# The Virtual Large Cardinal Hierarchy

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ABSTRACT. We continue the study of the virtual large cardinal hierarchy by analysing virtual versions of superstrong, Woodin, and Berkeley cardinals. Gitman and Schindler showed that virtualizations of strong and supercompact cardinals yield the same large cardinal notion. We provide various equivalent characterizations of virtually Woodin cardinals, including showing that On is virtually Woodin if and only if for every class *A*, there is a proper class of virtually *A*-extendible cardinals. We introduce the virtual Vopěnka principle for finite languages and show that it is not equivalent to the virtual Vopěnka principle (although the two principles are equiconsistent), but is equivalent to the assertion that On is virtually pre-Woodin, a weakening of virtually Woodin, which is equivalent to having for every class *A*, a weakly virtually *A*-extendible cardinal. We show that if there are no virtually Berkeley cardinals, then On is virtually Pre-Woodin (if and only if the virtual Vopěnka principle for finite languages holds). In particular, if the virtual Vopěnka principle holds and On is not Mahlo, then On is not virtually Woodin, and hence there is a virtually Berkeley cardinal.

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## 1 Introduction

The study of generic large cardinals, being cardinals that are critical points of elementary embeddings existing in generic extensions, goes back to the 1970's. At that time, the primary interest was the existence of *precipitous* and *saturated* ideals on small cardinals like  $\omega_1$  and  $\omega_2$ . Research in this area later moved to the study of more general generic embeddings, both defined on V, but also on rank-initial segments of V — these were investigated by e.g. [DL89] and [FG10].

The move to *virtual* large cardinals happened when [Sch00] introduced the *remark-able cardinals*, which it turned out later were precisely a virtualization of supercompactness. Various other virtual large cardinals were first investigated in [GS18]. The key difference between virtual large cardinals and generic versions of large cardinals studied earlier is that in the virtual case we require the embedding to be between sets with the target model being a subset of the ground model. These assumptions imply that virtual large cardinals are actual large cardinals: they are at least ineffable, but small enough to exist in *L*. These large cardinals are special because they allow us to work with embeddings as in the higher reaches of the large cardinal hierarchy while being consistent with V = L, which enables equiconsistencies at these "lower levels".

To take a few examples, [Sch00] has shown that the existence of a remarkable cardinal is equiconsistent with the statement that the theory of  $L(\mathbb{R})$  cannot be changed by proper forcing, which was improved to semi-proper forcing in [Sch04]. [Wil19] has shown that the existence of a virtually Vopěnka cardinal is equiconsistent with the hypothesis

$$\mathsf{ZF} + \lceil \mathbf{\Sigma}_2^1$$
 is the class of all  $\omega_1$ -Suslin sets $\rceil + \Theta = \omega_2$ ,

and [SW18] has shown that the existence of a virtually Shelah for supercompactness cardinal is equiconsistent with the hypothesis

 $ZFC + \lceil every universally Baire set of reals has the perfect set property \rceil$ .

Kunen's Inconsistency fails for virtual large cardinals in the sense that a forcing extension can have elementary embeddings  $j : V_{\alpha}^{V} \to V_{\alpha}^{V}$  with  $\alpha$  much larger than the supremum of the critical sequence. In the theory of large cardinals, Kunen's Inconsistency is for instance used to prove that requiring that  $j(\kappa) > \lambda$  in the definition of  $\kappa$  being  $\lambda$ -strong is superfluous. It turns out that the use of Kunen's Inconsistency in that argument is actually essential because versions of virtual strongness with and without that condition are not equivalent; see e.g. Corollary 3.11. The same holds for virtual versions of other large cardinals where this condition is used in the embeddings characterization. Each of these virtual large cardinals therefore has two non-equivalent versions, with and without the condition.

In this paper, we continue the study of virtual versions of various large cardinals.

In Section 3 we establish some new relationships between virtual large cardinals that were previously studied. We prove the Gitman-Schindler result, alluded to in [GS18], that virtualizations of strong and supercompact cardinals are equivalent to remarkability. We show how the existence of virtually supercompact cardinals without the  $j(\kappa) > \lambda$  condition is related to the existence of virtually rank-into-rank cardinals.

In Section 4 we study virtual versions of Woodin cardinals and introduce the virtual Vopěnka principle for finite languages. We provide various equivalent characterizations of virtually Woodin cardinals. It follows, from the equivalences, that On is virtually Woodin if and only if for every class *A* there is a virtually *A*-extendible cardinal (as defined in [GH19]), equivalently, there is a stationary class of virtually *A*-extendible cardinals. Recall from [GH19] that the virtual Vopěnka Principle holds if and only if for every class of weakly virtually *A*-extendible cardinals. It follows from arguments in [GH19] that the virtual Vopěnka Principle for finite languages holds if and only if for every class *A* there is a weakly virtually *A*-extendible cardinal. It follows from arguments in [GH19] that the virtual Vopěnka Principle for finite languages holds if and only if for every class *A* there is a weakly virtually *A*-extendible cardinal. We show that On is virtually Vopěnka for finite languages if and only if On is *faintly pre-Woodin*, a weakening of the notion of virtual Woodinness — see Definition 4.2.

In Section 5 we study a virtual version of *Berkeley* cardinals, a large cardinal known to be inconsistent with ZFC. We show that if there are no virtually Berkeley cardinals, then On is virtually Woodin if and only if On is faintly pre-Woodin if and only if the virtual Vopěnka principle for finite languages holds. In this situation, the virtual Vopěnka Principle for finite languages is equivalent to the virtual Vopěnka Principle. However, we will use virtual Berkeley cardinals to separate the two principles. It follows also that if the virtual Vopěnka Principle holds and On is not Mahlo (in particular, On is not virtually Woodin), then there is a virtually Berkeley cardinal, but as pointed out by the anonymous referee, it is possible to have that the virtual Vopěnka Principle holds and On is Mahlo, but On is not virtually Woodin.

#### 2 Preliminaries

We will denote the class of ordinals by On. For sets X and Y we denote by  ${}^{X}Y$  the set of all functions from X to Y. For an infinite cardinal  $\kappa$ , we let  $H_{\kappa}$  be the set of sets X such that the cardinality of the transitive closure of X has size less than  $\kappa$ . The symbol  $\frac{i}{2}$  will denote a contradiction and  $\mathscr{P}(X)$  will denote the power set of X. We will say that an elementary embedding  $j : \mathcal{M} \to \mathcal{N}$  is generic if it exists in a forcing extension of V. We will use  $H_{\lambda}$  and  $V_{\lambda}$  to denote these sets as defined in the ground model V, while the  $H_{\lambda}$  or  $V_{\lambda}$  of any other universe will have a superscript indicating which universe it comes from.

A key folklore lemma which we will frequently need when dealing with generic elementary embeddings is the following.

**LEMMA 2.1** (Countable Embedding Absoluteness). Let  $\mathcal{M}$  and  $\mathcal{N}$  be transitive sets and assume that  $\mathcal{M}$  is countable. Let  $\pi \colon \mathcal{M} \to \mathcal{N}$  be an elementary embedding,  $\mathcal{P}$  a transitive class with  $\mathcal{M}, \mathcal{N} \in \mathcal{P}$  and  $\mathcal{P} \models ZF^- + DC + \ulcorner\mathcal{M}$  is countable $\urcorner$ .<sup>1</sup> Then  $\mathcal{P}$  has an elementary embedding  $\pi^* \colon \mathcal{M} \to \mathcal{N}$  which agrees with  $\pi$  on any desired finite set and on the critical point if it exists.

The following proposition is an almost immediate corollary of Countable Embedding Absoluteness.

**PROPOSITION 2.2.** Let  $\mathcal{M}$  and  $\mathcal{N}$  be transitive models and assume that there is a generic elementary embedding  $\pi : \mathcal{M} \to \mathcal{N}$ . Then  $V^{\operatorname{Col}(\omega,\mathcal{M})}$  has an elementary embedding  $\pi^* : \mathcal{M} \to \mathcal{N}$  which agrees with  $\pi$  on any desired finite set and has the same critical point if it exists.

For proofs, see [GS18] (Section 3).

## 3 Virtually supercompact cardinals

In this section, we establish some relationships between virtual large cardinals related to virtually supercompact cardinals. We start with definitions of the relevant virtual large cardinal notions.

**DEFINITION 3.1.** Let  $\theta$  be a regular uncountable cardinal. Then a cardinal  $\kappa < \theta$  is

- faintly θ-measurable if, in a forcing extension, there is a transitive set N and an elementary embedding π: H<sub>θ</sub> → N with crit π = κ,
- faintly  $\theta$ -strong if it is faintly  $\theta$ -measurable,  $H_{\theta} = H_{\theta}^{\mathcal{N}}$  and  $\pi(\kappa) > \theta$ ,
- faintly  $\theta$ -supercompact if it is faintly  $\theta$ -measurable,  ${}^{<\theta} \mathcal{N} \cap V \subseteq \mathcal{N}$  and  $\pi(\kappa) > \theta$ .

We further replace "faintly" by **virtually** when  $\mathcal{N} \subseteq V$ , we attach a "**pre**" if we leave out the assumption  $\pi(\kappa) > \theta$ , and when we do not mention  $\theta$  we mean that it holds for all regular  $\theta > \kappa$ . For instance, a faintly pre-strong cardinal is a cardinal  $\kappa$  such that for all regular  $\theta > \kappa$ ,  $\kappa$  is faintly  $\theta$ -measurable with  $H_{\theta} \subseteq \mathcal{N}$ .

 $<sup>^{1}</sup>$ The theory ZF<sup>-</sup> consists of the axioms of ZF without the powerset axiom and with the collections scheme instead of the replacement scheme.

Observe that whenever we have a virtual large cardinal that has its defining property for all regular  $\theta$ , we can assume that the target of the embedding is an element of the ground model V and not just a subset of V. Suppose, for instance, that  $\kappa$  is virtually measurable and fix a regular  $\theta > \kappa$  and set  $\lambda := (2^{<\theta})^+$ . Take a generic elementary embedding  $\pi : H_{\lambda} \to \mathcal{M}_{\lambda}$  witnessing that  $\kappa$  is virtually  $\lambda$ -measurable. The restriction  $\pi \upharpoonright H_{\theta} : H_{\theta} \to \pi(H_{\theta})$  witnesses that  $\kappa$  is virtually  $\theta$ -measurable and the target model  $\mathcal{M}_{\theta} := \pi(H_{\theta})$  is in V because  $\mathcal{M}_{\lambda} \subseteq V$  by assumption. Thus, the weaker assumption that the target model  $\mathcal{M}_{\theta} \subseteq V$  only affects level-by-level virtual large cardinals. Indeed, as we will see in later sections, even further weakening the assumption  $\mathcal{N} \subseteq V$  to  $H_{\theta} = H_{\theta}^{\mathcal{N}}$  in the definition of virtually strong (or supercompact) cardinals yields the same notion (again, we do not know whether this holds level-by-level).

Small cardinals such as  $\omega_1$  can be generically measurable and hence faintly measurable. However, virtual large cardinals are large cardinals in the usual sense, as the following shows.

Recall from [GW11] that a cardinal  $\kappa$  is 1-iterable if for every  $A \subseteq \kappa$  there is a transitive  $\mathcal{M} \models \mathsf{ZFC}^-$  with  $\kappa, A \in \mathcal{M}$  and a weakly amenable  $\mathcal{M}$ -ultrafilter  $\mu$  on  $\kappa$  with a well-founded ultrapower. Recall that  $\mu$  is an  $\mathcal{M}$ -ultrafilter on  $\kappa$  if the structure  $\langle M, \in, \mu \rangle$  satisfies that  $\mu$  is a normal ultrafilter on  $\kappa$ , and such a  $\mu$  is weakly amenable if  $\mu \cap X \in \mathcal{M}$  for every  $X \in \mathcal{M}$  of  $\mathcal{M}$ -cardinality  $\leq \kappa$ . It is not difficult to see that an  $\mathcal{M}$ -ultrafilter  $\mu$  on  $\kappa$  with a well-founded ultrapower is weakly amenable if and only if the ultrapower embedding  $j : \mathcal{M} \to \mathcal{N}$  is  $\kappa$ -powerset preserving, meaning that  $\mathcal{M}$  and  $\mathcal{N}$  have the same subsets of  $\kappa$ . 1-iterable cardinals are weakly ineffable limits of ineffable cardinals, and hence, in particular, weakly compact [GW11].

# **PROPOSITION 3.2.** For any regular uncountable cardinal $\theta$ , every virtually $\theta$ -measurable cardinal is a 1-iterable limit of 1-iterable cardinals.

The proof is essentially the same as the proof of Theorem 4.8 in [GS18].

Schindler showed in [Sch14] that remarkable cardinals can be viewed as a version of virtual supercompact cardinals via Magidor's characterization of supercompactness. Later Gitman and Schindler showed in [GS18] that remarkables are precisely the virtually supercompacts, and indeed surprisingly, they are also precisely the virtually strongs. So, in particular, virtually strong and virtually supercompact cardinals are equivalent. We give the proof of the equivalences, which was omitted in [GS18], here.

**DEFINITION 3.3.** Let  $\theta$  be a regular uncountable cardinal. Then a cardinal  $\kappa < \theta$  is **virtually**  $\theta$ -supercompact ala Magidor if there are  $\bar{\kappa} < \bar{\theta} < \kappa$  and a generic elementary embedding  $\pi : H_{\bar{\theta}} \to H_{\theta}$  such that crit  $\pi = \bar{\kappa}$  and  $\pi(\bar{\kappa}) = \kappa$ .

**THEOREM 3.4** (G.-Schindler). For an uncountable cardinal  $\kappa$ , the following are equivalent.

- (i)  $\kappa$  is faintly strong.
- (ii)  $\kappa$  is virtually strong.
- (iii)  $\kappa$  is virtually supercompact.
- (iv)  $\kappa$  is virtually supercompact ala Magidor.

PROOF.  $(iii) \Rightarrow (ii) \Rightarrow (i)$  is simply by definition.

 $(i) \Rightarrow (iv)$ : Fix a regular uncountable  $\theta > \kappa$  and let  $\delta = (2^{<\theta})^+$ . By (i) there exists a generic elementary embedding  $\pi : H_{\delta} \to \mathcal{M}$  with  $\operatorname{crit} \pi = \kappa, \pi(\kappa) > \delta$ , and  $H_{\delta} = H_{\delta}^{\mathcal{M}}$ . We can restrict the embedding  $\pi$  to  $\pi : H_{\theta} \to H_{\pi(\theta)}^{\mathcal{M}}$ . Since  $H_{\theta}, H_{\pi(\theta)}^{\mathcal{M}} \in \mathcal{M}$ , Countable Embedding Absoluteness 2.1 implies that  $\mathcal{M}$  has a generic elementary embedding  $\pi^* : H_{\theta} \to H_{\pi(\theta)}^{\mathcal{M}}$  with  $\operatorname{crit} \pi^* = \kappa$  and  $\pi^*(\kappa) = \pi(\kappa) > \theta$ . Elementarity of  $\pi$  now implies that  $H_{\delta}$  has ordinals  $\bar{\kappa} < \bar{\theta} < \kappa$  and a generic elementary embedding  $\sigma : H_{\bar{\theta}} \to H_{\theta}$  with  $\operatorname{crit} \sigma = \bar{\kappa}$  and  $\sigma(\bar{\kappa}) = \kappa$ . This shows (iii).

 $(iv) \Rightarrow (iii)$ : Fix a regular uncountable  $\theta > \kappa$  and let  $\delta = (2^{<\theta})^+$ . By (iv)there exist ordinals  $\bar{\kappa} < \bar{\delta} < \kappa$  and a generic elementary embedding  $\pi : H_{\bar{\delta}} \to H_{\delta}$ with crit  $\pi = \bar{\kappa}$  and  $\pi(\bar{\kappa}) = \kappa$ . Let  $\pi(\bar{\theta}) = \theta$  (we can assume that  $\theta$  is in the range of  $\pi$  by taking it to be largest so that  $(2^{<\theta})^+ = \delta$ ). We will argue that  $\bar{\kappa}$  is virtually  $\bar{\theta}$ -supercompact in  $H_{\bar{\delta}}$ , so that by elementarity  $\kappa$  will be virtually  $\theta$ -supercompact in  $H_{\delta}$ , and hence also in V. Consider the restriction  $\sigma := \pi : H_{\bar{\theta}} \to H_{\theta}$ . Note that  $H_{\theta}$  is closed under  $<\bar{\theta}$ -sequences (and more) in V. We can assume without loss that  $\sigma$  lives in a  $\operatorname{Col}(\omega, H_{\bar{\theta}})$ -extension. Let  $\dot{\sigma}$  be a  $\operatorname{Col}(\omega, H_{\bar{\theta}})$ -name for  $\sigma$ . Now define

$$X := \bar{\theta} + 1 \cup \{ x \in H_{\theta} \mid \exists y \in H_{\bar{\theta}} \exists p \in \operatorname{Col}(\omega, H_{\bar{\theta}}) \, p \Vdash \dot{\sigma}(\check{y}) = \check{x} \} \in V.$$

Note that  $|X| = |H_{\bar{\theta}}| = 2^{<\bar{\theta}}$  and that ran  $\sigma \subseteq X$ . Now let  $\overline{X} \prec H_{\theta}$  be such that  $X \subseteq \overline{X}$  and  $\overline{X}$  is closed under  $<\bar{\theta}$ -sequences. Note that we can find such an  $\overline{X}$  of size  $(2^{<\bar{\theta}})^{<\bar{\theta}} = 2^{<\bar{\theta}}$ . Let  $\mathcal{M}$  be the transitive collapse of  $\overline{X}$ , so that  $\mathcal{M}$  is still closed under  $<\bar{\theta}$ -sequences and we still have that  $|\mathcal{M}| = 2^{<\bar{\theta}} < \bar{\delta}$ , making  $\mathcal{M} \in H_{\bar{\delta}}$ .

Countable Embedding Absoluteness 2.1 then implies that  $H_{\bar{\delta}}$  has a generic elementary embedding  $\sigma^* \colon H_{\bar{\theta}} \to \mathcal{M}$  with crit  $\sigma^* = \bar{\kappa}$  and the proof of Countable Embedding Absoluteness shows that we can ensure that  $\sigma^*(\bar{\kappa}) > \bar{\theta}$ . This verifies that  $\bar{\kappa}$  is virtually  $\bar{\theta}$ -supercompact in  $H_{\bar{\delta}}$ .

*Remark* 3.5. The above proof shows that if  $\kappa$  is faintly  $(2^{<\theta})^+$ -strong, then it is virtually  $\theta$ -supercompact, and if it is virtually  $(2^{<\theta})^+$ -supercompact ala Magidor, then it

is virtually  $\theta$ -supercompact. It is open whether they are equivalent level-by-level (see Question 6.1).

There are alternate possible virtualisations of strong cardinals that turn out to be weaker than our notion. Wilson has proposed a virtualisation of strongness for a cardinal  $\kappa$ defined by the existence of generic embeddings  $\sigma : H_{\theta} \to \mathcal{M}$  such that crit  $\sigma = \kappa$ ,  $\sigma(\kappa) > \theta$ ,  $H_{\theta} = H_{\theta}^{\mathcal{M}}$ , but  $\mathcal{M}$  is allowed to be ill-founded. His notion is just like our notion of faintly strong cardinals but the embeddings can have an ill-founded target.

Next, we define a virtualisation of the  $\alpha$ -superstrong cardinals.

**DEFINITION 3.6.** Let  $\theta$  be a regular uncountable cardinal and  $\alpha$  be an ordinal. Then a cardinal  $\kappa < \theta$  is faintly  $(\theta, \alpha)$ -superstrong if it is faintly  $\theta$ -measurable, as witnessed by an embedding  $\pi : H_{\theta} \to \mathcal{N}$  with crit  $\pi = \kappa$ ,  $H_{\theta} = H_{\theta}^{\mathcal{N}}$  and  $\pi^{\alpha}(\kappa) \leq \theta$ .<sup>2</sup> We replace "faintly" by virtually when  $\mathcal{N} \subseteq V$ , we say that  $\kappa$  is faintly  $\alpha$ -superstrong if it is faintly  $(\theta, \alpha)$ -superstrong for some  $\theta$ , and lastly  $\kappa$  is simply faintly superstrong if it is faintly 1-superstrong.

 $\dashv$ 

Recall that a cardinal  $\kappa$  is **virtually rank-into-rank** if there exists a cardinal  $\theta > \kappa$ and a generic elementary embedding  $\pi: H_{\theta} \to H_{\theta}$  with crit  $\pi = \kappa$ .

**PROPOSITION 3.7** (N.). A cardinal  $\kappa$  is virtually  $\omega$ -superstrong if and only if it is virtually rank-into-rank.

PROOF. Clearly every virtually rank-into-rank cardinal is virtually  $\omega$ -superstrong by definition. So suppose that  $\kappa$  is virtually  $\omega$ -superstrong, as witnessed by a generic elementary embedding  $\pi : H_{\theta} \to \mathcal{M}$  with  $\pi^{\omega}(\kappa) \leq \theta$ , and we let  $\lambda = \pi^{\omega}(\kappa)$ . First, observe that if  $a \in H_{\lambda}$ , then  $a \in H_{\pi^n(\kappa)}$  for some  $n < \omega$ , and hence by elementarity,  $\pi(a) \in H_{\pi^{n+1}(\kappa)} \subseteq H_{\lambda}$ . Thus, the restriction of  $\pi$  to  $H_{\lambda}$  maps into  $H_{\lambda}$ . We will argue that this map is elementary. Note that  $H_{\lambda}$  is the union of the elementary chain of the  $H_{\pi^n(\kappa)}$  for  $n < \omega$ . Thus,  $H_{\lambda} \models \varphi(a)$  implies that  $H_{\pi^n(\kappa)} \models \varphi(a)$  for some  $n < \omega$ , which implies that  $H_{\pi^{n+1}(\kappa)} \models \varphi(\pi(a))$  by elementarity of  $\pi$ , which finally implies that  $H_{\lambda} \models \varphi(\pi(a))$ . It follows that the restriction  $\pi^* : H_{\lambda} \to H_{\lambda}$  defined by  $\pi^*(a) = \pi(a)$  witnesses that  $\kappa$  is virtually rank-into-rank.

<sup>&</sup>lt;sup>2</sup>Here we set  $\pi^{\alpha}(\kappa) := \sup_{\xi < \alpha} \pi^{\xi}(\kappa)$  when  $\alpha$  is a limit ordinal.

**PROPOSITION 3.8** (N.). If  $\kappa$  is faintly superstrong, then  $H_{\kappa}$  has a proper class of virtually strong cardinals.

PROOF. Fix a regular  $\theta > \kappa$  and a generic elementary embedding  $\pi : H_{\theta} \to \mathcal{N}$  with crit  $\pi = \kappa$ ,  $H_{\theta} = H_{\theta}^{\mathcal{N}}$  and  $\pi(\kappa) \leq \theta$ . Let's argue that  $H_{\pi(\kappa)}(=H_{\pi(\kappa)}^{\mathcal{N}})$  thinks that  $\kappa$  is virtually strong. Fixing  $\kappa < \delta < \pi(\kappa)$ , we have that  $\pi \upharpoonright H_{\delta} : H_{\delta} \to H_{\pi(\delta)}^{\mathcal{N}}$ . In a  $\operatorname{Col}(\omega, H_{\delta})$ -extension of  $\mathcal{N}$ , there is an embedding  $\pi^* : H_{\delta} \to H_{\pi(\delta)}^{\mathcal{N}}$  with crit  $\pi^* = \kappa$  and  $\pi^*(\kappa) > \delta$ .

Following the proof of Theorem 3.4, we can build in  $\mathcal{N}, X \prec H_{\pi(\delta)}^{\mathcal{N}}$  with  $H_{\delta} \subseteq X$ of size  $|H_{\delta}|$  such that letting  $\mathcal{M}$  be the collapse of X, we get an embedding  $\sigma : H_{\delta} \to \mathcal{M}$ with crit  $\sigma = \kappa, \sigma(\kappa) > \delta, H_{\delta} \subseteq \mathcal{M}$ , and  $\mathcal{M} \in H_{\pi(\kappa)}$ , witnessing that  $\kappa$  is virtually  $\delta$ -strong in  $H_{\pi(\kappa)}$ . Now since  $H_{\kappa} \prec H_{\pi(\kappa)}$ , we have that  $H_{\kappa}$  thinks that there is a proper class of virtually strong cardinals.

The following theorem shows that the existence of "Kunen inconsistencies" is precisely what is stopping pre-strongness from being equivalent to strongness.

**THEOREM 3.9** (N.). Let  $\theta$  be a regular uncountable cardinal. Then a cardinal  $\kappa < \theta$  is virtually  $\theta$ -pre-strong if and only if one of the following holds.

- (i)  $\kappa$  is virtually  $\theta$ -strong, or
- (ii)  $\kappa$  is virtually  $(\theta, \omega)$ -superstrong.

PROOF. ( $\Leftarrow$ ) is trivial, so we show ( $\Rightarrow$ ). Let  $\kappa$  be virtually  $\theta$ -pre-strong. Assume (*i*) fails, meaning that there is a generic elementary embedding  $\pi \colon H_{\theta} \to \mathcal{N}$  for some transitive  $\mathcal{N} \subseteq V$  with  $H_{\theta} \subseteq \mathcal{N}$ , crit  $\pi = \kappa$  and  $\pi(\kappa) \leq \theta$ .

First, assume that there is some  $n < \omega$  such that  $\pi^n(\kappa) = \theta$ . The proof of Proposition 3.8 shows that  $\kappa$  is virtually strong in  $H_{\pi(\kappa)}$ . By elementarity,  $\kappa$  is virtually strong in  $H_{\pi(\kappa)}$ , and repeating this argument shows that  $\kappa$  is virtually strong in  $H_{\pi^n(\kappa)} = H_{\theta}$ . It also follows, by elementarity, that  $\pi(\kappa)$  is virtually strong in  $H_{\pi^2(\kappa)}$ , and by applying elementarity repeatedly, we get that  $\pi^n(\kappa) = \theta$  is virtually strong in  $\mathcal{N}$ . Note that the condition  $\pi^n(\kappa) = \theta$  implies that  $\theta$  is inaccessible in  $\mathcal{N}$ . Thus,  $H_{\theta}$  satisfies that there is no largest cardinal, and so by elementarity  $\mathcal{N}$  does not have a largest cardinal also.

Let  $\delta = (\theta^+)^{\mathcal{N}}$ . In particular,  $\theta$  is virtually  $\delta$ -strong in  $\mathcal{N}$ , and so  $\mathcal{N}$  has a generic elementary embedding  $\sigma : H^{\mathcal{N}}_{\delta} \to \mathcal{M}$  with crit  $\sigma = \theta$  and  $H_{\theta} \subseteq H^{\mathcal{N}}_{\delta} \subseteq \mathcal{M}$ . Thus,  $H_{\theta} \prec H^{\mathcal{M}}_{\sigma(\theta)}$ , from which it follows that  $\kappa$  is virtually strong in  $H^{\mathcal{M}}_{\sigma(\theta)}$ , and, in particular, virtually  $\theta$ -strong. But  $H^{\mathcal{M}}_{\sigma(\theta)}$  must be correct about this since  $H^{\mathcal{M}}_{\theta} = H^{\mathcal{N}}_{\theta} = H_{\theta}$ . But then  $\kappa$  is actually virtually  $\theta$ -strong, contradicting our assumption that (i) fails.

Next, assume that there is a least  $n < \omega$  such that  $\pi^{n+1}(\kappa) > \theta$ . In particular,  $\pi^n(\kappa) \leq \theta$ . Since  $\pi(\kappa) \leq \theta$ , we have as before that  $\kappa$  is virtually strong in  $H_{\pi^n(\kappa)}$ and that  $\pi^n(\kappa)$  is virtually strong in  $H^{\mathcal{N}}_{\pi^{n+1}(\kappa)}$ . Since  $\pi^{n+1}(\kappa)$  is inaccessible in  $\mathcal{N}$ ,  $\delta = (\theta^+)^{\mathcal{N}}$  exists. Thus, in  $H^{\mathcal{N}}_{\pi^{n+1}(\kappa)}$  there is some generic elementary embedding  $\sigma : H^{\mathcal{N}}_{\delta} \to \mathcal{M}$  with crit  $\sigma = \pi^n(\kappa), \sigma(\pi^n(\kappa)) > \delta$  and  $H_{\theta} \subseteq H^{\mathcal{N}}_{\delta} \subseteq \mathcal{M}$ . Thus, by elementarity, we get  $H_{\pi^n(\kappa)} \prec H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$ . Since, as we already argued,  $\kappa$  is virtually strong in  $H_{\pi^n(\kappa)}$  this means that  $\kappa$  is also virtually strong in  $H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$ , and as  $H^{\mathcal{M}}_{\theta} = H^{\mathcal{N}}_{\theta} = H_{\theta}$ , this means that  $\kappa$  is actually virtually  $\theta$ -strong, contradicting our assumption that (i) fails.

Finally, assume  $\pi^n(\kappa) < \theta$  for all  $n < \omega$  and let  $\lambda = \sup_{n < \omega} \pi^n(\kappa)$ . Since  $\lambda \le \theta$ , we have that  $\kappa$  is virtually  $(\theta, \omega)$ -superstrong by definition.

We then get the following consistency result.

**COROLLARY 3.10** (N.). For any uncountable regular cardinal  $\theta$ , the existence of a virtually  $\theta$ -strong cardinal is equiconsistent with the existence of a faintly  $\theta$ -measurable cardinal.

**PROOF.** The above Proposition 3.8 and Theorem 3.9 show that virtually  $\theta$ -pre-strongs are equiconsistent with virtually  $\theta$ -strongs. Let us now argue that if  $\kappa$  is faintly  $\theta$ measurable in L, then  $\kappa$  is virtually  $\theta$ -pre-strong in L. Suppose that  $\pi : L_{\theta} \to \mathcal{M}$  is a generic elementary embedding with  $\mathcal{M}$  transitive and crit  $\pi = \kappa$ . By elementarity,  $\mathcal{M}$ satisfies V = L, and hence by absoluteness of the construction of L and transitivity of  $\mathcal{M}, \mathcal{M} = L_{\beta}$  for some cardinal  $\beta \geq \theta$ . But then trivially we have  $L_{\theta} \subseteq \mathcal{M}$ .

COROLLARY 3.11 (N.). The following are equivalent.

- (i) For every regular uncountable cardinal θ, every virtually θ-pre-strong cardinal is virtually θ-strong.
- (ii) There are no virtually rank-into-rank cardinals.

**PROOF.** ( $\Leftarrow$ ): By Proposition 3.7 being virtually  $\omega$ -superstrong is equivalent to being virtually rank-into-rank. The above Theorem 3.9 then implies ( $\Leftarrow$ ).

 $(\Rightarrow)$ : Here we have to show that if there exists a virtually rank-into-rank cardinal, then there exists a  $\theta > \kappa$  and a virtually  $\theta$ -pre-strong cardinal which is not virtually  $\theta$ -strong. Let  $\langle \kappa, \theta \rangle$  be the lexicographically least pair such that  $\kappa$  is virtually rank-intorank as witnessed by a generic embedding  $\pi : H_{\theta} \to H_{\theta}$ , which trivially makes  $\kappa$ virtually  $\theta$ -pre-strong. If  $\kappa$  was also virtually  $\theta$ -strong, then we would have a generic elementary embedding  $\pi^* : H_\theta \to \mathcal{M}$  with  $\operatorname{crit} \pi^* = \kappa$ ,  $\pi^*(\kappa) > \theta$ , and  $\mathcal{M} \subseteq V$ . By Countable Embedding Absoluteness 2.1,  $\mathcal{M}$  sees that  $\kappa$  virtually rank-into-rank, but then, using elementarity, this reflects down below  $\kappa$ , showing that the pair  $\langle \kappa, \theta \rangle$  could not have been least.

We showed in Theorem 3.4 that faintly strong cardinals and virtually strong cardinals are equivalent and we showed in Corollary 3.10 that, in L, faintly measurable cardinals and virtually pre-strong cardinals are equivalent. As a final result of this section, we separate the faintly measurable and virtually measurable cardinals. The separation is trivial in general, as successor cardinals can be faintly measurable and are never virtually measurable, but the separation still holds true if we rule out this successor case.

We also show that a cardinal  $\kappa$  may not even be faintly  $\kappa^+$ -strong, but at the same time have the property that for every regular  $\theta$ , there is a generic embedding  $\pi : H_{\theta} \to \mathcal{M}$  with crit  $\pi = \kappa$ ,  $\pi(\kappa) > \theta$ , and  $H_{\theta} \subseteq \mathcal{M}$ . In particular, we don't get that  $H_{\theta} = H_{\theta}^{\mathcal{M}}$ .

**THEOREM 3.12** (G.). If  $\kappa$  is virtually measurable, then there is a forcing extension V[G]in which  $\kappa$  is inaccessible and faintly measurable, but not virtually measurable. If we further assume that  $\kappa$  is virtually strong, then, in V[G], for every regular  $\theta$ , there are generic elementary embeddings  $\sigma : H_{\theta}^{V[G]} \to \mathcal{M}$  with crit  $\sigma = \kappa$ ,  $\sigma(\kappa) > \theta$ , and  $H_{\theta}^{V[G]} \subseteq \mathcal{M}$ .

PROOF. Suppose that  $\kappa$  is virtually measurable. This implies, in particular, that for every regular  $\theta > \kappa$ , we have generic elementary embeddings  $\pi : H_{\theta} \to \mathcal{M}$  with crit  $\pi = \kappa$  such that  $\mathcal{M} \in V$ . Thus, by Proposition 2.2, we can assume that each generic embedding  $\pi$  exists in a  $\operatorname{Col}(\omega, H_{\theta})$ -extension.

Let  $\mathbb{P}_{\kappa}$  be the Easton support iteration that adds a Cohen subset to every regular  $\alpha < \kappa$ , and let  $G \subseteq \mathbb{P}_{\kappa}$  be V-generic. Standard computations show that  $\mathbb{P}_{\kappa}$  preserves all inaccessible cardinals. In particular,  $\kappa$  remains inaccessible in V[G].

Fix a regular  $\theta \gg \kappa$  and let  $h \subseteq \operatorname{Col}(\omega, H_{\theta})$  be V[G]-generic. In V[h], we must have an elementary embedding  $\pi : H_{\theta} \to \mathcal{M}$  with  $\operatorname{crit} \pi = \kappa$  and  $\mathcal{M} \in V$ , and we can assume without loss that  $\mathcal{M}$  is countable. Obviously,  $\pi \in V[G][h]$ . Working in V[G][h], we will now lift  $\pi$  to an elementary embedding on  $H_{\theta}[G] = H_{\theta}^{V[G]}$ . To ensure that such a lift exists, it suffices to find in V[G][h] an  $\mathcal{M}$ -generic filter for  $\pi(\mathbb{P}_{\kappa})$ containing  $\pi''G$ .<sup>3</sup> Observe first that  $\pi''G = G$  since the critical point of  $\pi$  is  $\kappa$  and we can assume that  $\mathbb{P}_{\kappa} \subseteq V_{\kappa}$ . Next, observe that  $\pi(\mathbb{P}_{\kappa}) \cong \mathbb{P}_{\kappa} * \mathbb{P}_{\text{tail}}$ , where  $\mathbb{P}_{\text{tail}}$  is the forcing beyond  $\kappa$ . Since  $\mathcal{M}[G]$  is countable, we can build an  $\mathcal{M}[G]$ -generic filter  $G_{\text{tail}}$ 

<sup>&</sup>lt;sup>3</sup>This standard lemma is referred to in the literature as the lifting criterion.

for  $\mathbb{P}_{\text{tail}}$  in V[G][h]. Thus,  $G * G_{\text{tail}}$  is  $\mathcal{M}$ -generic for  $\pi(\mathbb{P}_{\kappa})$ , and so we can lift  $\pi$  to  $\pi : H_{\theta}[G] \to \mathcal{M}[G][G_{\text{tail}}]$ . Since  $\theta$  was chosen arbitrarily, we have just shown that  $\kappa$  is faintly measurable in V[G].

Since generic embeddings witnessing the virtual measurability of  $\kappa$  are  $\kappa$ -powerset preserving, it suffices to show that we cannot have generic  $\kappa$ -powerset preserving embeddings witnessing the faint measurability of  $\kappa$  in V[G]. Fix a regular  $\theta < \bar{\theta}$  and a generic elementary embedding  $\sigma : H_{\bar{\theta}}[G] \to \mathcal{N}$  with crit  $\sigma = \kappa$  and  $\mathscr{P}(\kappa)^{V[G]} = \mathscr{P}(\kappa)^{\mathcal{N}}$ . By elementarity,  $H^{\mathcal{N}}_{\sigma(\theta)} = \sigma(H_{\theta})[\sigma(G)]$  is a forcing extension of  $K = \sigma(H_{\theta})$ by  $\sigma(G) = G * \bar{G}_{\text{tail}} \subseteq \sigma(\mathbb{P}_{\kappa}) \cong \mathbb{P}_{\kappa} * \bar{\mathbb{P}}_{\text{tail}}$ . Thus, we have the restrictions  $\sigma : H_{\theta} \to K$ and  $\sigma : H_{\theta}[G] \to K[G][\bar{G}_{\text{tail}}]$ .

Let us argue that  $\mathscr{P}^{V[G]}(\kappa) \subseteq \mathscr{P}^{K[G]}(\kappa)$ , and hence we have equality. Suppose  $A \subseteq \kappa$  in V[G] and let  $\dot{A}$  be a nice  $\mathbb{P}_{\kappa}$ -name for A, which can be coded by a subset of  $\kappa$ . Since crit  $\sigma = \kappa$ , we have that  $\dot{A} \in K$ , and hence  $A = \dot{A}_G \in K[G]$ . But now it follows that the K[G]-generic for  $\operatorname{Add}(\kappa, 1)$ , the forcing at stage  $\kappa$  in  $\sigma(\mathbb{P}_{\kappa})$ , cannot be in V[G]. Thus, we have reached a contradiction, showing that  $\kappa$  cannot be virtually measurable in V[G].

Now assume further that  $\kappa$  is virtually strong. It suffices to simply note that  $G \in \mathcal{M}[G * G_{\text{tail}}]$  so that  $H_{\theta}[G] \subseteq \mathcal{N}[G * G_{\text{tail}}]$  as well, and since we lifted  $\pi$ , we still have  $\pi(\kappa) > \theta$ .

# 4 Virtual Woodin cardinals and virtual Vopěnka Principle

In this section we will analyse the virtualisations of Woodin cardinals, which can be seen as "boldface" variants of strong cardinals. We also introduce the "virtual Vopěnka Principle for finite languages", which it turns out is not equivalent to the virtual Vopěnka principle (see Theorem 5.16).

**DEFINITION 4.1.** Let  $\theta$  be a regular uncountable cardinal. Then a cardinal  $\kappa < \theta$  is faintly  $(\theta, A)$ -strong for a set A if there is a forcing extension containing a transitive set M, a set B and an elementary embedding  $\pi : (H_{\theta}, \in, A \cap H_{\theta}) \to (\mathcal{M}, \in, B)$  such that crit  $\pi = \kappa, \pi(\kappa) > \theta, H_{\theta} = H_{\theta}^{\mathcal{M}}$ , and  $B \cap H_{\theta} = A \cap H_{\theta}$ . We say that  $\kappa$  is faintly  $(\theta, A)$ -supercompact if we further have that  ${}^{<\theta} \mathcal{M} \cap V \subseteq \mathcal{M}$ .

**DEFINITION 4.2.** A cardinal  $\delta$  is faintly Woodin if, given any  $A \subseteq H_{\delta}$ , there exists a faintly  $(\langle \delta, A \rangle)$ -strong cardinal  $\kappa$ .

As before, for both of the above two definitions we substitute "faintly" for virtually when  $\mathcal{M} \subseteq V$ , and "strong", "supercompact", and "Woodin" for pre-strong, pre-supercompact, and pre-Woodin when we do not require that  $\pi(\kappa) > \theta$ .

**DEFINITION 4.3.** Let  $\theta$  be a regular uncountable cardinal. Then a cardinal  $\kappa < \theta$  is **virtually**  $(\theta, A)$ -**extendible** for a set A if there exists a generic elementary embedding  $\pi : (H_{\theta}, \in, A \cap H_{\theta}) \rightarrow (H_{\mu}, \in, A \cap H_{\mu})$  such that  $\operatorname{crit} \pi = \kappa$  and  $\pi(\kappa) > \theta$ . As usual, we substitute "extendible" for **pre-extendible** when we do not require that  $\pi(\kappa) > \theta$ .  $\dashv$ 

We note in the following proposition that, in analogy with Woodin cardinals, virtually Woodin cardinals are Mahlo. This property fails for virtually pre-Woodin cardinals since [Wil19], together with Theorem 4.11 below, shows that they can be singular.

#### PROPOSITION 4.4 (Virtualised folklore). Virtually Woodin cardinals are Mahlo.

**PROOF.** Let  $\delta$  be virtually Woodin. Note that  $\delta$  is a limit of weakly compact cardinals by Proposition 3.2, making  $\delta$  a strong limit. As for regularity, assume that we have a cofinal increasing function  $f: \alpha \to \delta$  with  $f(0) > \alpha$  and  $\alpha < \delta$  and note that f cannot have any closure points. Fix a virtually  $(<\delta, f)$ -strong cardinal  $\kappa < \delta$ . We claim that  $\kappa$  is a closure point for f, which will yield the desired contradiction.

Let  $\gamma < \kappa$  and choose a regular  $\theta \in (f(\gamma), \delta)$  above  $\kappa$ . We then have a generic elementary embedding  $\pi : (H_{\theta}, \in, f \cap H_{\theta}) \to (\mathcal{N}, \in, f^+)$  with  $H_{\theta} \subseteq \mathcal{N}, \mathcal{N} \subseteq V$ , crit  $\pi = \kappa, \pi(\kappa) > \theta$ , and  $f^+$  a function such that  $f^+ \cap H_{\theta} = f \cap H_{\theta}$ . But then  $f^+(\gamma) = f(\gamma) < \pi(\kappa)$  by our choice of  $\theta$ , so elementarity implies that  $f(\gamma) < \kappa$ , making  $\kappa$  a closure point for f, a contradiction. Thus,  $\delta$  is inaccessible.

Next, let us show that  $\kappa$  is Mahlo. Let  $C \subseteq \delta$  be a club and let  $\kappa < \delta$  be a virtually  $(<\delta, C)$ -strong cardinal. Let  $\theta \in (\min C, \delta)$  be above  $\kappa$  and let  $\pi : (H_{\theta}, \in, C \cap H_{\theta}) \rightarrow (\mathcal{N}, \in, C^+)$  be the associated generic elementary embedding having  $C^+ \cap \theta = C$ . Then for every  $\gamma < \kappa$  there exists an element of  $C^+$  below  $\pi(\kappa)$ , namely min C, so by elementarity  $\kappa$  is a limit of elements of C, making it an element of C. As  $\kappa$  is regular, this shows that  $\delta$  is Mahlo.

The well-known equivalence of the "function definition" and "A-strong" definition of Woodin cardinals holds for virtually Woodin cardinals, and the analogue of the equivalence between virtually strongs and virtually supercompacts allows us to strengthen this:



Figure 1: Proof strategy of Theorem 4.5, dotted lines are trivial implications.

**THEOREM 4.5** (D.-G.-N.). For an uncountable cardinal  $\delta$ , the following are equivalent.

- (i)  $\delta$  is virtually Woodin.
- (ii) For every  $A \subseteq H_{\delta}$  there exists a virtually  $(\langle \delta, A \rangle)$ -supercompact  $\kappa \langle \delta$ .
- (iii) For every  $A \subseteq H_{\delta}$  there exists a virtually  $(\langle \delta, A \rangle)$ -extendible  $\kappa \langle \delta$ .
- (iv) For every function  $f: \delta \to \delta$  there are regular cardinals  $\kappa < \theta < \delta$ , such that  $\kappa$  is a closure point of f, and a generic elementary embedding  $\pi: H_{\theta} \to \mathcal{M}$  such that crit  $\pi = \kappa$ ,  $H_{\theta} \subseteq \mathcal{M}$ ,  $\mathcal{M} \subseteq V$  and  $\pi(f \restriction \kappa)(\kappa) < \theta$ .
- (v) For every function  $f: \delta \to \delta$  there are regular cardinals  $\kappa < \theta < \delta$ , such that  $\kappa$  is a closure point of f, and a generic elementary embedding  $\pi: H_{\theta} \to \mathcal{M}$  such that crit  $\pi = \kappa$ ,  ${}^{<\theta} \mathcal{M} \cap V \subseteq \mathcal{M}$ ,  $\mathcal{M} \subseteq V$  and  $\pi(f \restriction \kappa)(\kappa) < \theta$ .
- (vi) For every function  $f: \delta \to \delta$  there are regular cardinals  $\bar{\theta} < \kappa < \theta < \delta$ , such that  $\kappa$  is a closure point of f, and a generic elementary embedding  $\pi: H_{\bar{\theta}} \to H_{\theta}$  with  $\pi(\operatorname{crit} \pi) = \kappa$ ,  $f(\operatorname{crit} \pi) < \bar{\theta}$  and  $f \upharpoonright \kappa \in \operatorname{ran} \pi$ .

**PROOF.** Firstly note that  $(iii) \Rightarrow (ii) \Rightarrow (i)$  and  $(v) \Rightarrow (iv)$  are simply by definition.

 $\underbrace{(i) \Rightarrow (iv)}_{(<\delta, f) \text{-strong and let } \theta < \delta \text{ be a regular cardinal such that } \sup_{\alpha \leq \kappa} f(\alpha) < \theta.$  Then there is a generic elementary embedding  $\pi \colon (H_{\theta}, \in, f \cap H_{\theta}) \to (\mathcal{M}, \in, f^+)$  such that  $H_{\theta} \subseteq \mathcal{M}, f \cap H_{\theta} = f^+ \cap H_{\theta}, \mathcal{M} \subseteq V$ , and  $\pi(\kappa) > \theta$ . Note that, by our choice of  $\theta, f \upharpoonright \kappa \in H_{\theta}$  and  $\pi(f \upharpoonright \kappa)(\kappa) = f^+(\kappa) = f(\kappa) < \theta$ .

So it suffices to show that  $\kappa$  is a closure point for f. Let  $\alpha < \kappa$ . Then

$$f(\alpha) = f^+(\alpha) = \pi(f \upharpoonright \kappa)(\alpha) = \pi(f \upharpoonright \kappa)(\pi(\alpha)) = \pi(f(\alpha)),$$

so  $\pi$  fixes  $f(\alpha)$  for every  $\alpha < \kappa$ . Now, if  $\kappa$  was not a closure point of f then, letting  $\alpha < \kappa$  be the least such that  $f(\alpha) \ge \kappa$ , we have

$$\theta > f(\alpha) = \pi(f(\alpha)) > \theta,$$

a contradiction. Note that we used that  $\pi(\kappa) > \theta$  here, so this argument would not work if we had only assumed  $\delta$  to be virtually pre-Woodin.

 $(iv) \Rightarrow (vi)$  Assume (iv) holds, let  $f: \delta \to \delta$  be given and define  $g: \delta \to \delta$  as  $g(\alpha) := (2^{<\gamma_{\alpha}})^+$ , where  $\gamma_{\alpha}$  is the least regular cardinal above  $|f(\alpha)|$ . By (iv) there is a  $\kappa < \delta$  which is a closure point of g (and so also a closure point of f), and there is a regular  $\lambda \in (\kappa, \delta)$  for which there is a generic elementary embedding  $\pi: H_{\lambda} \to \mathcal{M}$  with crit  $\pi = \kappa, H_{\lambda} \subseteq \mathcal{M}, \mathcal{M} \subseteq V$ , and  $\pi(g \upharpoonright \kappa)(\kappa) < \lambda$ .

Let  $\theta$  be the least regular cardinal above  $|\pi(f \restriction \kappa)(\kappa)|$ , and note that  $H_{\theta} \in H_{\lambda}$  by our definition of g. Thus, both  $H_{\theta}$  and  $H_{\pi(\theta)}^{\mathcal{M}}$  are elements of  $\mathcal{M}$ . An application of Countable Embedding Absoluteness 2.1 then yields that  $\mathcal{M}$  has a generic elementary embedding  $\pi^* \colon H_{\theta}^{\mathcal{M}} \to H_{\pi(\theta)}^{\mathcal{M}}$  such that  $\operatorname{crit} \pi^* = \kappa, \pi^*(\kappa) = \pi(\kappa), \pi(f \restriction \kappa) \in$  $\operatorname{ran} \pi^*$ , and  $\pi(f \restriction \kappa)(\kappa) < \theta$ . By elementarity of  $\pi, H_{\lambda}$  has an ordinal  $\bar{\theta} < \kappa$  and a generic elementary embedding  $\sigma \colon H_{\bar{\theta}} \to H_{\theta}$  with  $\sigma(\operatorname{crit} \sigma) = \kappa, f \restriction \kappa \in \operatorname{ran} \sigma$  and  $f(\operatorname{crit} \sigma) < \bar{\theta}$ , which is what we wanted to show.

 $\begin{array}{c} (vi) \Rightarrow (v) \\ \hline \text{Assume } (vi) \text{ holds and let } f \colon \delta \to \delta \text{ be given. Define } g \colon \delta \to \delta \\ \text{as } g(\alpha) := \langle (2^{<\gamma_{\alpha}})^+, f(\alpha) \rangle, \text{ where } \gamma_{\alpha} \text{ is the least regular cardinal above } |f(\alpha)|. \text{ In } \\ \text{particular, } g \text{ codes } f. \text{ By } (vi) \text{ there exist regular } \bar{\kappa} < \bar{\lambda} < \kappa < \lambda \text{ such that } \kappa \text{ is a } \\ \text{closure point of } g \text{ (so also a closure point of } f) \text{ and there exists a generic elementary } \\ \text{embedding } \pi \colon H_{\bar{\lambda}} \to H_{\lambda} \text{ with } \operatorname{crit} \pi = \bar{\kappa}, \pi(\bar{\kappa}) = \kappa, g(\bar{\kappa}) < \bar{\lambda}, \text{ and } g \upharpoonright \kappa \in \operatorname{ran} \pi. \\ \text{Since } f \text{ is definable from } g \text{ and } g \upharpoonright \kappa \in \operatorname{ran} \pi, \text{ it follows that } f \upharpoonright \kappa \in \operatorname{ran} \pi. \\ \text{So let } \\ \pi(\bar{f}) = f \upharpoonright \kappa \text{ with } \bar{f} \colon \bar{\kappa} \to \bar{\kappa}. \\ \text{Now observe that } \bar{f} = f \upharpoonright \bar{\kappa} \text{ since for } \alpha < \bar{\kappa}, \text{ we have } \\ f(\alpha) = \pi(\bar{f})(\alpha) = \pi(\bar{f}(\alpha)) = \bar{f}(\alpha). \\ \end{array}$ 

Let  $\overline{\theta}$  be the least regular cardinal above  $|f(\overline{\kappa})|$ . By the definition of g, we have  $H_{\overline{\theta}} \in H_{\overline{\lambda}}$ . Now, following the  $(iii) \Rightarrow (ii)$  direction in the proof of Theorem 3.4 we get that  $H_{\overline{\lambda}}$  has a generic elementary embedding  $\sigma : H_{\overline{\theta}} \to \mathcal{M}$  with  $\mathcal{M}$  closed under  $<\overline{\theta}$ -sequences from V, crit  $\sigma = \overline{\kappa}, \sigma(\overline{\kappa}) > \overline{\theta}$ , and  $\sigma(\overline{f})(\overline{\kappa}) < \overline{\theta}$ . Let  $\pi(\overline{\theta}) = \theta$  and  $\pi(\mathcal{M}) = \mathcal{N}$ . Now by elementarity of  $\pi$ , we get that there is a generic elementary embedding  $\sigma^* : H_{\theta} \to \mathcal{N}$  with crit  $\sigma^* = \kappa, \sigma^*(\kappa) > \theta$ , and  $\sigma^*(\pi(\overline{f}))(\kappa) = \sigma^*(f \upharpoonright \kappa)(\kappa) < \theta$ .

 $(vi) \Rightarrow (iii)$  Let C be the club of all cardinals  $\alpha$  such that

$$(H_{\alpha}, \in, A \cap H_{\alpha}) \prec (H_{\delta}, \in, A).$$

Let  $f: \delta \to \delta$  be given as  $f(\alpha) := \langle \gamma_0^{\alpha}, \gamma_1^{\alpha} \rangle$ , where  $\gamma_0^{\alpha}$  is the first limit point of C above  $\alpha$  and the  $\gamma_1^{\alpha}$  are chosen such that  $\{\gamma_1^{\alpha} \mid \alpha < \beta\}$  encodes  $A \cap H_{\beta}$  for inaccessible cardinals  $\beta < \delta$ .

Let  $\kappa < \delta$  be a closure point of f such that there are regular cardinals  $\bar{\theta} < \kappa < \theta$  and a generic elementary embedding  $\pi \colon H_{\bar{\theta}} \to H_{\theta}$  such that  $\pi(\operatorname{crit} \pi) = \kappa$ ,  $f(\operatorname{crit} \pi) < \bar{\theta}$ , and  $f \upharpoonright \kappa \in \operatorname{ran} \pi$ . Let  $\bar{\kappa} = \operatorname{crit} \pi$ . The same argument as above gives that  $\pi(f \upharpoonright \bar{\kappa}) =$  $f \upharpoonright \kappa$ . In particular, it follows that  $\bar{\kappa}$  is a closure point of f, and hence  $\bar{\kappa} \in C$ . We claim that  $\bar{\kappa}$  is virtually  $(\langle \delta, A \rangle)$ -extendible. Since  $\kappa \in C$  because it is a closure point of f, it suffices by the definition of C to show that

$$(H_{\kappa}, \in, A \cap H_{\kappa}) \models \lceil \bar{\kappa} \text{ is virtually } (A \cap H_{\kappa}) \text{-extendible} \rceil$$

Let  $\beta$  be the least element of C above  $\bar{\kappa}$ , and note that  $\beta$  is below  $\bar{\theta}$  since  $f(\bar{\kappa}) < \bar{\theta}$ , and the definition of f says that the first coordinate of  $f(\bar{\kappa})$  is a limit point of C above  $\bar{\kappa}$ . It then holds that

$$(H_{\bar{\kappa}}, \in, A \cap H_{\bar{\kappa}}) \prec (H_{\beta}, \in, A \cap H_{\beta})$$

as both  $\bar{\kappa}$  and  $\beta$  are elements of C. Since f encodes A in the manner previously described and  $\pi(f \upharpoonright \bar{\kappa}) = f \upharpoonright \kappa$ , we get that  $\pi(A \cap H_{\bar{\kappa}}) = A \cap H_{\kappa}$ , and thus

$$(H_{\kappa}, \in, A \cap H_{\kappa}) \prec (H_{\pi(\beta)}, \in, A^*)$$

for  $A^* := \pi(A \cap H_{\beta})$ . Now, as  $(H_{\gamma}, \in, A \cap H_{\gamma})$  and  $(H_{\pi(\gamma)}, \in, A^* \cap H_{\pi(\gamma)})$  are elements of  $H_{\pi(\beta)}$  for every  $\gamma < \beta$ , Countable Embedding Absoluteness 2.1 implies that  $H_{\pi(\beta)}$  sees that  $\bar{\kappa}$  is virtually  $(<\beta, A^*)$ -extendible. Since

$$(H_{\beta}, \in A \cap H_{\beta}) \prec (H_{\pi(\beta)}, \in A^*),$$

it follows that  $(H_{\beta}, \in A \cap H_{\beta})$  satisfies that  $\bar{\kappa}$  is virtually  $(\langle \beta, A \cap H_{\beta})$ -extendible. Finally, since

$$(H_{\beta}, \in, A \cap H_{\beta}) \prec (H_{\kappa}, \in, A \cap H_{\kappa}),$$

it follows that  $(H_{\kappa}, \in, A \cap H_{\kappa})$  satisfies that  $\bar{\kappa}$  is virtually  $(\langle \kappa, A \cap H_{\kappa})$ -extendible.

As a corollary of the proof, we now have that virtually Woodin cardinals and faintly Woodin cardinals are equivalent.

#### **PROPOSITION 4.6.** A cardinal $\delta$ is virtually Woodin if and only if it is faintly Woodin.

Indeed, the argument would work as well if we only assumed existence of generic embeddings  $\pi$  :  $(H_{\theta}, \in, A \cap H_{\theta}) \rightarrow (\mathcal{M}, \in, B)$  such that  $\operatorname{crit} \pi = \kappa, \pi(\kappa) > \theta,$  $H_{\theta} \subseteq \mathcal{M}$  and  $A \cap H_{\theta} = B \cap H_{\theta}$ . In other words, we do not need the a priori stronger assumption that  $H_{\theta} = H_{\theta}^{\mathcal{M}}$ . Using Proposition 4.6, it suffices to observe that if  $\pi : (H_{\theta}, \in, A) \to (\mathcal{M}, \in, B)$  is a faintly  $(\theta, A)$ -strong embedding such that A codes the sequence of  $H_{\lambda}$  for  $\lambda < \theta$ , then  $H_{\theta} = H_{\theta}^{\mathcal{M}}$ .

We should also observe that if  $\delta$  is virtually Woodin, then indeed for every  $A \subseteq H_{\delta}$ , we have stationary many virtually  $(\langle \delta, A \rangle)$ -extendible cardinals by an argument very similar to the proof that virtually Woodin cardinals are Mahlo.

Recall that the **virtual Vopěnka Principle** states that for every proper class C consisting of structures in a common language, there are distinct structures  $\mathcal{M}, \mathcal{N} \in C$  for which there is a generic elementary embedding  $\pi : \mathcal{M} \to \mathcal{N}$ . The second author and Hamkins showed in [GH19] that the virtual Vopěnka principle holds if and only if for every class A there is a proper class of weakly virtually A-extendible cardinals (in our terminology, these are (<On, A)-pre-extendible cardinals). It follows from Theorem 4.5 that if On is faintly Woodin, then the virtual Vopěnka Principle holds. However, since it is consistent that the virtual Vopěnka Principle holds and On is not Mahlo [GH19], the two assertions are not equivalent.

It turns out, however, that a weakening of the virtual Vopěnka Principle is equivalent to the assertion that On is faintly pre-Woodin. Our formal setting for working with classes throughout this article will be the second-order Godël-Bernays set theory GBC whose axioms consist of ZFC together with class axioms consisting of extensionality for classes, class replacement asserting that every class function when restricted to a set is a set, global choice asserting that there is a class well-order of sets, and a weak comprehension scheme asserting that every first-order formula defines a class.

**DEFINITION 4.7.** The virtual Vopěnka Principle for finite languages states that for every proper class C consisting of structures in a common *finite* language, there are distinct structures  $\mathcal{M}, \mathcal{N} \in C$  for which there is a generic elementary embedding  $\pi \colon \mathcal{M} \to \mathcal{N}$ .  $\dashv$ 

The arguments in [GH19] show:

**THEOREM 4.8.** The virtual Vopěnka Principle for finite languages holds if and only if for every class *A*, there is a weakly virtually *A*-extendible cardinal.

The Vopěnka Principle is equivalent to the Vopěnka Principle for finite languages. Indeed, the Vopěnka Principle can be restated in terms of the existence of elementary embeddings between elements of *natural sequences* [Kan08]. But as we will see in the next section, this equivalence relies once again relies on Kunen's Inconsistency. **DEFINITION 4.9.** Say that a class function  $f: \text{On} \to \text{On}$  is an indexing function if it satisfies that  $f(\alpha) > \alpha$  and  $f(\alpha) \le f(\beta)$  for all  $\alpha < \beta$ .

**DEFINITION 4.10.** Say that an On-sequence  $\vec{\mathcal{M}} = \langle \mathcal{M}_{\alpha} \mid \alpha < \mathsf{On} \rangle$  is **natural** if there exists an indexing function  $f^{\vec{\mathcal{M}}} \colon \mathsf{On} \to \mathsf{On}$  and unary relations  $R^{\vec{\mathcal{M}}}_{\alpha} \subseteq V_{f^{\vec{\mathcal{M}}}(\alpha)}$  such that  $\mathcal{M}_{\alpha} = (V_{f^{\vec{\mathcal{M}}}(\alpha)}, \in, \{\alpha\}, R^{\vec{\mathcal{M}}}_{\alpha})$  for every  $\alpha$ .

The following theorem shows that the virtual Vopěnka Principle for finite languages holds if and only if On is virtually pre-Woodin if and only if On is faintly pre-Woodin if and only if for every class A there is an weakly virtually A-extendible cardinal. In particular, we get that virtually pre-Woodin and faintly pre-Woodin cardinals  $\delta$  are equivalent, and both are equivalent to the assertion that for every  $A \subseteq H_{\delta}$ , there is a  $(<\delta, A)$ -pre-extendible cardinal. In the next section, in Theorem 5.16, we will separate the virtual Vopěnka Principle for finite languages from the virtual Vopěnka principle.

**THEOREM 4.11** (D.-G.-N.). The following are equivalent.

- (i) The virtual Vopěnka Principle for finite languages holds.
- (ii) For every class A, there is a  $(\langle On, A \rangle)$ -pre-extendible cardinal.
- (iii) For any natural On-sequence  $\vec{\mathcal{M}}$  there exists a generic elementary embedding  $\pi: \mathcal{M}_{\alpha} \to \mathcal{M}_{\beta}$  for some  $\alpha < \beta$ .
- (iv) On is virtually pre-Woodin.
- ( $\upsilon$ ) On is faintly pre-Woodin.

**PROOF.** By Theorem 4.8, (i) and (ii) are equivalent. (i)  $\Rightarrow$  (iii) and (iv)  $\Rightarrow$  (v) are trivial.

 $(v) \Rightarrow (i)$ : Assume that On is faintly pre-Woodin and fix some class C of structures in a common language. Let  $\kappa$  be  $(\langle \mathsf{On}, \mathcal{C} \rangle)$ -pre-strong. Fix some regular  $\theta > \kappa$  such that  $H_{\theta}$  has a structure M from C of the  $\kappa$ -th rank among elements of C, and fix a generic elementary embedding

$$\pi \colon (H_{\theta}, \in, \mathcal{C} \cap H_{\theta}) \to (\mathcal{N}, \in, \mathcal{C}^*)$$

with crit  $\pi = \kappa$ ,  $H_{\theta} = H_{\theta}^{\mathcal{N}}$  (note that actually  $H_{\theta} \subseteq \mathcal{N}$  suffices here), and  $\mathcal{C} \cap H_{\theta} = \mathcal{C}^* \cap H_{\theta}$ . Now we have that  $\pi : M \to \pi(M)$  is an elementary embedding (note that the language is finite and therefore can be assumed to be fixed by  $\pi$ ), and  $\pi(M) \neq M$  because  $\pi(M)$  is a structure in  $\mathcal{C}^*$  of rank  $\pi(\kappa)$  among elements of  $\mathcal{C}^*$ . The structure  $(\mathcal{N}, \in, \mathcal{C}^*)$  believes that both M and  $\pi(M)$  are elements of  $\mathcal{C}^*$ , and, by Countable Embedding Absoluteness 2.1,  $\mathcal{N}$  has a generic elementary embedding between M and  $\pi(M)$ . Therefore,  $(\mathcal{N}, \in, \mathcal{C}^*)$  satisfies that there is a generic elementary embedding

between two distinct elements of  $C^*$ , and hence  $(H_\theta, \in, C \cap H_\theta)$  satisfies that there is a generic elementary embedding between two distinct elements of C, and it must be correct about this.

 $(iii) \Rightarrow (iv)$ : Assume (iii) holds and assume that On is not virtually pre-Woodin, which means that there exists some class A for which there are no virtually  $(\langle \mathsf{On}, A \rangle)$ pre-strong cardinals. Define a function  $f : \mathsf{On} \to \mathsf{On}$  by setting  $f(\alpha)$  to be the least regular  $\eta > \alpha$  such that  $\alpha$  is not virtually  $(\eta, A)$ -pre-strong. Also define  $g : \mathsf{On} \to \mathsf{On}$ by setting  $g(\alpha)$  to be the least strong limit cardinal above  $\alpha$  which is a closure point of f. Note that g is an indexing function, so we can let  $\vec{\mathcal{M}}$  be the natural sequence induced by g and  $R_{\alpha} := A \cap H_{g(\alpha)}$ . (ii) supplies us with  $\alpha < \beta$  and a generic elementary embedding

$$\pi \colon (H_{g(\alpha)}, \in, A \cap H_{g(\alpha)}) \to (H_{g(\beta)}, \in, A \cap H_{g(\beta)}).$$

Let us argue that  $\pi$  is not the identity map, so that it must have a critical point. Since  $g(\alpha)$  is a closure point of f, the structure  $(H_{g(\alpha)}, \in, A \cap H_{g(\alpha)})$  can define f correctly. So it must satisfy that f does not have a strong limit closure point above  $\alpha$ , but the structure  $(H_{g(\beta)}, \in, A \cap H_{g(\beta)})$  does have such a closure point, namely  $g(\alpha)$ . Thus,  $\pi$  must have a critical point. Now since  $g(\alpha)$  is a closure point of f, it holds that  $f(\operatorname{crit} \pi) < g(\alpha)$ , so fixing a regular  $\theta \in (f(\operatorname{crit} \pi), g(\alpha))$  we get that  $\operatorname{crit} \pi$  is virtually  $(\theta, A)$ -pre-strong, contradicting the definition of f. Hence On is virtually pre-Woodin.

### 5 Virtual Berkeley cardinals

Berkeley cardinals were introduced by W. Hugh Woodin at the University of California, Berkeley around 1992, as a large cardinal candidate that would potentially be inconsistent with ZF. They trivially imply Kunen's Inconsistency and are therefore at least inconsistent with ZFC, but they have not to date been shown to be inconsistent with ZF. In the virtual setting the virtually Berkeley cardinals, like all the other virtual large cardinals, are small large cardinals that are downwards absolute to L.

The theorems of this section show that virtually Berkeley cardinals are precisely the large cardinals which separate virtually pre-Woodin cardinals from virtually Woodin cardinals analogously to how rank-into-rank cardinals separate virtually strong cardinals from virtually pre-strong cardinals. To show this, we will argue that the virtualisation of the notion of the Vopěnka filter behaves like the original one if and only if there are no virtually Berkeley cardinals. We also show that if the virtual Vopěnka Principle holds and On is not Mahlo, then there is a virtually Berkeley cardinal. Finally, we use virtually Berkeley cardinals to separate the virtual Vopěnka Principle and the virtual Vopěnka principle for finite languages.

**DEFINITION 5.1.** A cardinal  $\delta$  is virtually proto-Berkeley if for every transitive set  $\mathcal{M}$  such that  $\delta \subseteq \mathcal{M}$  there exists a generic elementary embedding  $\pi \colon \mathcal{M} \to \mathcal{M}$  with  $\operatorname{crit} \pi < \delta$ .

If crit  $\pi$  can be chosen arbitrarily large below  $\delta$ , then  $\delta$  is virtually Berkeley, and if crit  $\pi$  can be chosen as an element of any club  $C \subseteq \delta$  we say  $\delta$  is virtually club Berkeley.

Suprisingly, it turns out that the virtually club Berkeley cardinals are precisely the  $\omega$ -Erdös cardinals. This follows from Lemmata 2.5 and 2.8 in [Wil18].

**THEOREM 5.2** (Wilson). An  $\omega$ -Erdős is equivalent to a virtually club Berkeley. The least such is also the least virtually Berkeley cardinal.<sup>4</sup>

**PROPOSITION 5.3.** Virtually (proto)-Berkeley cardinals and virtually club Berkeley cardinals are downward absolute to L. If  $0^{\#}$  exists, then every Silver indiscernible is virtually club Berkeley.

PROOF. Downward absoluteness to L follows by Countable Embedding Absoluteness 2.1. Suppose  $0^{\#}$  exists and  $\delta$  is a limit Silver indiscernible. Fix a transitive set  $\mathcal{M} \in L$  such that  $\delta \subseteq \mathcal{M}$  and a club  $C \subseteq \delta$  in L. Let  $\lambda \gg \operatorname{rank}(\mathcal{M})$  be a regular cardinal in V, so that every element of  $L_{\lambda}$  is definable from indiscernibles below  $\lambda$ .

In V, we can define a shift of indicernibles embedding  $\pi : L_{\lambda} \to L_{\lambda}$  with  $\operatorname{crit} \pi \in C$  fixing indiscernibles involved in the definition of  $\mathcal{M}$  (these are easy to avoid since there are finitely many). It follows that  $\pi(\mathcal{M}) = \mathcal{M}$ . By Countable Embedding Absoluteness 2.1, L must have such a generic elementary embedding  $\sigma : L_{\lambda} \to L_{\lambda}$  and it restricts to  $\sigma : \mathcal{M} \to \mathcal{M}$ . Thus,  $\delta$  is virtually club Berkeley, but then so is every Silver indiscernible.

Virtually (proto-)Berkeley cardinals turn out to be equivalent to their "boldface" versions. The proof is a straightforward virtualisation of Lemma 2.1.12 and Corollary 2.1.13 in [Cut17].

<sup>&</sup>lt;sup>4</sup>Note that this also shows that virtually club Berkeley cardinals and virtually Berkeley cardinals are equiconsistent, which is an open question in the non-virtual context.

**PROPOSITION 5.4** (Virtualised Cutolo). If  $\delta$  is virtually proto-Berkeley, then for every transitive set  $\mathcal{M}$  such that  $\delta \subseteq \mathcal{M}$  and every subset  $A \subseteq \mathcal{M}$  there exists a generic elementary embedding  $\pi : (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$  with  $\operatorname{crit} \pi < \delta$ . If  $\delta$  is virtually Berkeley then we can furthermore ensure that  $\operatorname{crit} \pi$  is arbitrarily large below  $\delta$ .

**PROOF.** Let  $\mathcal{M}$  be transitive with  $\delta \subseteq \mathcal{M}$  and  $A \subseteq \mathcal{M}$ . Let

$$\mathcal{N} := \mathcal{M} \cup \{A, \{\{A, x\} \mid x \in \mathcal{M}\}\} \cup \{\{A, x\} \mid x \in \mathcal{M}\}$$

and note that  $\mathcal{N}$  is transitive. Further, both A and  $\mathcal{M}$  are definable in  $\mathcal{N}$  without parameters: the set A is defined as the unique set such that there is a set B (namely  $\{\{A, x\} \mid x \in \mathcal{M}\}$ ) all of whose elements are pairs of the form  $\{A, x\}$ , and every set x in  $\mathcal{N}$  which does not contain A and which is equal to neither A nor B, must satisfy that  $\{A, x\} \in B$ . M is defined in N from A as the class containing exactly all elements x such that A is not an element of the transitive closure of x.

But this means that a generic elementary embedding  $\pi: \mathcal{N} \to \mathcal{N}$  fixes both  $\mathcal{M}$  and A, giving us a generic elementary  $\sigma: (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$  with  $\operatorname{crit} \sigma = \operatorname{crit} \pi$ , yielding the desired conclusion.

**COROLLARY** 5.5. If there is a model of ZFC with a virtually Berkeley cardinal, then there is a model of ZFC in which the virtual Vopěnka principle for finite languages holds and On is not Mahlo.

**PROOF.** Using Theorem 5.7, it suffices to show that there is a model of ZFC with a virtually Berkeley cardinal in which On is not Mahlo. Take any model with a virtually Berkeley cardinal, call it  $\delta$ . If On is not Mahlo there, then we are done. Otherwise, On is Mahlo, so we can let  $\kappa$  be the least inaccessible cardinal above  $\delta$ . Note that  $\delta$  is virtually Berkeley in  $H_{\kappa} \models$  ZFC and On is not Mahlo in  $H_{\kappa}$ . Thus,  $H_{\kappa}$  is a model of ZFC with a virtually Berkeley cardinal in which On is not Mahlo.

The following is a straight-forward virtualisation of the usual definition of the Vopěnka filter (see e.g. [Kan08]).

**DEFINITION 5.6.** The virtual Vopěnka filter F on On consists of those classes X for which there is an associated natural On-sequence  $\vec{\mathcal{M}}^X$  such that  $\operatorname{crit} \pi \in X$  for any  $\alpha < \beta$  and any generic elementary  $\pi \colon \mathcal{M}^X_{\alpha} \to \mathcal{M}^X_{\beta}$ .

Theorem 4.11 shows that the virtual Vopěnka filter is proper if and only if the virtual Vopěnka principle for finite languages holds. The proof of Proposition 24.14 in [Kan08] also shows that if the virtual Vopěnka principle for finite languages holds, then the virtual Vopěnka filter is normal. However, as we will see shortly, unlike the Vopěnka filter, the virtual Vopěnka filter might be proper, but not uniform. This means that there could be an ordinal  $\delta$  such that every natural sequence of structures has a generic elementary embedding between two of its structures with critical point below  $\delta$ .

Standard proofs of the uniformity of the Vopěnka filter fail in the virtual context once again because of the absence of Kunen's Inconsistency. In these arguments, given an On-length sequence  $\langle \mathcal{A}_{\alpha} \mid \alpha < \mathsf{On} \rangle$  of structures in a common language of size some  $\gamma$ , we come up with a natural sequences  $\langle \mathcal{M}_{\alpha} \mid \alpha < \mathsf{On} \rangle$  such that  $\mathcal{M}_{\alpha}$  codes  $\mathcal{A}_{\alpha}$ . Then given an elementary embedding  $\pi : \mathcal{M}_{\alpha} \to \mathcal{M}_{\beta}$  with crit  $\pi = \kappa$ , we would like to argue that  $\pi$  restricts to an elementary embedding  $\pi : \mathcal{A}_{\alpha} \to \mathcal{A}_{\beta}$ , but this requires that the common language is fixed by  $\pi$ . By elementarity of  $\pi$ , we have that  $\mathcal{A}_{\beta}$  is a structure in the language of size  $\pi(\gamma)$ , and hence  $\pi(\gamma) = \gamma$ .

In the presence of Kunen's Inconsistency, it must then be the case that  $\gamma < \kappa$ , which implies that the common language is fixed. But no argument like this will be possible in the virtual context. Indeed, we will show that if the virtual Vopěnka filter is proper, then it is uniform if and only if there are no virtually Berkeley cardinals if and only if On is virtually Woodin.

**THEOREM 5.7** (G.-N.). If  $\delta$  is a virtually proto-Berkeley cardinal, then for every natural sequence  $\vec{\mathcal{M}} = \langle \mathcal{M}_{\alpha} \mid \alpha < On \rangle$ , there are ordinals  $\alpha < \beta$  and an elementary embedding  $\pi : \mathcal{M}_{\alpha} \to \mathcal{M}_{\beta}$  with crit  $\pi = \kappa < \delta$ . In particular, the virtual Vopěnka principle for finite languages holds and the Vopěnka filter is proper but not uniform.

PROOF. Fix a natural sequence of models  $\vec{\mathcal{M}} = \langle \mathcal{M}_{\alpha} \mid \alpha < \mathsf{On} \rangle$ . Let  $\theta > \delta$  be a cardinal such that  $\vec{M} \upharpoonright \theta \subseteq V_{\theta}$ . Since  $\delta$  is virtually proto-Berkeley, there is an elementary embedding  $\pi : (V_{\theta}, \in, \vec{M} \cap V_{\theta}) \to (V_{\theta}, \vec{M} \cap V_{\theta})$  with crit  $\pi = \kappa < \delta$ . Then the restriction  $\pi : \mathcal{M}_{\kappa} \to \mathcal{M}_{\pi(\kappa)}$  is an elementary embedding with critical point  $\kappa < \delta$ .

**COROLLARY 5.8.** If there is a model of ZFC with a virtually Berkeley cardinal, then there is a model of ZFC in which the virtual Vopěnka principle for finite languages holds and On is not Mahlo.

**PROOF.** Using Theorem 5.7, it suffices to show that there is a model of ZFC with a virtually Berkeley cardinal in which On is not Mahlo. Take any model with a virtually

Berkeley cardinal, call it  $\delta$ . If On is not Mahlo there, then we are done. Otherwise, On is Mahlo, so we can let  $\kappa$  be the least inaccessible cardinal above  $\delta$ . Note that  $\delta$  is virtually Berkeley in  $H_{\kappa} \models$  ZFC and On is not Mahlo in  $H_{\kappa}$ . Thus,  $H_{\kappa}$  is a model of ZFC with a virtually Berkeley cardinal in which On is not Mahlo.

**LEMMA** 5.9 (N.). Suppose that the virtual Vopěnka principle for finite languages holds and that there are no virtually Berkeley cardinals. Then the virtually Vopěnka filter F on On contains every class club C.

PROOF. The crucial extra property we get by assuming that there are no virtually Berkeley cardinals is that F becomes uniform, i.e., contains every tail  $(\delta, \mathsf{On}) \subseteq \mathsf{On}$ . Indeed, assume that  $\delta$  is the least cardinal such that  $(\delta, \mathsf{On}) \notin F$  (note that  $(\gamma, \mathsf{On}) \in F$ up to at least the first inaccessible cardinal because a critical point of an elementary embedding  $\pi : V_{\alpha} \to V_{\beta}$  must be inaccessible).

Let  $\mathcal{M}$  be a transitive set with  $\delta \subseteq \mathcal{M}$  and  $\gamma < \delta$  a cardinal. As  $(\gamma, \mathsf{On}) \in F$  by minimality of  $\delta$ , we may fix a natural sequence  $\vec{\mathcal{N}}$  witnessing this. Let  $\vec{\mathcal{M}}$  be the natural sequence induced by the indexing function  $f: \mathsf{On} \to \mathsf{On}$  given by

$$f(\alpha) := \max(\operatorname{rank}(\mathcal{N}_{\alpha}) + 5, \operatorname{rank}(\mathcal{M}) + 5)$$

and unary relations  $R_{\alpha} := \{\langle \mathcal{M}, \vec{\mathcal{N}}_{\alpha} \rangle\}$ , where  $\vec{\mathcal{N}}_{\alpha} := \langle N_{\beta} \mid \beta \leq \alpha \rangle$ . We can also code in a constant for  $\delta$  ensuring that any elementary embedding between structures from  $\vec{\mathcal{M}}$  must fix  $\delta$ . Suppose  $\pi : \mathcal{M}_{\alpha} \to \mathcal{M}_{\beta}$  is a generic elementary embedding with crit  $\pi < \delta$ , which exists as  $(\delta, \mathsf{On}) \notin F$  and  $\pi$  must fix  $\delta$ . Since  $\pi$  respects the relations  $R_{\alpha}$ , we must have  $\pi \upharpoonright \mathcal{M} : \mathcal{M} \to \mathcal{M}$  and  $\pi(\vec{\mathcal{N}}_{\alpha}) = \vec{\mathcal{N}}_{\beta}$ . We also get that crit  $\pi > \gamma$ , as

$$\pi \upharpoonright \mathcal{N}_{\operatorname{crit} \pi} \colon \mathcal{N}_{\operatorname{crit} \pi} \to \mathcal{N}_{\pi(\operatorname{crit} \pi)}$$

is an embedding between two structures in  $\mathcal{N}$  and  $(\gamma, \mathsf{On}) \in F$ . This means that  $\delta$  is virtually Berkeley, a contradiction. Thus  $\operatorname{crit} \pi > \delta$ , implying that  $(\delta, \mathsf{On}) \in F \notin$ .

From here the proof of Lemma 8.11 in [Jec06] shows what we wanted.

**THEOREM 5.10** (N.). If there are no virtually Berkeley cardinals, then On is virtually pre-Woodin if and only if On is virtually Woodin.

PROOF. Assume On is virtually pre-Woodin, so the virtual Vopěnka principle for finite languages holds by Theorem 4.11. Let F be the virtual Vopěnka filter, which must be proper. By Lemma 5.9, F contains every club C. Assume towards a contradiction that for some class A, there are no virtually (<On, A)-extendible cardinals.

Define an indexing function  $f: On \to On$  by  $f(\alpha)$  is the least  $\eta > \alpha$  such that  $\alpha$  is not virtually  $(\eta, A)$ -extendible. Let C be the club of closure points of f. Define relations  $R_{\alpha}$  to code  $A \cap V_{f(\alpha)}$  and  $C \cap V_{f(\alpha)}$ , and let  $\vec{M} = \langle \mathcal{M}_{\alpha} \mid \alpha < On \rangle$  be the associated natural sequence of models. Since  $C \in F$ , there are ordinals  $\alpha < \beta$  and an elementary embedding  $\pi : \mathcal{M}_{\alpha} \to \mathcal{M}_{\beta}$  with crit  $\pi = \kappa \in C$ . It follows that  $\pi(\kappa) \in C$  as well, and hence it is a closure point of f. Thus,  $f(\kappa) < \pi(\kappa)$ . Now consider the restriction

$$\pi: (H_{f(\kappa)}, A \cap H_{f(\kappa)}) \to (H_{f(\pi(\kappa))}, \in, A \cap H_{f(\pi(\kappa))}),$$

which clearly witnesses that  $\kappa$  is virtually  $(f(\kappa), A)$ -extendible, contradicting the definition of f. Thus, for every class A, there is a virtually (< On, A)-extendible cardinal, which implies, by Theorem 4.5, that On is virtually Woodin. <sup>5</sup>

We get the following immediate corollaries from Theorem 4.11 and Proposition 4.4:

**COROLLARY 5.11.** If the virtual Vopěnka Principle for finite languages holds and On is not Mahlo, then there is a virtually Berkeley cardinal.

**COROLLARY 5.12.** The existence of a virtually pre-Woodin cardinal is equiconsistent with the existence of a virtually Woodin cardinal.

By Corollaries 5.11 and 5.8 we then get the following corollary.

COROLLARY 5.13. The following are equiconsistent:

- (i) There is a virtually Berkeley cardinal.
- (ii) The virtual Vopěnka principle for finite languages holds and On is not Mahlo.

Next, we observe that even the assumption On is virtually Woodin is not enough to guarantee that the virtual Vopěnka principle is uniform.

<sup>&</sup>lt;sup>5</sup>We would like to thank the anonymous referee for suggesting this simple proof.

**THEOREM 5.14** (G.-N.). It is consistent that On is virtually Woodin, but the virtual Vopěnka filter is not uniform.

**PROOF.** Let *V* be a universe in which On is virtually Woodin, yet there is a virtually Berkeley cardinal, for instance, *L* under the assumption of  $0^{\#}$ . Then, by Theorem 5.7, the virtual Vopěnka filter cannot be uniform.

We would like to thank the anonymous referee for pointing out the following separation result.

**THEOREM** 5.15. It is consistent that the virtual Vopěnka Principle holds and On is Mahlo, but On is not virtually Woodin.

**PROOF.** Let V be a universe with a virtually strong cardinal and an  $\omega$ -Erdös cardinal above it. Assume that  $\lambda$  is the least virtually strong cardinal. Since there is an  $\omega$ -Erdös cardinal above  $\lambda$ ,  $V_{\lambda}$  has a proper class of  $\omega$ -Erdös cardinals, each of which is, in particular, virtually Berkeley.

To show that the virtual Vopěnka principle holds in  $V_{\lambda}$ , we have to show that for every  $A \subseteq V_{\lambda}$ , there is a proper class of virtually  $(\langle \lambda, A \rangle)$ -pre-strong cardinals. Fix  $\gamma \langle \lambda \rangle$  and let  $\gamma \langle \delta \rangle \rangle$  be a virtually Berkeley cardinal (since these are unbounded in  $\lambda$ ). We will show that for every  $A \subseteq \lambda$ , there is a virtually  $(\langle \lambda, A \rangle)$ -pre-strong cardinal  $\kappa$  above  $\gamma$ . Using that  $\delta$  is virtually Berkeley, for every cardinal  $\theta \geq \delta$  (below  $\lambda$ ) there exists a generic elementary embedding

$$\pi_{\theta} \colon (H_{\theta}, \in, A \cap H_{\theta}) \to (H_{\theta}, \in, A \cap H_{\theta})$$

with  $\gamma < \operatorname{crit} \pi_{\theta} < \delta$ . By the pigeonhole principle we thus get some  $\gamma < \kappa < \delta$  which is the critical point of unboundedly many  $\pi_{\theta}$  below  $\lambda$ , showing that  $\kappa$  is virtually  $(<\lambda, A)$ -pre-strong.

Thus, the virtual Vopěnka Principle holds in  $V_{\lambda}$ , and  $\lambda$  is at least weakly compact (since it is a virtual large cardinal), so, in particular, is Mahlo. But obviously On cannot be virtually Woodin by the leastness property of  $\lambda$ .

Finally, we separate the virtual Vopěnka Principle for finite languages from the virtual Vopěnka Principle.

**THEOREM 5.16** (G.-N.). It is consistent that the virtual Vopěnka Principle for finite languages holds, but the virtual Vopěnka Principle fails.

**PROOF.** Let *V* be a universe with a virtually Berkeley cardinal  $\delta$  and an inaccessible cardinal above it. Let  $\lambda$  be the least inaccessible cardinal above  $\delta$ . It is not difficult to see that  $\delta$  remains virtually Berkeley in  $V_{\lambda}$ , and so the virtual Vopěnka Principle for finite languages holds in  $V_{\lambda}$  by Theorem 5.7. The virtual Vopěnka Principle fails in  $V_{\lambda}$  because it implies, in particular, that there is a proper class of inaccessible cardinals.

# 6 Questions

We noted in Remark 3.5 that we proved that if  $\kappa$  is faintly  $(2^{<\theta})^+$ -strong, then it is virtually  $\theta$ -supercompact, and if it is virtually  $(2^{<\theta})^+$ -supercompact ala Magidor, then it is virtually  $\theta$ -supercompact. We therefore ask if they are really equivalent for each  $\theta$ , or if this kind of "catching up" is necessary:

**QUESTION 6.1.** Are virtually  $\theta$ -strong cardinals, virtually  $\theta$ -supercompacts and virtually  $\theta$ -supercompacts ala Magidor all equivalent, for every regular  $\theta$ ?

We showed in Corollary 5.13 that a virtually Berkeley cardinal is equiconsistent with the Vopěnka principle for finite languages and On not being Mahlo. This naturally leads to the following question, asking whether this also holds for the virtual Vopěnka Principle:

**QUESTION 6.2.** Does Con(ZFC + there exists a virtually Berkeley cardinal) imply Con(GBC + the virtual Vopěnka Principle + On is not Mahlo)?

Question 1.7 in [Wil18] asks whether the existence of a non- $\Sigma_2$ -reflecting weakly remarkable cardinal always implies the existence of an  $\omega$ -Erdős cardinal. Here a weakly remarkable cardinal is a rewording of a virtually pre-strong cardinal. Furthermore, Wilson also showed that a non- $\Sigma_2$ -reflecting virtually pre-strong cardinal is equivalent to a virtually pre-strong cardinal which is not virtually strong. We can therefore reformulate Wilson's question to the following equivalent question.

**QUESTION 6.3** (Wilson). If there exists a virtually pre-strong cardinal which is not virtually strong, is there then a virtually Berkeley cardinal?

Wilson also showed, in [Wil18], that his question has a positive answer in L, which in particular shows that they are equiconsistent. Our results above at least give a partially positive result:

**COROLLARY 6.4.** If for every class A there exists a virtually A-pre-strong cardinal, and for some class A there is no virtually A-strong cardinal, then there exists a virtually Berkeley cardinal.

**PROOF.** The assumption implies by definition that **On** is virtually pre-Woodin but not virtually Woodin, so Theorem 5.10 supplies us with the desired result.

The assumption that there is a virtually A-pre-strong cardinal for every class A in the above corollary may seem a bit strong, but Theorem 5.7 shows that this is necessary, which might lead one to think that Question 6.3 could have a negative answer.

# A Chart of virtual large cardinals



Figure 2: Relative consistency implications between some virtual large cardinals. The  $\cong$  signs indicate equiconsistency, a solid line indicates that the two are not equiconsistent, and a dashed line indicates that we do not know whether they are equiconsistent.



Figure 3: Direct implications between some virtual large cardinals. The equals signs indicate equivalence, and a solid line indicates that the two are not equivalent.

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