Virtual large cardinal principles

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Virtual properties

Suppose $\mathcal{P}$ is a set-theoretic property asserting the existence of elementary embeddings between some first-order structures. We will say that $\mathcal{P}$ holds virtually if embeddings for structures from V witnessing $\mathcal{P}$ exist in set-forcing extensions of V.

- virtual large cardinals
  - applied to (very) large cardinals: supercompact, extendible, rank-into-rank, etc.
  - form a hierarchy between ineffable cardinals and ω-Erdős cardinals
  - measure consistency strength of other virtual properties
  - measure consistency strength of assertions from other contexts
- virtual forcing axioms
  - weak versions of PFA, SCFA, resurrection axioms
- Generic Vopěnka's Principle
  - consistent with $V = L$
  - equiconsistent with virtual large cardinals

Absoluteness lemma for countable embeddings

Lemma: (Silver) Suppose M and N are first-order structures such that

- M is countable,
- there is an elementary embedding $j : M \to N$.

Suppose W is a transitive (set or class) model of ZFC^- such that

- $M, N \in W$,
- M is countable in W.

Then for any finite $\bar{a} \subseteq M$, W has an elementary $j^* : M \to N$ agreeing with j on $\bar{a}$, and (where applicable) $\mathrm{crit}(j) = \mathrm{crit}(j^*)$.

Proof:

- Enumerate $M = \{a_n \mid n < \omega\}$ in W. Let $M \restriction n = \{a_i \mid i < n\}$.
- Let T be the tree of all partial finite isomorphisms

$$f : M \restriction n \to N,$$

  satisfying the requirements, ordered by extension.
- M elementarily embeds into N if and only if T has a cofinal branch.
- T is ill-founded in V, and hence in W. □

Examples of virtual embeddings

Proposition: There is a virtual isomorphism between $(\mathbb{R}, <)$ and $(\mathbb{Q}, <)$.

Proof: Go to the $\mathrm{Coll}(\omega, \mathbb{R})$-extension. □

Call an embedding $j : V_\alpha \to V_\alpha$ Kunen if $\alpha \gg$ supremum of the critical sequence of j.

Proposition: Assuming $0^\#$, L has virtual Kunen embeddings.

Proof:

- Let $\{i_\xi \mid \xi \in \mathrm{ORD}\}$ be the Silver indiscernibles.
- Let $j : L \to L$ be such that $j(i_n) = i_{n+1}$ for $n \in \omega$ and $j(i_\xi) = i_\xi$ for $\xi \geq \omega$.
- Let $i_\gamma = \alpha \gg i_\omega$ so that $j(\alpha) = \alpha$.
- The restriction $j : L_\alpha \to L_\alpha$ is elementary.
- $j : L_\alpha \to L_\alpha$ is in the forcing extension $V[H]$ by $\mathrm{Coll}(\omega, L_\alpha)$.
- In $L[H]$ there is $j^* : L_\alpha \to L_\alpha$ with $\mathrm{crit}(j^*) \leq i_0$ and $j^*(i_\omega) = i_\omega$.
- The supremum of the critical sequence of j^* is at most i_ω. □

Moral: Kunen's Inconsistency does not hold for virtual embeddings!

Virtual properties and collapse extensions

Lemma: Suppose M and N are first-order structures and some set-forcing extension has an elementary $j : M \to N$. Then for every finite $\bar{a} \subseteq M$, $V^{\mathrm{Coll}(\omega, M)}$ has an elementary $j^* : M \to N$ agreeing with j on $\bar{a}$ and (where applicable) $\mathrm{crit}(j) = \mathrm{crit}(j^*)$.

Proof: Suppose a set-forcing extension $V[G]$ has an elementary $j : M \to N$.

- Let $|M|^V = \delta$.
- Consider a further extension $V[G][H]$ by $\mathrm{Coll}(\omega, \delta)$.
- $j \in V[G][H]$ and M is countable in $V[G][H]$.
- $V[H] \subseteq V[G][H]$ has the elementary $j^* : M \to N$ (by Absoluteness Lemma). □

Virtual properties and determined games

Suppose M and N are first-order structures in a common language.
Let $G(M,N)$ be an ω-length Ehrenfeucht-Fraïssé type game:

- Stage n: player I plays some $a_n \in M$ and player II plays some $b_n \in N$.
- Player II wins if for every $n \in \omega$ and formula $\varphi(x_0,\ldots,x_n)$,
$$M \models \varphi(a_0,\ldots,a_n) \leftrightarrow N \models \varphi(b_0,\ldots,b_n),$$
and otherwise player I wins.
- If player II loses, she must do so in finitely many steps.
- $G(M,N)$ is closed, and hence determined by the Gale-Stewart Theorem.

Lemma: (Schindler) The following are equivalent.

(1) Player II has a winning strategy in $G(M,N)$.
(2) M elementarily embeds into N in $V^{\mathrm{Coll}(\omega,M)}$.

Proof:
(1) $\Rightarrow$ (2): A winning strategy for player II, remains winning in $V^{\mathrm{Coll}(\omega,M)}$ because no new finite sequences are added.
(2) $\Rightarrow$ (1): Fix $p \Vdash$ "$\tau : \check{M} \to \check{N}$ is an elementary embedding".

- To every finite $\bar{a}$ from M, associate $p_{\bar{a}} \Vdash \tau(\bar{a}) = \bar{b}$ below p so that: if $\bar{a}'$ extends $\bar{a}$, then $p_{\bar{a}'} \leq p_{\bar{a}}$.
- A winning strategy for player II: play $\bar{b}$ in response to $\bar{a}$. $\square$

Virtual large cardinals

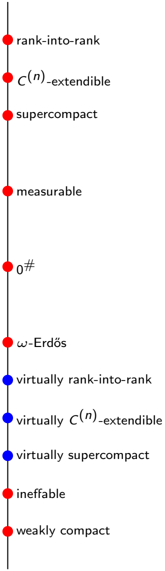
Suppose $\mathcal{P}$ is a (very) large cardinal property

- supercompact
- $C^{(n)}$-extendible
- rank-into-rank

asserting the existence of elementary embeddings $j : V_\alpha \to V_\beta$ satisfying a list of properties.
A cardinal is virtually $\mathcal{P}$ if the embeddings characterizing $\mathcal{P}$ exist in set-forcing extensions of V.

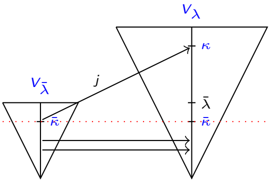
Virtual large cardinals are mini versions of their actual counterparts.

- Silver indiscernibles are virtual large cardinals.
- Virtual large cardinals lie between ineffable and ω-Erdős cardinals.
- Virtual large cardinals are downward absolute to L.
- Relationships between virtual large cardinals mirror their actual counterparts.



Magidor's characterization of supercompact cardinals

Theorem: (Magidor) A cardinal κ is supercompact iff for every $\lambda > \kappa$, there is $\bar{\lambda} < \kappa$ such that there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$.



Remarkable cardinals

Definition: (Schindler) A cardinal κ is remarkable if for every $\lambda > \kappa$, there is $\bar{\lambda} < \kappa$ such that in a set-forcing extension there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$.

Remarkable cardinals are virtually supercompact by Magidor's characterization!

Lemma: The following are equivalent for a cardinal κ.

- κ is remarkable.
- For every $\lambda > \kappa$, there is $\alpha > \lambda$ and a transitive M with $M^\lambda \subseteq M$ such that in a set-forcing extension there is an elementary $j : V_\alpha \to M$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.
- For every $\lambda > \kappa$, there is $\alpha > \lambda$ and a transitive M with $V_\lambda \subseteq M$ such that in a set-forcing extension there is an elementary $j : V_\alpha \to M$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

Moral:

- Closure requirements do not increase strength of virtual large cardinals.
- Robust virtual large cardinals are characterized by $j : V_\alpha \to V_\beta$.

Theorem: (Schindler) The assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing is equiconsistent with a remarkable cardinal.

$C^{(n)}$-extendible cardinals

Let $C^{(n)}$ be the class of all Σ_n-correct δ such that $V_\delta \prec_{\Sigma_n} V$.

Definition:

- A cardinal κ is extendible if for every $\alpha > \kappa$, there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.
- (Bagaria) A cardinal κ is $C^{(n)}$-extendible if for every $\alpha > \kappa$, there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $j(\kappa) \in C^{(n)}$.

Theorem:

- Extendible cardinals are $C^{(1)}$-extendible.
- (Bagaria, G., Schindler) The $C^{(n)}$-extendible cardinals form a hierarchy: a $C^{(n+1)}$-extendible cardinal is a limit of $C^{(n)}$-extendible cardinals.
- (Bagaria, G., Schindler) A cardinal κ is $C^{(n)}$-extendible if and only if for every $\kappa < \lambda \in C^{(n+1)}$, there is $\bar{\lambda} < \kappa$ also in $C^{(n+1)}$ such that there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$.

Theorem: (G., Hamkins, Tsaprounis) A cardinal κ is $C^{(n)}$-extendible if and only if for every $\kappa < \alpha \in C^{(n)}$, there is $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $\beta \in C^{(n)}$.

Weakly $C^{(n)}$-extendible cardinals

Call a cardinal κ weakly $C^{(n)}$-extendible if for every $\kappa < \alpha \in C^{(n)}$, there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $\beta \in C^{(n)}$.

Theorem: Weakly $C^{(n)}$-extendible cardinals are $C^{(n)}$-extendible.

Proof: Suppose κ is weakly extendible. Fix $\alpha > \kappa$.
Suppose for every $\xi < \eta$, there is $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \xi$.
But there is no embedding j with $j(\kappa) > \eta$.

- Let $\bar{\alpha}$ be large enough so $V_{\bar{\alpha}}$ sees that η is least counterexample for κ and α.
- Let $j : V_{\bar{\alpha}} \to V_{\bar{\beta}}$ with $\text{crit}(j) = \kappa$.
- By elementarity, $V_{\bar{\beta}}$ satisfies that $j(\eta)$ is least counterexample for $j(\kappa)$ and $j(\alpha)$.
- Suppose $j(\eta) > \eta$.
  - There is $h : V_{j(\alpha)} \to V_\delta$ with $\text{crit}(h) = j(\kappa)$ and $h(j(\kappa)) > \eta$.
  - $h \circ j : V_\alpha \to V_\delta$ has $\text{crit}(h \circ j) = \kappa$ and $h \circ j(\kappa) = h(j(\kappa)) > \eta$.
- So $j(\eta) = \eta$.
- Restrict $j : V_{\eta+2} \to V_{\eta+2}$ violating Kunen's Inconsistency. □

Virtually $C^{(n)}$-extendible cardinals

Definition: (Bagaria, G., Schindler)

- A cardinal κ is virtually extendible if for every $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.
- A cardinal κ is virtually $C^{(n)}$-extendible if for every $\alpha > \kappa$, in a set-forcing extension there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $j(\kappa) \in C^{(n)}$.

Theorem:

- Virtually extendible cardinals are virtually $C^{(1)}$-extendible.
- The virtually $C^{(n)}$-extendible cardinals form a hierarchy: a virtually $C^{(n+1)}$-extendible cardinal is a limit of virtually $C^{(n)}$-extendible cardinals.
- A cardinal κ is virtually $C^{(n)}$-extendible if and only if for every $\kappa < \lambda \in C^{(n+1)}$, there is $\bar{\lambda} < \kappa$ also in $C^{(n+1)}$ such that in a set-forcing extension there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$.
- A cardinal κ is virtually $C^{(n)}$-extendible if for every $\kappa < \alpha \in C^{(n)}$, in a set-forcing extension there is $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $\beta \in C^{(n)}$.

Virtually rank-into-rank cardinals

Definition: (G., Schindler) A cardinal κ is virtually rank-into-rank if in a set-forcing extension there is an elementary $j : V_\lambda \to V_\lambda$ with $\mathrm{crit}(j) = \kappa$.

Recall that we can have $\lambda \gg$ supremum of the critical sequence of j.

Theorem: If κ is (virtually) rank-into-rank, then V_κ is a model of proper class many (virtually) $C^{(n)}$-extendible cardinals for every $n \in \omega$.

Virtually weakly $C^{(n)}$-extendible cardinals

Definition: A cardinal κ virtually weakly $C^{(n)}$-extendible if for every $\kappa < \alpha \in C^{(n)}$, in a set-forcing extension there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $\beta \in C^{(n)}$.

Theorem: If κ is virtually weakly $C^{(n)}$-extendible, but not virtually $C^{(n)}$-extendible, then κ is virtually rank-into-rank.

Corollary: Virtually weakly $C^{(n)}$-extendible cardinals are equiconsistent with virtually $C^{(n)}$-extendible cardinals.

Other virtual large cardinals

Definition: (Schindler, Wilson) A cardinal κ is virtually Shelah for supercompactness if for every function $f : \kappa \to \kappa$, there is $\bar{\lambda} < \kappa$ and $\lambda > \kappa$ such that in a set-forcing extension there is an elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$, $f(\text{crit}(j)) \leq \bar{\lambda}$, and $f \in \text{ran}(j)$.

Theorem: (Schindler, Wilson) The assertion that every universally Baire set of reals has the perfect set property is equiconsistent with a virtually Shelah for supercompactness cardinal.

Definition: (G., Schindler) A cardinal κ is virtually n-huge* if there is $\alpha > \kappa$ such that in a set-forcing extension there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$.

- n-huge cardinals do not have a robust definition for virtualization.
- n-huge* cardinals hierarchy is intertwined with the n-huge cardinals hierarchy.
- If κ is virtually rank-into-rank, then V_κ is a model of a proper class of virtually n-huge* cardinals for every $n \in \omega$.
- If κ is virtually huge*, then V_κ is a model of a proper class of virtually $C^{(n)}$-extendible cardinals for every $n \in \omega$.

Vopěnka's Principle

Vopěnka's Principle: Every proper class of first-order structures in a fixed language has two structures which elementarily embed.

- second-order assertion formalizable in Gödel-Bernays set theory, GBC.
- $\mathrm{VP}(\Sigma_n)$: Vopěnka's Principle for Σ_n-definable (with parameters) classes.

Vopěnka's Scheme: Scheme asserting $\mathrm{VP}(\Sigma_\mathrm{n})$ for every $n \in \omega$.

Theorem: (Hamkins)

- There are models of GBC in which Vopěnka's Scheme holds, but Vopěnka's Principle fails.
- Over GBC, Vopěnka's Principle and Vopěnka's Scheme are equiconsistent and have the same first-order consequences.

Theorem: (Bagaria) The following are equivalent.

- Vopěnka's Scheme holds.
- For every $n \in \omega$, there is a proper class of $C^{(n)}$-extendible cardinals.

Generic Vopěnka's Principle

Generic Vopěnka's Principle: Every proper class of first-order structures in a fixed language has two structures which elementarily embed in some set-forcing extension.

$\mathrm{gVP}(\Sigma_n)$: Generic Vopěnka's Principle for Σ_n-definable (with parameters) classes.

Generic Vopěnka's Scheme: Scheme asserting $\mathrm{gVP}(\Sigma_\mathrm{n})$ for every $n \in \omega$.

Theorem: (G., Hamkins) The following are equivalent.

- Generic Vopěnka's Scheme holds.
- For every $n \in \omega$, there is a proper class of virtually weakly $C^{(n)}$-extendible cardinals.

Generic Vopěnka's Scheme from virtual large cardinals

Theorem: (G., Hamkins) If for every $n \in \omega$, there is a proper class of virtually weakly $C^{(n)}$-extendible cardinals, then Generic Vopěnka's Scheme holds.

Proof:

- Let $\mathcal{M}$ be a proper class of structures in language $\mathcal{L}$ defined by a Σ_n-formula $\varphi(x, a)$.
- Let $\kappa > \text{rk}(a), \text{rk}(\mathcal{L})$ be virtually weakly $C^{(n)}$-extendible.
- Choose $\kappa < \alpha \in C^{(m)}$ for some $m \gg n$.
- In $V^{\text{Coll}(\omega, V_\alpha)}$ there is
$$j : V_\alpha \to V_\beta$$
with $\text{crit}(j) = \kappa$ and $\beta \in C^{(n)}$.
- Let $M \in \mathcal{M}$ be any structure of the κ-th rank in $\mathcal{M}$.
- V_α agrees that M has κ-th rank in $\mathcal{M}$.
- By elementarity, $V_\beta \models$ "$j(M)$ has $j(\kappa)$-th rank in $\mathcal{M}$".
  - $j(M) \in \mathcal{M}$
  - $M \neq j(M)$
  - The restriction $j : M \to j(M)$ is elementary. □

Virtual large cardinals from Generic Vopěnka's Scheme

Theorem: (G., Hamkins) If Generic Vopěnka's Scheme holds, then for every $n \in \omega$, there is a proper class of virtually weakly $C^{(n)}$-extendible cardinals.

Proof: Fix an ordinal γ.

- There is α_0 such that for all $\alpha_0 < \alpha \in C^{(n)}$, in $V^{\mathrm{Coll}(\omega, V_\alpha)}$ there is an elementary $j : V_\alpha \to V_\beta$ with $\alpha < \beta \in C^{(n)}$:
  - j is an inclusion map or
  - $\mathrm{crit}(j) > \gamma$.
- Suppose not. Then there are unboundedly many counterexamples.
- Let $\mathcal{M}$ be the class of structures $\langle V_\alpha, \in, \xi\rangle_{\xi \leq \gamma}$ such that:
  - $\alpha \in C^{(n)}$,
  - in $V^{\mathrm{Coll}(\omega, V_\alpha)}$ there is no $j : \langle V_\alpha, \in, \xi\rangle_{\xi \leq \gamma} \to \langle V_\beta, \in, \xi\rangle_{\xi \leq \gamma}$ with $\alpha < \beta \in C^{(n)}$.
- By assumption, $\mathcal{M}$ is a proper class.
- In $V^{\mathrm{Coll}(\omega, V_\alpha)}$ there is

$$j : \langle V_\alpha, \in, \xi\rangle_{\xi \leq \gamma} \to \langle V_\beta, \in, \xi\rangle_{\xi \leq \gamma}.$$

  Contradiction!

Virtual large cardinals from Generic Vopěnka's Scheme

- $S = \{\alpha \in C^{(n)} \mid \alpha > \alpha_0 \text{ and } \mathrm{cof}(\alpha) = \omega\}$ is a stationary class.
- Given $\alpha \in S$, let $\bar{\alpha}$ be least in $C^{(n)}$ above α.
- In $V^{\mathrm{Coll}(\omega, V_{\bar{\alpha}})}$ there is $j : V_{\bar{\alpha}} \to V_{\bar{\beta}}$ with $\bar{\alpha} < \bar{\beta} \in C^{(n)}$ such that:
  - $\alpha < j(\alpha)$,
  - $\gamma < \mathrm{crit}(j) < \alpha$ ($\mathrm{crit}(j)$ is inaccessible).
- Define $F : S \to \mathrm{ORD}$ by $F(\alpha)$ is least critical point of some such j.
  - F is regressive.
  - $F(\alpha) = \kappa$ unboundedly often.
- For every $\kappa < \alpha \in C^{(n)}$, in $V^{\mathrm{Coll}(\omega, V_\alpha)}$ there is $j : V_\alpha \to V_\beta$ such that:
  - $\mathrm{crit}(j) = \kappa$,
  - $\beta \in C^{(n)}$. □

Generic Vopěnka's Principle and ORD is Mahlo

Definition:

- ORD is Mahlo if every class club has a regular cardinal.
- ORD is definably Mahlo if every definable class club has a regular cardinal.

Lemma: If Vopěnka's Principle holds, then ORD is Mahlo.

Lemma: If for every $n \in \omega$, there is a proper class of virtually $C^{(n)}$-extendible cardinals, then ORD is definably Mahlo.

Proof: Suppose C is a class club defined by a Σ_n-formula $\varphi(x, a)$.

- Let $\kappa > \mathrm{rk}(a)$ be virtually $C^{(n)}$-extendible.
- Since $\kappa \in C^{(n)}$ (indeed $\kappa \in C^{(n+2)}$), C is unbounded in κ.
- $\kappa \in C$. □

Theorem: (G., Hamkins) The following are consistent.

- Generic Vopěnka's Principle holds, but ORD is not Mahlo.
- Generic Vopěnka's Scheme holds, but there is a Σ_2-definable (w/o parameters) class club avoiding regular cardinals.

Corollary: It is consistent that Generic Vopěnka's Scheme holds, but there are no remarkable cardinals.

Proof: Remarkable cardinals are in $C^{(2)}$. □

A model of Generic Vopenka's Principle where ORD is not Mahlo

Proof: Assume $0^{\#}$ exists.

- L together with its definable classes is a model of GBC.
- Generic Vopěnka's Principle holds in L.
  - Silver indiscernibles are virtually $C^{(n)}$-extendible for every $n \in \omega$.
- In L, let $\mathbb{P}$ be the class forcing to add a class club avoiding regular cardinals.
  - Conditions: closed bounded sets of ordinals avoiding regular cardinals.
  - Order: end-extension.
  - $\mathbb{P}$ is $\leq \alpha$-distributive for every cardinal α.
  - $\mathbb{P}$ does not add sets.

Let $C \subseteq \mathrm{ORD}$ be L-generic for $\mathbb{P}$.

- The first-order part of $L[C]$ is L.
- The classes of $L[C]$ are definable from C over L.
- ORD is not Mahlo in $L[C]$.
- We show that Generic Vopěnka's Principle holds in $L[C]$.

A model of Generic Vopenka's Principle where ORD is not Mahlo

Key Lemma: Given an ordinal δ and $n \in \omega$, there is an ordinal θ such that:

- $L_\theta \prec_{\Sigma_n} L$
- $C \cap \theta$ is L_θ-generic for $\mathbb{P}^{L_\theta}$
- in $L^{\mathrm{Coll}(\omega, L_\theta)}$ there is
$$j : (L_\theta, \in, C \cap \theta) \to (L_\theta, \in C \cap \theta)$$
with $\mathrm{crit}(j) > \delta$.

Let $\mathcal{M}$ be a proper class of structures in language $\mathcal{L}$ defined by a Σ_m-formula $\varphi(x, a, C)$ in $L[C]$.

Fix $\delta > \mathrm{rk}(a), \mathrm{rk}(\mathcal{L})$ and $n \gg m$.

By Key Lemma, there is an ordinal θ:

- $(L_\theta, \in C \cap \theta) \prec_{\Sigma_n} (L, \in, C)$.
- In $L^{\mathrm{Coll}(\omega, L_\theta)}$ there is $j : (L_\theta, \in, C \cap \theta) \to (L_\theta, \in C \cap \theta)$ with $\mathrm{crit}(j) = \kappa > \delta$.
- Let $M \in \mathcal{M}$ be a structure of the κ-th rank in $\mathcal{M}$.
- $(L_\theta, \in, C \cap \theta)$ agrees that M has κ-th rank in $\mathcal{M}$.
- By elementarity, $(L_\theta, \in, C \cap \theta) \models j(M)$ has $j(\kappa)$-th rank in $\mathcal{M}$.
  - $j(M) \in \mathcal{M}$
  - $M \neq j(M)$
  - The restriction $j : M \to j(M)$ is elementary. □

Proof of Key Lemma

Fix $\delta \in \mathrm{ORD}$ and $n \in \omega$.

Let $D_{\delta,n}$ be the class of conditions $\bar{c} \in \mathbb{P}$ for which there is an ordinal θ such that:

- $L_\theta \prec_{\Sigma_n} L$,
- $\bar{c} \cap \theta$ is L_θ-generic for $\mathbb{P}^{L_\theta}$,
- in $L^{\mathrm{Coll}(\omega, L_\theta)}$ there is an elementary $j : (L_\theta, \in, \bar{c} \cap \theta) \to (L_\theta, \in, \bar{c} \cap \theta)$ with $\mathrm{crit}(j) > \delta$.

We show that $D_{\delta,n}$ is dense in $\mathbb{P}$. Fix $d \in \mathbb{P}$.

- Let $\kappa_0 > \sup(d), \delta$ be an uncountable cardinal of V.
- Let $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots < \kappa_\omega < \kappa_{\omega+1}$ be successive Silver indiscernibles.
- Let θ be least above κ_ω such that $L_\theta \prec L_{\kappa_{\omega+1}} (\prec L)$.
- Fix $\theta_0 < \theta_1 < \cdots < \theta_n < \cdots$ cofinal in θ.
- The intersection D_n of all Σ_n-definable with parameters from L_{θ_n} dense open sets of $\mathbb{P}^{L_\theta}$ is dense open.
- Let c be the L-least set of ordinals L_θ-generic for $\mathbb{P}^{L_\theta}$.
- Let $h : L \to L$ be such that $h(\kappa_n) = \kappa_{n+1}$ and all other indiscernibles are fixed.
- $\mathrm{crit}(h) = \kappa_0$ (indiscernibles below κ_0 generate L_{κ_0}).
- $h(\theta) = \theta$ and $h(c) = c$ (since $h(d) = d$, $h(\kappa_\omega) = \kappa_\omega$, $h(\kappa_{\omega+1}) = \kappa_{\omega+1}$).
- $h : (L_\theta, \in, c) \to (L_\theta, \in, c)$ with $\mathrm{crit}(h) = \kappa_0 > \delta$.
- In $L^{\mathrm{Coll}(\omega, L_\theta)}$ there is $j : (L_\theta, \in, c) \to (L_\theta, \in, c)$ with $\mathrm{crit}(j) = \kappa_0 > \delta$.
- Let $\bar{c} = c \cup \{\theta\}$ (θ is singular). $\square$

A model of Generic Vopěnka's Scheme with a bad Σ_2-definable class club

Proof: Assume $0^{\#}$ exists.

- Let $L[C]$ be a forcing extension by $\mathbb{P}$.
- Force to code C into the continuum pattern.

Let $\mathbb{Q}$ be the Easton-support class product forcing the failure of GCH at $\aleph_{\alpha+1}$ iff $\alpha \in C$.

- Let $G \subseteq \mathbb{Q}$ be $L[C]$-generic.
- Let $L[G] = \{\tau_G \mid \tau \text{ is a } \mathbb{Q}\text{-name}\}$ (first-order part of the forcing extension by $\mathbb{Q}$).
- C is Σ_2-definable (w/o parameters) in $L[G]$.
- Generic Vopěnka's Scheme holds in $L[G]$.

A model of Generic Vopěnka's Scheme with a bad Σ_2-definable class club

Iteration $\mathbb{P} * \dot{\mathbb{Q}}$:

Conditions: pairs (c, q)

- $c \in \mathbb{P}$
- q is a condition in corresponding Easton-support product coding c.

Let $\mathbb{Q}_\theta$ consist of conditions in $\mathbb{Q}$ with support contained in θ.

- $\mathbb{Q}_\theta \in L$.
- Let G_θ be the restriction of G to $\mathbb{Q}_\theta$.
- Let $G^*_\theta = L_\theta \cap G_\theta$.

Key Lemma: Given an ordinal $\delta \in \mathrm{ORD}$ and $n \in \omega$, there is an ordinal θ such that:

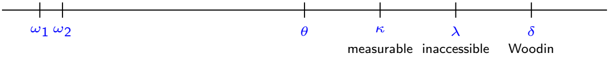
- $L_\theta \prec_{\Sigma_n} L$.
- $C \cap \theta$ is L_θ-generic for dense subsets of $\mathbb{P}^{L_\theta}$ Σ_n-definable in L_θ.
- G^*_θ is $\mathbb{Q}^{L_\theta[C\cap\theta]}$-generic for dense subsets of $\mathbb{Q}^{L_\theta[C\cap\theta]}$ Σ_n-definable in $L_\theta[C \cap \theta]$.
- In $L[G]^{\mathrm{Coll}(\omega, L_\theta)}$ there is an elementary $j : L_\theta[G^*_\theta] \to L_\theta[G^*_\theta]$ with $\mathrm{crit}(j) > \delta$.

Observe that $L_\theta[G^*_\theta] \prec_{\Sigma_m} L[G]$ for $m \ll n$ and repeat argument. $\square$

Virtual Kunen embeddings in cardinal preserving extensions

Question: Is it consistent to have virtual Kunen embeddings in forcing extensions preserving ω_1, or all cardinals $\leq \omega_n$, or all cardinals below the least inaccessible cardinal, etc?

Theorem: (Woodin) Suppose δ is a Woodin cardinal. Let $\theta < \kappa < \lambda < \delta$ be such that κ is measurable and λ is inaccessible. Then there is a forcing extension $V[G]$ with $V_\theta^{V[G]} = V_\theta$ in which there is a virtual Kunen embedding $j : V_\lambda \to V_\lambda$.



Theorem: (Woodin) Suppose δ is a Woodin cardinal. Let $\theta < \kappa < \lambda < \delta$ be such that κ is measurable and λ is inaccessible. Let $W = V[G]$ be a forcing extension by Prikry forcing on κ. Then W has a forcing extension $W[H]$ with $W_\lambda^\theta \subseteq W_\lambda$ in $W[H]$ in which there is a virtual Kunen embedding $j : W_\lambda \to W_\lambda$.

Question: Is it consistent to have a virtual Kunen embedding $j : V_\lambda \to V_\lambda$ in a forcing extension not adding ω-sequences?

Stationary tower forcing

Definition: Suppose X is a nonempty set.

- A set $C \subseteq P(X)$ is a club in $P(X)$ if there exists a function $F : [X]^{<\omega} \to X$ such that $C = \{x \mid F[[x]^{<\omega}] \subseteq x\}$.
- A set $S \subseteq P(X)$ is stationary in $P(X)$ if for every function $F : [X]^{<\omega} \to X$ there exists a $x \in S$ such that $F[[x]^{<\omega}] \subseteq x$.

Observations: Suppose X is a nonempty set.

- A set $S \subseteq P(X)$ is stationary in $P(X)$ if and only if every structure $M = (X, \ldots)$ in a countable language has an elementary substructure in S.
- If S is stationary in $P(X)$, then $\bigcup S = X$.

Definition: Suppose $\bigcup S \neq \emptyset$. We say that S is stationary if S is stationary in $\bigcup S$.

Stationary tower forcing $\mathbb{P}_{<\delta}$: Suppose δ is inaccessible.

- **Conditions**: stationary $S \in V_\delta$
- **Order**: $S \leq T$ whenever $\bigcup T \subseteq \bigcup S$ and for each $x \in S$, $x \cap (\bigcup T) \in T$.

Theorem: (Woodin) Suppose δ is a Woodin cardinal and $G \subseteq \mathbb{P}_{<\delta}$ is V-generic. Then in $V[G]$, there is an elementary $j : V \to M$ with $M^{<\delta} \subseteq M$ such that for $S \in G$, $j\text{"}(\bigcup S) \in j(S)$.

Proof of Woodin's Theorem

Theorem: (Woodin) Suppose δ is a Woodin cardinal. Let $\theta < \kappa < \lambda < \delta$ be such that κ is measurable and λ is inaccessible. Then there is a forcing extension $V[G]$ with $V_\theta^{V[G]} = V_\theta$ in which there is a virtual Kunen embedding $j : V_\lambda \to V_\lambda$.

Proof:

- Assume $|V_\theta| = \theta$.
- Let $S \subseteq P(V_\lambda)$ consist of $X \prec V_\lambda$ satisfying the following properties:
  - $\theta \subseteq X$
  - $|X| = \lambda$
  - Let $M_X \cong X$ be the collapse. There is $A \subseteq \theta$ such that M_X is definable in $L[A]$.
- S is stationary in $P(V_\lambda)$.

  Fix $F : [[V_\lambda]^{<\omega}] \to V_\lambda$. Let $\kappa, F \in Z_0 \prec V_{\lambda^*}$ with $|Z_0| = \theta$. Let $\pi : M_{Z_0} \cong Z_0$ be the collapse with $\pi(\bar{\kappa}) = \kappa$.

  Let $A \subseteq \theta$ code M_{Z_0}. In $L[A]$, iterate M_{Z_0} by a measure on $\bar{\kappa}$ λ-many times. Each iterate $M_{Z_\alpha} \cong Z_\alpha \prec V_{\lambda^*}$.
- Let $S \in G \subseteq \mathbb{P}_{<\delta}$ be V-generic. So that j " $V_\lambda \in j(S)$.
  - $j(\theta) \subseteq j$ " V_λ. So $\text{crit}(j) > \theta$.
  - $|j$ " $V_\lambda| = j(\lambda) = \lambda$. So λ remains inaccessible in $V[G]$.
  - There is $A \subseteq j(\theta) = \theta$ such that V_λ is definable in $L[A]$.
- $V_\theta^{V[G]} = V_\theta^M = V_\theta$ since V_θ is coded by a subset of θ.
- $A^\# \in M$ by elementarity.
- There is an elementary $h : L[A] \to L[A]$ with $\text{crit}(h) < \lambda$ and $h(V_\lambda) = V_\lambda$.
- $h : V_\lambda \to V_\lambda$. □

Thank you!