Set theory without powerset

Victoria Gitman

vgitman@nylogic.org
https://victoriagitman.github.io

Models and Sets Seminar

November 10th, 2021

Set theory without powerset

Start with ZFC.

ZF^-:

- Remove powerset.
- Replace the Replacement scheme with the Collection Scheme.

ZFC^-:

- Replace AC with the Well-Ordering Principle: every set can be well-ordered.

Models

- H_{κ^+}: collection of all sets with transitive closure of size $\leq\kappa$ for a cardinal κ
- A forcing extension of a model of ZFC by pretame class forcing.
  - $\mathbb{P} = \Pi_{\xi\in\mathrm{Ord}}\,\mathrm{Add}(\omega, 1)$.
- Every model of Kelley-Morse Set Theory with the Choice Scheme gives rise to a model of ZFC^- with the largest cardinal κ whose V_κ is the sets of the model and whose subsets of V_κ are the classes.

The axioms explained

Theorem: (Szczepaniak) There is a model of ZF^- in which AC holds (every family of sets has a choice function), but the Well-Ordering Principle fails.

Question: Does Zorn's Lemma imply AC over ZF^-?

Consider the following theory.

$\mathrm{ZFC}-$:

- Remove powerset
- Replace AC with the Well-Ordering Principle.

Theorem: (G., Hamkins, Johnstone, Zarach) There are models of $\mathrm{ZFC}-$ in which:

- Collection fails,
- ω_1 is singular,
- every set of reals is countable, but ω_1 exists,
- Łoś-Theorem fails for ultrapowers.

Second-order set theory

Second-order set theory has two sorts of objects: sets and classes.

Syntax: Two-sorted logic

- Separate variables and quantifiers for sets and classes
- Convention: upper-case letters for classes, lower-case letters for sets

Semantics: A model is a triple $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle$.

- V consists of the sets.
- $\mathcal{C}$ consists of the classes.
- Every set is a class: $V \subseteq \mathcal{C}$.
- $C \subseteq V$ for every $C \in \mathcal{C}$.

Second-order axioms

- Sets: ZFC
- Classes:
  - Extensionality
  - Replacement: If F is a function and a is a set, then $F \upharpoonright a$ is a set.
  - Global well-order: There is a class well-order of sets.

Gödel-Bernays set theory GBC:

Comprehension scheme for first-order formulas:

If $\varphi(x, A)$ is a first-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Kelley-Morse set theory KM:

Full comprehension:
If $\varphi(x, A)$ is a second-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Choice principles for classes

Choice Scheme CC:

Given a second-order formula $\varphi(x, X, A)$, if for every set x, there is a class X witnessing $\varphi(x, X, A)$, then there is a class Y collecting witnesses for every x on its slices $Y_x = \{y \mid \langle y, x\rangle \in Y\}$ so that $\varphi(x, Y_x, A)$ holds.

Dependent Choice Scheme DC_δ: (δ regular or $\delta = \mathrm{Ord}$)

Given a second-order formula $\varphi(X, Y, A)$, if for every class X, there is a class Y such that $\varphi(X, Y, A)$ holds, then there is a class Z such that for every $\xi < \delta$, $\varphi(Z \upharpoonright \xi, Z_\xi, A)$ holds.

> *"We can make δ-many dependent choices over any definable relation on classes without terminal nodes."*

Models of Kelley-Morse set theory (with choice principles)

Proposition: Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal. Then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM} + \mathrm{CC} + \mathrm{DC}_{\mathrm{Ord}}$.

Consider the following theory.

$\mathrm{ZFC}^-_{\mathrm{I}}$:

- ZFC^-
- There is the largest cardinal κ.
- κ is inaccessible: κ is regular and for all $\alpha < \kappa$, 2^α exists and $2^\alpha < \kappa$.
    - V_κ exists.
    - $V_\kappa \models \mathrm{ZFC}$.

Proposition: Suppose $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with the largest cardinal κ, then $\langle V_\kappa, \in, P(V_\kappa)\rangle \models \mathrm{KM} + \mathrm{CC}$.

Theorem: (G., Hamkins) There are models of KM in which CC fails for a Π^0_1-assertion and ω-many choices.

Bi-interpretability of $\mathrm{KM}+\mathrm{CC}$ and $\mathrm{ZFC}^-_\mathrm{I}$

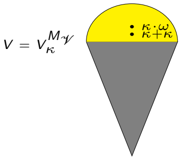
Theorem: (Marek) The theory $\mathrm{KM}{+}\mathrm{CC}$ is bi-interpretable with the theory $\mathrm{ZFC}^-_\mathrm{I}$.

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}{+}\mathrm{CC}$.

- View each extensional well-founded class relation $R \in \mathcal{C}$ as coding a transitive set.
  - $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord}\cdot\omega$
  - $V \cup \{V\}$
- Define a membership relation E on the collection of all such relations R (modulo isomorphism).
- Let $\langle M_{\mathscr{V}}, \mathsf{E}\rangle$, the companion model of $\mathscr{V}$, be the resulting first-order structure.
  - $M_{\mathscr{V}}$ has the largest cardinal $\kappa \cong \mathrm{Ord}^{\mathscr{V}}$.
  - $V_\kappa^{M_{\mathscr{V}}} \cong V$.
  - $\mathcal{P}(V_\kappa)^{M_{\mathscr{V}}} \cong \mathcal{C}$.
  - $\langle M_{\mathscr{V}}, \mathsf{E}\rangle \models \mathrm{ZFC}^-_\mathrm{I}$.

Suppose $M \models \mathrm{ZFC}^-_\mathrm{I}$ with the largest cardinal κ.

- $V = V_\kappa^M$
- $\mathcal{C} = \{X \in M \mid X \subseteq V_\kappa^M\}$
- $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}{+}\mathrm{CC}$
- $M_{\mathscr{V}} \cong M$ is the companion model of $\mathscr{V}$.



First-order Dependent Choice Scheme

Dependent Choice Scheme: DC_δ-scheme (δ regular)
Given a formula $\varphi(x, y, a)$, if for every b, there is a c such that $\varphi(b, c, a)$ holds, then there is a function f with domain δ such that for all $\xi < \delta$, $\varphi(f \upharpoonright \xi, f(\xi), a)$ holds.

> *"We can make δ-many dependent choices over any definable relation without terminal nodes."*

$\mathrm{DC}_{<\mathrm{Ord}}$-scheme: the DC_δ-scheme for every regular δ.

Proposition: In ZFC, the $\mathrm{DC}_{<\mathrm{Ord}}$-scheme holds.

Proof: Fix a regular δ and a relation $\varphi(x, y, a)$ without terminal nodes.
Fix a V_γ, with $\mathrm{cof}(\gamma) \geq \delta$, such that V_γ reflects $\varphi(x, y, a)$ and $\forall x \exists y \varphi(x, y, a)$.

- $V_\gamma^{<\delta} \subseteq V_\gamma$.
- Use a well-ordering of V_γ together with closure to construct f. $\square$

Models

- H_{κ^+}.
- Pretame forcing extensions of ZFC-models: pretame forcing preserves $\mathrm{DC}_{<\mathrm{Ord}}$-scheme.
- A model $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM} + \mathrm{CC} + \mathrm{DC}_\delta$ if and only if its companion model $M_{\mathscr{V}} \models \mathrm{ZFC}^-_\mathrm{I} + \mathrm{DC}_\delta$-scheme.

Applications of the Dependent Choice Scheme

Proposition: (G.) In ZFC^-, TFAE for a regular δ such that $\gamma^{<\delta}$ exists for every γ.

- DC_δ-scheme
- Every formula $\psi(x, a)$ reflects to a transitive model m such that $m^{<\delta} \subseteq m$.

Corollary:

- DC_ω-scheme is equivalent to the assertion that every formula reflects to a transitive set.
- In $\mathrm{ZFC}^-_\mathrm{I}$, the DC_δ-scheme holds if and only if every formula reflects to a transitive model closed under $<\delta$-sequences.

Theorem: (Folklore) In ZFC^-, TFAE.

- $\mathrm{DC}_{<\mathrm{Ord}}$-scheme
- The partial order $\mathrm{Add}(\mathrm{Ord}, 1)$ is Ord-distributive (doesn't add sets).
- Global well-order can be forced without adding sets.

Proposition: In $\mathrm{ZFC}^- + \mathrm{DC}_\delta$-scheme, every class surjects onto δ.

Proposition: In $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Ord}}$-scheme, every class is big: surjects onto every ordinal.

ZFC^- and Kunen's Inconsistency

Proposition: If there is an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ with $\mathrm{crit}(j) < \lambda$ (I_1), then there is an elementary embedding $j^* : H_{\lambda^+} \to H_{\lambda^+}$.

- Kunen's Inconsistency does not hold for ZFC^-.
- H_{λ^+} satisfies $\mathrm{DC}_{<\mathrm{Ord}}$.
- j^* is not cofinal.

Theorem: (Matthews) Kunen's Inconsistency holds for cofinal self-embeddings of models of $\mathrm{ZFC}^-+\mathrm{DC}_{<\mathrm{Ord}}$-scheme.

Question: Does Kunen's Inconsistency hold for cofinal self-embeddings models of ZFC^-?

Failures of the Dependent Choice Scheme

Theorem: (Friedman, G., Kanovei) There is a model $M \models \mathrm{ZFC}^-$ in which the DC_ω-scheme fails.

- Force with a tree iteration of Jensen's forcing along the tree $\omega_1^{<\omega}$.
- N is a symmetric submodel of the forcing extension.
- $M = H_{\omega_1}^N$
- $M \models$ "ω is the largest cardinal."

Work in progress: (G., Friedman) There is a model of $\mathrm{ZFC}_\mathrm{I}^-$ in which the DC_ω-scheme fails. Therefore, there is a model of $\mathrm{KM} + \mathrm{CC}$ in which DC_ω fails.

- Use a generalization of Jensen's forcing for an inaccessible cardinal.

More later...

ZFC^- and the Intermediate Model Theorem

Intermediate Model Theorem: (Solovay)

- If $V \models \mathrm{ZFC}$, $V[G]$ is a set forcing extension, and $W \models \mathrm{ZFC}$ such that $V \subseteq W \subseteq V[G]$, then $W = V[H]$ is a set forcing extension.
- If $V \models \mathrm{ZF}$, $V[G]$ is a set forcing extension, and $V[a] \models \mathrm{ZF}$ such that $a \subseteq V$ and $V[a] \subseteq V[G]$, then $V[a]$ is a set forcing extension.

Theorem: (Antos, G., Friedman) If $M \models \mathrm{ZFC}^-$, $M[G]$ is a set forcing extension, and $M[a] \models \mathrm{ZFC}^-$ such that $a \subseteq M$ and $M[a] \subseteq M[G]$, then $M[a] = M[H]$ is a set forcing extension.

Proof Sketch: Every poset $\mathbb{P} \in M$ densely embeds into a class complete Boolean algebra. $\square$.

Failure of the Intermediate Model Theorem

Theorem: (Antos, G., Friedman) If $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with the largest cardinal κ and $G \subseteq \mathrm{Add}(\kappa, 1)$ is M-generic, then there is a model $N \models \mathrm{ZFC}^-$ such that:

- $M \subseteq N \subseteq M[G]$,
- N is not a set forcing extension,
- if $M \models \mathrm{DC}_\kappa$-scheme, then $N \models \mathrm{DC}_\kappa$-scheme.

Proof Sketch:

- $G \subseteq \mathrm{Add}(\kappa, \kappa) \cong \mathrm{Add}(\kappa, 1)$
- $G_\xi = G \restriction \xi$ is the restriction of G to the first ξ-many coordinates of the product.
- $N = \bigcup_{\xi < \kappa} M[G_\xi]$
- Use an automorphism argument to show that N satisfies Collection.

ZFC^- and ground model definability

Theorem: (Laver, Woodin) A model $V \models \mathrm{ZFC}$ is uniformly definable with parameters from V in all its set forcing forcing extensions.

Theorem: (G., Johnstone) If $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with the largest cardinal κ, then M is uniformly definable in its forcing extensions by any poset in V_κ.

Theorem: (G., Johnstone)

- Suppose κ is regular and uncountable and $W = V[G]$ is a forcing extension by $\mathrm{Add}(\omega, \kappa^+)$. Then $M = H^W_{\kappa^+}$ is not definable in its forcing extensions by $\mathrm{Add}(\omega, 1)$.
  - $P(\omega)$ is a proper class in M.
- Suppose κ is inaccessible and $W = V[G]$ is a forcing extension by $\mathrm{Add}(\kappa, \kappa^+)$. Then $M = H^W_{\kappa^+}$ is not definable in its forcing extensions by $\mathrm{Add}(\kappa, 1)$.
  - $M \models \mathrm{ZFC}^-_{\mathrm{I}}$.

Theorem: (Woodin) If there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$ (I_0), then $M = H_{\lambda^+}$ is not definable in its forcing extension by $\mathrm{Add}(\omega, 1)$.

- $P(\omega) \in M$.

Question: What is the consistency strength of having a model $M \models \mathrm{ZFC}^-$ which is not definable in a forcing extension by $\mathbb{P} \in M$ with $P(\mathbb{P}) \in M$?

ZFC^- and HOD

Theorem: The inner model HOD (hereditarily ordinal definable sets) is definable in any model $V \models \mathrm{ZFC}$.

Proof: A set a is in HOD if and only if there is α such that a is ordinal definable over V_α. $\square$

Question: Is HOD definable in models of ZFC^-?

- ZFC^- does not imply the existence of a hierarchy.

Question: Is HOD definable in models of $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Ord}}$-scheme?

Strange models of ZFC^-

Set-up

$V \models \mathrm{ZFC} + \mathrm{CH}$

$\mathbb{P} = \mathrm{Add}(\omega, \omega) \cong \mathrm{Add}(\omega, 1)$.

$G \subseteq \mathrm{Add}(\omega, \omega)$ is V-generic.

$G_n = G \upharpoonright n$ is the restriction of G to the first n-many coordinates of the product.

$N = \bigcup_{n<\omega} V[G_n]$.

Theorem: (Zarach)

- $N \models \mathrm{ZFC}^-$: automorphism argument
- V and N have the same cardinals and cofinalities
- $P(\omega)$ does not exist in N
- $P(\omega)$ is a small class in N: $P(\omega)$ does not surject onto ω_2
- $N \models \mathrm{DC}_\omega$-scheme
- $N \models \neg\mathrm{DC}_{\omega_2}$-scheme

Question: Does the DC_{ω_1}-scheme hold in N?

Generalizing Zarach's construction

Set-up

$V \models \mathrm{ZFC} + 2^\delta = \delta^+$ (δ regular cardinal)

$\mathbb{P} = \mathrm{Add}(\delta, \delta) \cong \mathrm{Add}(\delta, 1)$.

$G \subseteq \mathrm{Add}(\delta, \delta)$ is V-generic.

$G_\xi = G \restriction \xi$ is the restriction of G to the first ξ-many coordinates of the product.

$N = \bigcup_{\xi < \delta} V[G_\xi]$.

Theorem: (G., Matthews)

- $N \models \mathrm{ZFC}^-$: automorphism argument
- V and N have (almost) the same cardinals and cofinalities
- $P(\delta)$ does not exist in N
- $P(\delta)$ is a small class in N: $P(\delta)$ does not surject onto δ^{++}
- $N \models \mathrm{DC}_\delta$-scheme (uses $N^{<\delta} \subseteq N$ in $V[G]$)
- $N \models \neg\mathrm{DC}_{\delta^{++}}$-scheme

Jensen's forcing

$\mathbb{J}$: (Jensen)

- subposet of Sacks forcing: perfect trees ordered by $\subseteq$
- constructed using $\diamondsuit$
- ccc
- adds a unique generic real

Products and iterations of Jensen's forcing have "uniqueness of generics" properties.

$\mathbb{J}^{<\omega}$: finite-support ω-length product of the $\mathbb{J}$.

Theorem: (Lyubetsky, Kanovei) If $G \subseteq \mathbb{J}^{<\omega}$ is V-generic, then in $V[G]$, the V-generic reals for $\mathbb{J}$ are precisely the "slices" G_n of G.

Suppose κ is inaccessible.

$\mathbb{J}_\kappa$ (G., Friedman)

- subposet of κ-Sacks forcing: perfect κ-trees ordered by $\subseteq$
- constructed using $\diamondsuit_{\kappa^+}(\text{cof}(\kappa))$
- κ^+-cc
- adds a unique generic subset of κ

$\mathbb{J}_\kappa^{<\kappa}$: bounded-support κ-length product of $\mathbb{J}_\kappa$.

Theorem: (G. Friedman) If $G \subseteq \mathbb{J}_\kappa^{<\kappa}$ is V-generic, then in $V[G]$, the V-generic reals for $\mathbb{J}_\kappa$ are precisely the "slices" G_ξ of G.

Getting the DC_ω-scheme to hold and the DC_{ω_1}-scheme to fail

Set-up

$L \models \mathrm{ZFC}$

$\mathbb{P} = \prod_{n<\omega} \mathbb{J}^{<\omega} \cong \mathbb{J}^{<\omega}$.

$G \subseteq \mathbb{P}$ is L-generic.

$G_n = G \restriction n$ is the restriction of G to the first n-many coordinates of the product.

$N = \bigcup_{n<\omega} V[G_n]$.

Theorem: (G., Matthews)

- $N \models \mathrm{ZFC}^-$
- L and N have the same cardinals and cofinalities
- $P(\omega)$ does not exist in N
- $P(\omega)$ is a small class in N: $P(\omega)$ does not surject onto ω_2
- $N \models \mathrm{DC}_\omega$-scheme
- $N \models \neg\mathrm{DC}_{\omega_1}$-scheme
  - for every countable sequence of Jensen's reals, there is a Jensen's real distinct from it
  - there is no ω_1-length sequence of distinct Jensen's reals

Getting the DC_κ-scheme to holds and the DC_{κ^+}-scheme to fail

Theorem: (G., Matthews) If κ is inaccessible, then there is a model of ZFC^- in which the DC_κ-scheme holds, but DC_{κ^+}-scheme fails.

Same construction using $\mathbb{J}_\kappa$ instead of $\mathbb{J}$.

A different model of $\mathrm{ZFC}^- + \neg\mathrm{DC}_\omega$-scheme

Theorem: (G., Matthews) There is a model of $N \models \mathrm{ZFC}^-$ such that:

- $P(\omega)$ does not exist.
- Every class is big.
- There are unboundedly many cardinals.
- DC_ω-scheme fails.

Proof Sketch: Force with the tree iteration $\mathbb{P}$ of Jensen's forcing along the tree $\mathrm{Ord}^{<\omega}$. Let $G \subseteq \mathbb{P}$ be V-generic.

- $\mathbb{P}$ has the ccc, and hence is pretame.
- $V[G] \models \mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Ord}}$-scheme.
- $N = \bigcup L[G_T]$, where T is a certain set subtree of $\mathrm{Ord}^{<\omega}$, $\mathbb{P}_T$ is the restriction of $\mathbb{P}$ to T, and G_T is the restriction of G to $\mathbb{P}_T$.