1. Naturalist approach

Suppose that $\mathcal{V} = \langle V, S\rangle$ is a β-model of KM^+. Let $M_{\mathcal{V}}$ be the model of ZFC^- with the largest inaccessible cardinal κ that is constructed in $\mathcal{V}$ such that $V_\kappa^{M_{\mathcal{V}}} = V$ and the subsets of $V_\kappa^{M_{\mathcal{V}}}$ are the classes of V. Since $\mathcal{V}$ is a β-model, $M_{\mathcal{V}}$ is a transitive model of ZFC^-. Let $\mathbb{P}$ be a tame class forcing in $\mathcal{V}$, so that forcing with $\mathbb{P}$ preserves KM^+ (see Observations section). We would like to argue that the forcing extension $\mathcal{V}[G] = \langle V[G], S[G]\rangle$ is precisely the KM^+-model we get back from $M_{\mathcal{V}}[G]$. First, let's argue that the classes of $S[G]$ are precisely $V_{\kappa+1}^{M_{\mathcal{V}}[G]}$. Suppose $A \in S[G]$, then $A = \dot{A}_G$ for a class name $\dot{A} \in S$. But then $\dot{A} \in M_{\mathcal{V}}$ and so $\dot{A}_G \in V_{\kappa+1}^{M_{\mathcal{V}}[G]}$. Next, suppose $A \in V_{\kappa+1}^{M_{\mathcal{V}}[G]}$. Then A has a nice name $\dot{A}$ for a subset of V_κ. But then $\dot{A}_G \in S[G]$. Now suppose $B \in M_{\mathcal{V}}[G]$ is any element. Then B is a coded by a subset of κ, which must be in $S[G]$ by a previous argument and so B will be in the ZFC^--model constructed from $\mathcal{V}[G]$.

The argument should generalize to even non-transitive models of KM^+. For such models, the forcing extension $\mathcal{V}[G]$ consists of equivalence classes of names $[\sigma]_G$ where $\sigma \sim \tau$ whenever there is $p \in G$ such that $p \Vdash \sigma = \tau$. Since whether a condition p forces an atomic statement about names σ and τ depends only on names of lower rank, it follows that the forcing relation for atomic formulas up to $V_{\kappa+1}$ in $M_{\mathcal{V}}$ matches the forcing relation in $\mathcal{V}$. This gives $S[G] = V_{\kappa+1}^{M_{\mathcal{V}}[G]}$.

So it does not make a difference whether we consider a forcing extension of $\mathcal{V}$ or a forcing extension of $M_{\mathcal{V}}$.

If $\mathbb{P}$ is a set forcing, then it can be densely embedded into a complete Boolean algebra.

Theorem 1.1 ([HKS]). *Suppose $\langle M, S\rangle \models$ GBC and $\mathbb{B} \in S$ is a complete Boolean algebra. Then $\mathbb{B}$ has the* ORD*-cc.*

So unless the class forcing $\mathbb{P}$ has the ORD-cc, there is no hope that it can be densely embedded into a complete Boolean algebra in a model of second-order set theory.

Theorem 1.2 ([HKS]). *If $\mathbb{P}$ has the* ORD*-cc, then it can be densely embedded into a complete Boolean algebra.*

So the condition is in fact both necessary and sufficient.

Suppose $\mathcal{V}$ is a model of GBC. Let's argue that every class partial order $\mathbb{P}$ densely embeds into a definable hyperclass Boolean algebra $\mathbb{B}$ which is complete for all classes. Let $\mathbb{B}$ consist of those classes which are regular open cuts of $\mathbb{P}$. Usual arguments show that $\mathbb{B}$ is a Boolean algebra and $\mathbb{P}$ densely embeds into $\mathbb{B}$. Given any class $A \subseteq \mathbb{B}$, $\mathbb{B}$ has $\bigwedge A$.

Given the much stronger second-order theory KM^+, we can argue that every antichain of $\mathbb{B}$ is coded in $\mathcal{V}$ and that B is complete for definable sub-collections. The comprehension of KM immediately gives the extra completeness. So let's argue that every definable antichain is coded (this will use class choice). Let $\varphi(x, C)$ an antichain $\mathcal{A}$ of $\mathbb{B}$. Let $X = \{p \in \mathbb{P} \mid \exists P \in \mathcal{A}, p \leq P\}$ (which exists by comprehension) and let $A \subseteq X$ be a maximal antichain of X. Now for every $p \in A$, choose some $P \in \mathcal{A}$ with $p \leq P$ (using class choice). The resulting coded collection $\bar{\mathcal{A}}$ clearly has the property that every element of $\mathcal{X}$ is compatible to something in it. So $\bar{\mathcal{A}} = \mathcal{A}$.

Question 1.3. Do we have counterexamples for weaker theories?

So let's work in a (β)-model $\mathcal{V} = \langle V, S\rangle$ of KM^+ (that is KM together with class choice). Let $M_{\mathcal{V}}$ be the model of ZFC^- with the largest inaccessible cardinal κ with which $\mathcal{V}$ is bi-interpretable. Let $\mathbb{P}$ be a class forcing notion in $\mathcal{V}$ and let $\mathbb{B}$ the hyperclass Boolean algebra described above. In $M_{\mathcal{V}}$, $\mathbb{P}$ is a set of size κ and $\mathbb{B}$ is a definable class Boolean algebra which is set-complete and has the ORD-cc.

Now we work in $M_{\mathcal{V}}$ with the class forcing notion $\mathbb{B}$. We define the class of $M_{\mathcal{V}}^{\mathbb{B}}$ of $\mathbb{B}$-names as usual. Now let's define the Boolean values of atomic formulas by recursion.

$$[[\tau \in \sigma]] = \bigvee_{\langle \nu, b\rangle \in \sigma} [[\nu = \tau]] \wedge b$$

$$[[\tau = \sigma]] = [[\tau \subseteq \sigma]] \wedge [[\sigma \subseteq \tau]]$$

$$[[\tau \subseteq \sigma]] = \bigwedge_{\nu \in \mathrm{dom}(\tau)} [[\nu \in \tau]] \to [[\nu \in \sigma]]$$

Note that this is a first-order recursion since it is set-like. Next, using completeness, we can define the Boolean values $[[\varphi(\sigma)]]$ for any first-order formula φ. So as in the case of set forcing, we get a Boolean-valued model. We would also like to ensure that the Boolean valued model is full, meaning that $[[\exists x \varphi(x, \sigma)]] = [[\varphi(\tau, \sigma)]]$ for some name τ for all existential assertions. This is usually proved by first proving the mixing lemma.

Lemma 1.4 (Mixing lemma)**.** *If $A \subseteq \mathbb{B}$ is an antichain and $\langle \tau_A \mid a \in A\rangle$ is any sequence of names indexed by A, then there is a name τ such that $a \leq [[\tau = \tau_A]]$ for each $a \in A$.*

Proof. Let $\tau = \{\langle \sigma, b \wedge a\rangle \mid \langle \sigma, b\rangle \in \tau_a \text{ and } a \in A\}$. □

If $[[\tau_a \neq \tau_b]] = 1$ for $a \neq b$ in A, then $a = [[\tau = \tau_a]]$.

Lemma 1.5. *The model $M_{\mathcal{V}}^{\mathbb{B}}$ is full.*

Proof. Let $b = [[\exists x \varphi(x, \sigma)]] = \bigvee_{\tau \in M_{\mathcal{V}}^{\mathbb{B}}} [[\varphi(\tau, \sigma)]]$. Let $D = \{p \in \mathbb{P} \mid \exists \tau\, p \leq [[\varphi(\tau, \sigma)]]\}$. Observe that D is open dense below b. So let A be a maximal antichain of D. It is easy to see that $\bigvee A = b$. Now for each $a \in A$, we can choose some τ_a such that $a = [[\varphi(\tau_a, \sigma)]]$. The mixed name now works. □

The next theorem says that $\mathbb{B}$ and $\mathbb{P}$ have the same forcing extensions of $M_{\mathcal{V}}$.

Theorem 1.6 (Theorem 5.3 in [HKS])**.** *$\mathbb{P}$ and $\mathbb{B}$ have the same forcing extensions.*

Proof. Clearly if $G \subseteq \mathbb{P}$ is M-generic, then the upward closure of G is M-generic for $\mathbb{B}$. Now suppose that G is M-generic for $\mathbb{B}$. We know that $G \cap \mathbb{P}$ is M-generic for $\mathbb{P}$. So we need to argue that $M[G] = M[G \cap \mathbb{P}]$. Fix a $\mathbb{B}$-name σ. We will argue that $\sigma_G \in M[G \cap \mathbb{P}]$. Let $\mathrm{tc}(\sigma) = \bigcup \{\{p\} \cup \mathrm{tc}(\tau) \mid \langle \tau, p\rangle \in \sigma\}$. For every $q \in \mathrm{tc}(\sigma) \cap \mathbb{B}$, let $D_q = \{p \in \mathbb{P} \mid p \leq q \vee p \perp q\}$, and observe that D_q is dense. Now we define inductively for every name $\tau \in \mathrm{tc}(\sigma)$, the name

$$\bar{\tau} = \{\langle \bar{\mu}, r\rangle \mid \exists s [\langle \mu, s\rangle \in \tau \wedge r \in D_q \wedge r \leq s]\}.$$

But then $\bar{\sigma}$ is a $\mathbb{P}$-name and $\bar{\sigma}_{G \cap \mathbb{P}} = \sigma_G$. □

Next, we want to turn a Boolean valued model into a two-valued model. But for this we need a definable ultrafilter on $\mathbb{B}$, which there is no reason to suppose exists. We can solve this problem by strengthening the theory further. We will force to add a global well-order and then force again to make it definable from a subset of κ as in [AF17]. The resulting model has the form $L_\alpha[A]$ for some $A \subseteq \kappa$ and the same V_κ as the original model, but obviously it has new subsets of κ and so in the corresponding KM^+-model we have added classes. Now we obviously have definable ultrafilters for $\mathbb{B}$ and so we have a definable model W which is, by usual arguments, elementarily equivalent to $M_\mathcal{V}$, indeed it has a submodel $\bar{W} = \check{M}$ into which $M_\mathcal{V}$ elementarily embeds and W is a forcing extension of $\bar{W}$ by a W-generic filter G.

2. Intermediate model theorem

Theorem 2.1 ([Fri99], recent results of Hamkins and Reitz)**.** *Vopenka's intermediate model theorem can fail for class forcing. If V is a model of* ZFC *and $V[G]$ is an extension by class forcing and $V \subseteq N \subseteq V[G]$ as a* ZFC*-model, then N need not be a forcing extension of V. Moreover the theorem can even fail for* ORD*-cc forcing for which a Boolean completion exists.*

Theorem 2.2. *Vopenka's intermediate theorem can fail for models of* KM^+*. There is a model $\mathcal{V}$ of* KM *and a class forcing in $\mathcal{V}$ with an intermediate model $\mathcal{V} \subseteq \mathcal{N} \subseteq \mathcal{V}[G]$, where $\mathcal{N} \models \mathrm{KM}$, but $\mathcal{N}$ is not a forcing extension of $\mathcal{V}$ by any class forcing in $\mathcal{V}$.*

Proof. Using a construction of Gitman and Hamkins from [GHJ], there is a model $V \models \mathrm{ZFC}$ with an inaccessible cardinal κ and ω-many κ-Souslin trees with special properties. The special properties give that the forcing extension $V[G]$ by the full-support ω-length product of the κ-Suslin trees has a symmetric submodel N such that $\langle V_\kappa^N, V_{\kappa+1}^N\rangle$ is a model of KM, but not KM^+. The forcing, the product of the κ-Souslin trees is obviously a class forcing of $\langle V_\kappa, V_{\kappa+1}\rangle$, so $\langle V_\kappa^N, V_{\kappa+1}^N\rangle$ is an intermediate model between $\langle V_\kappa, V_{\kappa+1}\rangle$ and its forcing extension $\langle V_\kappa[G], V_{\kappa+1}[G]\rangle$. But $\langle V_\kappa^N, V_{\kappa+1}^N\rangle$ cannot be a forcing extension because forcing preserves KM^+. □

Theorem 2.3. *Suppose M is a model of* ZFC^- *with the largest cardinal κ and $\mathbb{B}$ is a definable class of M which is a complete* ORD*-cc Boolean algebra. Suppose $N \models \mathrm{ZFC}^-$ is an intermediate model between M and $M[G]$ such that*

- *N is definable in $M[G]$,*
- *M is definable in N,*
- *N has a definable global well-order.*

Then $N = M[\mathbb{D} \cap G]$ for some definable complete subalgebra $\mathbb{D}$ of M.

Proof. Suppose $X \subseteq \mathbb{B}$ is a class. Let us say that a set well-founded tree T of elements of $\mathbb{B}$ is an *X-tree* if the leaves of T are elements of X, if an element $b \in T$ has a single successor, then it is $-b$, if it has multiple successors, then it is the join of them. Let $b \in \mathbb{B}_X$ if there is an X-tree with b as the root. It is easy to see that $\mathbb{B}_X$ is a complete subalgebra of $\mathbb{B}$. We will say that $\mathbb{B}_X$ is *generated* by X.

Let A be a class of ordinals coding all subsets of $\mathbb{B}$ in N. Each subset of $\mathbb{B}$ in N can be coded by a set of ordinals and they can all be coded together using global choice in N. Note this uses the definability of M in N. Since A is also definable in $M[G]$ (by definability of N in $M[G]$, we can choose a class $\mathbb{B}$-name $\dot{A}$ for A.

Let $X=\{[[\check{\alpha}\in\dot{A}]]\mid\alpha\in\mathrm{ORD}^M\}$, which is a class in M. Let $\mathbb{D}$ be a complete subalgebra of $\mathbb{B}$ generated by X in M. We will argue that $N=M[\mathbb{D}\cap G]$.

First, let's argue that $M[\mathbb{D}\cap G]\subseteq N$. Observe that $X\cap G=\{[[\alpha\in\dot{A}]]\mid\alpha\in A\}$ and that $\mathbb{D}\cap G$ is definable from $X\cap G$ in N (this uses M is definable in N because we need to recognize the X-trees from M). Also, A is definable from $X\cap G$.

Now suppose that $a\in N$. We can assume without loss that a is a set of ordinals. Let $\dot{a}$ be some name for a such that $\mathbb{1}_{\mathbb{B}}\Vdash\dot{a}\subseteq\beta$ for a fixed ordinal β. Let $x=\{[[z\in\dot{a}]]\mid z\in\beta\}$. We have $x\cap G=\{[[z\in\dot{a}]]\mid z\in a\}$ is coded in A. So $x\cap G$ is coded in A. But then a is definable from $x\cap G$ and $\dot{a}$. □

Note that if $N[\mathbb{D}\cap G]$ and M is definable in N, then N is definable in $M[G]$.

Question 2.4. Can we improve this result?

3. Observations

Lemma 3.1. KM$^+$ *is preserved by tame forcing.*

Proof. Let $V[G]$ be a forcing extension by a tame class forcing $\mathbb{P}$. Suppose $V[G]$ satisfies $\forall\alpha\exists X\varphi(\alpha,X,A)$. So there is $p\in\mathbb{P}$ such that $p\Vdash\forall\alpha\exists X\varphi(\alpha,X,\dot{A})$ where $\dot{A}$ is a $\mathbb{P}$-name for A. Fix an ordinal α. Let's argue that there is a class $\mathbb{P}$-name $\dot{X}_\alpha$ such that $p\Vdash\varphi(\alpha,\dot{X}_\alpha,\dot{A})$. Let $\mathcal{D}$ be the dense class of conditions q below p for which there is a class $\mathbb{P}$-name $\dot{X}_q$ such that $q\Vdash\varphi(\alpha,\dot{X}_q,\dot{A})$. The class $\mathcal{D}$ exists by comprehension. Let $\mathcal{A}$ be a maximal antichain of $\mathcal{D}$. We construct $\mathcal{A}$ as follows. Let $\mathcal{W}$ be a well-order of $\mathcal{D}$. Call $f:\beta\to\mathcal{D}$ be a good map if $f(0)$ is the least element of $\mathcal{W}$, and given $\gamma<\beta$, $f(\gamma)$ is the least element of $\mathcal{W}$ incompatible with all $f(\xi)$ for $\xi<\gamma$. Clearly for every ordinal β, there is a good map f and it is unique. So we can union up all the good maps f and the range of the union is a maximal antichain of $\mathcal{D}$. Now, using KM$^+$, we can pick for every $q\in\mathcal{D}$, a class $\mathbb{P}$-name $\dot{X}_q$ such that $q\Vdash\varphi(\alpha,\dot{X}_q,\dot{A})$. After this, we do the usual mixing argument to build the name $\dot{X}_\alpha$. Finally, again using KM$^+$, we pick for every α, a class $\mathbb{P}$-name $\dot{X}_\alpha$ and put them all together to form a name for the sequence of choices in $V[G]$. □

Lemma 3.2. KM^{++} *is preserved by tame forcing.*

Proof. Let $V[G]$ be a forcing extension by a tame class forcing $\mathbb{P}$. Suppose $V[G]$ satisfies $\forall X\exists Y\varphi(X,Y,A)$. So there is a $p\in\mathbb{P}$ such that $p\Vdash\forall X\exists Y\varphi(X,Y,\dot{A})$, where $\dot{A}$ is a $\mathbb{P}$-name for A. Let $\dot{X}_0=\check{V}$. Define a relation $\mathcal{R}$ on classes of V such that $X\mathcal{R}Y$ if X is not a $\mathbb{P}$-name and Y is a $\mathbb{P}$-name or X is a $\mathbb{P}$-name and $p\Vdash\varphi(X,Y,\dot{A})$. Let's argue that $\mathcal{R}$ has no terminal nodes. Fix a class X. If X is not a $\mathbb{P}$-name, we are done. Otherwise, by the same process as above, we can build a witnessing $\mathbb{P}$-name Y such that $p\Vdash\varphi(X,Y,\dot{A})$. Thus, by KM^{++}, there is an ω-sequence of witnessing names for classes. □

References

[AF17] Carolin Antos and Sy-David Friedman. Hyperclass forcing in Morse-Kelley class theory. *J. Symb. Log.*, 82(2):549–575, 2017.

[Fri99] Sy D. Friedman. Strict genericity. In *Models, algebras, and proofs (Bogotá, 1995)*, volume 203 of *Lecture Notes in Pure and Appl. Math.*, pages 51–56. Dekker, New York, 1999.

[GHJ] Victoria Gitman, Joel David Hamkins, and Thomas Johnstone. Kelley-Morse set theory and choice principles for classes. In preparation.

[HKS] Peter Holy, Regula Krapf, and Philipp Schlicht. Characterizations of pretameness and the Ord-cc. Preprint.