model and its set-forcing extension is also its set-forcing extension, can fail [AFG21]. If there is a non-trivial elementary embedding $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$, namely the large cardinal axiom I_1 holds, then it gives rise to an elementary embedding $j^+: H_{\lambda^+} \rightarrow H_{\lambda^+}$ which witnesses that Kunen's Inconsistency can fail for models of ZFC^- [Mat22]. It is an open question whether HOD , the collection of all hereditarily ordinal definable sets, is definable in models of ZFC^- .

One of the main themes of this article is the various ways in which the scheme version of dependent choice can fail in models of ZFC^- .

Definition 1.2. The DC_δ -scheme, for an infinite cardinal δ , asserts for every formula $\varphi(x, y, a)$ that if for every set x , there is a set y such that $\varphi(x, y, a)$ holds, then there is a function f on δ such that for every $\xi < \delta$, $\varphi(f \upharpoonright \xi, f(\xi), a)$ holds.

The $DC_{<Ord}$ -scheme is the scheme asserting that the DC_δ -scheme holds for every cardinal δ .

In other words, the DC_δ schemes states that we can make δ -many dependent choices along any definable relation without terminal nodes. The DC_δ -scheme generalizes the dependent choice axiom DC_δ which makes the analogous assertion for set relations. The $DC_{<Ord}$ -scheme follows from ZFC by reflecting the definable relation in question to some V_α , and then using a well-ordering of V_α to obtain the sequence of dependent choices. It follows that the $DC_{<Ord}$ -scheme holds in every structure H_{κ^+} . Also, by the preservation of the DC_δ -scheme, for any δ , by pretame forcing (see Proposition 2.7), forcing extensions of a model of ZFC by pretame forcing satisfy the the $DC_{<Ord}$ -scheme.

The DC_δ -schemes have numerous applications. Over ZFC^- , the DC_ω -scheme is equivalent to the *reflection principle* which is the assertion that every formula reflects to a transitive set [FGK19] (see Theorem 2.1). Although, there is no known reformulation of the DC_δ -scheme for uncountable δ in terms of a reflection principle, such a reformulation exists under mild existence of power set assumptions (see Theorem 2.2). Over ZFC^- , for a regular cardinal δ , the DC_δ -scheme implies that every proper class surjects onto δ . It is not difficult to see that in a model of ZFC the class partial order $Add(Ord, 1)$, whose conditions are partial functions on the ordinals ordered by extension, forces a global well-order without adding sets. The forcing $Add(Ord, 1)$ does not add sets because it is set-distributive. In a model of ZFC^- , the forcing $Add(Ord, 1)$ is set-distributive if and only if the $DC_{<Ord}$ -scheme

holds. Thus, in a model of $ZFC^- + DC_{<Ord}$ -scheme we can force a global well-order without adding sets, and actually conversely if we can force a global well-order without adding sets using some forcing, then $Add(Ord, 1)$ must be set-distributive. Thus, over models of ZFC^- , the $DC_{<Ord}$ -scheme is equivalent to being able to force a global well-order without adding sets. We will see another application of the $DC_{<Ord}$ -scheme shortly to establishing a form of Kunen's Inconsistency for models of ZFC^- .

Friedman et al. [FGK19] showed that the DC_ω -scheme can fail in a model of ZFC^- . The counterexample model is the H_{ω_1} of a symmetric submodel of a forcing extension by the iteration of Jensen's forcing along the tree $\omega_1^{<\omega}$ (see section 2.4 for details on Jensen's forcing and this result), in particular, ω is the largest cardinal in this model.

In this article we obtain the following failures of the various DC_δ -schemes in models of ZFC^- .

Theorem 3.5. *Suppose that $V \models ZFC + CH$. Then every Cohen forcing extension of V has a proper class transitive submodel satisfying ZFC^- in which the DC_ω -scheme holds, but the DC_{ω_2} -scheme fails.*

The model was constructed by Zarach in [Zar82], but we show that the DC_{ω_2} -scheme fails there. The status of the DC_{ω_1} -scheme in this model is open. Note that this model, unlike the counterexample model of [FGK19], must have unboundedly many cardinals by virtue of being a proper class transitive submodel of a model of ZFC.

Theorem 4.5. *Suppose that $V \models ZFC + 2^\delta = \delta^+$ for some regular cardinal δ . Then every forcing extension of V by the poset $Add(\delta, 1)$ has a proper class transitive submodel satisfying ZFC^- in which the DC_δ -scheme holds, but the $DC_{\delta^{++}}$ -scheme fails.*

The model above is constructed by generalizing Zarach's construction, however in these models we are unable to determine the status of the $DC_{\delta^{++}}$ -scheme. On the other hand, we are able to remove this gap by using Jensen's forcing instead.

Theorem 5.1. *Every forcing extension of L by the ω -length finite support product of Jensen's forcing has a proper class submodel satisfying ZFC^- in which the DC_ω -scheme holds, but the DC_{ω_1} -scheme fails.*

Theorem 5.2. *Suppose that κ is inaccessible in L . Every forcing extension of L by the κ -length bounded support product of the inaccessible Jensen's forcing has a proper class transitive submodel satisfying ZFC^- in which the DC_κ -scheme holds, but the DC_{κ^+} -scheme fails.*

The inaccessible Jensen's forcing is a generalization of Jensen's forcing to a subposet of the κ -Sacks forcing, for an inaccessible cardinal κ , that was recently developed in [FG21] (see section 2.4 for details). Using this idea of union models we extend the result of [FGK19] to obtain a model of ZFC^- in which the reflection principle fails and for which there are unboundedly many cardinals.

Theorem 6.3, 6.4. *Every forcing extension of L by the iteration of Jensen's forcing along the class tree $Ord^{<\omega}$ has a proper class transitive submodel N satisfying ZFC^- , with unboundedly many cardinals, in which the DC_ω -scheme fails.*

The results of this article were originally motivated by an open question from the work of the second author on Kunen's Inconsistency in models of ZFC^- [Mat22]. Suppose that $W \models \text{ZFC}^-$ and $A \subseteq W$. We will say that $W \models \text{ZFC}_A^-$ if W continues to satisfy ZFC^- in the language expanded by a predicate for A .

Theorem 1.3 ([Mat22]). *Suppose that $W \models \text{ZFC}^-$. There is no non-trivial, cofinal, Σ_0 -elementary embedding $j: W \rightarrow W$ such that $V_{\text{crit}(j)}$ exists in W and $W \models \text{ZFC}_j^-$.*

Thus, in particular, the elementary embedding $j^+: H_{\lambda^+} \rightarrow H_{\lambda^+}$ resulting from an I_1 -embedding $j: V_{\lambda+1} \rightarrow V_{\lambda+1}$ cannot be cofinal. Moreover, the second author showed that if the model W additionally satisfies the $\text{DC}_{<\text{Ord}}$ -scheme, then the existence of $V_{\text{crit}(j)}$ follows from the other assumptions [Mat22]. Thus, we have:

Theorem 1.4 ([Mat22]). *Suppose that $W \models \text{ZFC}^- + \text{DC}_{<\text{Ord}}$ -scheme. There is no non-trivial, cofinal, Σ_0 -elementary embedding $j: W \rightarrow W$ such that $W \models \text{ZFC}_j^-$.*

As a step toward determining whether the existence of $V_{\text{crit}(j)}$ is truly necessary for Theorem 1.3, the second author asked:

Question 1.5. Suppose that $W \models \text{ZFC}_j^-$ for a non-trivial, cofinal $j: W \rightarrow M$ with $M \subseteq W$. Does $V_{\text{crit}(j)}$ exist in W , does $P(\omega)$ exist in W ?

We answer the question negatively here using models of ZFC^- in which the DC_δ -scheme fails for some δ .

Theorem 7.2. *There is a model $W \models \text{ZFC}^-$ in which $\mathcal{P}(\omega)$ does not exist and which has a definable, non-trivial, cofinal, elementary embedding $j: W \rightarrow M \subseteq W$.*

2. PRELIMINARIES

2.1. Reflection. Although, the connection between the DC_δ -schemes and reflection is not necessary for any of the arguments in this article, we nevertheless devote this section to exploring this unexpected connection. The connection also lends to the title of the paper.

Theorem 2.1. *Over ZFC^- , the DC_ω -scheme is equivalent to the scheme asserting for every formula $\psi(\vec{x}, a)$ that there is a transitive set M with $a \in M$ reflecting $\psi(\vec{x}, a)$.*

We sketch the proof of this result, which has appeared in [FGK19].

Proof. First, observe that the DC_ω -scheme is equivalent to the DC_ω^* -scheme asserting that for every definable relation $\varphi(x, y, a)$ without terminal nodes, there is a sequence $\langle b_i \mid i < \omega \rangle$ such that $\varphi(b_i, b_{i+1}, a)$ holds for every $i < \omega$.

Suppose that W satisfies ZFC^- and the reflection assertion. Suppose that $\varphi(x, y, a)$ is a relation without terminal nodes. Let M be a transitive set with $a \in M$ which reflects $\varphi(x, y, a)$ together with the assertion that $\varphi(x, y, a)$ has no terminal nodes. Since M is a set, there is a well-ordering w of M in W . Let b_0 be the least element of M according to w . Let b_1 be the least element of M according to w such that $M \models \varphi(b_0, b_1, a)$, which exists since M knows that $\varphi(x, y, a)$ has no terminal nodes. Given that we have chosen $b_n \in M$, let b_{n+1} be the least element of M according to w such that $M \models \varphi(b_n, b_{n+1}, a)$. Clearly, the sequence $\langle b_i \mid i < \omega \rangle$ witnesses the DC_ω^* -scheme for $\varphi(x, y, a)$.

Next, suppose that $W \models \text{ZFC}^- + \text{DC}_\omega$ -scheme. The result will follow from induction on formulas with the only critical case being those formulas of the form $\exists x\psi(x, u)$. So suppose that the statement has been proven for $\psi(x, u)$. Observe that, by collection and the induction hypothesis, for any set A there is a transitive set A^ψ containing A which reflects $\psi(x, u)$ and such that

$$\forall u \in A \exists x \psi(x, u) \longrightarrow \forall u \in A \exists x \in A^\psi \psi(x, u).$$

Fix a set a . Let the formula $\varphi(x, y, a)$ assert that whenever x is a sequence of some finite length n such that $x_0 = \{a\}$, and $x_{i+1} = x_i^\psi$, then $y = x_{n-1}^\psi$. By the above argument, this relation has no terminal nodes. Using the DC_ω -scheme, the union of an ω -sequence of dependent choices along $\varphi(x, y, a)$ is a transitive set reflecting $\exists x\psi(x, u)$ and containing a . \square

It is worth noting why the above argument does not also show that the reflection principle implies the DC_δ -scheme for uncountable cardinals. The issue is that the transitive set M need not be closed under infinite sequences which are elements of W . Therefore, if we reflect our formula to some arbitrary set M our attempt to externally choose the b_α may fail because we cannot ensure at limit stages that our collection of previous choices forms a set in M .

Unfortunately, we do not know whether there is reformulation of the DC_δ -scheme for uncountable δ in terms of some reflecting principle. However, we do have the following weaker result.

Theorem 2.2. *Suppose that W satisfies ZFC^- and δ is a regular cardinal in W such that $\gamma^{<\delta}$ exists for every cardinal γ . Then in W , the DC_δ -scheme holds if and only if for every formula $\psi(\vec{x}, a)$, there is a transitive set M with $a \in M$ and $M^{<\delta} \subseteq M$ reflecting $\psi(\vec{x}, a)$.*

Proof. Suppose that W satisfies the reflection assertion. Fix a definable relation $\varphi(x, y, a)$ without terminal nodes and let M be a transitive set with $a \in M$ and $M^{<\delta} \subseteq M$ which reflects $\varphi(x, y, a)$ and the assertion that this relation has no terminal nodes. We construct a sequence of δ -many dependent choices as in the proof of Theorem 2.1, using the closure of M to get through the limit stages in the construction.

Suppose next that the DC_δ -scheme holds in W . We will say that a set A^ψ is a δ -transitive closure of a set A for a formula ψ if it is a transitive set containing A which reflects ψ , is closed under existential witnesses for ψ from W , and closed under $<\delta$ -sequences. We need the assumption that $\gamma^{<\delta}$ exists for every cardinal γ to ensure that every set can be closed under $<\delta$ -sequences. From here the argument proceeds exactly as in the proof of Theorem 2.1. \square

2.2. Big classes. Given a cardinal δ , let us say that a class is δ -big if it surjects onto δ . We will say that a class is simply big if it surjects onto every cardinal. It is easy to see that proper classes don't need to be big in weak set theories. For example, consider the model $L_{\aleph_\omega^L}$, which satisfies Kripke-Platek set theory, KP. The cardinals of $L_{\aleph_\omega^L}$ is a proper class from the point of view of this model, but this class obviously cannot surject onto \aleph_1^L because externally we know that it is countable. We will see in Sections 3 and 4 that proper classes do not need to be big in models of ZFC^- either. However, $\text{ZFC}^- + \text{DC}_{<\text{Ord}}$ -scheme implies that every proper class is big.

Proposition 2.3. *In a model of ZFC^- , the DC_δ -scheme implies that every proper class is δ -big. It follows that over ZFC^- , the $\text{DC}_{<\text{Ord}}$ -scheme implies that every proper class is big.*

Proof. Let $W \models \text{ZFC}^- + \text{DC}_\delta$ -scheme for some regular cardinal δ . Consider a proper class \mathcal{A} defined by a formula $\psi(x, a)$. Let $\varphi(x, y, a)$ be a formula asserting that whenever x is a function on an ordinal ξ such that $x(\eta) \in \mathcal{A}$ for all $\eta < \xi$, then $y \in \mathcal{A}$ and $y \neq x(\eta)$ for any $\eta < \xi$. Since \mathcal{A} is a proper class, the relation $\varphi(x, y, a)$ has no terminal nodes. Thus, by the DC_δ -scheme, there is a function f on δ such that for all $\xi < \delta$, $\varphi(f \upharpoonright \xi, f(\xi), a)$ holds. The function f gives a subset of \mathcal{A} of cardinality δ . \square

The model constructed in [FGK19] to show that the DC_ω -scheme can fail also shows that the converse to Proposition 2.3 does not hold. Recall that the model is the H_{ω_1} of a model $W \models \text{ZF}$. Additionally, $W \models \text{AC}_\omega$ (see [FGK19]). But AC_ω implies that every set surjects onto ω , the largest cardinal of the model. Thus, in that model, every class is big. In Section 6, we will strengthen this by showing that the DC_ω -scheme can fail in a model of ZFC^- with unboundedly many cardinals in which every proper class is big. On the other hand, adding small proper classes will be one of our main tools in this article for constructing models of ZFC^- with various violations of the DC_δ -scheme.

Theorem 1.4 from the introduction was obtained by showing that whenever we have a model $W \models \text{ZFC}^-$, in which every proper class is big and $j: W \rightarrow M \subseteq W$ is an elementary embedding with a critical point, $V_{\text{crit}(j)}$ exists in W [Mat22]. We will quickly reprove the theorem here to emphasize the exact assumptions and demonstrate how big classes are used in the proof.

Theorem 2.4. *Suppose that $W \models \text{ZFC}^-$ and every proper class is big in W . If $j: W \rightarrow M \subseteq W$ is an elementary embedding with critical point κ , then V_κ exists in W .*

Note that we are not assuming that $W \models \text{ZFC}_j^-$ or that j is cofinal.

Proof. First, observe that if $\alpha < \kappa$ and $A \subseteq \alpha$, then $j(A) = A$. Next, let's argue that $\mathcal{P}(\alpha)$ exists for every $\alpha < \kappa$. Fix $\alpha < \kappa$. Suppose towards a contradiction that $\mathcal{P}(\alpha)$ is a proper class. Then by our assumption that every proper class is big, there is a surjection from $\mathcal{P}(\alpha)$ onto κ . Applying collection, we can obtain a set $B \subseteq \mathcal{P}(\alpha)$ for which there is a surjection $h: B \rightarrow \kappa$. By elementarity, $j(h): j(B) \rightarrow j(\kappa)$ is a surjection onto $j(\kappa)$. Observe that $b \in B$ if and only if $b = j(b) \in j(B)$, and hence $B = j(B)$. Thus, $j(h): B \rightarrow j(\kappa)$. Also, for every $b \in B$, $j(h)(b) = j(h)(j(b)) = j(h(b)) = h(b)$. Thus, the range of $j(h)$ is κ contradicting that $j(h)$ is a surjection onto $j(\kappa)$. Now, a standard argument shows that $|\mathcal{P}(\alpha)| < \kappa$ for every $\alpha < \kappa$, and that κ is regular.

Next, let's argue that V_α exists for every $\alpha \leq \kappa$. Suppose inductively that we have shown that V_α exists and $|V_\alpha| = \beta < \kappa$. Then we can use a bijection $f: \beta \rightarrow V_\alpha$ and the previously shown fact that $\mathcal{P}(\beta)$ exists and $|\mathcal{P}(\beta)| < \kappa$ to argue that $V_{\alpha+1}$ exists and $|V_{\alpha+1}| < \kappa$. We then use collection to argue that V_λ exist for limit λ , and use the regularity of κ to argue that $|V_\lambda| < \kappa$ for $\lambda < \kappa$. \square

2.3. Class forcing. In Section 6, we use class forcing to construct a model of ZFC^- with unboundedly many cardinals and all big proper classes in which the DC_ω -scheme fails. Here, we briefly summarize the relevant properties of class forcing which we shall use in that argument.

Class forcing is best interpreted when working over a model of some second-order set theory. Second-order set theory is formalized in a two-sorted logic with separate sorts (variables and quantifiers) for sets and classes. Thus, unlike in first-order set theory, in this setting classes are actual elements of the model and not just objects of the meta-theory. Models of second-order set theory are triples $\mathcal{W} = \langle W, \in, \mathcal{C} \rangle$ where W is the sets of the model, \mathcal{C} is the classes, and \in is the membership relation between sets, as well as between sets and classes, letting us know of which sets each of the classes is composed. Let GB^- denote the second-order set theory whose axioms for sets are ZF^- and whose axioms for classes consist of extensionality, the class collection axiom asserting that for every class relation whose domain is restricted to a set, there is a set of witnesses of the relation's image, and the first-order comprehension scheme asserting that every first-order definable collection of sets is a class. Furthermore, we let GBC^- be GB^- with the global well-order axiom, which asserts that there is a bijection between W and Ord . By replacing ZF^- with ZF or ZFC in the theory GB^- , we obtain the Gödel-Bernays set theories of GB and GBC respectively. Every model of ZFC^- with a definable global well-order is naturally a model of GBC^- and every model of ZFC with a definable global well-order is naturally a model of GBC . By forcing with $Add(Ord, 1)$, we can show that every model of ZFC has a class forcing extension with the same sets and a global well-order. Thus, every model of ZFC has a class forcing extension with the same sets that is a model of GBC . As explained in the introduction, the analogous fact is true only for models ZFC^- provided that the $DC_{<Ord}$ -scheme holds.

A class forcing notion in a model $\mathcal{W} = \langle W, \in, \mathcal{C} \rangle \models GB^-$ is a partial order $\mathbb{P} \in \mathcal{C}$. Suppose that $\mathcal{W} = \langle W, \in, \mathcal{C} \rangle \models GB^-$ and $\mathbb{P} \in \mathcal{C}$ is class forcing notion. We say that $G \subseteq \mathbb{P}$ is \mathcal{W} -generic if G meets every dense class $D \subseteq \mathbb{P}$ from \mathcal{C} . The forcing extension is $\mathcal{W}[G] = \langle W[G], \in, \mathcal{C}[G] \rangle$, where $W[G]$ is the collection of all interpretations of (the usual) \mathbb{P} -names by G , and $\mathcal{C}[G]$ is the collection of all interpretations of the class \mathbb{P} -names by G , where a *class \mathbb{P} -name* is a class whose elements are pairs $\langle \dot{x}, p \rangle$ where \dot{x} is a \mathbb{P} -name and $p \in \mathbb{P}$.

In a number of significant ways, class forcing does not behave as nicely as set forcing. It is easy to see, for example by forcing with $Coll(\omega, Ord)$ (conditions are finite functions from ω to the ordinals ordered by extension) to collapse Ord to ω , that class forcing need not preserve replacement to the forcing extension. The forcing relations for a class forcing notion need not be definable (or more generally need not be a class). For example, a model of GBC whose classes are definable collections, can have a class forcing notion for which the forcing relation on atomic formulas is not definable [HKL⁺16]. However, there is a class of well-behaved class forcing notions, the pretame forcings, which avoid these pathological behaviors.

Definition 2.5. Suppose that $\mathcal{W} = \langle W, \in, \mathcal{C} \rangle \models GB^-$. A notion of class forcing $\mathbb{P} \in \mathcal{C}$ is *pretame* if for every $p \in \mathbb{P}$ and any sequence of classes $\langle D_i \mid i \in I \rangle \in \mathcal{C}$, with $I \in W$, such that each D_i is dense below p , there is a condition $q \leq p$ and a sequence $\langle d_i \mid i \in I \rangle \in W$ such that for every $i \in I$, $d_i \subseteq D_i$ and d_i is predense below q .

Theorem 2.6.

- (1) (Friedman [Fri00]) *Pretame class forcing notions preserve GB^- to the forcing extension.*
- (2) (Stanley [HKS18]) *Pretame class forcing notions have definable forcing relations.*

In the context of models of second-order set theory, let's redefine the DC_δ -scheme to assert that we can make δ -many dependent choices over every class (not just definable) relation without terminal nodes. In particular, all our results will follow for models in which the only classes are the definable collections. Although, we will not make use of the following proposition in the rest of the article, the result fits into our analysis of the $\text{DC}_{<\text{Ord}}$ -scheme.

Proposition 2.7. *Suppose that $\mathcal{W} = \langle W, \in, \mathcal{C} \rangle \models \text{GB}^- + \text{DC}_\delta$ -scheme for some regular cardinal δ . Then every pretame forcing extension of \mathcal{W} in which δ remains regular satisfies the DC_δ -scheme.*

Proof. Suppose that $\mathbb{P} \in \mathcal{C}$ is a pretame forcing notion. Let $G \subseteq \mathbb{P}$ be \mathcal{W} -generic. Let $R \in \mathcal{C}[G]$ be a class relation without terminal nodes. Let \dot{R} be a class \mathbb{P} -name for R and let $p \in \mathbb{P}$ be a condition forcing that \dot{R} does not have terminal nodes. Given a sequence x of \mathbb{P} -names of length some ordinal ξ , let $\dot{x}^{(\xi)}$ denote the canonical \mathbb{P} -name for a sequence of length ξ whose η -th element, for $\eta < \xi$, is the interpretation of $x(\eta)$. Let $\varphi(x, \dot{y}, p, \dot{R}, \mathbb{P})$ be a formula asserting, over \mathcal{W} , that whenever x is a sequence of \mathbb{P} -names of some ordinal length ξ and p forces that $\dot{x}^{(\eta)} \dot{R} x(\eta)$, then \dot{y} is a \mathbb{P} -name and p forces that $\dot{x}^{(\xi)} \dot{R} \dot{y}$. Since p forces that \dot{R} has no terminal nodes, the relation given by φ has no terminal nodes either. Thus, by the DC_δ -scheme, we can make δ -many choices along the relation given by φ . Let f be the function with domain δ witnessing this. Then $f_G^{(\delta)}$ witnesses that we can make δ -many dependent choices over the relation R . \square

The next proposition gives a useful criterion for pretameness.

Proposition 2.8. *Suppose that $\mathcal{W} = \langle W, \in, \mathcal{C} \rangle \models \text{GB}^- + \text{DC}_\delta$ -scheme, for a regular cardinal δ , and $\mathbb{P} \in \mathcal{C}$ is a class forcing notion with the δ -cc. Then \mathbb{P} is pretame.*

Proof. We will argue that every dense class $D \subseteq \mathbb{P}$ has a set maximal antichain contained in it, these antichains will then witness pretameness. Suppose towards a contradiction that there is no set maximal antichain contained in D . Let $\varphi(x, y, \mathbb{P}, D)$ be a formula asserting, over \mathcal{W} , that whenever x is a sequence of incompatible elements of D of some ordinal length ξ , then $y \in D$ and y is incompatible with all elements of the sequence x . Since there is no set maximal antichain contained in D , the relation given by φ has no terminal nodes. Thus, we can make δ -many dependent choices along it, which contradicts our assumption that \mathbb{P} has the δ -cc. \square

2.4. Jensen's forcing. In any universe $V \models \text{ZFC} + \diamond$ we can construct a subposet \mathbb{J} of Sacks forcing (elements are perfect trees ordered by the subtree relation) with the following two key properties.

Theorem 2.9 (Jensen [Jen70]).

- (1) *The poset \mathbb{J} has the ccc.*
- (2) *Suppose that r is a V -generic real for \mathbb{J} . Then in $V[r]$, r is the unique V -generic real for \mathbb{J} .*

Such a poset \mathbb{J} was first constructed by Jensen in L [Jen70]. In L , \mathbb{J} can be carefully constructed to yield the additional property that the collection of all L -generic reals for \mathbb{J} is Π_2^1 -definable in the forcing extension. Jensen used this to show that in a \mathbb{J} -forcing extension $L[r]$, r is a Σ_3^1 -definable non-constructible real [Jen70]. This is the lowest possible such complexity because Π_2^1 -definable reals must be constructible by Shoenfield's absoluteness.

Before we proceed, let us introduce a general notation for the product of μ many copies of a forcing with support less than δ , which we will use throughout this article.

Definition 2.10. Let \mathbb{P} be a forcing notion and let $\delta \leq \mu$ be regular cardinals. Let $\prod_{\mu}^{(\delta)} \mathbb{P}$ denote the product forcing of μ many copies of \mathbb{P} with $< \delta$ support.

Observe that we can treat conditions in $\prod_{\mu}^{(\delta)} \mathbb{P}$ as functions $f: \mu \rightarrow \mathbb{P}$ such that $\{\xi \in \mu \mid f(\xi) \neq \mathbb{1}\} < \delta$. Lyubetsky and Kanovei showed that the poset $\prod_{\omega}^{(\omega)} \mathbb{J}$ has the ccc and a uniqueness of generics property.

Theorem 2.11 (Lyubetsky, Kanovei [KL17]). *Suppose that $V \models \text{ZFC} + \diamond$.*

- (1) *The poset $\prod_{\omega}^{(\omega)} \mathbb{J}$ has the ccc.*
- (2) *Suppose that $G \subseteq \prod_{\omega}^{(\omega)} \mathbb{J}$ is V -generic. Then in $V[G]$, the V -generic reals for \mathbb{J} are precisely the ω -many reals coming from the slices of G .*

Indeed, an application of the Δ -system lemma shows that any length finite-support product of \mathbb{J} has these two properties.

We will say that a forcing iteration \mathbb{P}_n , of length n , is an *iteration of subposets of Sacks forcing* if every initial segment of \mathbb{P}_n forces that the next poset in the iteration is a subposet of the Sacks forcing of that extension. In universes where \mathbb{J} can be constructed, we can construct iterations \mathbb{J}_n , for any $n < \omega$, of subposets of Sacks forcing with the following key properties [FGK19].

- (1) If $m < n$, then $\mathbb{J}_n \upharpoonright m = \mathbb{J}_m$.
- (2) \mathbb{J}_n has the ccc.
- (3) Suppose that $\langle r_1, \dots, r_n \rangle$ is a V -generic sequence of reals for \mathbb{J}_n . Then in $V[\langle r_1, \dots, r_n \rangle]$, this is the unique V -generic sequence of reals for \mathbb{J}_n .

Let $\vec{\mathbb{J}} = \langle \mathbb{J}_n \mid n < \omega \rangle$. Let X be any set or class and consider the tree $X^{< \omega}$ of finite sequences from X ordered by extension. Let $\mathcal{T} \subseteq X^{< \omega}$ be a sub-tree. Let $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ be the (possibly class) poset whose elements are functions f_T on a finite subtree T of \mathcal{T} such that for nodes s on level n of T , $f_T(s) \in \mathbb{J}_n$ and for nodes $s < t$ in T , we have that $f_T(t) \upharpoonright \text{len}(s) = f_T(s)$. The ordering is given by $f_T \leq g_S$ provides that T extends S and for every node $s \in S$, we have that $f_T(s) \leq g_S(s)$. We call the poset $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$, an iteration of Jensen's forcing along the tree \mathcal{T} .

Theorem 2.12 ([FGK19]). *Suppose that X is any set or class.*

- (1) *The poset $\mathbb{P}(\vec{\mathbb{J}}, X^{<\omega})$ has the ccc.*
- (2) *Suppose that $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, X^{<\omega})$ is V -generic. Then the V -generic sequences $\langle r_1, \dots, r_n \rangle$ for \mathbb{J}_n in $V[G]$ are precisely the sequences added by nodes of $X^{<\omega}$ on level n .*

Proposition 2.13. *Suppose that X is any set or class. The poset $\mathbb{P}(\vec{\mathbb{J}}, X^{<\omega})$ is pretame.*

Proof. This follows by combining Theorem 2.12 with Proposition 2.8. \square

In [FG21], Friedman and the first author showed that in a universe $V \models \text{ZFC} + \diamond(\text{Cof}(\kappa))$ for an inaccessible cardinal κ , Jensen's construction of \mathbb{J} can be generalized to obtain a subposet \mathbb{J}^κ of κ -Sacks forcing with the following key properties.

Theorem 2.14 (Friedman, G. [FG21]).

- (1) *\mathbb{J}^κ has the κ^+ -cc.*
- (2) *Suppose that A is a V -generic subset of κ for \mathbb{J}^κ . Then in $V[A]$, A is the unique V -generic subset of κ for \mathbb{J}^κ .*

Theorem 2.15 ([FG21]). *Suppose that $V \models \text{ZFC} + \diamond(\text{Cof}(\kappa))$ for an inaccessible cardinal κ .*

- (1) *The poset $\prod_{\kappa}^{(\kappa)} \mathbb{J}^\kappa$ has the κ^+ -cc.*
- (2) *Suppose that $G \subseteq \prod_{\kappa}^{(\kappa)} \mathbb{J}^\kappa$ is V -generic. Then in $V[G]$, the V -generic subsets of κ for \mathbb{J}^κ are precisely the κ -many subsets of κ coming from the slices of G .*

3. ZARACH'S UNION MODELS OF ZFC^-

In [Zar82], Zarach gave a general construction for producing interesting models of ZFC^- as unions of models of ZFC arising as transitive submodels of a carefully chosen forcing extension. Because of the style in which it was presented, we have decided to rewrite his construction using modern notation.

Suppose that $V \models \text{ZFC}$. Let $\mathbb{P} \in V$ be a poset such that $\mathbb{Q} = \prod_{\omega}^{(\omega)} \mathbb{P}$ is isomorphic to \mathbb{P} . Let us call an automorphism π of \mathbb{Q} *coordinate-switching* if there is an automorphism $\tilde{\pi}$ of ω such that, for any condition p , $\pi(p) = q$, where q is defined by $q(i) = p(\tilde{\pi}^{-1}(i))$, namely π simply switches coordinates according to $\tilde{\pi}$.

Let $G \subseteq \mathbb{Q}$ be V -generic. For $n < \omega$, let

- (1) G_n be the restriction of G to the first n coordinates,
- (2) $G_{\{n\}}$ be the restriction of G to the n -th coordinate,
- (3) $G_{n,\text{tail}}$ be the tail of G after n ($G = G_n \times G_{n,\text{tail}}$).

Let $V_{[n]} = V[G_n]$. Let $W_G^V = \bigcup_{n < \omega} V_{[n]}$.

By uniform ground model definability ([Lav07], [Woo04]), W_G^V is definable in $V[G]$ from the generic filter G , say by the formula $\varphi(x, \mathbb{P}, G)$. Thus, to every formula $\psi(x)$, there corresponds a formula $\psi^W(x, y, z)$ such that for every $a \in W_G^V$, $W_G^V \models \psi(a)$ if and only if $V[G] \models \psi^W(a, \mathbb{P}, G)$. Observe that if π is a coordinate-switching automorphism of \mathbb{Q} , then $\varphi(x, \mathbb{P}, G)$ and $\varphi(x, \mathbb{P}, \pi''G)$ both define W_G^V .

in $V[G]$. Hence also, for every formula $\psi(x)$, we have that for every $a \in W_G^V$, $V[G]$ satisfies $\psi^W(a, \mathbb{P}, G)$ if and only if $V[G]$ satisfies $\psi^W(a, \mathbb{P}, \pi^*G)$. Let \dot{G} be the canonical \mathbb{Q} -name for the generic filter.

Proposition 3.1. *If $p \Vdash \psi^W(\check{a}, \check{\mathbb{P}}, \dot{G})$, then $\mathbf{1} \Vdash \psi^W(\check{a}, \check{\mathbb{P}}, \dot{G})$.*

Proof. Suppose for a contradiction that $p \Vdash \psi^W(\check{a}, \check{\mathbb{P}}, \dot{G})$ and $q \Vdash \neg\psi^W(\check{a}, \check{\mathbb{P}}, \dot{G})$. Let n be above the domains of p and q . Let π be a coordinate-switching automorphism that switches the coordinates in the domain of p to some coordinates above n . Then $\pi(p)$ and q are clearly compatible, and $\pi(p) \Vdash \psi^W(\check{a}, \check{\mathbb{P}}, \pi(\dot{G}))$. Since $\pi(\dot{G})_{\pi^*G} = G$ (the image of π^*G under the coordinate-switching automorphism π^{-1}), by our argument above, $\pi(p) \Vdash \psi^W(\check{a}, \check{\mathbb{P}}, \dot{G})$ as well, which is the desired contradiction. \square

In order to show that $W_G^V \models \text{ZFC}^-$, we shall prove that $W_G^V = \bigcup_{n < \omega} W_{[n]}^V$ where $W_{[n]} \prec W_G^V$ for each $n < \omega$. To do this, fix an isomorphism $h: \mathbb{P} \cong \mathbb{Q} = \prod_{\omega}^{(\omega)} \mathbb{P}$.

Since each $G_{\{n\}}$ is V -generic for \mathbb{P} , it follows that $G^{(n)} := h^*G_{\{n\}}$ is V -generic for \mathbb{Q} . As before, given $G^{(n)}$, for each $m < \omega$ we can obtain generics $G_m^{(n)}$, $G_{\{m\}}^{(n)}$, and $G_{m,\text{tail}}^{(n)}$. Now, for $n, m < \omega$, let

$$(V_{[n]})_{[m]} = V[G_n][G_m^{(n)}] = V_{[n]}[G_m^{(n)}]$$

and let $W_{G^{(n)}}^{V_{[n]}} = \bigcup_{m < \omega} (V_{[n]})_{[m]}$. Thus, we have:

$$V_{[n]} \subseteq (V_{[n]})_{[1]} \subseteq (V_{[n]})_{[2]} \subseteq \cdots \subseteq (V_{[n]})_{[m]} \subseteq \cdots \subseteq \bigcup_{m < \omega} (V_{[n]})_{[m]} = W_{G^{(n)}}^{V_{[n]}} \subseteq V_{[n+1]}.$$

For every $m, n < \omega$, let $H^{(n,m)}$ be the $(V_{[n]})_{[m]}$ -generic filter for \mathbb{Q} obtained from $G_{m,\text{tail}}^{(n)}$ via the isomorphism of \mathbb{Q} with its tail after m . Thus, for every $m, n < \omega$, we have that

$$V_{[n+1]} = V_{[n]}[G_m^{(n)} \times G_{m,\text{tail}}^{(n)}] = (V_{[n]})_{[m]}[H^{(n,m)}]$$

is a \mathbb{Q} -forcing extension of $(V_{[n]})_{[m]}$. Moreover, this yields $W_{G^{(n)}}^{V_{[n]}} = W_{H^{(n,m)}}^{(V_{[n]})_{[m]}}$ in $V_{[n+1]}$.

We shall now argue that $W_{G^{(n)}}^{V_{[n]}} \prec W_G^V$ for every $n < \omega$. This extremely powerful key lemma will yield many of our desired results.

Lemma 3.2. *For every $n < \omega$, $W_{G^{(n)}}^{V_{[n]}} \prec W_G^V$.*

Proof. Fix a formula $\psi(x)$ and $a \in W_{G^{(n)}}^{V_{[n]}}$ such that $W_{G^{(n)}}^{V_{[n]}} \models \psi(a)$. Since $W_{G^{(n)}}^{V_{[n]}}$ is the union of $(V_{[n]})_{[m]}$ for $m < \omega$, let m be such that $a \in (V_{[n]})_{[m]}$. Thus, by Proposition 3.1, $\mathbf{1} \Vdash \psi^W(\check{a}, \check{\mathbb{P}}, \dot{G})$ over $(V_{[n]})_{[m]}$. By the above argument, we have $(V_{[n]})_{[m]}[H^{(n,m)}] = V_{[n+1]}$, and thus

$$(V_{[n]})_{[m]}[H^{(n,m)}][G_{n,\text{tail}}] = V[G].$$

Via the obvious isomorphism of $\mathbb{P} \times \mathbb{Q}$ and \mathbb{Q} , we can view $H^{(n,m)} * G_{n,\text{tail}}$ as a $(V_{[n]})_{[m]}$ -generic for \mathbb{Q} with $H^{(n,m)}$ being the generic on the first coordinate.

Consider the model $W_{H^{(n,m)} * G_{n,\text{tail}}}^{(V_{[n]})_{[m]}}$ obtained from the generic $H^{(n,m)} * G_{n,\text{tail}}$. We have

$$\begin{aligned} W_{H^{(n,m)} * G_{n,\text{tail}}}^{(V_{[n]})_{[m]}} &= (V_{[n]})_{[m]}[H^{(n,m)}] \cup \bigcup_{i < \omega} (V_{[n]})_{[m]}[H^{(n,m)}][G_{n+1+i}] \\ &= \bigcup_{i < \omega} V_{[n+i]} \\ &= W_G^V. \end{aligned}$$

But we already showed that $(V_{[n]})_{[m]}$ satisfies that $\mathbb{1} \Vdash \psi^W(\check{a}, \check{\mathbb{P}}, \check{G})$. Thus, $W_G^V \models \psi(a)$. \square

Next, let us see what theory the model W_G^V satisfies. Observe right away that W_G^V cannot be a model of the power set axiom because $\mathcal{P}(\mathbb{P})$ does not exist in W_G^V .

Theorem 3.3 (Zarach [Zar96]). $W_G^V \models \text{ZFC}^-$.

Proof. It is clear that W_G^V satisfies extensionality, empty set, pairing, unions, infinity, and the foundation scheme. Also, W_G^V satisfies the well-ordering principle because any set in W_G^V is in some $V_{[n]} \models \text{ZFC}$. So it remains to argue that W_G^V satisfies the separation and collection schemes. First, let's do separation. Fix a formula $\psi(x, y)$ and some $a, b \in W_G^V$. We need to argue that

$$c = \{x \in a \mid W_G^V \models \psi(x, b)\}$$

is in W_G^V . Let n be large enough so that $a, b \in V_{[n]}$. By Lemma 3.2, $W_{G^{(n)}}^{V_{[n]}} \prec W_G^V$. Thus,

$$c = \{x \in a \mid W_{G^{(n)}}^{V_{[n]}} \models \psi(x, b)\}.$$

But $W_{G^{(n)}}^{V_{[n]}}$ is a definable submodel of $V_{[n+1]} \subseteq W_G^V$, and $V_{[n+1]} \models \text{ZFC}$. Thus, $c \in V_{[n+1]}$ by separation, and hence $c \in W_G^V$.

Next, let's do collection. Suppose that $\psi(x, y, z)$ is a formula and $a, b \in W_G^V$ such that

$$W_G^V \models \forall x \in a \exists y \psi(x, y, b).$$

Let n be large enough so that $a, b \in V_{[n]}$. By Lemma 3.2, $W_{G^{(n)}}^{V_{[n]}} \prec W_G^V$, and so $W_{G^{(n)}}^{V_{[n]}} \models \forall x \in a \exists y \psi(x, y, b)$. It follows that $V_{[n+1]}$ has a collecting set for $\psi(x, y, b)$, and hence so does W_G^V . \square

Theorem 3.4 (Zarach [Zar96]). $W_G^V \models \text{DC}_\omega\text{-scheme}$.

Proof. Fix a formula $\psi(x, y, a)$ defining over W_G^V a relation without terminal nodes. Let n be large enough so that $a \in V_{[n]}$. By Lemma 3.2, $W_{G^{(n)}}^{V_{[n]}} \prec W_G^V$. It follows that $\psi(x, y, a)$ defines a relation without terminal nodes over $W_{G^{(n)}}^{V_{[n]}}$. Let $\psi^*(x, y, a)$ be a relation defined over $V_{[n+1]}$ by

$$\text{if } x \in W_{G^{(n)}}^{V_{[n]}} \text{, then } y \in W_{G^{(n)}}^{V_{[n]}} \text{ and } W_{G^{(n)}}^{V_{[n]}} \models \psi(x, y, a).$$

Clearly, ψ^* is a relation without terminal nodes over $V_{[n+1]}$. Thus, by the DC_ω -scheme in $V_{[n+1]}$, there is in $V_{[n+1]}$ a function f on ω that is a sequence of ω -many dependent choices over $\psi^*(x, y, a)$. But clearly, since every initial segment of f

is in $W_{G^{(n)}}^{V_{[n]}}$, as it is closed under finite sequences, we have that for all $m < \omega$, $W_{G^{(n)}}^{V_{[n]}} \models \psi(f \upharpoonright m, f(m), a)$. Thus, by elementarity, $W_G^V \models \psi(f \upharpoonright m, f(m), a)$, for every $m < \omega$, as well. \square

Theorem 3.5. *Suppose that $V \models \text{CH}$ and $\mathbb{P} = \text{Add}(\omega, 1)$ is the Cohen poset. Then*

- (1) $W_G^V \models \text{ZFC}^- + \text{DC}_\omega$ -scheme.
- (2) W_G^V has the same cardinals and cofinalities as V .
- (3) $\mathcal{P}(\omega)$ is not ω_2 -big in W_G^V .
- (4) The DC_{ω_2} -scheme fails in W_G^V .

Proof. Clearly, $\text{Add}(\omega, 1) \cong \prod_{\omega}^{(\omega)} \text{Add}(\omega, 1)$. Item (1) follows from the theorems

of Zarach above. Since $\text{Add}(\omega, 1)$ has the ccc, V and $V[G]$ have the same cardinals and cofinalities, and hence so do V and $W_G^V \subseteq V[G]$. Clearly, by CH in $V[G]$, W_G^V cannot have a surjection from $\mathcal{P}(\omega)$, a proper class in W_G^V , onto ω_2 . Thus, by Proposition 2.3, the DC_{ω_2} -scheme fails in W_G^V . \square

Question 3.6. Does the model W_G^V from Theorem 3.5 satisfy the DC_{ω_1} -scheme?

4. GENERALIZED UNION MODELS

In this section, we will generalize Zarach's construction using products

$$\mathbb{Q} = \prod_{\mu}^{(\delta)} \mathbb{P}$$

for regular cardinals $\delta \leq \mu$ to obtain failures of the DC_δ -scheme for larger cardinals δ . The construction generalizes in a straightforward manner so we will just summarize the results here.

Suppose that $V \models \text{ZFC}$. Let $\mathbb{P} \in V$ be a poset and let $\delta \leq \mu$ be regular cardinals such that $\mathbb{Q} = \prod_{\mu}^{(\delta)} \mathbb{P}$ is isomorphic to \mathbb{P} . Let us call an automorphism π of

\mathbb{Q} *coordinate-switching* if, as before, it acts by switching coordinates according to some automorphism $\tilde{\pi}$ of μ .

Let $G \subseteq \mathbb{Q}$ be V -generic. For $\xi < \mu$, let

- (1) G_ξ be the restriction of G to the first ξ coordinates,
- (2) $G_{\{\xi\}}$ be the restriction of G to the ξ -th coordinate,
- (3) $G_{\xi, \text{tail}}$ be the tail of G after ξ ($G = G_\xi \times G_{\xi, \text{tail}}$).

Let $V_{[\xi]} = V[G_\xi]$. Let $W_G^V = \bigcup_{\xi < \mu} V_{[\xi]}$.

Let W_G^V be defined in $V[G]$ by the formula $\varphi(x, \langle \mathbb{P}, \mu, \delta \rangle, G)$, and for every formula $\psi(x)$ and $a \in W_G^V$, let the formula $\psi^W(x, \langle \mathbb{P}, \mu, \delta \rangle, G)$ be such that $W_G^V \models \psi(x)$ if and only if $V[G] \models \psi^W(a, \langle \mathbb{P}, \mu, \delta \rangle, G)$. As before, if π is a coordinate-switching automorphism of \mathbb{Q} , then both $\varphi(x, \langle \mathbb{P}, \mu, \delta \rangle, G)$ and $\varphi(x, \langle \mathbb{P}, \mu, \delta \rangle, \pi''G)$ define W_G^V in $V[G]$. Also, for every formula $\psi(x)$, we have that for every $a \in W_G^V$, $V[G]$ satisfies $\psi^W(a, \langle \mathbb{P}, \mu, \delta \rangle, G)$ if and only if $V[G]$ satisfies $\psi^W(a, \langle \mathbb{P}, \mu, \delta \rangle, \pi''G)$. By an analogous automorphism argument as before, we get that if some condition $p \Vdash \psi^W(\check{a}, \langle \check{\mathbb{P}}, \check{\mu}, \check{\delta} \rangle, \check{G})$, where \check{G} is the canonical \mathbb{Q} -name for the generic filter, then $\mathbb{1} \Vdash \psi^W(\check{a}, \langle \check{\mathbb{P}}, \check{\mu}, \check{\delta} \rangle, \check{G})$.

As before, we shall write W_G^V as the union of a sequence of models $W_{G^{(\xi)}}^{V_{[\xi]}}$, each of which is an elementary submodel of W_G^V . To do this, fix an isomorphism $h: \mathbb{P} \cong \mathbb{Q} = \prod_{\mu}^{(\delta)} \mathbb{P}$. Define the V -generic filters $G^{(\xi)} = h''G_{\{\xi\}}$, for $\xi < \mu$. From $G^{(\xi)}$, for $\nu < \mu$, we obtain $G_{\nu}^{(\xi)}$, $G_{\{\nu\}}^{(\xi)}$, and $G_{\nu, \text{tail}}^{(\xi)}$. Let

$$(V_{[\xi]})_{[\nu]} = V[G_{\xi}][G_{\nu}^{(\xi)}] = V_{[\xi]}[G_{\nu}^{(\xi)}]$$

for $\nu < \mu$, and let

$$W_{G^{(\xi)}}^{V_{[\xi]}} = \bigcup_{\nu < \mu} (V_{[\xi]})_{[\nu]}.$$

Thus, we have:

$$V_{[\xi]} \subseteq (V_{[\xi]})_{[1]} \subseteq (V_{[\xi]})_{[2]} \subseteq \cdots \subseteq (V_{[\xi]})_{[\nu]} \subseteq \cdots \subseteq \bigcup_{\nu < \mu} (V_{[\xi]})_{[\nu]} = W_{G^{(\xi)}}^{V_{[\xi]}} \subseteq V_{[\xi+1]}.$$

An analogous argument to the proof of Lemma 3.2 yields.

Lemma 4.1. *For every $\xi < \mu$, $W_{G^{(\xi)}}^{V_{[\xi]}} \prec W_G^V$.*

Observe that W_G^V cannot be a model of the power set axiom because $\mathcal{P}(\mathbb{P})$ does not exist in W_G^V . Lemma 4.1 gives:

Theorem 4.2. $W_G^V \models \text{ZFC}^-$.

Theorem 4.3. *Suppose that \mathbb{P} is $<\delta$ -closed. Then $(W_G^V)^{<\delta} \subseteq W_G^V$ in $V[G]$.*

Proof. Since \mathbb{P} is $<\delta$ -closed, the product \mathbb{Q} is $<\delta$ -closed as well. Since $\mu \geq \delta$ was regular in V , it follows that the cofinality of μ in $V[G]$ is at least δ . Suppose that $f: \gamma \rightarrow W_G^V$ in $V[G]$ for some $\gamma < \delta$. Then there is some $\xi < \mu$ such that the range of f is contained in $V_{[\xi]} = V[G_{\xi}]$ by cofinality considerations. But since the tail of the product after ξ is $<\delta$ -closed, it follows that $f \in V_{[\xi]}$, and hence $f \in W_G^V$. \square

Theorem 4.4. *Suppose that \mathbb{P} is $<\delta$ -closed. Then $W_G^V \models \text{DC}_{\delta}$ -scheme.*

Proof. Fix a formula $\psi(x, y, a)$ defining over W_G^V a relation without terminal nodes. Let ξ be large enough so that $a \in V_{[\xi]}$. By Lemma 4.1, $W_{G^{(\xi)}}^{V_{[\xi]}} \prec W_G^V$. It follows that $\psi(x, y, a)$ defines a relation without terminal nodes over $W_{G^{(\xi)}}^{V_{[\xi]}}$. Let $\psi^*(x, y, a)$ be a relation defined over $V_{[\xi+1]}$ by whenever $x \in W_{G^{(\xi)}}^{V_{[\xi]}}$, then $y \in W_{G^{(\xi)}}^{V_{[\xi]}}$ and $W_{G^{(\xi)}}^{V_{[\xi]}} \models \psi(x, y, a)$. Thus, by the DC_{δ} -scheme in $V_{[\xi+1]}$, there is in $V_{[\xi+1]}$ a function f on δ that is a sequence of δ -many dependent choices over $\psi^*(x, y, a)$. Now we use the $<\delta$ -closure of $W_{G^{(\xi)}}^{V_{[\xi]}}$ (Theorem 4.3) to confirm that every initial segment of f is in $W_{G^{(\xi)}}^{V_{[\xi]}}$. Thus, for all $\nu < \delta$, $W_{G^{(\xi)}}^{V_{[\xi]}} \models \psi(f \upharpoonright \nu, f(\nu), a)$. So now by elementarity, $W_G^V \models \psi(f \upharpoonright \nu, f(\nu), a)$, for every $\nu < \delta$, as well. \square

Given a regular cardinal δ , let $\text{Add}(\delta, 1)$ be the generalized Cohen poset adding a subset to δ with conditions of size less than δ .

Theorem 4.5. *Suppose that $V \models 2^\delta = \delta^+$ for some regular cardinal δ and let $\mathbb{P} = \text{Add}(\delta, 1)$. Then*

- (1) $W_G^V \models \text{ZFC}^- + \text{DC}_\delta\text{-scheme}$.
- (2) W_G^V has the same cardinals and cofinalities as V , with the possible exception of δ^+ .
- (3) $\mathcal{P}(\delta)$ is not δ^{++} -big in W_G^V .
- (4) The $\text{DC}_{\delta^{++}}$ -scheme fails in W_G^V .

Proof. Clearly, $\text{Add}(\delta, 1) \cong \prod_{\delta}^{(\delta)} \text{Add}(\delta, 1)$. Item (1) follows from the theorems

above. Since $\text{Add}(\delta, 1)$ is $<\delta$ -closed and has at most δ^{++} -cc (by $2^\delta = \delta^+$), V and $V[G]$ have the same cardinals and cofinalities with the possible exception of δ^+ , and hence so do V and $W_G^V \subseteq V[G]$. Clearly, since $2^\delta = \delta^+$ holds in $V[G]$, W_G^V cannot have a surjection from $\mathcal{P}(\delta)$, a proper class in W_G^V , onto δ^{++} . Thus, by Proposition 2.3, the $\text{DC}_{\delta^{++}}$ -scheme fails in W_G^V . \square

Question 4.6. Does the model W_G^V from Theorem 4.5 satisfy the DC_{δ^+} -scheme?

5. DC_δ -SCHEME + $\neg\text{DC}_{\delta^+}$ -SCHEME

In the previous sections, we obtained models of ZFC^- with a gap of one cardinal between the DC_δ -scheme holding and failing. In this section, we will obtain models of ZFC^- , where for a regular cardinal δ , the DC_δ -scheme holds, while the DC_{δ^+} -scheme fails.

We work in L . Let $\mathbb{P} = \prod_{\omega}^{(\omega)} \mathbb{J}$ and let $\mathbb{Q} = \prod_{\omega}^{(\omega)} \mathbb{P}$. Clearly, $\mathbb{P} \cong \mathbb{Q}$, and so

theorems from Section 3 apply. Let $G \subseteq \mathbb{Q}$ be L -generic. By Theorem 2.11 (2), $L[G]$ has only ω -many L -generic reals for \mathbb{J} . Consider the model W_G^L constructed as in Section 3.

Theorem 5.1.

- (1) $W_G^L \models \text{ZFC}^- + \text{DC}_\omega\text{-scheme}$.
- (2) W_G^L has the same cardinals and cofinalities as L .
- (3) The DC_{ω_1} -scheme fails in W_G^L .

Proof. (1) follows from the theorems of Section 3. (2) follows because \mathbb{Q} has the ccc by Theorem 2.11 (1). So it remains to prove (3). Consider the relation $\psi(x, y)$ over W_G^L , which holds if whenever x is a function f with domain α for some countable ordinal α such that $f(\xi)$ is L -generic for \mathbb{J} for all $\xi < \alpha$, then y is an L -generic real for \mathbb{J} that is distinct from all of the $f(\xi)$. Note that the definability of ψ uses that we work in L , a definable submodel of W_G^L . Let's argue next that $\psi(x, y)$ has no terminal nodes. Suppose that $f \in W_G^L$ is a function on some ordinal α whose range consists of L -generic reals for \mathbb{J} . Since $f \in L_n$ for some $n < \omega$, there must be an L -generic real, r , for \mathbb{J} in $V_{[n+1]}$ that is distinct from all the L -generic reals of L_n , and hence from the range of f . Thus, $\psi(f, r)$ holds in W_G^L . Now suppose that f is a function giving ω_1 -dependent choices along $\psi(x, y)$. Clearly, the range of f consists of ω_1 -many distinct L -generic reals for \mathbb{J} , but such a sequence cannot exist in $L[G]$. Thus, we have a counterexample relation to the DC_{ω_1} -scheme in W_G^L . \square

Next, we work in L with an inaccessible cardinal δ . Let $\mathbb{P} = \prod_{\delta}^{(\delta)} \mathbb{J}^{\delta}$ and let

$\mathbb{Q} = \prod_{\delta}^{(\delta)} \mathbb{P}$. Clearly, $\mathbb{P} \cong \mathbb{Q}$, and so the theorems from Section 4 apply. Let

$G \subseteq \mathbb{Q}$ be L -generic. By Theorem 2.14 (2), $L[G]$ has only δ -many L -generic reals for \mathbb{J}^{δ} . Consider the model W_G^L constructed as in Section 4.

Theorem 5.2.

- (1) $W_G^L \models \text{ZFC}^- + \text{DC}_{\delta}\text{-scheme}$
- (2) W_G^L has the same cardinals and cofinalities as L .
- (3) The DC_{δ^+} -scheme fails in W_G^L .

Proof. (1) follows from the theorems of Section 4. (2) follows because \mathbb{Q} has the δ^+ -cc by Theorem 2.14 (2). The proof of (3) is completely analogous to the proof of (3) in Theorem 5.1. \square

6. A LARGE MODEL WHERE THE DC_{ω} -SCHEME FAILS

In this section we shall provide a union model in the style of Zarach for which the DC_{ω} -scheme fails. Unlike the small model of [FGK19], this model will have unboundedly many cardinals.

We work in the constructible universe L . We will force with the class tree iteration $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$. Let $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ be L -generic. By Proposition 2.13, $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ is pretame, and hence $V[G] \models \text{ZFC}^-$. Although, we won't make use of this fact, let's also note that $V[G] \models \text{DC}_{<\text{Ord}}$ -scheme by Proposition 2.7.

Extending our earlier terminology, we will call an automorphism π of $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ *tree-switching* if there is an automorphism $\tilde{\pi}$ of $\text{Ord}^{<\omega}$ such that for any condition p $\pi(p) = q$, where $q(t) = p(\tilde{\pi}^{-1}(t))$, namely π switches the nodes of $\text{Ord}^{<\omega}$ according to $\tilde{\pi}$.

Fix a set tree $\mathcal{T} \subseteq \text{Ord}^{<\omega}$. Let $G_{\mathcal{T}} \subseteq G$ consist of all functions $f_T \in \mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ with $T \subseteq \mathcal{T}$ a finite subtree. Let's argue that $G_{\mathcal{T}}$ is L -generic for $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$. It suffices to show that every maximal antichain A of $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ remains maximal in $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$. Fix a maximal antichain A of $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$. Take any $f_S \in \mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$. Let $S^* = S \cap \mathcal{T}$ and let $f_{S^*} = f_S \upharpoonright S^*$, so that $f_{S^*} \in \mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$. By the maximality of A in $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$, there is $f_T \in A$ compatible with f_{S^*} . But then clearly f_T is compatible with f_S as well. Thus, $G_{\mathcal{T}}$ is an L -generic for $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ and therefore $L[G_{\mathcal{T}}] \models \text{ZFC}$.

Let \mathbb{T} consist of all infinite trees $\mathcal{T} \subseteq \text{Ord}^{<\omega}$ such that \mathcal{T} does not have a cofinal branch. Let $W = \bigcup_{\mathcal{T} \in \mathbb{T}} L[G_{\mathcal{T}}]$. We will show below that $W \models \text{ZFC}^- + \neg \text{DC}_{\omega}$ -scheme. But first we need some technical preliminaries.

Proposition 6.1. *Suppose that \mathcal{S}_1 and \mathcal{S}_2 are subtrees of $\text{Ord}^{<\omega}$, and π is a tree-switching automorphism of $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ such that $\tilde{\pi}'' \mathcal{S}_1 = \mathcal{S}_2$. Then $L[(\pi^{-1})'' G]_{\mathcal{S}_1} = L[G_{\mathcal{S}_2}]$.*

Proof. The automorphism π^{-1} restricts to an isomorphism from $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_2)$ to $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_1)$ and the image of $G_{\mathcal{S}_2}$ under this isomorphism is $(\pi^{-1})'' G_{\mathcal{S}_2}$. Thus,

$$L[(\pi^{-1})'' G]_{\mathcal{S}_1} \cong L[G_{\mathcal{S}_2}],$$

and hence $L[(\pi^{-1})'' G]_{\mathcal{S}_1} = L[G_{\mathcal{S}_2}]$. \square

Observe that, since L is definable in $L[G]$, there is a formula $\varphi(x, \vec{J}, G)$ defining W in $L[G]$. Thus, for every formula $\psi(x)$, there is a corresponding formula $\psi^W(x, y, z)$ such that for every $a \in W$, $W \models \psi(a)$ if and only if $L[G] \models \psi^W(a, \vec{J}, G)$. Next, let's argue that if π is any tree-switching automorphism of $\mathbb{P}(\vec{J}, \text{Ord}^{<\omega})$, then $\varphi(x, \vec{J}, \pi''G)$ also defines W in $L[G]$. Fix $\mathcal{T} \in \mathbb{T}$. By Proposition 6.1, $L[(\pi''G)_{\tilde{\pi}^{-1}''\mathcal{T}}] = L[G_{\mathcal{T}}]$. Thus, $\bigcup_{\mathcal{T} \in \mathbb{T}} L[G_{\mathcal{T}}] = \bigcup_{\mathcal{T} \in \mathbb{T}} L[(\pi''G)_{\mathcal{T}}]$. Hence also, for every formula $\psi(x)$, we have that for every $a \in W$, $W \models \psi(a)$ if and only if $L[G] \models \psi^W(a, \vec{J}, \pi''G)$.

Proposition 6.2. *Suppose that for some formula $\psi(x)$, $p \Vdash \psi^W(\dot{a})$, where \dot{a} is a $\mathbb{P}(\vec{J}, \mathcal{T})$ -name, for some tree $\mathcal{T} \in \mathbb{T}$. Then $p \restriction \mathcal{T} \Vdash \psi^W(\dot{a})$.*

Proof. Suppose towards a contradiction that $p \restriction \mathcal{T}$ does not force $\psi^W(\dot{a})$. Then there is a condition $q \leq p \restriction \mathcal{T}$ such that $q \Vdash \neg \psi^W(\dot{a})$. Let π be a tree-switching automorphism such that $\tilde{\pi}$ fixes \mathcal{T} and moves the nodes in $\text{dom}(p) \setminus \mathcal{T}$ so that

$$\text{dom}(q) \cap \text{dom}(\pi(p)) = \text{dom}(p \restriction \mathcal{T}) \subseteq \mathcal{T}.$$

We have $\pi(p) \Vdash \psi^W(\dot{a})$ because $\pi(\dot{a}) = \dot{a}$ and tree-switching automorphisms don't affect W . But this is impossible because clearly $\pi(p)$ and q are compatible. \square

Theorem 6.3. $W \models \text{ZFC}^-$.

Proof. It is clear that W satisfies extensionality, empty set, pairing, unions, infinity, the foundation scheme, and the well-ordering principle. We will be done if we can argue that W satisfies the replacement and collection schemes (separation will then follow). We will verify collection because the same argument will yield replacement as well.

Since W satisfies the well-ordering principle, it suffices to verify instances of collection for ordinals. So suppose that

$$W \models \forall \xi < \delta \exists y \psi(\xi, y, a).$$

Let $a \in L[G_{\mathcal{T}}]$ for some $\mathcal{T} \in \mathbb{T}$, and let \dot{a} be a $\mathbb{P}(\vec{J}, \mathcal{T})$ -name for a . Let $p \Vdash \theta^W(\dot{a})$, where $\theta(\dot{a}) := \forall \xi < \delta \exists y \psi(\xi, y, \dot{a})$. By Proposition 6.2, we can assume without loss of generality that $\text{dom}(p) \subseteq \mathcal{T}$. Given a tree $\mathcal{S} \in \mathbb{T}$, let $\dot{G}_{\mathcal{S}}$ be the canonical $\mathbb{P}(\vec{J}, \mathcal{S})$ -name for the generic filter. Observe that if π is a tree-switching automorphism, then $\pi(\dot{G}_{\mathcal{S}}) = \dot{G}_{\tilde{\pi}''\mathcal{S}}$.

Before giving the details of the proof, we sketch the idea behind the argument: For each $\xi < \delta$ there is some tree $\mathcal{S}_{\xi} \in \mathbb{T}$ such that $W \models \exists y \in L[G_{\mathcal{S}_{\xi}}] \psi(\xi, y, a)$. The aim is to find some tree $\mathcal{R} \in \mathbb{T}$ such that, for each $\xi \in \delta$, $W \models \exists y \in L[G_{\mathcal{R}}] \psi(\xi, y, a)$.

Since $\bigcup_{\xi < \delta} \mathcal{S}_{\xi}$ may contain a cofinal branch, we want to find a sequence of tree-switching automorphisms π_{ξ} such that $W \models \exists y \in L[G_{\pi_{\xi}''\mathcal{S}_{\xi}}] \psi(\xi, y, a)$ and $\pi_{\xi}''\mathcal{S}_{\xi}$ and $\pi_{\eta}''\mathcal{S}_{\eta}$ are disjoint modulo \mathcal{T} for any $\xi, \eta < \delta$.

However, we are unable to determine these automorphisms in the ground model and therefore it need not be the case that $\bigcup_{\xi < \delta} \pi_{\xi}''\mathcal{S}_{\xi} \in \mathbb{T}$ (in L). To avoid this issue we shall use the ccc to recursively construct in L countable sequences of trees $\langle \mathcal{S}_{\xi}^{(\alpha)} \mid \alpha < \beta_{\xi} \rangle$ for $\xi < \delta$ such that $\mathcal{S}_{\xi}^{(\alpha)}$ and $\mathcal{S}_{\eta}^{(\gamma)}$ are pairwise disjoint modulo \mathcal{T} for every α, ξ, γ and η . This will be done in such a way that, for $\mathcal{T}_{\xi} = \bigcup_{\alpha < \beta_{\xi}} \mathcal{S}_{\xi}^{(\alpha)}$,

$$W \models \forall \xi < \delta \exists y \in L[G_{\mathcal{T}_{\xi}}] \psi(\xi, y, a).$$

We will then be able to take $\mathcal{R} := \bigcup_{\xi < \delta} \mathcal{T}_\xi \in \mathbb{T}$ as our witnessing tree. So that this can be easily modified into a proof for replacement, we will also explicitly construct names y_ξ for each $\xi < \delta$.

In order to do this, for every $\xi < \delta$, let D_ξ be the dense class of conditions q below p for which there is some $\mathcal{S} \in \mathbb{T}$ such that $\mathcal{T} \subseteq \mathcal{S}$ and a $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S})$ -name \dot{y} such that

$$q \Vdash \dot{y} \in L[\dot{G}_\mathcal{S}] \wedge \psi^W(\check{\xi}, \dot{y}, \dot{a}).$$

Choose any condition $q_0^{(0)} \in D_0$ (using global choice in L) and fix $\dot{y}_0^{(0)}$ and $\mathcal{S}_0^{(0)}$ such that

$$q_0^{(0)} \Vdash \dot{y}_0^{(0)} \in L[\dot{G}_{\mathcal{S}_0^{(0)}}] \wedge \psi^W(\check{0}, \dot{y}_0^{(0)}, \dot{a}).$$

Next, assuming this is possible, choose any condition $q_1^* \in D_0$ incompatible with $q_0^{(0)}$, and fix \dot{y}_1^* and \mathcal{S}_1^* such that

$$q_1^* \Vdash \dot{y}_1^* \in L[\dot{G}_{\mathcal{S}_1^*}] \wedge \psi^W(\check{0}, \dot{y}_1^*, \dot{a}).$$

Let π_1 be a tree-switching automorphism such that $\tilde{\pi}_1$ fixes \mathcal{T} and $\mathcal{S}_0^{(1)} := \tilde{\pi}_1'' \mathcal{S}_1^*$ satisfies $\mathcal{S}_0^{(1)} \cap \mathcal{S}_0^{(0)} \subseteq \mathcal{T}$. Now, we have that

$$\pi_1(q_1^*) \Vdash \pi_1(\dot{y}_1^*) \in L[\dot{G}_{\mathcal{S}_0^{(1)}}] \wedge \psi^W(\check{0}, \pi_1(\dot{y}_1^*), \dot{a}).$$

Also, since $\text{dom}(p) \subseteq \mathcal{T}$ and $\tilde{\pi}_1$ fixes \mathcal{T} , $\pi_1(q_1^*) \leq p$. Let $q_0^{(1)} = \pi_1(q_1^*) \cup q_1^*$. Thus, it is still case that $q_0^{(1)} \in D_0$ and $q_0^{(1)}$ is not compatible with $q_0^{(0)}$. Let $\dot{y}_0^{(1)} = \pi_1(\dot{y}_1^*)$. Continuing in this manner, we keep building a sequence of mutually incompatible conditions $q_0^{(\alpha)} \in D_0$ such that

$$q_0^{(\alpha)} \Vdash \dot{y}_0^{(\alpha)} \in L[\dot{G}_{\mathcal{S}_0^{(\alpha)}}] \wedge \psi^W(\check{0}, \dot{y}_0^{(\alpha)}, \dot{a})$$

and $\mathcal{S}_0^{(\alpha)} \cap \bigcup_{\gamma < \alpha} \mathcal{S}_0^{(\gamma)} \subseteq \mathcal{T}$. This process must terminate after β_0 -many steps for a countable β_0 because the poset $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ has the ccc. Let $A_0 = \{q_0^{(\alpha)} \mid \alpha < \beta_0\}$ be the resulting maximal antichain contained in D_0 . Let $\mathcal{T}_0 = \bigcup_{\alpha < \beta_0} \mathcal{S}_0^{(\alpha)}$, and observe that by the disjointness of the $\mathcal{S}_0^{(\alpha)}$ modulo \mathcal{T} , we have that \mathcal{T}_0 cannot have an infinite branch and therefore $\mathcal{T}_0 \in \mathbb{T}$.

Let \dot{z}_0 be the mixed name of the names $\dot{y}_0^{(\alpha)}$ over the antichain A_0 . Namely,

$$\dot{z}_0 = \bigcup_{\alpha < \beta_0} \left\{ \langle \dot{x}, r \rangle \mid r \leq q_0^{(\alpha)}, r \in \mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_0^{(\alpha)}), r \Vdash \dot{x} \in \dot{y}_0^{(\alpha)}, \dot{x} \in \text{dom}(\dot{y}_0^{(\alpha)}) \right\}.$$

Note that we can include the condition $r \in \mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_0^{(\alpha)})$ because if $r \Vdash \dot{x} \in \dot{y}_0^{(\alpha)}$ for some $\dot{x} \in \text{dom}(\dot{y}_0^{(\alpha)})$, then $r \upharpoonright \mathcal{S}_0^{(\alpha)}$ also forces this by Proposition 6.2. Finally, observe that \dot{z}_0 is a $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T}_0)$ -name and $W \models (\dot{z}_0)_G \in L[G_{\mathcal{T}_0}] \wedge \psi(\check{0}, (\dot{z}_0)_G, \dot{a})$.

Next, we repeat the process for D_1 , building a maximal antichain

$$A_1 = \{q_1^{(\alpha)} \mid \alpha < \beta_1\}$$

contained in D_1 and trees $\mathcal{S}_1^{(\alpha)}$ such that

$$q_1^{(\alpha)} \Vdash \dot{y}_1^{(\alpha)} \in L[\dot{G}_{\mathcal{S}_1^{(\alpha)}}] \wedge \psi^W(\check{1}, \dot{y}_1^{(\alpha)}, \dot{a}).$$

At the same time, we ensure that for any $\alpha < \beta_1$, $\mathcal{S}_1^{(\alpha)} \cap (\bigcup_{\gamma < \alpha} \mathcal{S}_1^{(\gamma)} \cup \mathcal{T}_0) \subseteq \mathcal{T}$. Let $\mathcal{T}_1 = \bigcup_{\alpha < \beta_1} \mathcal{S}_1^{(\alpha)}$ and observe that $\mathcal{T}_0 \cup \mathcal{T}_1$ is in \mathbb{T} . Let \dot{z}_1 be the mixed name of the names $\dot{y}_1^{(\alpha)}$ over the antichain A_1 , and observe that \dot{z}_1 is a $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T}_1)$ -name.

We continue the process for every D_ξ , and let $\mathcal{R} = \bigcup_{\xi < \delta} \mathcal{T}_\xi$, which is in \mathbb{T} by construction. Let \dot{z} be the canonical name for a sequence of length δ obtained from the names \dot{z}_ξ for $\xi < \delta$. Then \dot{z} is a $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{R})$ -name. By construction, for every $\xi < \delta$, $W \models \psi(\check{\xi}, \dot{z}_G(\xi), a)$, so \dot{z}_G witnesses this instance of collection. \square

Theorem 6.4. $W \models \neg \text{DC}_\omega\text{-scheme}$.

Proof. Consider the definable class tree whose domain is

$$\{\vec{r} \mid \vec{r} \text{ is } L\text{-generic for } \mathbb{J}_n \text{ for some } n\}$$

ordered by extension in W . Clearly, the tree relation has no terminal nodes. Thus, if we can show that it doesn't have an infinite branch, we will have a violation of the DC_ω -scheme. So suppose that $b \in W$ is an infinite branch through this class tree. Then $b \in L[G_\mathcal{S}]$ for some tree $\mathcal{S} \in \mathbb{T}$. Since \mathcal{S} does not have an infinite branch by the definition of \mathbb{T} , there must be some \vec{r}_n , the element of b on level n , which is L -generic for \mathbb{J}_n but not in \mathcal{S} . However, this is impossible by Theorem 2.12 (2). \square

The model W is also interesting because even though the DC_ω -scheme fails, every proper class in W is big. Before we prove this we need the following lemma.

Lemma 6.5. *Suppose that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T} \in \mathbb{T}$ are such that $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{T}$. Then*

$$L[G_{\mathcal{S}_1}] \cap L[G_{\mathcal{S}_2}] = L[G_\mathcal{T}].$$

Proof. Since we are dealing with models of ZFC, it suffices to show that every set of ordinals in $L[G_{\mathcal{S}_1}] \cap L[G_{\mathcal{S}_2}]$ is in $L[G_\mathcal{T}]$. Suppose that $A \in L[G_{\mathcal{S}_1}] \cap L[G_{\mathcal{S}_2}]$ is a subset of an ordinal α . Let \dot{x} be a nice $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_1)$ -name for A , namely $\dot{x} = \bigcup_{\xi < \alpha} \{\check{\xi}\} \times A_\xi$, where the A_ξ are antichains of $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_1)$. Similarly, let $\dot{y} = \bigcup_{\xi < \alpha} \{\check{\xi}\} \times B_\xi$ be a nice $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_2)$ -name for A . Fix a condition $p \in \mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_1 \cup \mathcal{S}_2)$ forcing that $\dot{x} = \dot{y}$. Let $p = p_1 \cup p_2$, where $p_1 = p \upharpoonright \mathcal{S}_1$ and $p_2 = p \upharpoonright \mathcal{S}_2$. We will work below this condition p . By shrinking the set A , we can assume without loss of generality that p does not decide $\check{\xi} \in \dot{x}$ for any $\xi < \alpha$. We can also assume that conditions in all A_ξ are compatible with p .

Next, let's argue that if some condition $q \leq p_1$ in $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_1)$ decides $\check{\xi} \in \dot{x}$, then $p_1 \cup q \upharpoonright \mathcal{T}$ already decides $\check{\xi} \in \dot{x}$. Suppose that $q \Vdash \check{\xi} \in \dot{x}$ (the case $q \Vdash \check{\xi} \notin \dot{x}$ will be the same). We will first show that $p_2 \cup q \upharpoonright \mathcal{T}$ forces that $\check{\xi} \in \dot{y}$. So, suppose that this is not the case. Then fix $r \leq p_2 \cup q \upharpoonright \mathcal{T}$ in $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_2)$ such that $r \Vdash \check{\xi} \notin \dot{y}$. But then $q \cup r \leq q$ in $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_1 \cup \mathcal{S}_2)$, and $q \cup r \Vdash \check{\xi} \notin \dot{y}$ in $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_1 \cup \mathcal{S}_2)$ (by absoluteness for atomic forcing formulas), and also $q \cup r \Vdash \check{\xi} \in \dot{x}$ in $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{S}_1 \cup \mathcal{S}_2)$. But this is a contradiction because $q \cup r \leq p_1 \cup p_2 \cup q \upharpoonright \mathcal{T} \leq p$ and so must force $\dot{x} = \dot{y}$. Thus, $p_2 \cup q \upharpoonright \mathcal{T} \Vdash \check{\xi} \in \dot{y}$. But now essentially the same argument on the \mathcal{S}_1 -side with \dot{x} shows that $p_1 \cup q \upharpoonright \mathcal{T} \Vdash \check{\xi} \in \dot{x}$.

Let $\dot{x}^* = \bigcup_{\xi < \alpha} \{\check{\xi}\} \times A_\xi^*$, where $A_\xi^* = \{q \upharpoonright \mathcal{T} \mid q \in A_\xi\}$, and note that \dot{x}^* is a $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ -name. We claim that $\dot{x}_G^* = \dot{x}_G$. Suppose that $\xi \in \dot{x}_G$. Then there is $q \in G \cap A_\xi$. Thus, $q \upharpoonright \mathcal{T} \in G \cap A_\xi^*$, and hence $\xi \in \dot{x}_G^*$. Next, suppose that $\xi \notin \dot{x}_G$. Then there is $q \leq p$ in G such that $q \Vdash \check{\xi} \notin \dot{x}$. By the above argument, the condition $p_1 \cup q \upharpoonright \mathcal{T}$ also forces $\check{\xi} \notin \dot{x}$. Thus, conditions incompatible with some $a \in A_\xi$ are

dense below $p_1 \cup q \upharpoonright \mathcal{T}$. But then, by our assumption that p , and therefore p_1 , is compatible with all conditions in every A_ξ , it follows that conditions incompatible with some $a \in A_\xi^*$ are dense below $q \upharpoonright \mathcal{T}$. Thus, $q \upharpoonright \mathcal{T} \Vdash \check{\xi} \notin \dot{x}^*$. Since $q \upharpoonright \mathcal{T} \in G$, it follows that $\xi \notin \dot{x}_G^*$. \square

Theorem 6.6. *Every proper class in W is big.*

Proof. Suppose that a formula $\psi(x, a)$ defines a proper class \mathcal{A} in W and $a \in L[G_{\mathcal{T}}]$ for some $\mathcal{T} \in \mathbb{T}$. Fix a cardinal δ and recall that, since $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ has the ccc, all of our models have the same cardinals. We need to verify that there is a surjection from \mathcal{A} onto δ . First, suppose that $\mathcal{A} \cap L[G_{\mathcal{T}}]$ is a proper class. In this case, there must be some ordinal α such that $\mathcal{A} \cap L[G_{\mathcal{T}}]_\alpha$ has size at least δ in $L[G]$. Let $A = \mathcal{A} \cap L[G_{\mathcal{T}}]_\alpha$, which exists in W by separation. Since W can enumerate A by the well-ordering principle, let $f: \beta \rightarrow A$ for some ordinal β , and observe that by our assumption on the size of A , $\beta \geq \delta$.

So let's assume now that $\mathcal{A} \cap L[G_{\mathcal{T}}]$ is not a proper class. As in Theorem 6.3, we begin by sketching the idea behind the argument. Since \mathcal{A} is a proper class in W , there will be some tree \mathcal{S}_0 extending \mathcal{T} for which $W \models \exists x \in L[G_{\mathcal{S}_0}] \setminus L[G_{\mathcal{T}}] \psi(x, a)$. The aim is to find some sequence $\langle \mathcal{S}_\xi \mid \xi \in \delta \rangle$ each of which is isomorphic to \mathcal{S}_0 , pairwise disjoint modulo \mathcal{T} , and such that

$$W \models \exists x \in L[G_{\mathcal{S}_\xi}] \setminus L[G_{\mathcal{T}}] \psi(x, a).$$

However, since these trees cannot be determined in the ground model we will again use the ccc to recursively construct countable sequences of trees $\langle \mathcal{S}_\xi^{(\alpha)} \mid \alpha < \beta_\xi \rangle$ for $\xi < \delta$ such that $\mathcal{S}_\xi^{(\alpha)} \cap \mathcal{S}_\eta^{(\gamma)} \subseteq \mathcal{T}$ for each α, ξ, γ and η and, for $\mathcal{T}_\xi = \bigcup_{\alpha < \beta_\xi} \mathcal{S}_\xi^{(\alpha)}$, we have

$$W \models \exists x \in L[G_{\mathcal{T}_\xi}] \setminus L[G_{\mathcal{T}}] \psi(x, a).$$

So, let \dot{a} be a $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ -name for a . Fix a condition

$$p \Vdash \exists x \notin L[\dot{G}_{\mathcal{T}}] \psi^W(x, \dot{a}).$$

By Proposition 6.2, we can assume without loss of generality that $p \in \mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$. Let D be the dense class of conditions q below p forcing for some tree $\mathcal{S} \in \mathbb{T}$ that $\exists x \in L[\dot{G}_{\mathcal{S}}] \psi^W(x, \dot{a})$. Following the proof of Theorem 6.3, build a maximal antichain $A_0 = \{q_0^{(\alpha)} \mid \alpha < \beta_0\}$ contained in D such that

$$q_0^{(\alpha)} \Vdash \exists x \in L[\dot{G}_{\mathcal{S}_0^{(\alpha)}}] \psi^W(x, \dot{a}),$$

where $\mathcal{S}_0^{(\alpha)} \cap \bigcup_{\gamma < \alpha} \mathcal{S}_0^{(\gamma)} \subseteq \mathcal{T}$. Let $\mathcal{T}_0 = \bigcup_{\alpha < \beta_0} \mathcal{S}_0^{(\alpha)}$. Next, we repeat the process, constructing a maximal antichain $A_1 = \{q_1^{(\alpha)} \mid \alpha < \beta_1\}$ contained in D such that

$$q_1^{(\alpha)} \Vdash \exists x \in L[\dot{G}_{\mathcal{S}_1^{(\alpha)}}] \psi^W(x, \dot{a}),$$

and we have $\mathcal{S}_1^{(\alpha)} \cap (\bigcup_{\gamma < \alpha} \mathcal{S}_1^{(\gamma)} \cup \mathcal{T}_0) \subseteq \mathcal{T}$. We continue this process, constructing maximal antichains $A_\xi = \{q_\xi^{(\alpha)} \mid \alpha < \beta_\xi\}$ contained in D such that

$$q_\xi^{(\alpha)} \Vdash \exists x \in L[\dot{G}_{\mathcal{S}_\xi^{(\alpha)}}] \psi^W(x, \dot{a}),$$

maintaining the disjointness of the trees modulo \mathcal{T} .

Let $\mathcal{R} = \bigcup_{\xi < \delta} \mathcal{T}_\xi$ which is in \mathbb{T} by the disjointness of the trees $S_\xi^{(\alpha)}$ modulo \mathcal{T} . Finally, by Lemma 6.5, for each $\xi < \delta$

$$A \cap L[G_{\mathcal{T}_\xi}] \setminus \bigcup_{\eta < \xi} A \cap L[G_{\mathcal{T}_\eta}] \neq \emptyset$$

and therefore the model $L[G_{\mathcal{R}}]$ contains at least δ -many elements of \mathcal{A} . \square

We should point out that the construction given above fails if we replace $\text{Ord}^{<\omega}$ with $\alpha^{<\omega}$ for some cardinal α . Indeed, the model W constructed analogously in a forcing extension $L[G]$ by an L -generic $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \alpha^{<\omega})$ fails to satisfy the following instance of collection. We analogously let \mathbb{T} be the collection of all subtrees of $\alpha^{<\omega}$ of size α which do not have an infinite branch. In this case, \mathbb{T} is a set from L , and hence in the union model W . The model W satisfies that for every $\mathcal{T} \in \mathbb{T}$, there is a constellation of Jensen reals along \mathcal{T} . More formally, there is a map $F_{\mathcal{T}}$ with domain \mathcal{T} such that nodes of length n get mapped to sequences of L -generic reals for \mathbb{J}_n and the sequences on longer nodes end-extend sequences on shorter nodes. Suppose towards a contradiction that there is a collecting set C for this instance of collection. C must then be in some $L[G_{\mathcal{T}}]$ with $\mathcal{T} \in \mathbb{T}$. Let \mathcal{S} be a tree in \mathbb{T} of rank higher than \mathcal{T} . This ensures that there is no tree isomorphism between \mathcal{T} and a subtree of \mathcal{S} . But then by Theorem 2.12 (2), $L[G_{\mathcal{T}}]$ cannot contain the map $F_{\mathcal{S}}$.

We should also note that instead of forcing over L , we could have forced over H_λ^L for some regular, uncountable λ . The model $H_\lambda^L \models \text{ZFC}^-$ and for it $\mathbb{P}(\vec{\mathbb{J}}, \lambda^{<\omega})$ is a pretame forcing. Thus, if $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \lambda^{<\omega})$ is H_λ -generic, then $H_\lambda[G] \models \text{ZFC}^-$. The rest of the arguments in this section then go through.

In Theorem 6.6, we proved that W is a model of ZFC^- in which every proper class is big. While this is a very desirable property for our model to satisfy, we can make one final observation for this section. This is that, by combining the construction with that of Section 4, it is possible to produce a model of ZFC^- in which the DC_ω -scheme fails and in which there are proper classes that are not big.

Corollary 6.7. *It is possible to produce a model of ZFC^- with unboundedly many cardinals in which the DC_ω -scheme fails and there is a proper class that is not big.*

Proof. We start with a model V of $\text{ZFC} + V = L$ to ensure that \diamond holds and that we have ground model definability. Take a generic $H \subseteq \text{Add}(\omega_1, 1)$ and consider W_H^V from Theorem 4.5. This is a model of $\text{ZFC}^- + \text{DC}_{\omega_1}$ -scheme in which $\mathcal{P}(\omega_1)$ does not surject onto ω_3 . By Proposition 2.8, since the DC_{ω_1} -scheme holds, any ccc class forcing over W_H^V is pretame. Specifically, the class tree iteration $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ remains pretame in W_H^V . Next, note that H added no new subsets of ω , which in particular means that \diamond holds in W_H^V . From this, it is easy to see that $\mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ satisfies all the necessary properties mentioned in Section 2.4 and, for any generic G , W_H^V is definible in $W_H^V[G]$. So, let $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \text{Ord}^{<\omega})$ be generic and consider $W = \bigcup_{\mathcal{T} \in \mathbb{T}} W_H^V[G_{\mathcal{T}}]$ as before. By following the previous analysis, it is clear that $W \models \text{ZFC}^- + \neg \text{DC}_\omega$ -scheme.

Finally, while $\mathcal{P}(\omega_1)$ is now a big proper class in W , $\mathcal{P}(\omega_1) \cap W_H^V$ is not. This is because $\mathcal{P}(\omega_1) \cap W_H^V$ is a set of cardinality ω_2 in $L[H]$ and, since the second forcing doesn't collapse cardinals, this must still be true in $L[H][G]$. Therefore this class cannot possibly surject onto ω_3 in W . Thus, W is our desired model of ZFC^- with a proper class that is not big. \square

7. EMBEDDINGS WITH $\mathcal{P}(\omega)$ A PROPER CLASS

In this section, we show that there is a model of ZFC^- having a definable elementary embedding j with a critical point in which $\mathcal{P}(\omega)$ does not exist.

Suppose that $V \models \text{ZFC}$ and κ is a measurable cardinal in V with a normal measure U . Let $j: V \rightarrow M$ be the ultrapower map by U . Let $\mathbb{P} = \text{Add}(\omega, 1)$, $\mathbb{Q} = \prod_{\omega}^{(\omega)} \mathbb{P}$, and $G \subseteq \mathbb{Q}$ be V -generic. We construct W_G^V in $V[G]$ as in Section 3.

A folklore result, known as the Lifting Criterion, states that given an elementary embedding $j: V \rightarrow M$, a poset $\mathbb{P} \in V$, a V -generic filter $G \subseteq \mathbb{P}$ and an M -generic filter $H \subseteq j(\mathbb{P})$, we can lift (extend) the embedding j to $j: V[G] \rightarrow M[H]$ if and only if $j''G \subseteq H$. In case we can lift, the lift j is given by $j(\dot{x}_G) = j(\dot{x})_H$. Thus, by the Lifting Criterion, using that $j(\mathbb{Q}) = \mathbb{Q}$ and $j''G = G$, we can lift j to the embedding $j: V[G] \rightarrow M[G]$. Moreover, we can show that j is the ultrapower map by the measure U_ω generated by U , namely $A \in U_\omega$ if and only if there is $B \in U$ such that $B \subseteq A$. This also shows that, for every $n < \omega$, j lifts to $j_n: V_{[n]} \rightarrow M_{[n]}$ ($V_{[n]} = V[G_n]$, $M_{[n]} = M[G_n]$), and is the ultrapower map by the measure U_n generated by U in $V_{[n]}$. Thus,

$$U \subseteq U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots \subseteq U_\omega.$$

By the definability of W_G^V in $V[G]$, we can restrict the lift $j: V[G] \rightarrow M[G]$ to an elementary embedding

$$j^W: W_G^V \rightarrow W_G^M,$$

where W_G^M is constructed in $M[G]$ analogously to W_G^V .

Proposition 7.1. $j^W = \bigcup_{n < \omega} j_n$.

Proof. Observe that it suffices to show that the lift $j_n: V_{[n]}[G_{n,\text{tail}}] \rightarrow M_{[n]}[G_{n,\text{tail}}]$ is the lift $j: V[G] \rightarrow M[G]$. But this is clear because, by our observation above, the lift of j_n is the ultrapower map by the measure generated by U_n , which is clearly U_ω . \square

Let E be the membership relation modulo U_ω . Observe that U_ω -equivalence and E are both definable in W_G^V from the set $U \in W_G^V$. Fix a function $f: \kappa \rightarrow W_G^V$ in some $V_{[n]}$. Since $j^W = \bigcup_{n < \omega} j_n$, we know that whenever $g: \kappa \rightarrow W_G^V$ from W_G^V is an E -member of f , then g is U_ω -equivalent to some $g^*: \kappa \rightarrow V_{[n]}$ with $g^* \in V_{[n]}$. Thus, $V_{[n]}$ has a set X^f consisting of functions $g: \kappa \rightarrow V_{[n]}$ that are E -members of f such that any function $h: \kappa \rightarrow W_G^V$ from W_G^V that is an E -member of f is U_ω -equivalent to a function in X^f . Thus, in W_G^V , given any function $f: \kappa \rightarrow W_G^V$, we can associate to it the set X^f .

Now let's provide a definition of j in W_G^V . Fix a function $f: \kappa \rightarrow W_G^V$, and let $b \in V[G]$ be the image of $[f]_{U_\omega}$ under the transitive collapse. Working in W_G^V , let $X_0^f = X^f$. Now suppose inductively that we are given X_n^f , and let $X_{n+1}^f = \bigcup_{g \in X_n^f} X^g$. Let $X_\omega^f = \bigcup_{n < \omega} X_n^f$. It should be clear that the transitive collapse of $\langle X_\omega^f, E \rangle$ (modulo U_ω) is the transitive closure of b , from which we can compute b . Thus, W_G^V can compute $j(a)$ by computing $X_\omega^{c_a}$, where $c_a: \kappa \rightarrow \{a\}$ is the constant function.

Putting together all of the above, we obtain the following result.

Theorem 7.2.

- (1) $W_G^V \models \text{ZFC}^- + \text{DC}_\omega\text{-scheme}$.
- (2) *The DC_{ω_2} -scheme fails in W_G^L .*
- (3) $\mathcal{P}(\omega)$ (and therefore V_α for $\alpha > \omega$) does not exist in W_G^V .
- (4) W_G^V has a definable elementary embedding j with a critical point.

We can also use the construction of Section 3 to produce other examples of the limitations to Theorem 2.4. For instance, it is easy to produce a model $W \models \text{ZFC}^-$ with a cofinal elementary embedding $j: W \rightarrow W$ having a critical point (but W won't satisfy ZFC_j^-).

Start with a transitive model $M \models \text{ZFC}$ for which there is an elementary embedding $j: M \rightarrow M$ with critical point some ordinal κ (the consistency strength of this assumption is below $0^\#$). Let $\mathbb{P} = \text{Add}(\omega, 1)$ in M , force with $\mathbb{Q} = \prod_{\omega}^{(\omega)} \mathbb{P}$, and let

$G \subseteq \mathbb{Q}$ be M -generic. First, we lift j to an elementary embedding $j: M[G] \rightarrow M[G]$, and then restrict j to $j^W: W_G^M \rightarrow W_G^M$.

We can also use the construction of Section 3 to produce a model $W \models \text{ZFC}_j^-$ such that $\mathcal{P}(\omega)$ is a proper class and where $j: W \rightarrow W$ is an elementary embedding with a critical point, but not cofinal.

Suppose that $V \models \text{ZFC} + \text{I}_1$ and fix an elementary embedding

$$j: V_{\lambda+1} \rightarrow V_{\lambda+1}$$

with critical point $\kappa < \lambda$. Let

$$j^+: H_{\lambda+} \rightarrow H_{\lambda+}$$

be the elementary embedding obtained from j . Let $\mathbb{P} = \text{Add}(\omega, 1)$, $G \subseteq \mathbb{Q} = \prod_{\omega}^{(\omega)} \mathbb{P}$ be V -generic, and W_G^V be constructed in $V[G]$ as in Section 3. Let

$$j^+: H_{\lambda+}[G] \rightarrow H_{\lambda+}[G]$$

be the lift of j^+ given by $j^+(\dot{x}_G) = j^+(\dot{x})_G$. Let

$$N_G^V = \bigcup_{n < \omega} H_{\lambda+}[G_n]$$

be the proper class of W_G^V consisting of sets whose transitive closure has size at most λ . Let

$$j^N: N_G^V \rightarrow N_G^V$$

be the restriction of j^+ to N_G^V . Then the embedding j^N can be defined by the formula $\varphi^j(x, y, \mathbb{P}, j, \lambda, G)$ asserting that:

- $x \in W_G^V$ has transitive closure of size at most λ ,
- there is a \mathbb{Q} -name $\dot{x} \in H_{\lambda+}^V$ such that $x = \dot{x}_G$, and
- $y = j^+(\dot{x})_G$.

Next, let's argue that if π is any coordinate-switching automorphism of \mathbb{Q} , then $\varphi^j(x, y, \mathbb{P}, j, \lambda, \pi''G)$ also defines j^N . Suppose that for some $a \in N_G^V$, $j^+(a) = b$. Let $a = \dot{x}_G$ with $\dot{x} \in H_{\lambda+}^V$. Then $b = j^+(\dot{x})_G$. Clearly, $a = (\pi(\dot{x}))_{\pi''G}$. Now,

$$j^+(\pi(\dot{x})) = j^+(\pi)(j^+(\dot{x})) = \pi(j^+(\dot{x})).$$

Thus, $j^+(\pi(\dot{x}))_{\pi''G} = j^+(\dot{x})_G = b$. Thus, using G or $\pi''G$ both yield the same embedding j^N . From this we immediately get the following strengthening of the

key elementarity lemma from Section 3, where we let $j_n^+ : H_{\lambda^+}^{V_{[n]}} \rightarrow H_{\lambda^+}^{V_{[n]}}$ be the lift of j^+ in $V_{[n]}$ and j_n^N be its restriction to $N_{G^{(n)}}^{V_{[n]}}$.

Lemma 7.3. *For every $n < \omega$, $\langle W_{G^{(n)}}^{V_{[n]}}, j_n^N \rangle \prec \langle W_G^V, j^N \rangle$.*

Using the arguments from Section 3, Lemma 7.3 gives:

Theorem 7.4. $W_G^V \models \text{ZFC}_{j^N}^-$, and hence $N_G^V \models \text{ZFC}_{j^N}^-$.

Proof. It follows from Lemma 7.3 that $W_G^V \models \text{ZFC}_{j^N}^-$. It remains to observe that any instance of collection for a set whose transitive closure has size at most λ can also be assumed to have transitive closure of size at most λ . \square

The following question is still open.

Question 7.5. Is the following situation consistent: There is a non-trivial, cofinal, elementary embedding $j : W \rightarrow W$ such that $W \models \text{ZFC}_j^-$?

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