Virtual Vopěnka's Principle

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Accessible categories and their connections

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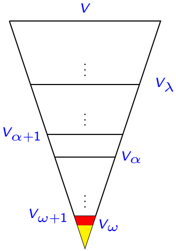
Set-theoretic universes

Axioms of set theory: ZFC Zermelo-Fraenkel set theory with Choice (or ZF)

Iterative conception of set: We think of a set-theoretic universe as built up by a transfinite process along the ordinals ORD.



- $V_0 = \emptyset$
- $V_{\alpha+1}$ consists of all subsets of V_α.
- $V_\lambda = \bigcup_{\alpha<\beta} V_\alpha$ for a limit λ.
- $V = \bigcup_{\alpha\in\mathrm{ORD}} V_\alpha$.

Examples

- V_ω has all the natural numbers.
- $V_{\omega+1}$ has all the reals.
- $V_{\omega+2}$ has all the subsets of reals.

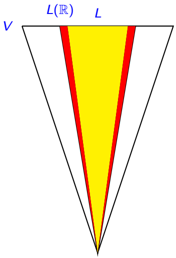
Canonical universes of set theory

Canonical universes are highly constructible from bottom up.

Suppose V is a universe of set theory.

Gödel's constructible universe L.

- $L_0 = \emptyset$
- $L_{\alpha+1}$ consists of all definable subsets of L_α.
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ.
- $L = \bigcup_{\alpha\in\mathrm{ORD}} L_\alpha$.



L of the reals: $L(\mathbb{R})$

- $L_0(\mathbb{R}) = \mathbb{R}$
- $\vdots$

Theorem:

- (Gödel) If $V \models \mathrm{ZF}$, then $L \models \mathrm{ZFC}$.
- If $V \models \mathrm{ZFC}$, then $L(\mathbb{R}) \models \mathrm{ZF} + \mathrm{DC}$ (Axiom of Dependent Choices).

Classes in set-theoretic universes

Classes are collections of sets that are too large to be sets themselves.

In naive set theory, a lack of distinction between sets and classes led to Russell's Paradox.

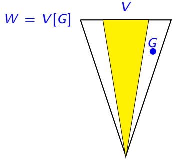
Definition: A class in a set-theoretic universe is a definable (with parameters) collection of sets. A proper class is proper is a class that is not a set.

Examples of (proper) classes

- The whole universe V.
- The constructible universe L.
- The ordinals ORD.
- The collection of all initial segments $\{V_\alpha \mid \alpha \in \mathrm{ORD}\}$.

Forcing: building set-theoretic universes

The forcing construction: (Cohen) Build a universe W extending an existing universe V.



- Fix a forcing notion: partial order $\mathbb{P} \in V$.
- Define a collection $V^{\mathbb{P}}$ of names for elements of W.
  - Each element of W has a name $\tau \in V^{\mathbb{P}}$.
  - An element of W can have more than one name.
- Let $G \notin V$ be a generic filter on $\mathbb{P}$: G meets every dense set $D \in V$ of $\mathbb{P}$.
- The forcing extension $W = V[G] = \{\tau_G \mid \tau \in V^{\mathbb{P}}\}$ consists of the "interpretation" of all names in $V^{\mathbb{P}}$ by G.
  - $V \subseteq V[G]$
  - $G \in V[G]$

The forcing relation $p \Vdash \varphi(\tau)$

- $p \in \mathbb{P}$, $\tau \in V^{\mathbb{P}}$
- φ is a first-order formula

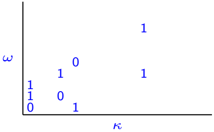
Whenever G is a generic filter and $p \in G$, then $V[G] \models \varphi(\tau_G)$.

The Forcing Theorem: (Cohen) For every first-order formula $\varphi(x)$, the relation $p \Vdash \varphi(\tau)$ is definable.

Examples of forcing notions

$\mathbb{P} = \mathrm{Add}(\omega, \kappa)$ for a cardinal κ.

- Elements: partial functions $f : \omega \times \kappa \to \{0, 1\}$ with finite domain.
- Order: extension.
- Generic filter $G : \omega \times \kappa \to \{0, 1\}$ with $G_\alpha \neq G_\beta$ for any $\alpha < \beta < \kappa$.
- If $\kappa > \omega_1$, then CH fails in $V[G]$.



$\mathbb{P} = \mathrm{Coll}(\omega, \kappa)$ for a cardinal κ.

- Elements: partial functions $f : \omega \to \kappa$ with finite domain.
- Order: extension.
- Generic filter $G : \omega \xrightarrow{\text{onto}} \kappa$.
- The cardinal κ of V becomes a countable ordinal in $V[G]$.

Strong axioms

Measuring strength

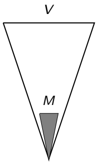
- Many important set-theoretic properties, such as CH, are independent of ZFC.
- We can rank properties based on consistency strength.
- Properties A and B are equiconsistent: there is a model of $\mathrm{ZFC}+\mathrm{A}$ if and only if there is a model of $\mathrm{ZFC}+\mathrm{B}$.
    - CH and $\neg\mathrm{CH}$ are equiconsistent.
- Property A is stronger than property B: if there is a model of $\mathrm{ZFC}+\mathrm{A}$, then there is a model of $\mathrm{ZFC}+\mathrm{B}$, but not conversely.

Large cardinal axioms

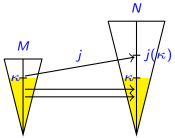
- A natural hierarchy of set theoretic properties against which the consistency strength of all other properties can be measured.
- Characterized by the existence of very large infinite objects.
- Characterized by the existence of elementary embeddings.

Elemetary embeddings of models of set-theory

- A set M is transitive if "it doesn't have holes", whenever $a \in M$ and $b \in a$, then $b \in M$.



- Suppose $j : M \to N$ is an elementary embedding between transitive sets M and N. An ordinal κ is the critical point of j, $\text{crit}(j) = \kappa$, if κ is "the first ordinal moved by j", $j(\kappa) > \kappa$, but $j(\alpha) = \alpha$ for all $\alpha < \kappa$.



In practice,

- M will satisfy (a large fragment of) ZFC,
- κ will be a large cardinal.

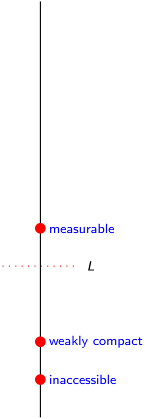
Large cardinals

A cardinal κ is inaccessible if for every $\alpha < \kappa$, every function $f : \alpha \to \kappa$ is bounded and $P(\alpha)$ has size less than κ.

- $V_\kappa \models \mathrm{ZFC}$.

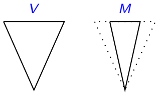
A cardinal κ is weakly compact if (it is inaccessible) and every tree T of height κ with levels of size less than κ has a cofinal branch.

- "König's Lemma holds at κ".
- There are many inaccessible cardinals below κ.
- Every $M \models \mathrm{ZFC}$ of size κ with $\kappa \in M$ has an elementary embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$.



A cardinal κ is measurable if there is a κ-complete ultrafilter on κ.

- There are many weakly compact cardinals below κ.
- There is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$.

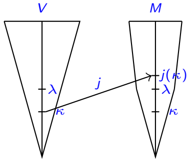


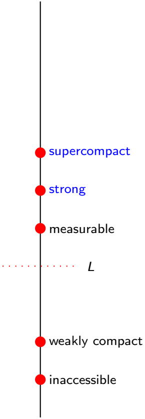
- **Theorem**: (Scott) There are no measurable cardinals in L.

(*) All models of set theory are assumed to be transitive.

Larger large cardinals

A cardinal κ is strong if for every $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $j(\kappa) > \lambda$.





- M is "close" to V.
- Characterized by existence of certain ultrafilters.
- For every $\lambda > \kappa$, there is $\alpha > \lambda$ and an elementary embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $V_\lambda \subseteq N$, and $j(\kappa) > \lambda$.

A cardinal κ is supercompact if for every $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $M^\lambda \subseteq M$, (and $j(\kappa) > \lambda$).

- Characterized by existence of certain ultrafilters.
- For every $\lambda > \kappa$, there is $\alpha > \lambda$ and an elementary embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $N^\lambda \subseteq N$, and $j(\kappa) > \lambda$.

Even larger large cardinals

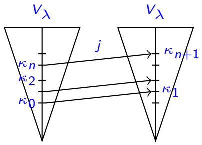
A cardinal κ is extendible if for every $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ (and $j(\kappa) > \alpha$).

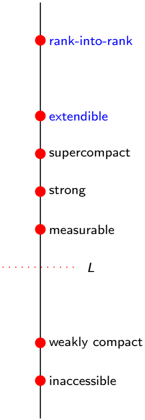
- There are many supercompact cardinals below an extendible κ.

A cardinal κ is rank-to-rank if there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$.

Theorem: (Kunen's Inconsistency) There cannot be a non-trivial elementary embedding $j : V \to V$.

- Suppose $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$.
- The critical sequence of j: $\kappa_0 = \kappa$, $k_{n+1} = j(\kappa_n)$.
- $\gamma = \sup_{n<\omega} \kappa_n$.
- Then $\lambda = \gamma$ or $\lambda = \gamma + 1$.





Forcing and large cardinals

Indestructibility of large cardinals by forcing.

- **Question**: Does κ retain its large cardinal property in a forcing extension $V[G]$ by $\mathbb{P}$?
- **Theorem**: (Lévy-Solovay) Large cardinals are indestructible by small forcing.

Large cardinal strength of forcing axioms.

- Forcing axiom: An assertion that for a certain class of partial orders, the universe has almost generic filters.

Large cardinal strength of forcing absoluteness.

- Forcing absoluteness: An assertion that some properties of the universe cannot be changed by forcing.
- **Therem**: (Woodin) If there is a proper class of supercompact cardinals, then the theory of $L(\mathbb{R})$ cannot be changed by forcing.

Virtual large cardinals.

- Elementary embeddings (between set models) implied by a large cardinal axiom exist in some set-forcing extension.

Vopěnka's Principle

Vopěnka's Principle: Every proper class of structures in a common language has at least two structures which elementarily embed.

- proper class: think of structures as being indexed by ordinals
- common language: groups, vector spaces, sets, metric spaces, etc.
- infinitely many assertions: cannot be expressed as a single axiom
  - For a first-order formula $\varphi(x)$, VP_φ asserts that if $\varphi(x)$ defines a proper class of structures, then there are $x \neq y$ such that $\varphi(x)$ and $\varphi(y)$ and there is an elementary embedding $j : x \to y$.
  - Vopěnka's Principle consists of all VP_φ.
- large cardinal principle
- applied in other areas of mathematics

Large cardinal teaser: Consider the class of all structures $\langle V_\alpha, \in\rangle$. Vopěnka's Principle implies that there are ordinals $\alpha < \beta$ such that there is an elementary embedding $j : V_\alpha \to V_\beta$.

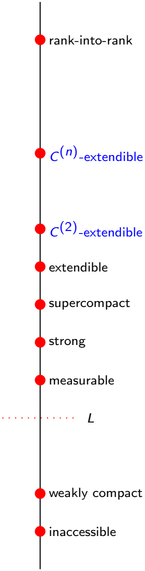
$C^{(n)}$-extendible cardinals

$C^{(n)}$ is the class of all ordinals α such that $V_\alpha \prec_{\Sigma_n} V$.

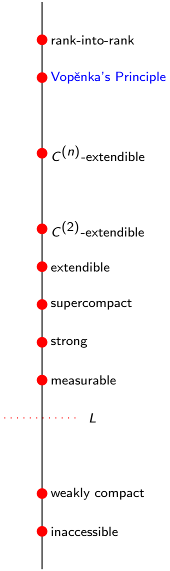
- V_α reflects V for formulas $\underbrace{\exists\forall\exists\cdots}_{n\text{-many}}$.
- α must be a cardinal.
- **Lévy Reflection**: $C^{(n)}$ is a proper class.

(Bagaria) A cardinal κ is $C^{(n)}$-extendible, for $n > 0$, if for every $\kappa < \alpha \in C^{(n)}$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $\beta \in C^{(n)}$ (and $j(\kappa) > \alpha$).

- Extendible cardinals are $C^{(1)}$-extendible.
- $C^{(n)}$-extendible cardinals form a hierarchy between supercompact cardinals and rank-into-rank cardinals.
- The assumption $j(\kappa) > \alpha$ is superfluous.
  - Either there is $h : V_\alpha \to V_{\bar\beta}$ with $h(\kappa) > \alpha$, or
  - there is $h : V_\lambda \to V_\lambda$ with $\lambda > \gamma + 1$ for $\gamma = \sup_{n<\omega}\kappa_n$.
  - The second possibility contradicts Kunen's Inconsistency.



Vopěnka's Principle and large cardinals



Theorem: (Bagaria) Vopěnka's Principle holds if and only if for every $0 < n < \omega$, there is a proper class of $C^{(n)}$-extendible cardinals.

- Proof makes use of Kunen's Inconsistency.

Remarkable cardinals

Theorem: (Woodin) If there is a supercompact cardinal, then there is a model in which the theory of $L(\mathbb{R})$ cannot be changed by forcing.

Question: What is the consistency strength of the assertion:

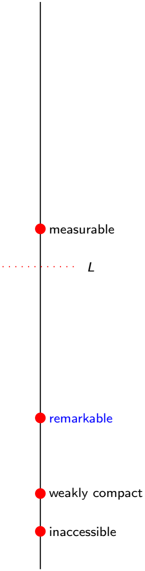
"The theory of $L(\mathbb{R})$ cannot be changed by proper forcing."?

- Proper partial orders is a very useful subclass of partial orders that preserve ω_1.

Theorem: (Schindler) Consistency strength of the assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing is a remarkable cardinal.

(Schindler) A cardinal κ is remarkable if for every $\lambda > \kappa$, there is $\alpha > \lambda$ and a model N with $N^\lambda \subseteq N$ such that **in a forcing extension** there is an elementary embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

- virtually supercompact!
- many very different characterizations.



Virtual elementary embeddings

There is a virtual elementary embedding between structures M and N if some forcing extension has an elementary embedding $j : M \to N$.

Example: A virtual elementary embedding from the reals $\mathbb{R}$ to the rationals $\mathbb{Q}$.

- Let $2^\omega = \gamma$ be the cardinality of $\mathbb{R}$.
- Force with $\mathrm{Coll}(\omega, \gamma)$ to make $\mathbb{R}$ countable in the forcing extension $V[G]$.
- In $V[G]$, $\mathbb{R}^V$ is a countable dense linear order without endpoints.

Example: In L, there can be unboundedly many α such that there is a virtual elementary embedding $j : V_\alpha \to V_\alpha$ with $\alpha \gg \gamma$, where γ is the supremum of the critical sequence.

- Kunen's Inconsistency does not hold for virtual embeddings.
- Consistency strength is very low.

Absoluteness Lemma for virtual embeddings

Lemma: (Folklore) Suppose M and N are first-order structures. If in some forcing extension $V[H]$ there is an elementary embedding $h : M \to N$, then any forcing extension $V[G]$ by the forcing $\mathrm{Coll}(\omega, |M|)$ (making M countable) there is an elementary embedding $j : M \to N$.

- Only forcing extensions making M countable matter.
- j can agree with h on finitely many values.
- j can agree with h on the critical point.

A game characterization of virtual embeddings

Suppose M and N are first-order structures in a common language.
Let $G(M, N)$ be a game played for ω-many steps:

- Player I plays elements $a_n \in M$.
- Player II plays elements $b_n \in N$.
- Players I and II alternate moves.

I	a_0		a_1		a_2		$\dots$		a_n	
II		b_0		b_1		b_2		$\dots$		b_n

- Player II wins if for every $n \in \omega$ and formula $\varphi(x_0, \dots, x_n)$

$$M \models \varphi(a_0, \dots, a_n) \leftrightarrow N \models \varphi(b_0, \dots, b_n),$$

  the map sending a_i to b_i for $i \leq n$ is a finite partial isomorphism between M and N.
- Otherwise, Player I wins.

Theorem: (Schindler) There is a virtual elementary embedding between M and N if and only if Player II has a winning strategy in the game $G(M, N)$.

Virtual large cardinals

A cardinal κ is virtually $C^{(n)}$-extendible if for every $\kappa < \alpha \in C^{(n)}$ **in a forcing extension** there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$, $\beta \in C^{(n)}$, and $j(\kappa) > \alpha$.

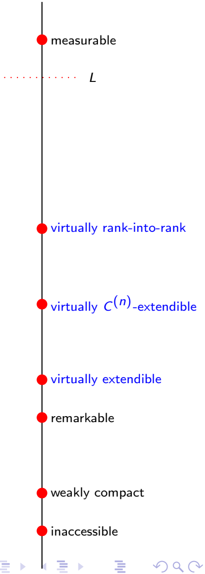
A cardinal κ is **weakly** virtually $C^{(n)}$-extendible if for every $\kappa < \alpha \in C^{(n)}$ in a forcing extension there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $\beta \in C^{(n)}$.

A cardinal κ is virtually rank-into-rank if **in a forcing extension** there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$.

- Kunen's Inconsistency fails: λ can be much larger than the supremum of the critical sequence.
- If a **weakly** virtually $C^{(n)}$-extendible κ is not virtually $C^{(n)}$-extendible, then κ is virtually rank-into-rank.

Theorem: (G., Schindler)

- Virtual large cardinals are compatible with L.
- The hierarchy of the virtual large cardinals mirrors that of their original counterparts.



Virtual Vopěnka's Principle

Virtual Vopěnka's Principle: Every proper class of structures in a common language has at least two structures which elementarily embed in a forcing extension.

Theorem: (Bagaria, G., Schindler) Virtual Vopěnka's Principle is equiconsistent with the scheme asserting that for every $0 < n < \omega$ there is a proper class of virtually $C^{(n)}$-extendible cardinals.

Theorem: (Bagaria, G., Schindler) Virtual Vopěnka's Principle is compatible with L.

Theorem: (G., Hamkins) Virtual Vopěnka's Principle holds if and only if for every $0 < n < \omega$ there is a proper class of **weakly** virtually $C^{(n)}$-extendible cardinals.

Theorem: (G., Hamkins) There is a model in which Virtual Vopěnka's Principle holds, but there are no virtually extendible cardinals.

Corollary: A **weakly** virtually $C^{(n)}$-extendible cardinal may not be virtually $C^{(n)}$-extendible.

Indestructibility of Virtual Vopěnka's Principle

Theorem: (Brooke-Taylor) Vopěnka's Principle is indestructible by a large class of forcing notions: ORD-length Easton-support iterations.

Question: Is Virtual Vopěnka's Principle indestructible by all ORD-length Easton-support iterations?

Strong virtual large cardinals

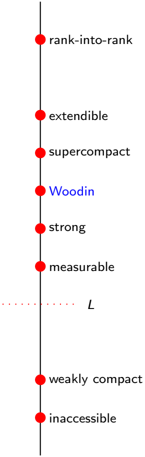
If there is a virtual elementary embedding between M and N, then there is an elementary embedding $j : M \to N$ in a forcing extension where M becomes countable.

Question (Boney): What is the consistency strength of the assertion:

> *"In a forcing extension preserving a large initial segment of the universe there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $crit(j) = \kappa$ and λ much larger than the supremum of the critical sequence."?*

Theorem: (Woodin) If there is a Woodin cardinal κ, then for every $\theta < \lambda < \kappa$, there is a forcing extension preserving V_θ in which there is an elementary embedding $j : V_\lambda \to V_\lambda$ with λ much larger than the supremum of the critical sequence.

Theorem: (Woodin) If there is a Woodin cardinal κ, then there is a model in which for every $\theta < \lambda < \kappa$, there is a forcing extension in which $V_\lambda^\theta \subseteq V_\lambda$ and there is elementary embedding $j : V_\lambda \to V_\lambda$ with λ much larger than the supremum of the critical sequence.



Strong virtual large cardinals: questions

The proof of Kunen's Inconsistency gives:

Theorem: There cannot be an elementary embedding $j : M \to V$ with $M \subseteq V$ and $M^\omega \subseteq M$ in V.

Question: What is the consistency strength of the assertion:

> *"There is a forcing extension not adding ω-sequences, in which there is an elementary embedding $j : V_\lambda \to V_\lambda$ with λ much larger than the supremum of the critical sequence."?*

Arguments that the hierarchy of the virtual large cardinals mirrors the hierarchy of their original counterparts relies on the Absoluteness Lemma for virtual embeddings.

Question: Under some appropriate definition, do strong virtual large cardinals form a hierarchy?

Strong Virtual Vopěnka's Principle

Strong Virtual Vopěnka's Principle: Every proper class of first-order structures in a common language has two structures which elementarily embed in:

- a forcing extension preserving ω_1, or
- a forcing extension preserving a large initial segment V_α of the universe, or
- a forcing extension not adding ω-sequences.

Question: What are the consistency strengths of the various strong Vopěnka's Principles?