Ramsey cardinals and the continuum function

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This is joint work with Brent Cody (Virginia Commonwealth University).

The continuum function

is first studied by Cantor who shows that $2^\alpha > \alpha$.

In 1877, Cantor puts forth the Continuum Hypothesis (CH):

$$2^\omega = \omega_1.$$

In 1904, König presents a false proof that continuum is not an $\aleph$.
But hidden inside is the famous König's Inequality:

$$\text{cf}(2^\alpha) > \alpha \text{ for every cardinal } \alpha.$$

Example: We cannot have $2^\omega = \aleph_\omega$.

In 1905, Jourdain states the Generalized Continuum Hypothesis (GCH):

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \text{ for every ordinal } \alpha.$$

Definition: The (class) continuum function F maps every (regular) cardinal α to 2^α.

Note: By König's Inequality, $\text{cf}(F(\alpha)) > \alpha$.

Question: What other restrictions apply to the continuum function?

Continuum Hypothesis resolved?

In 1938, Gödel constructs L, the smallest inner model of set theory, and shows that the GCH holds in L.

In 1963, Cohen invents forcing and uses it to show that $\neg$CH is consistent with ZFC.

Soon after, Solovay shows that if $V \models \text{GCH}$ and κ a cardinal with $\text{cf}(\kappa) > \omega$, then there is a (cardinality and) cofinality preserving forcing extension in which:

$$2^\omega = \kappa.$$

König's Inequality turns out to be the sole restriction on 2^ω!

Easton's Theorem

In 1970, Easton shows that König's Inequality is the sole restriction on the continuum function on the regular cardinals.

Definition: A (class) function on the regular cardinals is an Easton function if:

- $\alpha < \beta \longrightarrow F(\alpha) \leq F(\beta)$,
- $\mathrm{cf}(F(\alpha)) > \alpha$ (König's Inequality).

Theorem (Easton, 1970)

If $V \models \mathrm{GCH}$ and F is an Easton function, then there is a cofinality preserving forcing extension in which:

$$2^\alpha = F(\alpha) \text{ for every regular cardinal } \alpha.$$

The situation with singular cardinals is much more complex, for example:

Theorem (Silver, 1975): Let α be a singular cardinal of uncountable cofinality. If $2^\delta = \delta^+$ for all cardinals $\delta < \alpha$, then $2^\alpha = \alpha^+$.

But if we want to preserve large cardinals?

Large cardinals and the continuum function

Large cardinals affect the continuum function in both obvious and subtle ways.

Inaccessible cardinals:

- Any inaccessible κ is a closure point of F ($F\,"\,\kappa \subseteq \kappa$).
- If $V \models \mathrm{GCH}$, κ is inaccessible, and F is an Easton function with a closure point at κ, then there is a (cofinality preserving) forcing extension in which:
  - $2^\alpha = F(\alpha)$ for every regular cardinal α,
  - κ remains inaccessible.

  (by the proof of Easton's Theorem)
- Result extends to a class of inaccessible cardinals.

Weakly compact cardinals:

- (Folklore) If $V \models \mathrm{GCH}$, κ is weakly compact, and F is an Easton function with a closure point at κ, then there is a (cofinality preserving) forcing extension in which:
  - $2^\alpha = F(\alpha)$ for every regular cardinal α,
  - κ remains weakly compact.

  (needs a different forcing because Easton product destroys weak compactness over L, stay tuned for sketch of proof)
- Result extends to a class of weakly compact cardinals.
- A weakly compact κ can be the first regular cardinal at which GCH holds or the first at which GCH fails.

Large cardinals and the continuum function (continued)

Strongly unfoldable cardinals:

An inaccessible cardinal κ is strongly unfoldable if for every ordinal θ and every transitive set M of size κ with $\kappa \in M \models \mathrm{ZFC}$ and $M^{<\kappa} \subseteq M$ there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ such that $\theta \leq j(\kappa)$ and $V_\theta \subseteq N$.

- A strongly unfoldable κ cannot be the first regular cardinal at which GCH holds or the first at which GCH fails.
- If GCH holds below a strongly unfoldable κ, then GCH holds.

(these facts follow easily from the definition)

- If $V \models \mathrm{GCH}$, κ is strongly unfoldable, and F is an Easton function defined above κ, then there is a forcing extension in which:
  - $2^\alpha = F(\alpha)$ for every regular cardinal $\alpha > \kappa$,
  - κ remains strongly unfoldable.

(first make κ indestructible by all $<\kappa$-closed κ^+-preserving forcing (Hamkins, Johnstone, 2010))

Remarkable cardinals:

A cardinal κ is remarkable if in $V^{\mathrm{Coll}(\omega,<\kappa)}$, for every cardinal $\lambda > \kappa$, there is some $X \prec H_\lambda^V$ such that $|X| = \omega$, $X \cap \kappa \in \kappa$, and there is some V-cardinal $\overline{\lambda}$ such that $X \cong H_{\overline{\lambda}}^V$.

- A remarkable κ cannot be the first regular cardinal at which GCH holds (or fails).
- If GCH holds below a remarkable κ, then GCH holds.

(these facts follow easily from the definition)

Large cardinals and the continuum function (continued)

Ramsey cardinals:

- (Cody, G., 2012) If $V \models \mathrm{GCH}$, κ is Ramsey, and F is an Easton function with a closure point at κ, then there is a (cofinality preserving) forcing extension in which:
  - $2^\alpha = F(\alpha)$ for every regular cardinal α,
  - κ remains Ramsey.
- Result extends to a class of Ramsey cardinals.
- A Ramsey κ can be the first regular cardinal at which GCH holds or the first at which GCH fails.

Large cardinals and the continuum function (continued)

Measurable cardinals:

- A measurable κ cannot be the first regular cardinal at which GCH fails.
- (Levinski, 1995) A measurable κ can be the first regular cardinal at which GCH holds.
- (Gitik, 1993) A measurable κ at which GCH fails has the consistency strength of a measurable cardinal of Mitchell order $o(\kappa) = \kappa^{++}$.

Woodin cardinals:

A cardinal δ is Woodin if for every $A \subseteq V_\delta$ there are arbitrarily large $\kappa < \delta$ such that for all $\lambda < \delta$ there exists an elementary embedding $j : V \to M$ with critical point κ, such that $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $A \cap V_\lambda = j(A) \cap V_\lambda$.

- (Cody, 2011) If $V \models \mathrm{GCH}$, κ is Woodin, and F is an Easton function with a closure point at κ, then there is a (cofinality preserving) forcing extension in which:
  - $2^\alpha = F(\alpha)$ for every regular cardinal α,
  - κ remains Woodin.
- Result extends to a class of Woodin cardinals.
- A Woodin κ can be the first regular cardinal at which GCH holds or the first at which GCH fails.

Large cardinals and the continuum function (continued)

Supercompact cardinals:

A cardinal κ is supercompact if for every ordinal θ there is an elementary embedding $j : V \to M$ with critical point κ, such that $\theta < j(\kappa)$ and $M^\theta \subseteq M$.

- A supercompact κ cannot be the first regular cardinal at which GCH holds (or fails).
- If GCH holds below a supercompact κ, then GCH holds. (these facts follow easily from the definition)
- If $V \models \mathrm{GCH}$, κ is supercompact, and F is an Easton function defined above κ, then there is a forcing extension in which:
    - $2^\alpha = F(\alpha)$ for every regular cardinal $\alpha > \kappa$,
    - κ remains supercompact.

(first make κ indestructible by all $<\kappa$-directed closed forcing (Laver, 1978))

I_0-**axiom**:

For some λ, there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point below λ.

- (Dimonte, Friedman, 2013) If $V \models \mathrm{GCH}$, I_0 holds with its associated λ, and F is an Easton function such that $F \upharpoonright \lambda$ is definable over V_λ, then there is a (cofinality preserving) forcing extension in which:
    - $2^\alpha = F(\alpha)$ for every regular cardinal α,
    - I_0 holds with λ.

Large cardinals and the continuum function (singular cardinals)

Singular Cardinal Hypothesis (**SCH**):

The SCH holds if for all singular κ, we have $\kappa^{\mathrm{cf}(\kappa)} = \max\{\kappa^+, 2^{\mathrm{cf}(\kappa)}\}$.

- If SCH fails, then $0^\sharp$ exists.
- SCH holds above a supercompact cardinal.

A standard definition of Ramsey cardinals

that we won't care much about.

Definition (Erdős, Hajnal, 1962)

A cardinal κ is Ramsey if every coloring $f : [\kappa]^{<\omega} \to 2$ has a homogeneous set of size κ.

Question: Do Ramsey cardinals have a characterization in terms of the existence of elementary embeddings?

Answer: Yes!

Large cardinals and elementary embeddings

Measurable cardinals and most larger large cardinals κ are characterized by the existence of elementary embeddings $j : V \to M$, from the universe into an inner model, with critical point κ.

Example: A cardinal κ is measurable if there is $j : V \to M$ with critical point κ.

- WLOG j is the ultrapower by a countably complete ultrafilter U on κ. ($A \in U \leftrightarrow \kappa \in j(A)$)
- The ultrapower construction with U can be iterated ORD-many times to construct an ORD-length directed system of elementary embeddings of inner models.

  (take the ultrapower by image of ultrafilter from previous stage at successor stages and direct limits at limit stages)

Question: What type of embeddings characterize smaller large cardinals?

Answer: "They are small!"

Large cardinals and elementary embeddings (continued)

Many smaller large cardinals κ are characterized by the existence of elementary embeddings $j : M \to N$ with critical point κ such that:

- M, N transitive,
- $\kappa \in M \models \mathrm{ZFC}$ (or ZFC^-),
- $|M| = \kappa$.

Example: An inaccessible cardinal κ is weakly compact if for every $A \subseteq \kappa$, there is $j : M \to N$ as above with $A \in M$.

- WLOG $M^{<\kappa} \subseteq M$. (code M by subset of κ, put into $\overline{M}$ as above and restrict to $j : M \to j(M)$)
- WLOG j is the ultrapower by an M-ultrafilter U: countably complete ultrafilter from the perspective of $\langle M, \in, U\rangle$. ($A \in U \leftrightarrow \kappa \in j(A)$)
- If $M^\omega \not\subseteq M$, an M-ultrafilter U may not be countably complete.
- To iterate the ultrapower construction, U must be weakly amenable: if $|X|^M = \kappa$, then $U \cap X \in M$.
- Weak amenability is equivalent to $P(\kappa)^M = P(\kappa)^N$.
- Existence of weakly amenable M-ultrafilters is stronger than weak compactness.

Ramsey embeddings

Theorem (Mitchell, 1979)

A cardinal κ is Ramsey if and only if for every $A \subseteq \kappa$, there is a transitive $M \models \mathrm{ZFC}$ of size κ with $A, \kappa \in M$ and a weakly amenable, countably complete M-ultrafilter on κ.

- Weak amenability is necessary to iterate.
- Countable completeness ensures that all iterates are well-founded.
- Additionally assuming $M^{<\kappa} \subseteq M$ is stronger than Ramsey.
- WLOG $M = V^N_{j(\kappa)} \in N$.

Ramsey embedding: $j : M \to N$

- $M \models \mathrm{ZFC}$ has size κ, with $\kappa \in M$
- j is the ultrapower by a countably complete M-ultrafilter on κ
- $P(\kappa)^M = P(\kappa)^N$
- $M = V^N_{j(\kappa)}$, so $M \in N$
- M, N are internally approachable (stay tuned)

Customizing the continuum function

Fix an Easton function F.

Let $\mathbb{P}^F = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ be the following ORD-length Easton support iteration:

- If α is not a closure point of F, then $\dot{\mathbb{Q}}_\alpha$ is trivial.
- If α is a closure point of F, let $\overline{\alpha}$ be the next least closure point.
  Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the Easton support product

$$\prod_{\gamma \in [\alpha, \overline{\alpha}) \cap \mathrm{REG}} \mathrm{Add}(\gamma, F(\gamma))$$

  as defined in $V^{\mathbb{P}_\alpha}$.

Tagline: "Use an iteration of Easton support products between closure points of F."

Theorem (Menas, 1976)

If $V \models \mathrm{GCH}$, then any forcing extension by $\mathbb{P}^F$ is cofinality preserving and F is its continuum function on the regular cardinals.

Customizing the continuum function: Ramsey cardinals

Fix a Ramsey cardinal κ and an Easton function F with closure point at κ.

Question: How do we argue that κ remains Ramsey after forcing with $\mathbb{P}^F$?

Observe:

- A forcing that doesn't add subsets to κ cannot destroy Ramsey cardinals.
- The forcing following $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa))$ is $\leq \kappa$-distributive.
- It suffices to show that κ remains Ramsey in the forcing extension by:

$$\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa)).$$

Question: How do we show this?

The indestructibility toolkit: Lifting Criterion

Lifting Criterion: Suppose $j : M \to N$ is an elementary embedding of models of ZFC, $\mathbb{P} \in M$ is a poset and $G \subseteq \mathbb{P}$ is M-generic. Then j lifts to

$$j : M[G] \to N[H],$$

where $H = j(G) \subseteq j(\mathbb{P})$ is N-generic, if and only if

$$j\,"\,G \subseteq H.$$

Tagline: “To lift j, we need N-generic $H \subseteq j(\mathbb{P})$ such that $j\,"\,G \subseteq H$.”

Question: How do we obtain H?

The indestructibility toolkit: diagonalization criterions

Diagonalization Criterion: If $N \models \mathrm{ZFC}$ is transitive of size κ with $N^{<\kappa} \subseteq N$ and $\mathbb{Q}$ is a $<\kappa$-closed poset in N, then there is an N-generic for $\mathbb{Q}$.

Definition: A transitive $M \models \mathrm{ZFC}$ of size κ with $\kappa \in M$ is internally approachable if it is the union of an elementary chain

$$X_0 \prec X_1 \prec \cdots \prec X_n \prec \cdots \prec M,$$

such that:

- $X_i \in M$,
- $|X_i|^M = \kappa$,
- $X_i^{<\kappa} \subseteq X_i$ in M,
- (X_i need not be transitive).

Diagonalization Criterion*: (G., Johnstone, 2011) If $N \models \mathrm{ZFC}$ is internally approachable and $\mathbb{Q}$ is a $\leq\kappa$-distributive poset in N, then there is an N-generic for $\mathbb{Q}$.

The indestructibility toolkit: preserving Ramsey embeddings

Theorem: (G., Johnstone, 2011) Suppose that

- $M \models \mathrm{ZFC}$ and $\kappa \in M$,
- $j : M \to N$ is the ultrapower map by a countably complete M-ultrafilter U on κ,
- $\mathbb{P} \in M$ is a poset that is countably closed in V and $G \subseteq \mathbb{P}$ is V-generic,
- j lifts to $j : M[G] \to N[j(G)]$ in $V[G]$.

Then the lift j is the ultrapower by a countably complete $M[G]$-ultrafilter in $V[G]$.

Note: We must still argue separately that

$$P(\kappa)^{M[G]} = P(\kappa)^{N[j(G)]}.$$

Preserving weakly compact cardinals in $V^{\mathbb{P}^F}$

Theorem: (Folklore) If κ is weakly compact and F is an Easton function with a closure point at κ, then κ remains weakly compact in any forcing extension by $\mathbb{P}^F$.

Sketch of Proof:

It suffices to show that κ remains weakly compact in any forcing extension:

$$V[G][K] \text{ by } \mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa)).$$

Task: In $V[G][K]$, find for every $A \subseteq \kappa$, a model $M \models \mathrm{ZFC}$ of size κ with $\kappa, A \in M$, and $j : M \to N$ with critical point κ.

Observe:

- $\mathbb{P}^F_\kappa$ has size κ and the κ-cc.
- $\mathrm{Add}(\kappa, F(\kappa))$ has the κ^+-cc.
- $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa))$ has the κ^+-cc.

Preserving weakly compact cardinals in $V^{\mathbb{P}^F}$ (continued)

Sketch of Proof: (continued)

Strategy: (failing)

- Fix $A \subseteq \kappa \in V[G][K]$.
- Fix a nice $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa))$-name $\dot{A}$ such that $(\dot{A})_{G*K} = A$.

  ($\dot{A} = \bigcup_{\alpha<\kappa} \{\check{\alpha}\} \times A_\alpha$, where A_α is an antichain of $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa))$)
- Find $j : M \to N$ with critical point κ, where $|M| = \kappa$, $M^{<\kappa} \subseteq M$, such that: $\dot{A}$, $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa))$, $f = F \upharpoonright \kappa \in M$.
- Force over M with $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa))$.
- Lift j to $j : M[G][K] \to N[j(G)][j(K)]$.
- $(\dot{A})_{G*K} = A \in M[G][K]$.

Problem: $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa))$ could be too large to fit into M (of size κ).

Preserving weakly compact cardinals in $V^{\mathbb{P}^F}$ (continued)

Standard trick:

- Since $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa))$ has the κ^+-cc, at most κ-many conditions of $\mathrm{Add}(\kappa, F(\kappa))$ appear in $\dot{A}$.
- WLOG all conditions in $\dot{A}$ appear in the first coordinate of $\mathrm{Add}(\kappa, F(\kappa))$.

  (use an automorphism)
- Let g be the restriction of K to first coordinate of $\mathrm{Add}(\kappa, F(\kappa))$.
- g is $V[G]$-generic for $\mathrm{Add}(\kappa, 1)^{V[G]}$ and $\dot{A}_{G*g} = A$.

Strategy: (correct)

- Fix $A \subseteq \kappa \in V[G][K]$.
- Fix a nice $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, 1)$-name $\dot{A}$ such that $(\dot{A})_{G*g} = A$.
- Find $j : M \to N$ with critical point κ, where $|M| = \kappa$, $M^{<\kappa} \subseteq M$, such that: $\dot{A}, \mathbb{P}^F_\kappa, f \in M$.
- Force over M with $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, 1)$.
- Lift j to $j : M[G][g] \to N[j(G)][j(g)]$.
- $(\dot{A})_{G*g} = A \in M[G][g]$.

Preserving weakly compact cardinals in $V^{\mathbb{P}^F}$ (continued)

Sketch of Proof: (continued)

Step 1: Lift j to $j : M[G] \to N[j(G)]$

- $j(\mathbb{P}^F_\kappa) = \mathbb{P}^{j(f)}_{j(\kappa)} = \mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, j(f)(\kappa)) \times \prod_{\gamma\in(\kappa,\overline{\kappa})} \mathrm{Add}(\gamma, j(f)(\gamma)) * \mathbb{P}_{\text{tail}}$.
- $\mathrm{Add}(\kappa, j(f)(\kappa))^{N[G]} = \mathrm{Add}(\kappa, j(f)(\kappa))^{V[G]} \cong \mathrm{Add}(\kappa,\kappa)^{V[G]}$. $(N[G]^{<\kappa} \subseteq N[G]$ in $V[G])$
- Let $H \in V[G][K]$ be $V[G]$-generic for $\mathrm{Add}(\kappa, j(f)(\kappa))$ such that $H = g \times k$.
- Use Diag. Crit. to build $N[G][H]$-generic $\overline{H} \subseteq \prod_{\gamma\in(\kappa,\overline{\kappa})} \mathrm{Add}(\gamma, j(f)(\gamma))$.
- Use Diag. Crit. to build $N[G][H][\overline{H}]$-generic $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$.
- Let $j(G) = G * g \times k \times \overline{H} * G_{\text{tail}}$.

Step 2: Lift j to $j : M[G][g] \to N[j(G)][j(g)]$

- $j(\mathrm{Add}(\kappa, 1)) = \mathrm{Add}(j(\kappa), 1)^{N[j(G)]}$.
- j " $g = g \in \mathrm{Add}(j(\kappa), 1)^{N[j(G)]}$.
- Use Diag. Crit. to build $N[j(G)]$-generic $g^* \subseteq \mathrm{Add}(j(\kappa), 1)^{N[j(G)]}$ below the master condition g.

Note: $N[j(G)] = N[G * g \times k \times \overline{H} * G_{\text{tail}}]$ has subsets of κ that are not in $M[G][g]$.

This argument does not work for Ramsey cardinals!

Preserving Ramsey cardinals in $V^{\mathbb{P}^F}$

Theorem: (Cody, G., 2012) If κ is Ramsey and F is an Easton function with a closure point at κ, then κ remains Ramsey in any forcing extension by $\mathbb{P}^F$.

Sketch of Proof: (rough)

It suffices to show that κ remains Ramsey in any forcing extension:

$$V[G][K] \text{ by } \mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, F(\kappa)).$$

Task: In $V[G][K]$, find for every $A \subseteq \kappa$, a model $M \models \mathrm{ZFC}$ of size κ with $\kappa, A \in M$, and a weakly amenable countably complete M-ultrafilter on κ.

Strategy:

- Fix $A \subseteq \kappa \in V[G][g]$.
- Let g be the restriction of K to first coordinate of $\mathrm{Add}(\kappa, F(\kappa))$.
- Fix a nice $\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, 1)$-name $\dot{A}$ such that $(\dot{A})_{G*g} = A$.
- Fix a Ramsey embedding $j : M \to N$ with $\dot{A}, \mathbb{P}^F_\kappa, f = F \upharpoonright \kappa, V_\kappa \in M$.
- Force over M with ______?
- Lift j to ______?

Preserving Ramsey cardinals in $V^{\mathbb{P}^F}$

Sketch of Proof: (continued)

Strategy: (continued)

- Force over M with $(\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa,\kappa^+))^M$.
- Lift j to $j : M[G][H] \to N[j(G)][j(H)]$.
- The $M[G]$-generic H is obtained from K with g on first coordinate. (stay tuned)
- $(\dot{A})_{G*g} = A \in M[G][H]$.
- A careful choice of H and $j(G)$ will ensure that $M[G][H]$ and $N[j(G)]$ have same subsets of κ.
- Still have to argue that lift of j is the ultrapower by a countably complete ultrafilter!

Preserving Ramsey cardinals in $V^{\mathbb{P}^F}$ (continued)

Sketch of Proof: (continued)

Step 1: Lift j to $j : M[G] \to N[j(G)]$

- $j(\mathbb{P}^F_\kappa) = \mathbb{P}^{j(f)}_{j(\kappa)} = \mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, j(f)(\kappa)) * \mathbb{P}_{\text{tail}}$. ($\mathbb{P}_{\text{tail}}$ includes $\prod_{\gamma \in (\kappa, \overline{\kappa})} \mathrm{Add}(\gamma, j(f)(\gamma))$)
- $\mathrm{Add}(\kappa, j(f)(\kappa))^{N[G]} \cong \mathrm{Add}(\kappa, \kappa^+)^{M[G]}$. (this needs proof)
- Let $\tilde{H}$ be a "permutation" of H that is $N[G]$-generic for $\mathrm{Add}(\kappa, j(f)(\kappa))^{N[G]}$.
- Since $\tilde{H}$ is a "permutation" of H, no new subsets of κ are added.
- Use Diag. Crit.* to build $N[G][\tilde{H}]$-generic $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$. ($\leq\kappa$-distributive)
- Let $j(G) = G * \tilde{H} * G_{\text{tail}}$.

Step 2: Lift j to $j : M[G][H] \to N[j(G)][j(g)]$

- $j(\mathrm{Add}(\kappa, \kappa^+)) = \mathrm{Add}(j(\kappa), j(\kappa)^+)^{N[j(G)]}$.
- Because $\tilde{H}$ is a "permutation" of H, we have increasingly powerful master conditions in $N[j(G)]$.
- Use Diag. Crit.* to build $N[j(G)]$-generic $J \subseteq \mathrm{Add}(j(\kappa), j(\kappa)^+))^{N[j(G)]}$ below the increasingly powerful master conditions.

Preserving Ramsey cardinals in $V^{\mathbb{P}^F}$ (continued)

Sketch of Proof: (continued)

Step 3: Verify that $j : M[G][H] \to N[j(G)][j(g)]$ is the ultrapower by a countably complete $M[G][H]$-ultrafilter in $V[G][K]$.

- WLOG $\mathbb{P}^F_\kappa$ is countably closed, but...
- $\mathrm{Add}(\kappa, \kappa^+)^{M[G]}$ is not!
- Since H is obtained from K and $\mathrm{Add}(\kappa, F(\kappa))$ is countably closed...
- things work out!

Ramsey embeddings (recall)

Ramsey embedding: $j : M \to N$

- $M \models \mathrm{ZFC}$ has size κ, with $\kappa \in M$
- j is the ultrapower by a countably complete M-ultrafilter on κ
- $P(\kappa)^M = P(\kappa)^N$
- $M = V^N_{j(\kappa)}$, so $M \in N$
- M, N are internally approachable

Definition: A transitive $M \models \mathrm{ZFC}$ of size κ with $\kappa \in M$ is internally approachable if it is the union of an elementary chain

$$X_0 \prec X_1 \prec \cdots \prec X_n \prec \cdots \prec M,$$

such that:

- $X_i \in M$,
- $|X_i|^M = \kappa$,
- $X_i^{<\kappa} \subseteq X_i$ in M,
- (X_i need not be transitive).

The mystery behind H

Set-up:

- $j : M \to N$ is a Ramsey embedding.
- We force over M with $(\mathbb{P}^F_\kappa * \mathrm{Add}(\kappa,\kappa^+))^M$.
- $M[G] = \cup_{i<\omega}\overline{X}_i$ is internally approachable. $(\overline{X}_i = X_i[G])$

Constructing H:

- Partition $(\kappa^+)^M = \sqcup_{i<\omega} x_i$ with $x_i \in M[G]$.
  - $x_0 = \overline{X}_1 \cap (\kappa^+)^M$,
  - $x_i = (\overline{X}_{i+1} \setminus \overline{X}_i) \cap (\kappa^+)^M$.
- Define $\mathbb{Q}_i$: consists of all conditions in $\mathrm{Add}(\kappa,\kappa^+)^{M[G]}$ with domain $\subseteq x_i$.
- $\mathrm{Add}(\kappa,\kappa^+)^{M[G]}$ is isomorphic to the finite support product $\prod_{i<\omega}\mathbb{Q}_i$.
- $\mathbb{Q}_i \cong \mathrm{Add}(\kappa,1)^{V[G]}$ in $M[G]$ by φ_i. $(|\overline{X}_i| = \kappa$ in $M[G])$
- $\prod_{i<n}\mathbb{Q}_i \cong X_n \cap \mathrm{Add}(\kappa,\kappa^+)^{M[G]}$ is in $M[G]$.
- Let $\overline{H} \in V[G][K]$ be $V[G]$-generic for $\mathrm{Add}(\kappa,\omega)^{V[G]}$.
- Let H be all conditions in $\overline{H}$ with finite support.
- H is not $V[G]$-generic for the finite support product $\prod_{i<\omega}\mathbb{Q}_i$, but...
- H is $M[G]$-generic because every antichain is a subset of some $\overline{X}_i$.

The mystery behind $j(G)$

Set-up:

- $j(\mathbb{P}^F_\kappa) = \mathbb{P}^{j(f)}_{j(\kappa)} = \mathbb{P}^F_\kappa * \mathrm{Add}(\kappa, j(f)(\kappa)) * \mathbb{P}_{\text{tail}}$.
- $j(G) = G * \_\_\_ * G_{\text{tail}}$.
- $N[G] = \cup_{i<\omega} \overline{Y}_i$ is internally approachable.

Constructing $N[G]$**-generic for** $\mathrm{Add}(\kappa, j(f)(\kappa))^{N[G]}$:

- We partition $j(f)(\kappa)^N = \sqcup_{i<\omega} y_i$.
  - $y_0 = \overline{Y}_1 \cap j(f)(\kappa)^N$,
  - $y_i = (\overline{Y}_{i+1} \setminus \overline{Y}_i) \cap j(f)(\kappa)^N$.
- Define $\mathbb{R}_i$: consists of all conditions in $\mathrm{Add}(\kappa, j(f)(\kappa))^{N[G]}$ with domain $\subseteq y_i$.
- $\mathrm{Add}(\kappa, j(f)(\kappa))^{N[G]}$ is isomorphic to the finite support product $\prod_{i<\omega} \mathbb{R}_i$.
- $\mathbb{R}_i \cong \mathrm{Add}(\kappa, 1)^{V[G]}$ in $N[G]$ by ψ_i.
- $\prod_{i<n} \mathbb{R}_i \cong \overline{Y}_n \cap \mathrm{Add}(\kappa, j(f)(\kappa))^{N[G]}$ is in $N[G]$.
- In $V[G]$, define an isomorphism between $\prod_{i<\omega} \mathbb{Q}_i$ and $\prod_{i<\omega} \mathbb{R}_i$ using φ_i and ψ_i.
- Use the isomorphism to obtain $\tilde{H}$ from H.

Thank you!