

1. NATURALIST APPROACH

Suppose that $\mathcal{V} = \langle V, S \rangle$ is a β -model of KM^+ . Let $M_{\mathcal{V}}$ be the model of ZFC^- with the largest inaccessible cardinal κ that is constructed in \mathcal{V} such that $V_{\kappa}^{M_{\mathcal{V}}} = V$ and the subsets of $V_{\kappa}^{M_{\mathcal{V}}}$ are the classes of V . Since \mathcal{V} is a β -model, $M_{\mathcal{V}}$ is a transitive model of ZFC^- . Let \mathbb{P} be a tame class forcing in \mathcal{V} , so that forcing with \mathbb{P} preserves KM^+ (see Observations section). We would like to argue that the forcing extension $\mathcal{V}[G] = \langle V[G], S[G] \rangle$ is precisely the KM^+ -model we get back from $M_{\mathcal{V}}[G]$. First, let's argue that the classes of $S[G]$ are precisely $V_{\kappa+1}^{M_{\mathcal{V}}[G]}$. Suppose $A \in S[G]$, then $A = \dot{A}_G$ for a class name $\dot{A} \in S$. But then $\dot{A} \in M_{\mathcal{V}}$ and so $\dot{A}_G \in V_{\kappa+1}^{M_{\mathcal{V}}[G]}$. Next, suppose $A \in V_{\kappa+1}^{M_{\mathcal{V}}[G]}$. Then A has a nice name \dot{A} for a subset of V_{κ} . But then $\dot{A}_G \in S[G]$. Now suppose $B \in M_{\mathcal{V}}[G]$ is any element. Then B is coded by a subset of κ , which must be in $S[G]$ by a previous argument and so B will be in the ZFC^- -model constructed from $\mathcal{V}[G]$.

The argument should generalize to even non-transitive models of KM^+ . For such models, the forcing extension $\mathcal{V}[G]$ consists of equivalence classes of names $[\sigma]_G$ where $\sigma \sim \tau$ whenever there is $p \in G$ such that $p \Vdash \sigma = \tau$. Since whether a condition p forces an atomic statement about names σ and τ depends only on names of lower rank, it follows that the forcing relation for atomic formulas up to $V_{\kappa+1}$ in $M_{\mathcal{V}}$ matches the forcing relation in \mathcal{V} . This gives $S[G] = V_{\kappa+1}^{M_{\mathcal{V}}[G]}$.

So it does not make a difference whether we consider a forcing extension of \mathcal{V} or a forcing extension of $M_{\mathcal{V}}$.

If \mathbb{P} is a set forcing, then it can be densely embedded into a complete Boolean algebra.

Theorem 1.1 ([HKS]). *Suppose $\langle M, S \rangle \models \text{GBC}$ and $\mathbb{B} \in S$ is a complete Boolean algebra. Then \mathbb{B} has the ORD-cc.*

So unless the class forcing \mathbb{P} has the ORD-cc, there is no hope that it can be densely embedded into a complete Boolean algebra in a model of second-order set theory.

Theorem 1.2 ([HKS]). *If \mathbb{P} has the ORD-cc, then it can be densely embedded into a complete Boolean algebra.*

So the condition is in fact both necessary and sufficient.

Suppose \mathcal{V} is a model of GBC . Let's argue that every class partial order \mathbb{P} densely embeds into a definable hyperclass Boolean algebra \mathbb{B} which is complete for all classes. Let \mathbb{B} consist of those classes which are regular open cuts of \mathbb{P} . Usual arguments show that \mathbb{B} is a Boolean algebra and \mathbb{P} densely embeds into \mathbb{B} . Given any class $A \subseteq \mathbb{B}$, \mathbb{B} has $\bigwedge A$.

Given the much stronger second-order theory KM^+ , we can argue that every antichain of \mathbb{B} is coded in \mathcal{V} and that \mathbb{B} is complete for definable sub-collections. The comprehension of KM immediately gives the extra completeness. So let's argue that every definable antichain is coded (this will use class choice). Let $\varphi(x, C)$ an antichain \mathcal{A} of \mathbb{B} . Let $X = \{p \in \mathbb{P} \mid \exists P \in \mathcal{A}, p \leq P\}$ (which exists by comprehension) and let $A \subseteq X$ be a maximal antichain of X . Now for every $p \in A$, choose some $P \in \mathcal{A}$ with $p \leq P$ (using class choice). The resulting coded collection $\dot{\mathcal{A}}$ clearly has the property that every element of \mathcal{A} is compatible to something in it. So $\dot{\mathcal{A}} = \mathcal{A}$.

Question 1.3. Do we have counterexamples for weaker theories?

So let's work in a (β) -model $\mathcal{V} = \langle V, S \rangle$ of KM^+ (that is KM together with class choice). Let $M_{\mathcal{V}}$ be the model of ZFC^- with the largest inaccessible cardinal κ with which \mathcal{V} is bi-interpretable. Let \mathbb{P} be a class forcing notion in \mathcal{V} and let \mathbb{B} the hyperclass Boolean algebra described above. In $M_{\mathcal{V}}$, \mathbb{P} is a set of size κ and \mathbb{B} is a definable class Boolean algebra which is set-complete and has the ORD-cc.

Now we work in $M_{\mathcal{V}}$ with the class forcing notion \mathbb{B} . We define the class of $M_{\mathcal{V}}^{\mathbb{B}}$ of \mathbb{B} -names as usual. Now let's define the Boolean values of atomic formulas by recursion.

$$\begin{aligned} [[\tau \in \sigma]] &= \bigvee_{\langle \nu, b \rangle \in \sigma} [[\nu = \tau]] \wedge b \\ [[\tau = \sigma]] &= [[\tau \subseteq \sigma]] \wedge [[\sigma \subseteq \tau]] \\ [[\tau \subseteq \sigma]] &= \bigwedge_{\nu \in \text{dom}(\tau)} [[\nu \in \tau]] \rightarrow [[\nu \in \sigma]] \end{aligned}$$

Note that this is a first-order recursion since it is set-like. Next, using completeness, we can define the Boolean values $[[\varphi(\sigma)]]$ for any first-order formula φ . So as in the case of set forcing, we get a Boolean-valued model. We would also like to ensure that the Boolean valued model is full, meaning that $[[\exists x \varphi(x, \sigma)]] = [[\varphi(\tau, \sigma)]]$ for some name τ for all existential assertions. This is usually proved by first proving the mixing lemma.

Lemma 1.4 (Mixing lemma). *If $A \subseteq \mathbb{B}$ is an antichain and $\langle \tau_a \mid a \in A \rangle$ is any sequence of names indexed by A , then there is a name τ such that $a \leq [[\tau = \tau_a]]$ for each $a \in A$.*

Proof. Let $\tau = \{ \langle \sigma, b \wedge a \rangle \mid \langle \sigma, b \rangle \in \tau_a \text{ and } a \in A \}$. □

If $[[\tau_a \neq \tau_b]] = 1$ for $a \neq b$ in A , then $a = [[\tau = \tau_a]]$.

Lemma 1.5. *The model $M_{\mathcal{V}}^{\mathbb{B}}$ is full.*

Proof. Let $b = [[\exists x \varphi(x, \sigma)]] = \bigvee_{\tau \in M_{\mathcal{V}}^{\mathbb{B}}} [[\varphi(\tau, \sigma)]]$. Let $D = \{ p \in \mathbb{P} \mid \exists \tau p \leq [[\varphi(\tau, \sigma)]] \}$. Observe that D is open dense below b . So let A be a maximal antichain of D . It is easy to see that $\bigvee A = b$. Now for each $a \in A$, we can choose some τ_a such that $a = [[\varphi(\tau_a, \sigma)]]$. The mixed name now works. □

The next theorem says that \mathbb{B} and \mathbb{P} have the same forcing extensions of $M_{\mathcal{V}}$.

Theorem 1.6 (Theorem 5.3 in [HKS]). *\mathbb{P} and \mathbb{B} have the same forcing extensions.*

Proof. Clearly if $G \subseteq \mathbb{P}$ is M -generic, then the upward closure of G is M -generic for \mathbb{B} . Now suppose that G is M -generic for \mathbb{B} . We know that $G \cap \mathbb{P}$ is M -generic for \mathbb{P} . So we need to argue that $M[G] = M[G \cap \mathbb{P}]$. Fix a \mathbb{B} -name σ . We will argue that $\sigma_G \in M[G \cap \mathbb{P}]$. Let $\text{tc}(\sigma) = \bigcup \{ \{ p \} \cup \text{tc}(\tau) \mid \langle \tau, p \rangle \in \sigma \}$. For every $q \in \text{tc}(\sigma) \cap \mathbb{B}$, let $D_q = \{ p \in \mathbb{P} \mid p \leq q \vee p \perp q \}$, and observe that D_q is dense. Now we define inductively for every name $\tau \in \text{tc}(\sigma)$, the name

$$\bar{\tau} = \{ \langle \bar{\mu}, r \rangle \mid \exists s [\langle \mu, s \rangle \in \tau \wedge r \in D_q \wedge r \leq s] \}.$$

But then $\bar{\sigma}$ is a \mathbb{P} -name and $\bar{\sigma}_{G \cap \mathbb{P}} = \sigma_G$. □

Next, we want to turn a Boolean valued model into a two-valued model. But for this we need a definable ultrafilter on \mathbb{B} , which there is no reason to suppose exists. We can solve this problem by strengthening the theory further. We will force to add a global well-order and then force again to make it definable from a subset of κ as in [AF17]. The resulting model has the form $L_\alpha[A]$ for some $A \subseteq \kappa$ and the same V_κ as the original model, but obviously it has new subsets of κ and so in the corresponding KM^+ -model we have added classes. Now we obviously have definable ultrafilters for \mathbb{B} and so we have a definable model W which is, by usual arguments, elementarily equivalent to $M_{\mathcal{V}}$, indeed it has a submodel $\bar{W} = \bar{M}$ into which $M_{\mathcal{V}}$ elementarily embeds and W is a forcing extension of \bar{W} by a W -generic filter G .

2. INTERMEDIATE MODEL THEOREM

Theorem 2.1 ([Fri99], recent results of Hamkins and Reitz). *Vopenka's intermediate model theorem can fail for class forcing. If V is a model of ZFC and $V[G]$ is an extension by class forcing and $V \subseteq N \subseteq V[G]$ as a ZFC-model, then N need not be a forcing extension of V . Moreover the theorem can even fail for ORD-cc forcing for which a Boolean completion exists.*

Theorem 2.2. *Vopenka's intermediate theorem can fail for models of KM^+ . There is a model \mathcal{V} of KM and a class forcing in \mathcal{V} with an intermediate model $\mathcal{V} \subseteq \mathcal{N} \subseteq \mathcal{V}[G]$, where $\mathcal{N} \models \text{KM}$, but \mathcal{N} is not a forcing extension of \mathcal{V} by any class forcing in \mathcal{V} .*

Proof. Using a construction of Gitman and Hamkins from [GHJ], there is a model $V \models \text{ZFC}$ with an inaccessible cardinal κ and ω -many κ -Souslin trees with special properties. The special properties give that the forcing extension $V[G]$ by the full-support ω -length product of the κ -Suslin trees has a symmetric submodel N such that $\langle V_\kappa^N, V_{\kappa+1}^N \rangle$ is a model of KM , but not KM^+ . The forcing, the product of the κ -Souslin trees is obviously a class forcing of $\langle V_\kappa, V_{\kappa+1} \rangle$, so $\langle V_\kappa^N, V_{\kappa+1}^N \rangle$ is an intermediate model between $\langle V_\kappa, V_{\kappa+1} \rangle$ and its forcing extension $\langle V_\kappa[G], V_{\kappa+1}[G] \rangle$. But $\langle V_\kappa^N, V_{\kappa+1}^N \rangle$ cannot be a forcing extension because forcing preserves KM^+ . \square

Theorem 2.3. *Suppose M is a model of ZFC^- with the largest cardinal κ and \mathbb{B} is a definable class of M which is a complete ORD-cc Boolean algebra. Suppose $N \models \text{ZFC}^-$ is an intermediate model between M and $M[G]$ such that*

- N is definable in $M[G]$,
- M is definable in N ,
- N has a definable global well-order.

Then $N = M[\mathbb{D} \cap G]$ for some definable complete subalgebra \mathbb{D} of M .

Proof. Suppose $X \subseteq \mathbb{B}$ is a class. Let us say that a set well-founded tree T of elements of \mathbb{B} is an X -tree if the leaves of T are elements of X , if an element $b \in T$ has a single successor, then it is $-b$, if it has multiple successors, then it is the join of them. Let $b \in \mathbb{B}_X$ if there is an X -tree with b as the root. It is easy to see that \mathbb{B}_X is a complete subalgebra of \mathbb{B} . We will say that \mathbb{B}_X is *generated* by X .

Let A be a class of ordinals coding all subsets of \mathbb{B} in N . Each subset of \mathbb{B} in N can be coded by a set of ordinals and they can all be coded together using global choice in N . Note this uses the definability of M in N . Since A is also definable in $M[G]$ (by definability of N in $M[G]$, we can choose a class \mathbb{B} -name \dot{A} for A .

Let $X = \{[\check{\alpha} \in \dot{A}] \mid \alpha \in \text{ORD}^M\}$, which is a class in M . Let \mathbb{D} be a complete subalgebra of \mathbb{B} generated by X in M . We will argue that $N = M[\mathbb{D} \cap G]$.

First, let's argue that $M[\mathbb{D} \cap G] \subseteq N$. Observe that $X \cap G = \{[\alpha \in \dot{A}] \mid \alpha \in A\}$ and that $\mathbb{D} \cap G$ is definable from $X \cap G$ in N (this uses M is definable in N because we need to recognize the X -trees from M). Also, A is definable from $X \cap G$.

Now suppose that $a \in N$. We can assume without loss that a is a set of ordinals. Let \dot{a} be some name for a such that $\mathbb{1}_{\mathbb{B}} \Vdash \dot{a} \subseteq \beta$ for a fixed ordinal β . Let $x = \{[z \in \dot{a}] \mid z \in \beta\}$. We have $x \cap G = \{[z \in \dot{a}] \mid z \in a\}$ is coded in A . So $x \cap G$ is coded in A . But then a is definable from $x \cap G$ and \dot{a} . \square

Note that if $N[\mathbb{D} \cap G]$ and M is definable in N , then N is definable in $M[G]$.

Question 2.4. Can we improve this result?

3. OBSERVATIONS

Lemma 3.1. *KM⁺ is preserved by tame forcing.*

Proof. Let $V[G]$ be a forcing extension by a tame class forcing \mathbb{P} . Suppose $V[G]$ satisfies $\forall \alpha \exists X \varphi(\alpha, X, A)$. So there is $p \in \mathbb{P}$ such that $p \Vdash \forall \alpha \exists X \varphi(\alpha, X, \dot{A})$ where \dot{A} is a \mathbb{P} -name for A . Fix an ordinal α . Let's argue that there is a class \mathbb{P} -name \dot{X}_α such that $p \Vdash \varphi(\alpha, \dot{X}_\alpha, \dot{A})$. Let \mathcal{D} be the dense class of conditions q below p for which there is a class \mathbb{P} -name \dot{X}_q such that $q \Vdash \varphi(\alpha, \dot{X}_q, \dot{A})$. The class \mathcal{D} exists by comprehension. Let \mathcal{A} be a maximal antichain of \mathcal{D} . We construct \mathcal{A} as follows. Let \mathcal{W} be a well-order of \mathcal{D} . Call $f : \beta \rightarrow \mathcal{D}$ be a good map if $f(0)$ is the least element of \mathcal{W} , and given $\gamma < \beta$, $f(\gamma)$ is the least element of \mathcal{W} incompatible with all $f(\xi)$ for $\xi < \gamma$. Clearly for every ordinal β , there is a good map f and it is unique. So we can union up all the good maps f and the range of the union is a maximal antichain of \mathcal{D} . Now, using KM^+ , we can pick for every $q \in \mathcal{D}$, a class \mathbb{P} -name \dot{X}_q such that $q \Vdash \varphi(\alpha, \dot{X}_q, \dot{A})$. After this, we do the usual mixing argument to build the name \dot{X}_α . Finally, again using KM^+ , we pick for every α , a class \mathbb{P} -name \dot{X}_α and put them all together to form a name for the sequence of choices in $V[G]$. \square

Lemma 3.2. *KM⁺⁺ is preserved by tame forcing.*

Proof. Let $V[G]$ be a forcing extension by a tame class forcing \mathbb{P} . Suppose $V[G]$ satisfies $\forall X \exists Y \varphi(X, Y, A)$. So there is a $p \in \mathbb{P}$ such that $p \Vdash \forall X \exists Y \varphi(X, Y, \dot{A})$, where \dot{A} is a \mathbb{P} -name for A . Let $\dot{X}_0 = \dot{V}$. Define a relation \mathcal{R} on classes of V such that $X \mathcal{R} Y$ if X is not a \mathbb{P} -name and Y is a \mathbb{P} -name or X is a \mathbb{P} -name and $p \Vdash \varphi(X, Y, \dot{A})$. Let's argue that \mathcal{R} has no terminal nodes. Fix a class X . If X is not a \mathbb{P} -name, we are done. Otherwise, by the same process as above, we can build a witnessing \mathbb{P} -name Y such that $p \Vdash \varphi(X, Y, \dot{A})$. Thus, by KM^{++} , there is an ω -sequence of witnessing names for classes. \square

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