Baby measurable cardinals

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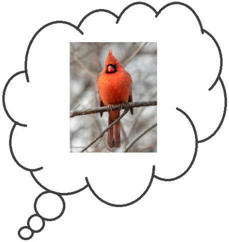
CUNY Logic Workshop

September 27, 2024



When an average person thinks of large cardinals.

When an average set theorist thinks of large cardinals.

When I think of large cardinals.

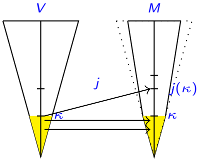
The larger of the large cardinals and elementary embeddings

A cardinal κ is measurable if there is an elementary embedding

$$j : V \to M$$

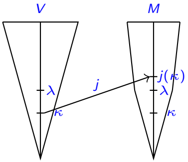
from the universe V of set theory into a transitive submodel M with $\text{crit}(j) = \kappa$.

- A set or a class A is transitive if whenever $a \in A$ and $b \in a$, then $b \in A$ (there are no holes).
- The critical point $\text{crit}(j)$ of an elementary embedding j is the first ordinal that is moved by j.
- If $\text{crit}(j) = \kappa$, then $V_{\kappa+1} \subseteq M$.



The larger of the large cardinals template

A cardinal κ is strong if for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subseteq M$.

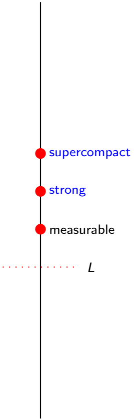


A cardinal κ is supercompact if for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$.

(for every $f : \lambda \to M$, $f \in M$)

Template: The closer M is to V the stronger the large cardinal notion.

Theorem: (Kunen's Inconsistency) The existence of a non-trivial elementary embedding $j : V \to V$ is inconsistent.



Elementary embeddings and ultrafilters

Suppose κ is a cardinal and $U \subseteq P(\kappa)$ is an ultrafilter.

- U is uniform if for every $\alpha < \kappa$, the tail set $\kappa \setminus \alpha \in U$.
- U is α-complete if whenever $\beta < \alpha$ and $A_\xi \in U$ for every $\xi < \beta$, then $\bigcap_{\xi<\beta} A_\xi \in U$.
- U is normal if whenever $A_\xi \in U$ for every $\xi < \kappa$, then the diagonal intersection $\Delta_{\xi<\kappa} A_\xi \in U$. $\Delta_{\xi<\kappa} A_\xi = \{\alpha < \kappa \mid \alpha \in \bigcap_{\xi<\alpha} A_\xi\}$

Theorem: The ultrapower of V by U is well-founded if and only if U is an ω_1-complete.

Observations:

- If U is ω_1-complete, then we get an elementary embedding

$$j_U : V \to M.$$

(M is the transitive collapse of the ultrapower.)

- If U is κ-complete, then $\mathrm{crit}(j_U) \geq \kappa$.
- If U is normal and uniform, then U is κ-complete.

Proposition: Suppose $j : V \to M$ is an elementary embedding with $\mathrm{crit}(j) = \kappa$. Then

$$U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$$

is a normal uniform ultrafilter. We call U the ultrafilter generated by κ via j.

Iterated ultrapowers

Suppose κ is a cardinal and $U \subseteq P(\kappa)$ is an ultrafilter.

The ultrapower construction with U can be iterated along the ordinals.

Let $V = M_0$ and $j_{01} : M_0 \to M_1$ be the ultrapower of V by U.

- Let $j_{12} : M_1 \to M_2$ be the ultrapower of M_1 by $j_{01}(U)$, which is an ultrafilter on $j_{01}(\kappa)$ in M_1.
- Let $j_{12} \circ j_{01} = j_{0,2} : M_0 \to M_2$.

Inductively, given $j_{\xi\gamma} : M_\xi \to M_\gamma$ for $\xi < \gamma < \delta$, define:

- if $\delta = \gamma + 1$, let $j_{\gamma,\delta} : M_\gamma \to M_\delta$ be the ultrapower of M_γ by $j_{0\gamma}(U)$.
- if δ is a limit, let M_δ be the direct limit of the system of iterated ultrapower embeddings constructed so far.

Theorem: (Gaifman) If U is ω_1-complete, then the iterated ultrapowers M_ξ for $\xi \in \mathrm{Ord}$ are well-founded.

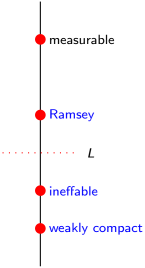
- If M_ξ is well-founded, then $M_{\xi+1}$ is well-founded ($j_{0\xi}(U)$ is ω_1-complete in M_ξ).
- It suffices to see that the countable limit stages M_ξ for $\xi < \omega_1$ are well-founded.

Punier large cardinals

A cardinal κ is weakly compact if every coloring $f : [\kappa]^2 \to 2$ of pairs of elements of κ in two colors has a homogeneous set of size κ.

A cardinal κ is ineffable if every coloring $f : [\kappa]^2 \to 2$ of pairs of elements of κ in two colors has a stationary homogeneous set.

A cardinal κ is Ramsey if every coloring $f : [\kappa]^{<\omega} \to 2$ of finite tuples of elements of κ in two colors has a homogeneous set of size κ.



Weak κ-models

Smaller large cardinals are characterized by the existence of elementary embeddings of small models of (a weak) set theory.

Suppose κ is an inaccessible cardinal.

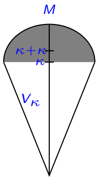
Definition:

- A weak κ-model is a transitive set $M \models \mathrm{ZFC}^-$ of size κ with $V_\kappa \in M$.

  ZFC^- is the theory ZFC without the powerset axiom with the collection scheme instead of the replacement scheme.
- A κ-model M is a weak κ-model such that $M^{<\kappa} \subseteq M$.

  This is the maximum possible closure for a model of size κ.
- A weak κ-model is simple if κ is the largest cardinal of M.

Example: If $M \prec H_{\kappa^+}$ has size κ, then M is a simple weak κ-model. $H_\theta = \{x \mid |\mathrm{TCl}(x)| < \theta\}$



Small ultrafilters

Suppose M is a weak κ-model.

Let $P^M(\kappa) = \{A \subseteq \kappa \mid A \in M\}$. $P^M(\kappa)$ typically won't be an element of M.

Definition: A set $U \subseteq P^M(\kappa)$ is an M-ultrafilter if the structure

$$\langle M, \in, U\rangle \models \text{“}U \text{ is a normal uniform ultrafilter on } \kappa.\text{”}$$

- U is an ultrafilter measuring $P^M(\kappa)$.
- U is closed under diagonal intersections $\Delta_{\xi<\kappa} A_\xi$ for sequences $\{A_\xi \mid \xi < \kappa\} \in M$.
- Typically, $U \notin M$.
- Typically, separation and collection will fail badly in the structure $\langle M, \in, U\rangle$.

  We will see why later on.

Definition: Suppose U is an M-ultrafilter.

- U is α-complete if whenever $\beta < \alpha$ and $A_\xi \in U$ for every $\xi < \beta$, then $\bigcap_{\xi<\beta} A_\xi \neq \emptyset$.
- U is good if the ultrapower of M by U is well-founded.

Small elementary embeddings

Suppose M is a weak κ-model.

Proposition:

- Suppose U is an M-ultrafilter. Then the ultrapower map

$$j_U : M \to N$$

  is an elementary embedding with $\text{crit}(j_U) = \kappa$. ($N$ may not be well-founded)
- Suppose $j : M \to N$ is an elementary embedding with $\text{crit}(j) = \kappa$. (N may not be well-founded)
  Then

$$U = \{A \in M \mid A \subseteq \kappa \text{ and } \kappa \in j(A)\}$$

  is an M-ultrafilter.
  We call U the M-ultrafilter generated by κ via j.
  If N is well-founded, then U is good.

Observations: Suppose U is an M-ultrafilter.

- If U is ω_1-complete, then U is good.

  We will see shortly that the converse fails.
- If M is a κ-model, then U is ω_1-complete.

Iterated small ultrapowers

Suppose M is a weak κ-model, U is an M-ultrafilter, and $j_U : M \to N$ is the ultrapower map.

To iterate the ultrapower construction, we need "$j_U(U)$".

Definition: An M-ultrafilter U is weakly amenable if for every $A \in M$ with $|A|^M \leq \kappa$, $U \cap A \in M$.

- If M is simple, then U is fully amenable.
- $j_U(U) = \{A \subseteq j(\kappa) \mid A = [f] \text{ and } \{\xi < \kappa \mid f(\xi) \in U\} \in U\}$.

Weakly amenable M-ultrafilters U are "partially internal to M".

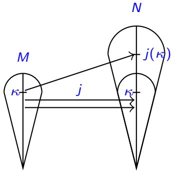
Weakly amenable M-ultrafilters

Suppose M is a weak κ-model and U is an M-ultrafilter.

Definition: An elementary embedding $j : M \to N$ with $\text{crit}(j) = \kappa$ is κ-powerset preserving if $P^M(\kappa) = P^N(\kappa)$. ($H^M_{\kappa^+} = H^N_{\kappa^+}$)

Proposition:

- If U is weakly amenable, then $j_U : M \to N$ is κ-powerset preserving. (N may not be well-founded)
  - If M is simple, then $M = H^N_{\kappa^+}$.



- If $j : M \to N$ is κ-powerset preserving, then U, the M-ultrafilter generated by κ via j, is weakly amenable.

Elementary embedding characterizations of weakly compact cardinals

Theorem: (Folklore) The following are equivalent for an inaccessible cardinal κ.

- κ is weakly compact.
- For every $A \subseteq \kappa$, there is a weak κ-model M, with $A \in M$, for which there is a good M-ultrafilter.
- For every $A \subseteq \kappa$, there is a κ-model M, with $A \in M$, for which there is an M-ultrafilter.
- For every $A \subseteq \kappa$, there is a κ-model $M \prec H_{\kappa^+}$, with $A \in M$, for which there is an M-ultrafilter.
- For every weak κ-model M, there is a good M-ultrafilter.

Question: Can we get weakly amenable M-ultrafilters?

We will see that the more "internal" the M-ultrafilter is to M, the stronger the associated large cardinal. This is the template for smaller large cardinals.

α-iterable cardinals

Suppose M is a weak κ-model.

Definition: An M-ultrafilter U is α-iterable if it is weakly amenable and has α-many well-founded iterated ultrapowers. U is iterable if it is α-iterable for every α.

(A 0-iterable M-ultrafilter may not be good)

Proposition: (Gaifman) If an M-ultrafilter U is ω_1-iterable, then U is iterable.

Theorem: (Kunen) If an M-ultrafilter U is ω_1-complete, then U is iterable.

Definition: (G., Welch) A cardinal κ is α-iterable, for $0 \leq \alpha \leq \omega_1$, if for every $A \subseteq \kappa$ there is a weak κ-model M, with $A \in M$, for which there is an α-iterable M-ultrafilter.

Observation: We can always assume by replacing M with $H^M_{\kappa^+}$ that M is simple.

Theorem:

- (G.) A 0-iterable cardinal κ is a limit of ineffable cardinals.
- (G., Welch) An α-iterable cardinal is a limit of β-iterable cardinals for all $\beta < \alpha$.
- (G., Welch) If $\alpha < \omega_1$, then an α-iterable cardinal is downward absolute to L.
- ω_1-iterable cardinals cannot exist in L.

Stronger consistency proof example

Theorem: A 0-iterable cardinal is a limit of weakly compact cardinals.

Proof: Suppose κ is 0-iterable.

- Fix a weak κ-model M for which there is a weakly amenable M-ultrafilter U.
- Let $j_U : M \to N$ be the ultrapower map by U.
- $H_{\kappa^+}^M = H_{\kappa^+}^N$ (N may not be well-founded).
- Fix a κ-model $\bar{M} \in N$.
- $\bar{M} \in M$.
- $U \cap \bar{M} \in M$ (by weak amenability) is an $\bar{M}$-ultrafilter.
- $U \cap \bar{M} \in N$.
- $N \models$ "κ is weakly compact".
- Given $\alpha < \kappa$,
$$N \models \text{"there is a weakly compact cardinal between } \alpha \text{ and } j(\kappa)\text{"}.$$
- By elementarity,
$$M \models \text{"there is a weakly compact cardinal between } \alpha \text{ and } \kappa\text{"}.$$
- M is correct because $V_\kappa \subseteq M$. □

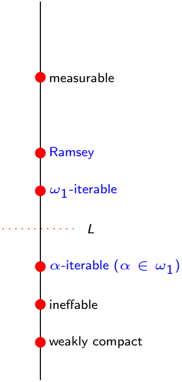
Elementary embedding characterization of Ramsey cardinals

Theorem: (Mitchell) A cardinal κ is Ramsey if and only if for every $A \subseteq \kappa$ there is a weak κ-model M, with $A \in M$, for which there is a weakly amenable ω_1-complete M-ultrafilter.

Corollary: A Ramsey cardinal is ω_1-iterable.

Theorem: (Sharpe, Welch) A Ramsey cardinal is a limit of ω_1-iterable cardinals.

Question: Can we strengthen the Ramsey embedding characterization by replacing weak κ-model with κ-model or κ-model elementary in H_{κ^+}, etc.?



Strongly and super Ramsey cardinals

Definition (G.):

- A cardinal κ is strongly Ramsey if for every $A \subseteq \kappa$ there is a κ-model M, with $A \in M$, for which there is a weakly amenable M-ultrafilter.
- A cardinal κ is super Ramsey if for every $A \subseteq \kappa$ there is a κ-model $M \prec H_{\kappa^+}$, with $A \in M$, for which there is a weakly amenable M-ultrafilter.

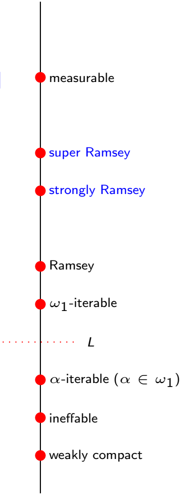
Theorem: (G.)

- A measurable cardinal is a limit of super Ramsey cardinals.
- A super Ramsey cardinal is a limit of strongly Ramsey cardinals.
- A strongly Ramsey cardinal is a limit of Ramsey cardinals.
- It is inconsistent for every κ-model to have a weakly amenable M-ultrafilter.

Question: Can we stratify by closure on the weak κ-model M?

Assuming $M^\omega \subseteq M$ in the characterization of Ramsey cardinals already pushes strength beyond Ramsey.

Question: Can we have elementary embeddings of models elementary in some large H_θ?



Switching to non-transitive models

Observation: If $M \prec H_\theta$ for $\theta > \kappa^+$ such that $\kappa \in M$ and $|M| = \kappa$, then M is not transitive. ($\kappa^+ \in M$, but $\kappa^+ \not\subseteq M$)

Definition:

- A basic weak κ-model is a set $M \models \mathrm{ZFC}^-$ of size κ such that:
  - $M \prec_{\Sigma_0} V$,
  - $V_\kappa \cup \{V_\kappa\} \subseteq M$.
- A basic κ-model is a basic weak κ-model M such that $M^{<\kappa} \subseteq M$.

α-Ramsey cardinals

Definition: (Holy, Schlicht) A cardinal κ is:

- α-Ramsey for a regular cardinal $\omega_1 \leq \alpha \leq \kappa$, if for every $A \subseteq \kappa$ and arbitrarily large regular θ, there is a basic weak κ-model $M \prec H_\theta$, with $A \in M$ and $M^{<\alpha} \subseteq M$, for which there is a weakly amenable M-ultrafilter.
- $<\alpha$-Ramsey if it is β-Ramsey for every $\beta < \alpha$.

Theorem: (Holy, Schlicht)

- A measurable cardinal is a limit of κ-Ramsey cardinals κ.
- A κ-Ramsey cardinal κ is a limit of super Ramsey cardinals.
- (G.) A strongly Ramsey cardinal is a limit of cardinals α that are $<\alpha$-Ramsey.
- If $\omega_1 \leq \beta < \alpha$, then an α-Ramsey cardinal κ is a limit of β-Ramsey cardinals $\bar{\kappa} > \beta$. (e.g. An ω_2-Ramsey cardinal is a limit of ω_1-Ramsey cardinals.)
- An ω_1-Ramsey cardinal is a limit of Ramsey cardinals.

- measurable
- κ-Ramsey
- super Ramsey
- strongly Ramsey
- $<\alpha$-Ramsey
- Ramsey
- ω_1-iterable
- L
- α-iterable
- ineffable
- weakly compact

Games with κ-models and small ultrafilters

Definition: (Holy, Schlicht) Fix regular $\omega_1 \leq \alpha \leq \kappa$ and regular $\theta > \kappa$. The game Ramsey$G^\theta_\alpha(\kappa)$ is played by the challenger and the judge for at most α-many steps.

- the challenger plays basic κ-models $M_\xi \prec H_\theta$
- the judge responds with M_ξ-ultrafilters U_ξ

c	M_0		M_1		M_2		$\ldots$		M_ξ		$\ldots$
j		U_0		U_1		U_2		$\cdots$		U_ξ	$\cdots$

- $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\xi \subseteq \cdots$
- $U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_\xi \subseteq \cdots$
- $\{\langle M_{\bar\xi}, \in, U_{\bar\xi}\rangle \mid \bar\xi < \xi\} \in M_\xi$

The judge wins if she can play for α-many steps and otherwise the challenger wins.

Observations: Suppose the judge wins a run of the game Ramsey$G^\theta_\alpha(\kappa)$.

- $M = \bigcup_{\xi<\alpha} M_\xi$ is closed under $<\alpha$-sequences.
- $U = \bigcup_{\xi<\alpha} U_\xi$ is a weakly amenable M-ultrafilter.

Definition: The game Ramsey$\bar{G}^\theta_\alpha(\kappa)$ is played like Ramsey$G^\theta_\alpha(\kappa)$, but now the judge plays structures $\langle N_\xi, \in, U_\xi\rangle$ such that N_ξ is a κ-model with $P^{M_\xi}(\kappa) \subseteq N_\xi$ and U_ξ is an N_ξ-ultrafilter.

Games and α-Ramsey cardinals

Theorem: (Holy, Schlicht) The existence of a winning strategy for either player in the games Ramsey$G_\alpha^\theta(\kappa)$ or Ramsey$\bar{G}_\alpha^\theta(\kappa)$ is independent of θ.

Theorem: (Holy, Schlicht) The following are equivalent.

- κ is α-Ramsey.
- The challenger doesn't have a winning strategy in the game Ramsey$G_\alpha^\theta(\kappa)$ for some/all θ.
- The challenger doesn't have a winning strategy in the game Ramsey$\bar{G}_\alpha^\theta(\kappa)$ for some/all θ.

Question: Can we formulate a natural large cardinal hierarchy between κ-Ramsey and measurable cardinals?

By making the M-ultrafilter U more and more "internal" to the weak κ-model M.

Fragments of ZFC^-

ZFC_n^-

- separation for Σ_n-formulas
- collection for Σ_n-formulas

KP_n

- separation for Δ_0-formulas
- collection for Σ_n-formulas

Theorem: (Folklore) KP_n proves:

- Δ_n-separation
- Σ_n-replacement
- Σ_n-recursion

Observations:

- $\mathrm{KP} = \mathrm{KP}_1 = \mathrm{KP}_0 = \mathrm{ZFC}_0^-$
- $\mathrm{KP}_{n+1} \to \mathrm{ZFC}_n \to \mathrm{KP}_n$

Weak amenability and Δ_0-separation in $\langle M, \in, U\rangle$

Suppose M is a simple weak κ-model and U is an M-ultrafilter.

Proposition: The structure $\langle M, \in, U\rangle \models \Delta_0$-separation if and only if U is weakly amenable.

Proof:

Suppose $\langle M, \in, U\rangle \models \Delta_0$-separation.

- Fix $A \in M$.
- $U \cap A = \{a \in A \mid a \in U\}$.
- Use separation on the formula $x \in U$ for the set A.

Suppose U is weakly amenable.

- Fix $A \in M$ and a Δ_0-formula $\varphi(x, b)$ (in the language with a predicate for U).
- Need to verify $\{x \in A \mid M \models \varphi(x, b)\} \in M$.
- Every quantifier in $\varphi(x, b)$ is bounded by x or b.
- Let $B \in M$ be the transitive closure of $A \cup b$.
- For $x \in A$, to evaluate $\varphi(x, b)$, it suffices to have $U \cap B$. $\square$

Question: Is weak amenability of U equivalent to $\langle M, \in, U\rangle \models \mathrm{KP}_0$?

Baby measurable cardinals (joint with Philipp Schlicht)

The following generalizes notions defined earlier by Bovykin and McKenzie.

Definition: A cardinal κ is:

1. faintly n-baby measurable if for every $A \subseteq \kappa$, there is a weak κ-model M, with $A \in M$, for which there is an M-ultrafilter such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_n$.

   (analogue: 0-iterable cardinal)
2. weakly n-baby measurable if (1) holds and U is good. (analogue: 1-iterable cardinal)
3. n-baby measurable if (2) holds and M is a κ-model. (analogue: strongly Ramsey cardinal)
4. $[n]$-baby measurable if (3) holds, but with KP_n instead of ZFC^-_n.
5. (faintly, weakly) baby measurable if ((1),(2)) (3) holds, but with ZFC^- instead of ZFC^-_n.

We can always assume by replacing M with $H^M_{\kappa^+}$ that M is simple.

Definition: A cardinal κ is:

- (α, n)-baby measurable for a regular cardinal $\omega_1 \leq \alpha \leq \kappa$, if for every $A \subseteq \kappa$ and arbitrarily large regular θ, there is a basic weak κ-model $M \prec H_\theta$, with $A \in M$ and $M^{<\alpha} \subseteq M$, for which there is an M-ultrafilter U such that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-_n$.

  (analogue: α-Ramsey cardinal)
- $({<}\alpha, n)$-baby measurable if it is (β, n)-baby measurable for every $\beta < \alpha$.
- α-baby measurable if we replace ZFC^-_n with ZFC^-.

Baby measurable games

Definition: Fix regular $\omega_1 \leq \alpha \leq \kappa$ and regular $\theta > \kappa$. The game $G_\alpha^{\theta,n}(\kappa)$ is played by the challenger and the judge for at most α-many steps.

- the challenger plays basic κ-models $M_\xi \prec H_\theta$
- the judge responds with structures $\langle N_\xi, \in, U_\xi\rangle$ such that N_ξ is a κ-model with $P^{M_\xi}(\kappa) \subseteq N_\xi$ and U_ξ is an N_ξ-ultrafilter.
- $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\xi \subseteq \cdots$
- $U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_\xi \subseteq \cdots$
- $\{\langle M_{\bar{\xi}}, \in, U_{\bar{\xi}}\rangle \mid \bar{\xi} < \xi\} \in M_\xi$

The judge wins if she can play for α-many steps with

- $M = \bigcup_{\xi<\alpha} M_\xi$,
- $U = \bigcup_{\xi<\alpha} U_\xi$,

and $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-_n$. $H^M_{\kappa^+} = \bigcup_{\xi<\alpha} N_\xi$
Otherwise the challenger wins.

The game $G_\alpha^\theta(\kappa)$ is played analogously, but for the judge to win, she needs $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$.

Games and (α, n)-measurable cardinals

Theorem: The existence of a winning strategy for either player in the games $G_\alpha^{\theta,n}(\kappa)$ or $G_\alpha^\theta(\kappa)$ is independent of θ.

Theorem: The following are equivalent.

- κ is (α, n)-baby measurable.
- The challenger doesn't have a winning strategy in the game $G_\alpha^{\theta,n}(\kappa)$ for some/all θ.

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- κ is α-baby measurable.
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Constructing a better model in $\langle M, \in, U\rangle$

Suppose M is a simple weak κ-model and U is an M-ultrafilter.

Lemma: If $\langle M, \in, U\rangle \models \mathrm{KP}_n(\mathrm{ZFC}_n^-)$ and $\alpha = \mathrm{Ord}^M$, then for any $A \in M$:

- $L_\alpha[A, U] \subseteq M$ is a weak κ-model,
- $A \in L_\alpha[A, U]$,
- $\langle L_\alpha[A, U], \in, U \cap L_\alpha[A, U]\rangle \models \mathrm{KP}_n(\mathrm{ZFC}_n^-)$.

If $\beta = (\kappa^+)^{L_\alpha[A,U]}$, then:

- $L_\beta[A, U]$ is a simple weak κ-model,
- $A \in L_\beta[A, U]$,
- $\langle L_\beta[A, U], \in, U \cap L_\beta[A, U]\rangle \models \mathrm{KP}_n(\mathrm{ZFC}_n^-)$.

Lemma: The structure $\langle L_\beta[A, U], \in, U \cap L_\beta[A, U]\rangle \models \mathrm{KP}_n$ has:

- Δ_1-definable global well-order
- Σ_{n+1}-definable Skolem functions for Σ_{n+1}-formulas
- $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec_{\Sigma_n} \langle L_\beta[A, U], \in, U \cap L_\beta[A, U]\rangle$ such that $\bar{M}$ is a κ-model.
  - $\bar{M} = L_{\bar{\beta}}[A, U]$ with $\bar{\beta} \leq \beta$
  - $\bar{M} \in L_\beta[A, U]$ or $\bar{M} = L_\beta[A, U]$

The $\langle M, \in, U\rangle$ toolbox

Suppose M is a weak κ-model and U is an M-ultrafilter.

Proposition: If M thinks that $\bar{M} \in M$ is a κ-model, then $\bar{M}$ is a κ-model. (Use $V_\kappa \in M$.)

Lemma: Suppose $\langle M, \in, U\rangle \models \mathrm{KP}_n$. If $\bar{M} \in M$ is transitive and $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle$, then $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \mathrm{KP}_n$. (This can fail for ZFC_n^-.)

Corollary: If $\langle M, \in, U\rangle \models \mathrm{KP}_n$, then there is a κ-model $\bar{M} \subseteq M$ such that $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \mathrm{KP}_n$.

Lemma: If $\langle M, \in, U\rangle \models \mathrm{KP}_{n+1}$ $(n \geq 1)$ and has a Δ_1-definable global well-order, then for every $A \in M$, there is a κ-model $\bar{M} \in M$, with $A \in \bar{M}$, such that:

- $\bar{M} \prec M$
- $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle$
- $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \mathrm{ZFC}_n^-$.

More tools

Suppose M is a weak κ-model and U is an M-ultrafilter.

Σ_n-reflection: For every Σ_n-formula $\varphi(x, b)$, there is a transitive set B, with $b \in B$, such that $B \models \varphi(x, b)$ if and only if $\varphi(x, b)$ holds.

(e.g. If for every $b \in M$, there is $\bar{M} \in M$, with $b \in \bar{M}$ such that $\bar{M} \prec_{\Sigma_n} M$.)

Lemma: If $\langle M, \in, U\rangle$ satisfies Σ_n-reflection ($n \geq 1$), then $\langle M, \in, U\rangle \models \mathrm{ZFC}_n^-$.

Corollary: If there are models $M_i \in M$ for $i < \omega$ such that

$$\langle M_0, \in, U \cap M_0\rangle \prec_{\Sigma_n} \langle M_1, \in, U \cap M_1\rangle \prec_{\Sigma_n} \cdots \prec_{\Sigma_n} \langle M_i, \in, U \cap M_i\rangle \prec_{\Sigma_n} \cdots \prec_{\Sigma_n} \langle M, \in, U\rangle,$$

then $\langle M, \in, U\rangle \models \mathrm{ZFC}_n^-$.

Lemma: If $\langle M, \in, U\rangle \models \mathrm{ZFC}_n^-$ ($n \geq 1$), then for every $A \in M$, there is a weak κ-model $\bar{M} \in M$, with $A \in \bar{M}$, such that $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \mathrm{KP}_n$.

The smallest baby measurable cardinal

Observation: The following large cardinals are equivalent.

- [0]-baby measurable
- [1]-baby measurable
- faintly 0-baby measurable
- weakly 0-baby measurable
- 0-baby measurable ($\mathrm{KP} = \mathrm{KP}_1 = \mathrm{KP}_0 = \mathrm{ZFC}_0^-$)

Theorem: A faintly 0-baby measurable cardinal is a limit of strongly Ramsey cardinals.

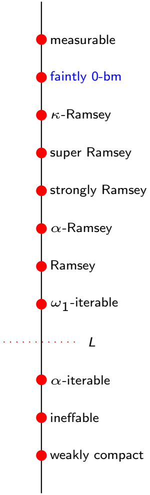
Proof idea: Inside $\langle M, \in, U\rangle \models \mathrm{KP}_1$, use Σ_1-recursion to construct

$$M_0 \prec M_1 \prec \cdots \prec M_\xi \prec \cdots \prec M$$

for $\xi < \kappa$ such that M_ξ is a κ-model and $U \cap M_\xi \in M_{\xi+1}$.

- $M_\kappa = \bigcup_{\xi<\kappa} M_\xi$
- $U \cap M_\kappa$ is weakly amenable
- κ is strongly Ramsey in M, and hence in N ($H^M_{\kappa^+} = H^N_{\kappa^+}$).

Theorem: If κ is faintly 0-baby measurable, then there is a model in which κ is κ-Ramsey.



The baby measurable cardinal hierarchy for $n \geq 1$

Theorem: A faintly n-baby measurable cardinal is a limit of $[n]$-baby measurable cardinals.

Theorem: A weakly n-baby measurable cardinal is a limit of faintly n-baby measurable cardinals.

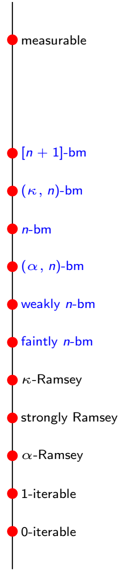
Theorem: A n-baby measurable cardinal is a limit of weakly n-baby measurable cardinals.

Theorem: An (κ, n)-baby measurable cardinal κ is a limit of n-baby measurable cardinals.

Theorem: If $\omega_1 \leq \beta < \alpha$, then an (α, n)-baby measurable cardinal κ is a limit of (β, n)-baby measurable cardinals $\bar{\kappa} > \beta$.

Theorem: An n-baby measurable cardinal is a limit of cardinals α that are $({<}\alpha, n)$-baby measurable.

Theorem: A faintly $[n+1]$-baby measurable cardinal is a limit of (α, n)-baby measurable cardinals α.



The baby measurable hierarchy sine n

Theorem: A faintly baby measurable cardinal is a limit of (α, n)-baby measurable cardinal α for every $n < \omega$.

Theorem: A weakly baby measurable cardinal is a limit of faintly baby measurable cardinals.

Theorem: A baby measurable cardinal is a limit of weakly baby measurable cardinals.

Theorem: A κ-baby measurable cardinal κ is a limit of baby measurable cardinals.

Theorem: If $\omega_1 \leq \beta < \alpha$, then an α-baby measurable cardinal κ is a limit of β-baby measurable cardinals $\bar{\kappa} > \beta$.

Theorem: A baby measurable cardinal is a limit of cardinals α that are $<\alpha$-baby measurable.