Applications of the Proper Forcing Axiom to Models of Peano Arithmetic

by

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Abstract

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In Chapter 1, new results are presented on Scott's Problem in the subject of models of Peano Arithmetic. Some forty years ago, Dana Scott showed that countable Scott sets are exactly the countable standard systems of models of PA, and two decades later, Knight and Nadel extended his result to Scott sets of size ω_1. Here it is shown that assuming the Proper Forcing Axiom, every arithmetically closed proper Scott set is the standard system of a model of PA. In Chapter 2, new large cardinal axioms, based on Ramsey-like embedding properties, are introduced and placed within the large cardinal hierarchy. These notions generalize the seldom encountered embedding characterization of Ramsey cardinals. I also show how these large cardinals can be used to obtain indestructibility results for Ramsey cardinals.

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Contents

Chapter 1

Scott's Problem for Proper Scott Sets

1.1 Introduction

Given a model of Peano Arithmetic (PA), we can consider the collection of all subsets of the natural numbers that arise as intersections of the definable sets of the model with its standard part. This collection, known as the *standard system* of the model, was introduced by Harvey Friedman [6], and has since proven to be one of the most effective tools in exploring the rich theory of nonstandard models of PA. Already in 1962, Dana Scott investigated collections of subsets of the natural numbers that are represented in complete extensions of PA [19]. A subset A of $\mathbb{N}$ is said to be *represented* in a complete theory $T \supseteq \text{PA}$ if there is a formula $\varphi(v_0)$ such that $n \in A$ if and only if $T \vdash \varphi(n)$. Translated into the later terminology, Scott was

looking at the standard systems of prime models of complete extensions of PA. In this context, Scott discovered certain set theoretic and computability theoretic properties of standard systems, which he was able to prove completely characterize countable standard systems. Two decades later, Knight and Nadel showed that this characterization, which came to be known as a *Scott set*, held for collections of size ω_1 as well. I will use the Proper Forcing Axiom (PFA) to find new instances under which the characterization can be extended to collections of size ω_2. I will also apply the forcing techniques I developed for this problem to other open questions concerning models of PA.

Definition 1.1 (Friedman, 1973)**.** Let M be a model of PA. The *standard system* of M, denoted $\mathrm{SSy}(M)$, is the collection of all subsets of the natural numbers that are intersections of the parametrically definable sets of M with its standard part $\mathbb{N}$. [6]

The standard system of a *nonstandard* model of PA can be equivalently defined using the notion of *coded* sets. Gödel used the Chinese Remainder Theorem to code finite sequences of natural numbers by a single natural number. With this coding, every natural number can be seen as coding a finite sequence of numbers. The n^{th} element of the sequence coded by a number m is denoted $(m)_n$. It turns out that this coding machinery extends

to any nonstandard model M of PA in the sense that every element of M can be viewed as coding an M-finite sequence of elements of M. Given a and b elements of M, we will similarly denote the a^{th} element of the sequence coded by b as $(b)_a$. For details on coding in nonstandard models of PA, see [12] (p. 53). We say that a set A of natural numbers is *coded* in a nonstandard model M of PA if there is an element $a \in M$ such that $(a)_n = 1$ if and only if $n \in A$. In this case, we also say that *a codes A* or *a is a code* for A. It is easy to check that the following proposition holds:

Proposition 1.2. *Let M be a nonstandard model of* PA*. Then the standard system of M is the collection of all subsets of the natural numbers that are coded in M.*

What properties characterize standard systems? Without reference to models of PA, a standard system is just a particular collection of subsets of the natural numbers. Can we come up with a list of elementary (set theoretic, computability theoretic) properties that a collection $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ must satisfy in order to be the standard system of some model of PA?

Definition 1.3 (Scott, 1962)**.** $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ is a *Scott set* if

1. $\mathfrak{X}$ is a Boolean algebra of sets.

2. If $A \in \mathfrak{X}$ and B is Turing computable from A, then $B \in \mathfrak{X}$.[1]

3. If T is an infinite binary tree coded in $\mathfrak{X}$, then $\mathfrak{X}$ codes a cofinal branch through T. [19]

For condition (3) to make sense, recall that we already fixed a method of coding finite binary sequences as natural numbers (see p. 2, bottom). We say that a collection of finite binary sequences is *coded* in $\mathfrak{X}$ if there is a set in $\mathfrak{X}$ whose elements are exactly the codes of sequences in this collection. Observe also that conditions (1) and (2) together imply that if $A_1, \ldots, A_n \in \mathfrak{X}$ and B is computable from $A_1 \oplus \cdots \oplus A_n$, then $B \in \mathfrak{X}$.

It is relatively easy to see that every standard system is a Scott set. Conversely, Dana Scott proved that every countable Scott set is the standard system of a model of PA [19].

Theorem 1.4 (Scott, 1962)**.** *Every countable Scott set is the standard system of a model of* PA*.*

I will give a proof of Theorem 1.4 in Section 1.2. Thus, the countable Scott sets are exactly the countable standard systems of models of PA. Scott's Theorem leads naturally to the following question:

[1] We say that a set B is Turing computable from a set A if there is a program which computes the characteristic function of B with oracle A.

Scott's Problem. Is every Scott set the standard system of a model of PA?

This question has, for decades, been a part of the folklore of models of PA. Since Scott considered only sets represented in theories, the issue of uncountable standard systems never arose for him. The question began to make sense only after Friedman's introduction of standard systems. In 1982, Knight and Nadel settled the question for Scott sets of size ω_1 [14].

Theorem 1.5 (Knight and Nadel, 1982)**.** *Every Scott set of size ω_1 is the standard system of a model of* PA*.*

It follows that Scott sets of size ω_1 are exactly the standard systems of size ω_1 of models of PA. I will give a proof of Theorem 1.5 in Section 1.3.

Corollary 1.6. *If* CH *holds, then Scott's Problem has a positive answer.*

Very little is known about Scott's Problem if CH fails, that is, for Scott sets of size larger than ω_1. I will use strong set theoretic hypothesis to find new techniques for building models of PA with a given Scott set as the standard system. Let us call a Scott set $\mathfrak{X}$ *proper* if the quotient Boolean algebra $\mathfrak{X}/\text{Fin}$ is a proper partial order. Proper posets were introduced by Shelah in the 1980's, and have since been studied extensively by set theorists (for definition, see Section 1.4). The Proper Forcing Axiom (PFA) was introduced

by Baumgartner [1] as a generalization of Martin's Axiom for proper posets. I will show that:

Theorem 1.7. *Assuming* PFA*, every arithmetically closed proper Scott set is the standard system of a model of* PA*.*

1.2 Scott's Theorem

In this section, I will give a proof of Scott's Theorem for countable Scott sets. I will start by proving the easy direction that every standard system is a Scott set. Whenever an object such as a tree or a cofinal branch through a tree is coded in a Scott set, I will say that the object is *in* the Scott set to simplify notation. The theorems and proofs below follow [12] (p. 172).

Theorem 1.8. *Every standard system is a Scott set.*

Proof. The standard system of $\mathbb{N}$ is the collection of all arithmetic sets, which is clearly a Scott set. So let M be a *nonstandard* model of PA. The standard system of M is a Boolean algebra of sets since the definable sets of M form a Boolean algebra. Let A be an element of $\mathrm{SSy}(M)$ and B be Turing computable from A. Fix a definable set A' of M such that $A = A' \cap \mathbb{N}$ and a Turing program p which computes B with oracle A. We can define the computation p in M with the definable set A' as an oracle. The result of the

computation will be a definable set in M, whose intersection with $\mathbb{N}$ will be exactly B. This follows since the computation on the standard part halts for every input and only uses the standard part of the oracle, which is precisely A. This shows that $\mathrm{SSy}(M)$ is closed under relative computability. Finally, suppose that T is an infinite binary tree in $\mathrm{SSy}(M)$. Let T' be a definable set in M such that $T' \cap \mathbb{N} = T$. Since $\mathbb{N}$ is not definable in M, there must be a nonstandard $c > \mathbb{N}$ in M such that below c the set T' still codes a binary tree. That is, if $a < c$ and $a \in T'$, then every binary initial segment b of a is in T' as well. Now we can take any *nonstandard* node d of this nonstandard binary tree and extend the branch all the way down through T. That is, we look at the set of all binary predecessors of d, which is clearly definable in M, and intersect it with $\mathbb{N}$ to obtain a cofinal branch through T. This completes the argument that every infinite binary tree in $\mathrm{SSy}(M)$ has a cofinal branch in $\mathrm{SSy}(M)$. □

The next three results are building up to the proof of Theorem 1.4. I will say that a theory T in a computable language is *coded* in $\mathfrak{X}$ if there is a set in $\mathfrak{X}$ having as elements exactly the codes of the sentences of T. Again, we say that a theory $T \in \mathfrak{X}$ if T is coded in $\mathfrak{X}$.

Lemma 1.9. *Let $\mathfrak{X}$ be a Scott set and let $T \in \mathfrak{X}$ be a consistent theory in a*

computable language. Then T has a consistent completion $S \in \mathfrak{X}$.

Proof. Use T to build a T-computable binary tree of finite approximations to a consistent completion. Since this tree is computable from T, it is in $\mathfrak{X}$ by property (2) of Scott sets. Any cofinal branch through this tree defines a consistent completion for T, and one such branch must be in $\mathfrak{X}$ by property (3) of Scott sets. □

The next theorem states that you can carry out the proof of the Completeness Theorem for theories in $\mathfrak{X}$ "inside" $\mathfrak{X}$.

Theorem 1.10. *Let $\mathfrak{X}$ be a Scott set and let $T \in \mathfrak{X}$ be a consistent theory in a computable language $\mathcal{L}$ having an infinite model. Then $\mathfrak{X}$ contains the elementary diagram for a model M of T whose universe is $\mathbb{N}$.*

Proof. Observe that if T has an infinite model, then we can extend T to the consistent theory $T' := T \cup \{\exists x_1, \ldots, x_n(\wedge_{i,j}\ x_i \neq x_j) \mid n \in \mathbb{N}\}$ having no finite models. Clearly T' is computable from T, and hence in $\mathfrak{X}$. Thus, we can assume without loss of generality that T itself has no finite models. This fact will be crucial in the later part of the proof. Define a sequence of languages $\mathcal{L}_i$ for $i \in \mathbb{N}$ such that $\mathcal{L}_0 = \mathcal{L}$ and $\mathcal{L}_{i+1}$ consists of $\mathcal{L}_i$ together with constants c_φ for every formula $\varphi(v_0) \in \mathcal{L}_i$. Let $\mathcal{L}^* = \cup_{i\in\mathbb{N}}\mathcal{L}_i$. Clearly $\mathcal{L}^*$ is computable. Define the theory T^* such that:

1. $T \subseteq T^*$.

2. For every formula $\varphi(v_0)$ in the language of $\mathcal{L}^*$, the formula $\exists v_0\, \varphi(v_0) \to \varphi(c_\varphi) \in T^*$.

Clearly T^* is computable from T, and hence in $\mathfrak{X}$. It is also clear that T^* is consistent. Let S be some consistent completion of T^* in $\mathfrak{X}$. We can use S to define the equivalence relation on the constants c_φ such that $c \sim d$ if and only if $c = d \in S$. By the usual Henkin construction, we build a model M of T out of the equivalence classes. Observe that there is an S-computable way of choosing a single representative from each equivalence class. Start by letting c_0 be the constant with least code in $\mathcal{L}^*$. Next, search for a constant c_1 with least code that is not equivalent to c_0. Since we assumed that T has no finite models, we are guaranteed to find c_1, and so on. It should now be clear that the elementary diagram of M is computable from the theory $S \in \mathfrak{X}$. □

Theorem 1.11. *Let $\mathfrak{X}$ be a countable Scott set and let $T \supseteq$ PA be a consistent theory such that $T \in \mathfrak{X}$. Then there is a model M of T whose standard system is exactly $\mathfrak{X}$.*

Proof. Enumerate $\mathfrak{X} = \{A_n \mid n \in \mathbb{N}\}$. Let $\mathcal{L}^{(0)}$ be the language $\mathcal{L}$ together with a new constant $a^{(0)}$. Let $T^{(0)}$ be the theory in $\mathcal{L}^{(0)}$ consisting of T

together with the sentences $\{(a^{(0)})_k = 1 \mid k \in A_0\}$ and $\{(a^{(0)})_k = 0 \mid k \notin A_0\}$. Clearly $T^{(0)}$ is consistent and $T^{(0)} \in \mathfrak{X}$ since it is computable from A_0 and T. Define the language $\mathcal{L}^{(0)*}$, the theory $T^{(0)*}$, and a completion $S^{(0)}$ in $\mathfrak{X}$ as above. Given $\mathcal{L}^{(n)*}$ and $S^{(n)}$ in $\mathfrak{X}$, let $\mathcal{L}^{(n+1)}$ be the language $\mathcal{L}^{(n)*}$ together with a new constant $a^{(n+1)}$ and let $T^{(n+1)}$ be the theory in $\mathcal{L}^{(n+1)}$ consisting of $S^{(n)}$ together with the sentences $\{(a^{(n+1)})_k = 0 \mid k \in A_{n+1}\}$ and $\{(a^{(n+1)})_k = 0 \mid k \notin A_{n+1}\}$. Again, define the language $\mathcal{L}^{(n+1)}$, the theory $T^{(n+1)*}$, and a completion $S^{(n+1)}$ in $\mathfrak{X}$. Let $S = \cup_{n\in\mathbb{N}} S^{(n)}$. Clearly S is a complete Henkin theory, and therefore we can define the corresponding Henkin model on the equivalence classes of the constants. Let $M \models T$ be this model. It remains to check that $\mathrm{SSy}(M) = \mathfrak{X}$. By construction, we have that $\mathfrak{X} \subseteq \mathrm{SSy}(M)$. So fixing $[c] \in M$, we need to show that $[c]$ codes a set in $\mathfrak{X}$. There is $n \in \mathbb{N}$ such that the constant c first appeared in the theory $T^{(n)*}$. Since $S^{(n)}$ was a completion of $T^{(n)*}$, it must be that $S^{(n)}$ already decided all the sentences of the form $(c)_k = 1$ and $(c)_k \neq 1$. Therefore the set $\{k \in \mathbb{N} \mid M \models ([c])_k = 1\} = \{k \in \mathbb{N} \mid (c)_k = 1 \in S^{(n)}\}$ is in $\mathfrak{X}$. This completes the proof that $\mathrm{SSy}(M) = \mathfrak{X}$. □

For the conclusion of Theorem 1.11, it actually suffices to assume that $T \cap \Sigma_n$ is in $\mathfrak{X}$ for every $n \in \mathbb{N}$. The proof is a fairly straightforward modifi-

cation of the proof of Theorem 1.11.

Corollary 1.12. *Every countable Scott set is the standard system of a model of* PA*.*

Proof. Let $\mathfrak{X}$ be a countable Scott set. Since $\mathfrak{X}$ is closed under relative computability, it must contain all computable sets. Therefore PA $\in \mathfrak{X}$, and the rest follows by Theorem 1.11. □

Observe finally that there does not appear to be a way to generalize this proof to uncountable Scott sets since after countably many steps the theory we obtain is no longer in $\mathfrak{X}$.

1.3 Ehrenfeucht's Lemma

In this section, I will introduce a theorem known as *Ehrenfeucht's Lemma* and use it to prove Theorem 1.5. I will then define generalizations of this theorem which will allow us to prove Theorem 1.7. Ehrenfeucht's Lemma is due to Ehrenfeucht in the 1970's.[2] The proof of Theorem 1.5 follows [21].

Theorem 1.13 (Ehrenfeucht's Lemma)**.** *If M is a countable model of* PA *whose standard system is contained in a Scott set $\mathfrak{X}$, then for any $A \in \mathfrak{X}$, there is an elementary extension $M \prec N$ such that $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}$.*

[2]Roman Kossak, personal communication.

Proof. First, we consider nonstandard M. Let $\mathfrak{X}$ be a Scott set such that SSy(M) $\subseteq \mathfrak{X}$ and let $A \in \mathfrak{X}$. Choose a countable Scott set $\mathfrak{Y} \subseteq \mathfrak{X}$ containing SSy(M) and A. Using the truth predicate for Σ_n-formulas, we can prove that the Σ_n-theory of M is in SSy(M) for every $n \in \mathbb{N}$. Since PA $\in$ SSy(M), it follows that the theory $T :=$ PA + "Σ_1-theory of M" is in SSy(M). By Theorem 1.11, we get a model M^* of T with SSy(M^*) $= \mathfrak{Y}$. By Friedman's Embedding Theorem (see [12], p. 160), since $M^* \models$ "Σ_1-theory of M" and SSy(M) $\subseteq \mathfrak{Y}$, we have $M \prec_{\Delta_0} M^*$. Close M under initial segment in M^* and call the resulting submodel N. Then $M \prec N$ since it is cofinal and Δ_0-elementary (by Gaifman's Embedding Theorem, see [12], p. 87). But also SSy(N) = SSy(M^*) $= \mathfrak{Y}$ as required since N is an initial segment of M^*. This completes the proof for nonstandard models. Let TA $= \{\varphi \mid \mathbb{N} \models \varphi\}$ denote *True Arithmetic*. It is clear that $\mathbb{N} \prec N$ if and only if $N \models$ TA. Recall that the standard system of $\mathbb{N}$ is the collection of all arithmetic sets. So suppose that $\mathfrak{X}$ is a Scott set containing all arithmetic sets and fix $A \in \mathfrak{X}$. It follows that TA $\cap \Sigma_n$ is in $\mathfrak{X}$ for every $n \in \mathbb{N}$. Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a countable Scott set containing A and TA$\cap\Sigma_n$ for every $n \in \mathbb{N}$. By the remark following Theorem 1.11, there exists a model $N \models$ TA whose standard system is exactly $\mathfrak{Y}$. Thus, $\mathbb{N} \prec N$ and $A \in$ SSy(N) $\subseteq \mathfrak{X}$. $\square$

We are now ready to prove Knight and Nadel's result.[3]

Corollary 1.14. *Every Scott set of size ω_1 is the standard system of a model of* PA.

Proof. Let $\mathfrak{X}$ be a Scott set of size ω_1 and enumerate $\mathfrak{X} = \{A_\xi \mid \xi < \omega_1\}$. The idea is to build up a model with the Scott set $\mathfrak{X}$ as the standard system in ω_1 steps by successively throwing in one more set at each step and using Ehrenfeucht's Lemma to stay within $\mathfrak{X}$. More precisely, we will define an elementary chain $M_0 \prec M_1 \prec \cdots \prec M_\xi \prec \cdots$ of length ω_1 of countable models of PA such that $\mathrm{SSy}(M_\xi) \subseteq \mathfrak{X}$ and $A_\xi \in \mathrm{SSy}(M_{\xi+1})$. Then clearly $M = \cup_{\xi<\omega_1} M_\xi$ will work. Let M_0 be any countable model of PA with $\mathrm{SSy}(M_0) \subseteq \mathfrak{X}$. Such M_0 exists by Scott's Theorem. Given M_ξ, by Ehrenfeucht's Lemma, there exists $M_{\xi+1}$ such that $M_\xi \prec M_{\xi+1}$, the set $A_\xi \in \mathrm{SSy}(M_{\xi+1})$, and $\mathrm{SSy}(M_{\xi+1}) \subseteq \mathfrak{X}$. At limit stages take unions. □

The key ideas in the proof of Theorem 1.5 can be summarized in the following definition and theorem:

Definition 1.15 (The κ-Ehrenfeucht Principle for Γ)**.** Let κ be a cardinal and Γ some collection of Scott sets. The *κ-Ehrenfeucht Principle for* Γ states that if M is a model of PA of size less than κ and $\mathfrak{X}$ is a Scott set in Γ such

[3]This is not Knight and Nadel's original proof.

that $\mathrm{SSy}(M) \subseteq \mathfrak{X}$, then for any $A \in \mathfrak{X}$, there is an elementary extension $M \prec N$ such that $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}$. If Γ is the collection of all Scott sets, we will say simply that the *κ-Ehrenfeucht Principle* holds.

Ehrenfeucht's Lemma becomes the ω_1-Ehrenfeucht Principle. Note also that we can freely assume that the elementary extension N given by the κ-Ehrenfeucht Principle has size less than κ, since if this is not the case, we can always take an elementary submodel N' of N such that $M \prec N'$ and $A \in \mathrm{SSy}(N')$.

Theorem 1.16. *If the κ-Ehrenfeucht Principle for Γ holds, then every Scott set in Γ of size κ is the standard system of a model of* PA.

Proof. We will mimic the proof of Theorem 1.5. Fix a Scott set $\mathfrak{X} \in \Gamma$ of size κ and enumerate $\mathfrak{X} = \{A_\xi \mid \xi < \kappa\}$. We will define an elementary chain $M_0 \prec M_1 \prec \cdots \prec M_\xi \prec \cdots$ of length κ of models of PA such that $\mathrm{SSy}(M_\xi) \subseteq \mathfrak{X}$ and $A_\xi \in \mathrm{SSy}(M_{\xi+1})$. Let M_0 be any countable model of PA with $\mathrm{SSy}(M_0) \subseteq \mathfrak{X}$. Suppose we have built $M_0 \prec M_1 \prec \cdots \prec M_\xi \prec \cdots \prec M_\alpha$ with the desired properties and each M_ξ has size less than κ. By the κ-Ehrenfeucht Principle, there exists $M_{\alpha+1}$ of size less than κ such that $M_\alpha \prec M_{\alpha+1}$, the set $A_\alpha \in \mathrm{SSy}(M_{\alpha+1})$, and $\mathrm{SSy}(M_{\alpha+1}) \subseteq \mathfrak{X}$. At limit stages take unions. $\square$

Thus, one approach to solving Scott's Problem would be to try to prove the ω_2-Ehrenfeucht Principle for some collection of Scott sets. However, proofs of Ehrenfeucht's Lemma hinge precisely on those techniques in the field of models of PA that appear to work only with countable models. As an example, Friedman's famous Embedding Theorem does not extend to uncountable models.[4] In what follows, I will mainly investigate the Ehrenfeucht principles. The results on Scott's Problem will follow as a corollary. Under PFA, I will show that the *ω_2-Ehrenfeucht Principle for arithmetically closed proper Scott sets* holds.

1.4 Set Theory and Scott's Problem

Since the result of Knight and Nadel, very little progress has been made on Scott's Problem until some recent work of Fredrik Engström [5]. It is not difficult to believe that Scott's Problem past ω_1 might have a set theoretic resolution. Engström followed a strategy, suggested more than a decade earlier by Joel Hamkins and others, to use forcing axioms to gain new insight into the problem. We saw that a positive answer to Scott's Problem follows from CH. It is a standard practice in set theoretic proofs that if a statement follows from CH, we try to prove it or its negation from ¬CH + Martin's

[4]Here ω_1-like models are an obvious counterexample.

Axiom. Martin's Axiom (MA) is a forcing axiom which asserts that for every c.c.c. poset $\mathbb{P}$ and every collection $\mathcal{D}$ of less than the continuum many dense subsets of $\mathbb{P}$, there is a filter on $\mathbb{P}$ that meets all of them. Such filters are often called *partially generic filters*. Engström tried to use Martin's Axiom to find new techniques for building models of PA whose standard system is a given Scott set.

Given a Scott set $\mathfrak{X}$, Engström chose the poset $\mathfrak{X}/\text{Fin}$, whose elements are infinite sets in $\mathfrak{X}$ ordered by almost inclusion. That is, for infinite A and B in $\mathfrak{X}$, we say that $A \leq B$ if and only if $A \subseteq_{\text{Fin}} B$. Observe that $\mathfrak{X}/\text{Fin}$ is forcing equivalent to forcing with the Boolean algebra $\mathfrak{X}$ modulo the ideal of finite sets. A familiar and thoroughly studied instance of this poset is $\mathcal{P}(\omega)/\text{Fin}$. If $\mathscr{P}$ is a property of posets and $\mathfrak{X}/\text{Fin}$ has $\mathscr{P}$, I will simply say that $\mathfrak{X}$ *has property* $\mathscr{P}$. For the theorem below, we define that a Scott set is *arithmetically closed* if whenever A is in it and B is arithmetically definable from A, then B is also in it (for a more extensive discussion, see Section 1.5).

Theorem 1.17 (Engström, 2004)**.** *Assuming Martin's Axiom, every arithmetically closed c.c.c. Scott set $\mathfrak{X}$ of size less than the continuum is the standard system of a model of* PA*.*

To obtain models for Scott sets for which we could not do so before,

Engström needed that there are uncountable Scott sets $\mathfrak{X}$ which are c.c.c.. Unfortunately:

Theorem 1.18. *A Scott set is c.c.c. if and only if it is countable.*

Proof. Let $\mathfrak{X}$ be a Scott set. If x is a finite subset of $\mathbb{N}$, let $\ulcorner x \urcorner$ denote the Gödel code of x. For every $A \in \mathfrak{X}$, define an associated $A' = \{\ulcorner A \cap n \urcorner \mid n \in \mathbb{N}\}$. Clearly A' is computable from A, and hence in $\mathfrak{X}$. Observe that if $A \neq B$, then $|A' \cap B'| < \omega$. Hence if $A \neq B$, we get that A' and B' are incompatible in $\mathfrak{X}/\mathrm{Fin}$. It follows that $\mathscr{A} = \{A' \mid A \in \mathfrak{X}\}$ is an antichain of $\mathfrak{X}/\mathrm{Fin}$ of size $|\mathfrak{X}|$. This shows that $\mathfrak{X}/\mathrm{Fin}$ always has antichains as large as the whole poset. □

Thus, the poset $\mathfrak{X}/\mathrm{Fin}$ has the worst possible chain condition, namely $|\mathfrak{X}|^+$-c.c.. Theorem 1.18 implies that no new instances of Scott's Problem can be obtained from Theorem 1.17.

I will borrow from Engström's work the poset $\mathfrak{X}/\mathrm{Fin}$. But my strategy will be different in two respects. First, instead of MA, I will use the poset together with the forcing axiom PFA, allowing me to get around the obstacle of Theorem 1.18. In Section 1.8, I will show that, unlike the case with c.c.c. Scott sets, uncountable proper Scott sets do exist. However, I will not be able to explicitly obtain any new instances of Scott's Problem. Second, my main

aim will be to obtain an extension of Ehrenfeucht's Lemma to uncountable models, while Engström's was to directly get a model whose standard system is a given Scott set. This approach will allow me to handle Scott sets of size continuum, which had not been possible with Egström's techniques.

Definition 1.19. Let λ be a cardinal, then H_λ is the set of all sets whose transitive closure has size less than λ.

Let $\mathbb{P}$ be a poset and λ be a cardinal greater than $2^{|\mathbb{P}|}$. Since we can always take an isomorphic copy of $\mathbb{P}$ on the cardinal $|\mathbb{P}|$, we can assume without loss of generality that $\mathbb{P}$ and $\mathcal{P}(\mathbb{P})$ are elements of H_λ. In particular, we want to ensure that if D is a dense subset of $\mathbb{P}$, then $D \in H_\lambda$. Let M be a countable elementary submodel of H_λ containing $\mathbb{P}$ as an element. If G is a filter on $\mathbb{P}$, we say that G is *M-generic* if for every maximal antichain $A \in M$ of $\mathbb{P}$, the intersection $G \cap A \cap M \neq \emptyset$. It must be explicitly specified what M-generic means in this context since the usual notion of generic filters makes sense only for transitive structures and M is not necessarily transitive. This definition of M-generic is closely related to the definition for transitive structures. To see this, let M^* be the Mostowski collapse of M and $\mathbb{P}^*$ be the image of $\mathbb{P}$ under the collapse. Let $G^* \subseteq \mathbb{P}^*$ be the pointwise image of $G \cap M$ under the collapse. Then G is M-generic if and only if G^* is M^*-generic for

$\mathbb{P}^*$ in the usual sense.

Definition 1.20. Let $\mathbb{P} \in H_\lambda$ be a poset and M be an elementary submodel of H_λ containing $\mathbb{P}$. Then a condition $q \in \mathbb{P}$ is *M-generic* if and only if every V-generic filter $G \subseteq \mathbb{P}$ containing q is M-generic.

Definition 1.21. A poset $\mathbb{P}$ is *proper* if for every $\lambda > 2^{|\mathbb{P}|}$ and every countable $M \prec H_\lambda$ containing $\mathbb{P}$, for every $p \in \mathbb{P} \cap M$, there is an M-generic condition below p.

It can be shown that it is actually equivalent to consider only some fixed $\lambda > 2^{|\mathbb{P}|}$ and to show that generic conditions exist only for a club of countable $M \prec H_\lambda$ [20] (p. 102). I will go back and forth between these equivalent definitions when proving whether a given poset is proper.

Definition 1.22. *The Proper Forcing Axiom* (PFA) is the assertion that for every proper poset $\mathbb{P}$ and every collection $\mathcal{D}$ of at most ω_1 many dense subsets of $\mathbb{P}$, there is a filter on $\mathbb{P}$ that meets all of them.

Proper forcing was invented by Shelah, who sought a class of ω_1-preserving forcing notions that would be preserved under countable support iterations (for an introduction to proper forcing see [9] (p. 601) or [20]). The two familiar classes of ω_1-preserving forcing notions, namely the c.c.c.

and countably closed forcing notions, turn out to be proper as well. The Proper Forcing Axiom, introduced by Baumgartner [1], is easily seen to be a generalization of Martin's Axiom since c.c.c. posets are proper and PFA decides the size of the continuum is ω_2. The later fact is a highly nontrivial result in [22]. In many respects, however, PFA is very much unlike MA. Not only does it decide the size of the continuum, the axiom also has large cardinal strength. The best known large cardinal upper bound on the consistency of PFA is a supercompact cardinal [1]. Much fruitful set theoretical work in recent years has involved PFA and its consequences.

1.5 Proof of Theorem 1.7

In this section, I will use PFA to prove the *ω_2-Ehrenfeucht Principle for arithmetically closed proper Scott sets*. Theorem 1.7 will follow as a corollary.

A filter G on the poset $\mathfrak{X}/\mathrm{Fin}$ is easily seen to be a filter on the Boolean algebra $\mathfrak{X}$. By extending G to a larger filter if necessary, we can assume without loss of generality that G is an ultrafilter. Recall that to prove the ω_2-Ehrenfeucht Principle, given a model M of size $\leq \omega_1$ and a Scott set $\mathfrak{X}$ such $\mathrm{SSy(M)} \subseteq \mathfrak{X}$, we need to find for every $A \in \mathfrak{X}$, an elementary extension N such that $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}$. The strategy will be to find ω_1 many dense subsets of $\mathfrak{X}/\mathrm{Fin}$ such that if G is a partially generic ultrafilter meeting all

of them, then the standard system of the ultrapower of M by G will stay within $\mathfrak{X}$. Thus, if $\mathfrak{X}$ is proper, we will be able to use PFA to obtain such an ultrafilter. I will also show that to every $A \in \mathfrak{X}$, there corresponds a set $B \in \mathfrak{X}$ such that whenever B is in an ultrafilter G, the set A will end up in the ultrapower of M by G.

Let $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$ and expand the language of arithmetic $\mathcal{L}_A$ to include unary predicates $\underline{A}$ for all $A \in \mathcal{S}$. Then the structure $\mathcal{A} = \langle \mathbb{N}, A \rangle_{A \in \mathcal{S}}$ is a structure of this expanded language with the natural interpretation. Since Scott sets are closed under relative computability, basic computability theory arguments show that if $\mathfrak{X}$ is a Scott set, the structure $\mathcal{A} = \langle \mathbb{N}, A \rangle_{A \in \mathfrak{X}}$ is closed under Δ_1-definability. That is, if B is Δ_1-definable in $\mathcal{A}$, then $B \in \mathfrak{X}$.

Definition 1.23. A collection $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$ is *arithmetically closed* if the structure $\mathcal{A} = \langle \mathbb{N}, A \rangle_{A \in \mathcal{S}}$ is closed under definability. That is, if B is definable in $\mathcal{A}$, then $B \in \mathcal{S}$.

A Scott set $\mathfrak{X}$ is *arithmetically closed* simply when it satisfies Definition 1.23. Observe actually that if $\mathcal{S}$ is arithmetically closed, then it is a Scott set. Thus, arithmetic closure subsumes the definition of a Scott set. An easy induction on the complexity of formulas establishes that if $\mathfrak{X}$ is a Boolean algebra of sets and $\mathcal{A} = \langle \mathbb{N}, A \rangle_{A \in \mathfrak{X}}$ is closed under Σ_1-definability, then $\mathfrak{X}$ is

arithmetically closed. Hence a Scott set is arithmetically closed if and only if it is closed under the Turing jump operation.

Definition 1.24. Say that $\langle B_n \mid n \in \mathbb{N}\rangle$ is *coded* in $\mathfrak{X}$ if there is $B \in \mathfrak{X}$ such that $B_n = \{m \in \mathbb{N} \mid \langle n, m\rangle \in B\}$. Given $\langle B_n \mid n \in \mathbb{N}\rangle$ coded in $\mathfrak{X}$ and $C \in \mathfrak{X}/\mathrm{Fin}$, say that C *decides* $\langle B_n \mid n \in \mathbb{N}\rangle$ if whenever U is an ultrafilter on $\mathfrak{X}$ and $C \in U$, then $\{n \in \mathbb{N} \mid B_n \in U\} \in \mathfrak{X}$. Call a Scott set $\mathfrak{X}$ *decisive* if for every $\langle B_n \mid n \in \mathbb{N}\rangle$ coded in $\mathfrak{X}$, the set $\mathscr{D} = \{C \in \mathfrak{X}/\mathrm{Fin} \mid C \text{ decides } \langle B_n \mid n \in \mathbb{N}\rangle\}$ is dense in $\mathfrak{X}/\mathrm{Fin}$.

Decisiveness is precisely the property of a Scott set which is required for our proof of Theorem 1.7. I will show below that decisiveness is equivalent to arithmetic closure.

Lemma 1.25. *The following are equivalent:*

(1) $\mathfrak{X}$ *is an arithmetically closed Scott set.*

(2) $\mathfrak{X}$ *is a decisive Scott set.*

(3) $\mathfrak{X}$ *is a Scott set such that for every* $\langle B_n \mid n \in \omega\rangle$ *coded in* $\mathfrak{X}$*, there is* $C \in \mathfrak{X}/\mathrm{Fin}$ *deciding* $\langle B_n \mid n \in \omega\rangle$.

Proof.

(1)$\Longrightarrow$(2):[5] Assume that $\mathfrak{X}$ is arithmetically closed. Fix $A \in \mathfrak{X}/\mathrm{Fin}$ and a

[5]Similar arguments have appeared in [5] and other places.

sequence $\langle B_n \mid n \in \mathbb{N}\rangle$ coded in $\mathfrak{X}$. We need to show that there is an element in $\mathfrak{X}/\mathrm{Fin}$ below A deciding $\langle B_n \mid n \in \mathbb{N}\rangle$. For every finite binary sequence s, we will define B_s by induction on the length of s. Let $B_\emptyset = A$. Given B_s where s has length n, define $B_{s1} = B_s \cap B_n$ and $B_{s0} = B_s \cap (\mathbb{N} - B_n)$. Define the binary tree $T = \{s \in 2^{<\omega} \mid B_s \text{ is infinite}\}$. Clearly T is infinite since if we split an infinite set into two pieces one of them must still be infinite. Since $\mathfrak{X}$ is arithmetically closed and T is arithmetic in A and $\langle B_n \mid n \in \mathbb{N}\rangle$, it follows that $T \in \mathfrak{X}$. Thus, $\mathfrak{X}$ contains a cofinal branch P through T. Define $C = \{b_n \mid n \in \mathbb{N}\}$ such that b_0 is least element of $B_\emptyset$ and b_{n+1} is least element of $B_{P\restriction n}$ that is greater than b_n. Clearly C is infinite and $C \subseteq A$. Now suppose U is an ultrafilter on $\mathfrak{X}$ and $C \in U$, then $B_n \in U$ if and only if $C \subseteq_{\mathrm{Fin}} B_n$. Thus, $\{n \in \mathbb{N} \mid B_n \in U\} = \{n \in \mathbb{N} \mid C \subseteq_{\mathrm{Fin}} B_n\} \in \mathfrak{X}$, since $\mathfrak{X}$ is arithmetically closed.

(2)$\Longrightarrow$(3): Clear.

(3)$\Longrightarrow$(1): It suffices to show that $\mathfrak{X}$ is closed under the Turing jump operation. Fix $A \in \mathfrak{X}$ and define a sequence $\langle B_n \mid n \in \omega\rangle$ by $k \in B_n$ if and only if the Turing program coded by n with oracle A halts on input n in less than k many steps. Clearly the sequence is computable from A, and hence coded in $\mathfrak{X}$. Let $H = \{n \in \mathbb{N} \mid \text{the program coded by } n \text{ with oracle } A \text{ halts on input } n\}$ be the halting problem for A. It should be clear that

$n \in H$ implies that B_n is cofinite and $n \notin H$ implies that $B_n = \emptyset$. Let $C \in \mathfrak{X}/\mathrm{Fin}$ deciding $\langle B_n \mid n \in \omega \rangle$ and U be any ultrafilter containing C, then $\{n \in \mathbb{N} \mid B_n \in U\} \in \mathfrak{X}$. But this set is exactly H. This shows that $H \in \mathfrak{X}$, and hence $\mathfrak{X}$ is closed under the Turing jump operation. □

Theorem 1.26. *Assuming* PFA*, the ω_2-Ehrenfeucht Principle for arithmetically closed proper Scott sets holds. That is, if $\mathfrak{X}$ is an arithmetically closed proper Scott set and M is a model of* PA *of size $\leq \omega_1$ whose standard system is contained in $\mathfrak{X}$, then for any $A \in \mathfrak{X}$, there is an elementary extension $M \prec N$ such that $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}$.*

Proof. I will build N using a variation on the ultrapower construction introduced by Kirby and Paris [13]. Fix a model M of PA and a Scott set $\mathfrak{X}$ such that $\mathrm{SSy}(M) \subseteq \mathfrak{X}$. Let G be some ultrafilter on $\mathfrak{X}$. If $f : \mathbb{N} \to M$, we say that f is *coded* in M when there is $a \in M$ such that $(a)_n = f(n)$ for all $n \in \mathbb{N}$. Given f and g coded in M, define $f \sim_G g$ if $\{n \in \mathbb{N} \mid f(n) = g(n)\} \in G$. The definition makes sense since clearly $\{n \in \mathbb{N} \mid f(n) = g(n)\} \in \mathrm{SSy}(M) \subseteq \mathfrak{X}$. The classical ultrapower construction uses an ultrafilter on $\mathcal{P}(\mathbb{N})$ and all functions from $\mathbb{N}$ to M. This construction uses only functions coded in M, and therefore needs only an ultrafilter on $\mathrm{SSy}(M) \subseteq \mathfrak{X}$. As in the classical construction, we get an equivalence relation and a well-defined $\mathcal{L}_A$ structure

on the equivalence classes. The proof relies on the fact that $\mathfrak{X}$ is a Boolean algebra. Call $\Pi_{\mathfrak{X}}M/G$ the collection of equivalence classes $[f]_G$ where f is coded in M. Also, as usual, we get:

Lemma 1.26.1. *Łoś' Lemma holds. That is,* $\Pi_{\mathfrak{X}}M/G \models \varphi([f]_G)$ *if and only if* $\{n \in \mathbb{N} \mid M \models \varphi(f(n))\} \in G$.

Proof. Classical argument. □

Lemma 1.26.2. *For every* $A \in \mathfrak{X}$*, there is* $B \in \mathfrak{X}/\mathrm{Fin}$ *such that if* G *is any ultrafilter on* $\mathfrak{X}$ *containing* B*, then* $A \in \mathrm{SSy}(\Pi_{\mathfrak{X}}M/G)$.

Proof. Let χ_A be the characteristic function of A. For every $n \in \mathbb{N}$, define $B_n = \{m \in \mathbb{N} \mid (m)_n = \chi_A(n)\}$. Then clearly each $B_n \in \mathfrak{X}$. Also $\langle B_n \mid n \in \mathbb{N}\rangle$ is coded in $\mathfrak{X}$ since the sequence is arithmetic in A. Observe that the intersection of any finite number of B_n is infinite. Let $B = \{b_n \mid n \in \mathbb{N}\}$ where b_0 is least element of B_0 and b_{n+1} is least element of $\cap_{m \leq n+1} B_m$ that is greater than b_n. Then $B \subseteq_{\mathrm{Fin}} B_n$ for all $n \in \mathbb{N}$. It follows that if G is any ultrafilter containing B, then G must contain all the B_n as well. Let G be an ultrafilter containing B. Let $id : \mathbb{N} \to \mathbb{N}$ be the identity function. I claim that $([id]_G)_n = \chi_A(n)$. It will follow that $A \in \mathrm{SSy}(\Pi_{\mathfrak{X}}M/G)$. But this is true since $([id]_G)_n = \chi_A(n)$ if and only if $\{m \in \mathbb{N} \mid (m)_n = \chi_A(n)\} = B_n \in G$. □

Lemma 1.26.2 tells us that if we want to add some set A to the standard system of the ultrapower that we are building, we just have to make sure that a correct set gets put into the ultrafilter. Thus, it easily follows that that we can build ultrapowers of M having any given element of $\mathfrak{X}$ in the standard system.

The crucial step of the construction is to find a family of size ω_1 of dense subsets of $\mathfrak{X}/\mathrm{Fin}$ such that if the ultrafilter meets all members of the family, the standard system of the ultrapower stays within $\mathfrak{X}$. It is here that we need all the extra hypothesis of Theorem 1.26 including the *decisiveness* of $\mathfrak{X}$.

Recall that a set E is in the standard system of a nonstandard model if and only if there is an element e such that $E = \{n \in \mathbb{N} \mid (e)_n = 1\}$ (Proposition 1.2), meaning E is coded in the model. Thus, we have to show that the sets coded by elements of $\Pi_{\mathfrak{X}} M/G$ are in $\mathfrak{X}$.

Lemma 1.26.3. *For every function f coded in M, there is a dense subset $\mathscr{D}_f$ of $\mathfrak{X}/\mathrm{Fin}$ such that if G meets $\mathscr{D}_f$, then $[f]_G \in \Pi_{\mathfrak{X}} M/G$ codes a set in $\mathfrak{X}$.*

Proof. Fix a function f coded in M and let $E_f = \{n \in \mathbb{N} \mid \Pi_{\mathfrak{X}} M/G \models ([f]_G)_n = 1\}$. By Łoś' Lemma, $\Pi_{\mathfrak{X}} M/G \models ([f]_G)_n = 1$ if and only if $\{m \in \mathbb{N} \mid (f(m))_n = 1\} \in G$. Define $B_{n,f} = \{m \in \mathbb{N} \mid (f(m))_n = 1\}$ and note that $\langle B_{n,f} \mid n \in \mathbb{N}\rangle$ is coded in $\mathrm{SSy}(M)$. Observe that $n \in E_f$ if and

only if $B_{n,f} \in G$. Thus, we have to make sure that $\{n \in \mathbb{N} \mid B_{n,f} \in G\} \in \mathfrak{X}$. Let $\mathscr{D}_f = \{C \in \mathfrak{X}/\mathrm{Fin} \mid C \text{ decides } \langle B_{n,f} \mid n \in \mathbb{N}\rangle\}$. Since $\mathfrak{X}$ is decisive, $\mathscr{D}_f$ is dense. Clearly if G meets $\mathscr{D}_f$, the set coded by $[f]_G$ will be in $\mathfrak{X}$. □

Now we can finish the proof of Theorem 1.26. Let $\mathcal{D} = \{\mathscr{D}_f \mid f \text{ is coded in } M\}$. Since M has size $\leq \omega_1$, the collection $\mathcal{D}$ has size $\leq \omega_1$ also. Assuming PFA guarantees that we can find an ultrafilter G meeting every $\mathscr{D}_f \in \mathcal{D}$. But this is precisely what forces the standard system of $\Pi_{\mathfrak{X}} M/G$ to stay inside $\mathfrak{X}$. □

Theorem 1.7 now follows directly from Theorem 1.26.

Proof of Theorem 1.7. Since PFA implies $2^\omega = \omega_2$ and Scott sets of size ω_1 are already handled by Knight and Nadel's result, we only need to consider Scott sets of size ω_2. But the result for these follows from Theorem 1.16 and the ω_2-Ehrenfeucht Principle established by Theorem 1.26. □

1.6 Extensions of Ehrenfeucht's Lemma

In this section, I will go through some results related to the question of extending Ehrenfeucht's lemma to models of size ω_1 (ω_2-Ehrenfeucht Principle).

Question 1.27. Does the ω_2-Ehrenfeucht Principle hold?

Theorem 1.26 shows that in a universe satisfying PFA, the ω_2-*Ehrenfeucht Principle for arithmetically closed proper Scott sets* holds. Next, I will use the same techniques to show that the κ-*Ehrenfeucht Principle for arithmetically closed Scott sets* holds for all κ if we only consider models with *countable* standard systems. For this argument, we do not need to use PFA or properness.

Theorem 1.28. *If M is a model of* PA *whose standard system is countable and contained in an arithmetically closed Scott set $\mathfrak{X}$, then for any $A \in \mathfrak{X}$, there is an elementary extension $M \prec N$ such that $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}$.*

Proof. Fix an arithmetically closed Scott set $\mathfrak{X}$ and a model M of PA such that $\mathrm{SSy}(M)$ is countable and contained in $\mathfrak{X}$. To mimic the proof of Theorem 1.26, we need to find an ultrafilter G on $\mathfrak{X}$ which meets the dense sets $\mathscr{D}_f = \{C \in \mathfrak{X}/\mathrm{Fin} \mid C \text{ decides } \langle B_{n,f} \mid n \in \mathbb{N}\rangle\}$. I claim that there are only countably many $\mathscr{D}_f$. If this is the case, then such an ultrafilter G exists without any forcing axiom assumption. Given $f : \mathbb{N} \to M$, let B_f code $\langle B_{n,f} \mid n \in \mathbb{N}\rangle$. There are possibly as many f as elements of M, but there can be only countably many B_f since each $B_f \in \mathrm{SSy}(M)$. It remains only to observe that $\mathscr{D}_f$ is determined by B_f. So there are as many $\mathscr{D}_f$ as there are different B_f. Thus, there are only countably many $\mathscr{D}_f$ in spite of the fact

that M can be arbitrarily large. □

The same idea can be used to extend Theorem 1.26 to show that the *κ-Ehrenfeucht Principle for arithmetically closed proper Scott sets* will hold for all κ if we consider only models whose standard system has size ω_1.

Theorem 1.29. *Assuming* PFA*, if $\mathfrak{X}$ is an arithmetically closed proper Scott set and M is a model of* PA *whose standard system has size ω_1 and is contained in $\mathfrak{X}$, then for any $A \in \mathfrak{X}$, there is an elementary extension $M \prec N$ such that $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}$.*

It is also an easy consequence of an amalgamation result for models of PA that the κ-Ehrenfeucht Principle holds for all κ for models with a *countable nonstandard elementary initial segment*. Neither PFA nor arithmetic closure is required for this result.

Theorem 1.30. *Suppose M_0, M_1, and M_2 are models of* PA *such that $M_0 \prec_{cof} M_1$ and $M_0 \prec_{end} M_2$. Then there is an amalgamation M_3 of M_1 and M_2 over M_0 such that $M_1 \prec_{end} M_3$ and $M_2 \prec_{cof} M_3$.* (see [15], p. 40)

Theorem 1.31. *Suppose M is a model of* PA *with a countable nonstandard elementary initial segment and $\mathfrak{X}$ is a Scott set such that $\mathrm{SSy}(M) \subseteq \mathfrak{X}$. Then for any $A \in \mathfrak{X}$, there is an elementary extension $M \prec N$ such that $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}$.*

Proof. Fix a set $A \in \mathfrak{X}$. Let K be a countable nonstandard elementary initial segment of M, then $\mathrm{SSy}(K) = \mathrm{SSy}(M)$. By Ehrenfeucht's Lemma, there is an extension $K \prec_{cof} K'$ such that $A \in \mathrm{SSy}(K') \subseteq \mathfrak{X}$. By Theorem 1.31, there is a model N, an amalgamation of K' and M over K, such that $K' \prec_{end} N$ and $M \prec_{cof} N$. It follows that $\mathrm{SSy}(K') = \mathrm{SSy}(N)$. Thus, $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}$. □

Corollary 1.32. *The κ-Ehrenfeucht Principle holds for ω_1-like models for all cardinals κ.*

These observations suggest that if the ω_2-Ehrenfeucht Principle fails to hold, one should look to models with an uncountable standard system for such a counterexample.

Question 1.33. Does the ω_2-Ehrenfeucht Principle hold for models with a countable standard system? That is, can we remove the assumption of *arithmetic closure* from Theorem 1.28?

1.7 Other Applications of $\mathfrak{X}$/Fin

It appears that $\mathfrak{X}$/Fin is a natural poset to use in several unresolved questions in the field of models of PA. In the previous sections, I used it to find new conditions for extending Ehrenfeucht's Lemma and Scott's Problem. Here I

will mention some other instances in which the poset naturally arises.

Definition 1.34. Let $\mathcal{L}$ be some language extending $\mathcal{L}_A$. We say that a model M of $\mathcal{L}$ satisfies PA* if M satisfies induction axioms in the expanded language. If $M \models \mathrm{PA}^*$, then $M \subseteq N$ is a *conservative extension* if it is a proper extension and every parametrically definable subset of N when restricted to M is also definable in M.

Gaifman showed in [7] that for any countable $\mathcal{L}$, every $M \models \mathrm{PA}^*$ in $\mathcal{L}$ has a conservative elementary extension. A result of George Mills shows that the statement fails for uncountable languages. Mills proved that every countable nonstandard model $M \models \mathrm{PA}^*$ in a countable language has an expansion to an uncountable language such that $M \models \mathrm{PA}^*$ in the expanded language but has no conservative elementary extension (see [15], p. 168). His techniques failed for the standard model, leaving open the question whether there is an expansion of the standard model $\mathbb{N}$ to some uncountable language that does not have a conservative elementary extension. This question has recently been answered by Ali Enayat, who demonstrated that there is always an uncountable arithmetically closed Scott set $\mathfrak{X}$ such that $\langle \mathbb{N}, A\rangle_{A\in\mathfrak{X}}$ has no conservative elementary extension [4]. This raises the question of whether we can say something general about Scott sets $\mathfrak{X}$ for which $\langle \mathbb{N}, A\rangle_{A\in\mathfrak{X}}$ has a

conservative elementary extension.

Theorem 1.35. *Assuming* PFA*, if* $\mathfrak{X}$ *is an arithmetically closed proper Scott set of size* ω_1*, then* $\langle \mathbb{N}, A\rangle_{A\in\mathfrak{X}}$ *has a conservative elementary extension.*

Proof. Let $\mathcal{L}_{\mathfrak{X}}$ be the language of arithmetic $\mathcal{L}_A$ together with predicates for sets in $\mathfrak{X}$. Let G be an ultrafilter on $\mathfrak{X}$. We define $\Pi_{\mathfrak{X}}\mathbb{N}/G$, the ultrapower of $\mathbb{N}$ by G, to consist of equivalence classes of functions coded in $\mathfrak{X}$. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be *coded* in a Scott set $\mathfrak{X}$ if there is a set in $\mathfrak{X}$ whose elements are exactly the codes for elements of the graph of f. We have to make this modification to the construction of the proof of Theorem 1.26 since the idea of functions coded in the model clearly does not make sense for $\mathbb{N}$. The usual arguments show that we can impose an $\mathcal{L}_{\mathfrak{X}}$ structure on $\Pi_{\mathfrak{X}}\mathbb{N}/G$ and Łoś' Lemma holds. I will show, by choosing G carefully, that $\langle \Pi_{\mathfrak{X}}\mathbb{N}/G, A'\rangle_{A\in\mathfrak{X}}$ is a conservative extension of $\langle \mathbb{N}, A\rangle_{A\in\mathfrak{X}}$ where $A' = \{[f]_G \in \Pi_{\mathfrak{X}}\mathbb{N}/G \mid \{n \in \mathbb{N} \mid f(n) \in A\} \in G\}$. Fix a set E definable in $\langle \Pi_{\mathfrak{X}}\mathbb{N}/G, A'\rangle_{A\in\mathfrak{X}}$ by a formula $\varphi(x, [f]_G)$. Observe that $n \in E \leftrightarrow \Pi_{\mathfrak{X}}\mathbb{N}/G \models \varphi(n, [f]_G) \leftrightarrow B^{\varphi,f}_n = \{m \in \mathbb{N} \mid \mathbb{N} \models \varphi(n, f(m))\} \in G$. Let $\mathscr{D}_{\varphi,f} = \{C \in \mathfrak{X}/\mathrm{Fin} \mid C \text{ decides } \langle B^{\varphi,f}_n \mid n \in \mathbb{N}\rangle\}$. The sets $\mathscr{D}_{\varphi,f}$ are dense since $\mathfrak{X}$ is decisive. Clearly if G meets all the $\mathscr{D}_{\varphi,f}$, the ultrapower $\langle \Pi_{\mathfrak{X}}\mathbb{N}/G, A'\rangle_{A\in\mathfrak{X}}$ will be a conservative extension of $\langle \mathbb{N}, A\rangle_{A\in\mathfrak{X}}$. Finally, since $\mathfrak{X}$ has size ω_1,

there are at most ω_1 many formulas φ of $\mathcal{L}_{\mathfrak{X}}$ and functions f coded in $\mathfrak{X}$, and hence at most ω_1 many dense sets $\mathscr{D}_{\varphi,f}$. So we can find the desired G by PFA. □

It should be observed here that the above construction yields another proof of Scott's Theorem for countable *arithmetically closed* Scott sets. Given a countable arithmetically closed Scott set $\mathfrak{X}$, we find a conservative extension for the model $\langle \mathbb{N}, A\rangle_{A\in\mathfrak{X}}$ by the above construction. Since $\mathfrak{X}$ is countable, we do not need PFA to find the ultrafilter. Finally, it is easy to see that the standard system of the reduct of the conservative extension to the language of PA has to be exactly $\mathfrak{X}$.

Another open question in the field of models of PA, for which $\mathfrak{X}/\mathrm{Fin}$ is relevant, involves the existence of minimal cofinal extensions for uncountable models.

Definition 1.36. Let M be a model of PA, then $M \prec N$ is a *minimal extension* if it is a proper extension and whenever $M \prec K \prec N$, either $K = M$ or $K = N$.

Theorem 1.37. *Every nonstandard countable model of* PA *has a minimal cofinal extension.* (see [15], p. 28)

Question 1.38. Does every uncountable model of PA have a minimal cofinal extension?

Gaifman showed that every model of PA, regardless of cardinality, has a minimal end extension [7].

Definition 1.39. Let $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ be a Boolean algebra. If U is an ultrafilter on $\mathfrak{X}$, we say that U is *Ramsey* if for every function $f : \mathbb{N} \to \mathbb{N}$ coded in $\mathfrak{X}$, there is a set $A \in U$ such that f is either 1-1 or constant on A.

Lemma 1.40. *If M is a nonstandard model of* PA *such that* $\mathrm{SSy}(M)$ *has a Ramsey ultrafilter, then M has a minimal cofinal extension.*[6]

Proof. Let U be a Ramsey ultrafilter on $\mathrm{SSy}(M)$. The strategy will be to show that the ultrapower $\Pi_{\mathrm{SSy}(M)}M/U$ is a minimal cofinal extension of M. The meaning of $\Pi_{\mathrm{SSy}(M)}M/U$ here is identical to the one in the proof of Theorem 1.26. First, observe that for any ultrafilter U, we have $\Pi_{\mathfrak{X}}M/U = Scl([id]_U, M)$, the Skolem closure of the equivalence class of the identity function together with elements of M. This holds since any $[f]_U = t([id]_U)$ where t is the Skolem term defined by f in M. Next, observe that such ultrapowers are always cofinal. To see this, fix $[f]_U \in \Pi_{\mathfrak{X}}M/U$ and let $a > f(n)$ for all $n \in \mathbb{N}$. Such a exists since f is coded in M. Clearly $[f]_U < [c_a]_U$

[6] Connections between minimal extensions and Ramsey ultrafilters appear widely in the literature, see, for example, [8] (ch. 0) on uses in set theory.

where $c_a(n) = a$ for all $n \in \mathbb{N}$. These observations hold for any Scott set $\mathfrak{X} \supseteq \mathrm{SSy}(M)$ and, in particular, for $\mathfrak{X} = \mathrm{SSy}(M)$. To show that the extension $\Pi_{\mathrm{SSy}(M)}M/U$ is minimal, we fix $M \prec K \prec \Pi_{\mathfrak{X}}M/U$ and show that $K = M$ or $K = \Pi_{\mathfrak{X}}M/U$. It suffices to see that $[id]_U \in Scl([f]_U, M)$ for every $[f]_U \in (\Pi_{\mathrm{SSy}(M)}M/U) - M$. Fix $f : \mathbb{N} \to M$ and define $g : \mathbb{N} \to \mathbb{N}$ such that $g(0) = 0$ and $g(n) = n$ if $f(n)$ is not equal to $f(m)$ for any $m < n$, or $g(n) = m$ where m is least such that $f(m) = f(n)$. Observe that $g \in \mathrm{SSy}(M)$. Also for any $A \subseteq \mathbb{N}$, the function g is 1-1 or constant on A if and only if f is. Since U is Ramsey, g is either constant or 1-1 on some set $A \in U$. Hence f is either constant or 1-1 on A as well. If f is constant on A, then $[f]_U \in M$. If f is 1-1 on A, let s be a Skolem term that is the inverse of f on A. Then clearly $s([f]_U) = [id]_U$. This completes the argument that $\Pi_{\mathrm{SSy}(M)}M/U$ is a minimal cofinal extension of M. □

The converse to the above theorem does not hold. That is, if M has a minimal cofinal extension it does not follow that there is a Ramsey ultrafilter on $\mathrm{SSy}(M)$. The following lemma will help us see this. Let M be a model of PA and let $\mathcal{L}_M$ be the language of PA together with constants for all elements of M. Let T_M be the theory of M in $\mathcal{L}_M$. If $M \prec N$ and $a \in N$, let $M(a)$ denote the Skolem closure of M and a in N. A complete type $p(x)$ is a type

over M if it is in $\mathcal{L}_M$ and consistent with T_M. A complete type $p(x)$ over M is *minimal* if for every model N of T_M, we can extend $p(x)$ to a complete type $q(x)$ over N such that whenever $N \prec K$ and $a \in K$ realizes $q(x)$, then $N(a)$ is a minimal end extension of N. Minimal types are due to Gaifman, who showed that they exist over every model M of PA (see [7]).

Lemma 1.41. *Every model of* PA *has an elementary end extension having a minimal cofinal extension.*[7]

Proof. Suppose M is a model of PA. Let $p(x)$ be a minimal type over M and extend M to $M(a)$ with a realizing $p(x)$. Thus, $M(a)$ is a minimal end extension of M. Since $M(a)$ is a model of T_M, the type $p(x)$ has an extension to a type $q(x)$ over $M(a)$ which generates minimal extensions. Extend $M(a)$ to $M(a)(b)$ with b realizing $q(x)$. Thus, $M(a)(b)$ is a minimal end extension of $M(a)$. Also, since $q(x)$ extends $p(x)$, we have that b realizes $p(x)$, and hence $M(b)$ is a minimal end extension of M. I claim that $M(a)(b)$ is a minimal cofinal extension of $M(b)$. Since $M(b)$ is a minimal end extension of M, we know that $M(a)(b) \neq M(b)$. To see that $M(b)$ is cofinal in $M(a)(b)$, let N be the closure of $M(b)$ under initial segment in $M(a)(b)$. Since $a < b$, it follows that $a \in N$, and since $b \in M(b)$, it follows that $b \in N$. Thus, $N = M(a)(b)$, and hence $M(b)$ is cofinal in $M(a)(b)$. To see that $M(a)(b)$ is a minimal

[7] I am grateful to Haim Gaifman for pointing this out.

extension of $M(b)$, suppose $M(b) \prec K \prec M(a)(b)$. If $K \neq M(b)$, then there is $c \in K$ such that $c \notin M(b)$. Since $c \in M(a)(b)$, there is a Skolem function $t(x, y)$ defined with parameters from M such that $c = t(a, b)$ in $M(a)(b)$. By elementarity, the model K must have the least x such that $t(x, b) = c$. Clearly $x \leq a$, which implies that $x \in M(a)$. Also clearly $x \notin M$, since if $x \in M$, then $c \in M(b)$. It follows that $x \in M(a) - M$. But this implies that $a \in K$ by the minimality of $M(a)$. Thus, $K = M(a)(b)$. This completes the proof that $M(a)(b)$ is a minimal cofinal extension of $M(b)$. □

Kunen has shown in [16] that it is consistent that there are no Ramsey ultrafilters on $\mathcal{P}(\mathbb{N})$. So suppose M is a model of PA whose standard system is $\mathcal{P}(\mathbb{N})$. Let N be an extension of M having a minimal cofinal extension. Thus, N has a minimal cofinal extension but there are no Ramsey ultrafilters on $\mathrm{SSy}(N) = \mathcal{P}(\mathbb{N})$.

Theorem 1.42. *Assuming* PFA*, Ramsey ultrafilters exist for proper Scott sets of size ω_1. Thus, if M is a model of* PA *and* $\mathrm{SSy}(M)$ *is proper of size ω_1, then M has a minimal cofinal extension.*

Proof. The existence of a Ramsey ultrafilter involves being able to meet a family of dense sets. To see this, fix $f : \mathbb{N} \to \mathbb{N}$ and observe that $\mathscr{D}_f = \{A \in \mathrm{SSy}(M)/\mathrm{Fin} \mid f \text{ is 1-1 on } A \text{ or } f \text{ is constant on } A\}$ is dense. Note

that to see that $\mathscr{D}_f$ is dense, we do not need to assume that $\mathrm{SSy}(M)$ is arithmetically closed. □

The proof of Theorem 1.42 shows that any M with a countable standard system will have a minimal cofinal extension since we do not need PFA to construct an ultrafilter meeting countably many dense sets.

1.8 When is $\mathfrak{X}$/Fin Proper?

In this section, we investigate proper Scott sets. Recall that if $\mathscr{P}$ is a property of posets and $\mathfrak{X}$/Fin has $\mathscr{P}$, I will simply say that $\mathfrak{X}$ *has property* $\mathscr{P}$. At this stage, we are left to answer an almost purely set theoretic question:

Question 1.43. Which Scott sets are proper?

Theorem 1.44. *If $\mathfrak{X}$ is countable or $\mathfrak{X} = \mathcal{P}(\mathbb{N})$, then $\mathfrak{X}$ is proper.*

Proof. The class of proper posets includes c.c.c. and countably closed posets. If $\mathfrak{X}$ is countable, then $\mathfrak{X}$ is c.c.c., and if $\mathfrak{X} = \mathcal{P}(\mathbb{N})$, then $\mathfrak{X}$ is countably closed. □

We are already in a better position than with c.c.c. Scott sets since we have an instance of an uncountable proper Scott set, namely $\mathcal{P}(\mathbb{N})$. This does not, however, give us a new instance of Scott's Problem since we already know

by compactness that there are models of PA with standard system $\mathcal{P}(\mathbb{N})$. I will also show that it is consistent with ZFC that there exist uncountable proper Scott sets $\mathfrak{X} \neq \mathcal{P}(\mathbb{N})$.

The easiest way to show that a poset is proper is to show that it is c.c.c. or countably closed. We already know that if $\mathfrak{X}$ is c.c.c., then it is countable (Theorem 1.18). So this condition gives us no new proper Scott sets. It turns out that neither does the countably closed condition.

Theorem 1.45. *If $\mathfrak{X}$ is a countably closed Scott set, then $\mathfrak{X} = \mathcal{P}(\mathbb{N})$.*

Proof. First, I claim that if $\mathfrak{X}$ is countably closed, then $\mathfrak{X}$ is arithmetically closed. I will show that for every sequence $\langle B_n \mid n \in \omega \rangle$ coded in $\mathfrak{X}$, there is $C \in \mathfrak{X}$ deciding $\langle B_n \mid n \in \omega \rangle$. This suffices by Theorem 1.25. Fix $\langle B_n \mid n \in \omega \rangle$ coded in $\mathfrak{X}$. Define a descending sequence $B_0^* \geq B_1^* \geq \cdots \geq B_n^* \geq \cdots$ of elements of $\mathfrak{X}/\mathrm{Fin}$ by induction on n such that $B_0^* = B_0$ and B_{n+1}^* is $B_n^* \cap B_{n+1}$ if this intersection is infinite or $B_n^* \cap (\mathbb{N} - B_{n+1})$ otherwise. By countable closure, there is $C \in \mathfrak{X}/\mathrm{Fin}$ below this sequence. Clearly C decides $\langle B_n \mid n \in \omega \rangle$. Therefore $\mathfrak{X}$ is arithmetically closed. Now I will show that every $A \subseteq \mathbb{N}$ is in $\mathfrak{X}$. Define $B_n = \{m \in \mathbb{N} \mid (m)_n = \chi_A(n)\}$ as before. Let $A_m = \cap_{n \leq m} B_n$ and observe that $A_0 \geq A_1 \geq \cdots \geq A_m \geq \ldots$ in $\mathfrak{X}/\mathrm{Fin}$. By countable closure, there exists $C \in \mathfrak{X}/\mathrm{Fin}$ such that $C \subseteq_{\mathrm{Fin}} A_m$

for all $m \in \mathbb{N}$. Thus, $C \subseteq_{\text{Fin}} B_n$ for all $n \in \mathbb{N}$. It follows that $A = \{n \in \mathbb{N} \mid \exists m \, \forall k \in C \text{ if } k > m, \text{ then } (k)_n = 1\}$. Thus, A is arithmetic in C, and hence $A \in \mathfrak{X}$ by arithmetic closure. Since A was arbitrary, this concludes the proof that $\mathfrak{X} = \mathcal{P}(\mathbb{N})$. □

The countable closure condition can be weakened slightly. If a poset is just strategically ω-closed, it is enough to imply properness.

Definition 1.46. Let $\mathbb{P}$ be a poset, then $\mathscr{G}_{\mathbb{P}}$ is the following infinite game between players I and II: Player I plays an element $p_0 \in \mathbb{P}$, and then player II plays $p_1 \in \mathbb{P}$ such that $p_0 \geq p_1$. Then player I plays $p_1 \geq p_2$ and player II plays $p_2 \geq p_3$. Player I and II alternate in this fashion for ω steps to end up with the descending sequence $p_0 \geq p_1 \geq p_2 \geq \ldots \geq p_n \geq \ldots$. Player II wins if the sequence has a lower bound in $\mathbb{P}$. Otherwise, player I wins. A poset $\mathbb{P}$ is *ω-strategically closed* if player II has a winning strategy in the game $\mathscr{G}_{\mathbb{P}}$.

Observe that a strategically ω-closed Scott set has to be arithmetically closed. To see this, suppose that $\mathfrak{X}$ is a strategically ω-closed Scott set and $\langle B_n \mid n \in \omega \rangle$ is a sequence coded in $\mathfrak{X}$. We will find $C \in \mathfrak{X}/\text{Fin}$ deciding the sequence by having player I play either B_n or $\mathbb{N} - B_n$ intersected with the previous move of player II at the n^{th} step of the game.

Question 1.47. Are there Scott sets that are strategically ω-closed but not

countably closed?

One might wonder at this point whether it is possibly the case that a Scott set is proper only when it is countable or $\mathcal{P}(\mathbb{N})$. I will show that it is at least consistent with ZFC that this is not the case. I will first show that in $L[g]$ where L is the constructible universe and g is L-generic for the Cohen real forcing, the Scott set $\mathcal{P}(\mathbb{N})^L$ is proper. Clearly $\mathcal{P}(\mathbb{N})^L \neq \mathcal{P}(\mathbb{N})^{L[g]}$ since $g \notin \mathcal{P}(\mathbb{N})^L$.

Lemma 1.48. *Suppose $\mathbb{Q}$ is a c.c.c. poset and $G \subseteq \mathbb{P}$ is V-generic for a countably closed poset $\mathbb{P}$. Then $\mathbb{Q}$ remains c.c.c. in $V[G]$.*

Proof. Suppose $\mathbb{Q}$ does not remain c.c.c. in $V[G]$. Fix a $\mathbb{P}$-name $\dot{A}$ and $r \in \mathbb{P}$ such that $r \Vdash$ "$\dot{A}$ is a maximal antichain of $\check{\mathbb{Q}}$ of size ω_1". Choose $p_0 \leq r$ and $a_0 \in \mathbb{Q}$ such that $p_0 \Vdash \check{a}_0 \in \dot{A}$. Suppose that we have defined $p_0 \geq p_1 \geq \cdots \geq p_\xi \geq \cdots$ for $\xi < \beta$ where β is some countable ordinal, together with a corresponding sequence $\langle a_\xi \mid \xi < \beta \rangle$ of elements of $\mathbb{Q}$ such that $p_\xi \Vdash \check{a}_\xi \in \dot{A}$ and $a_{\xi_1} \neq a_{\xi_2}$ for all $\xi_1 < \xi_2$. By countable closure of $\mathbb{P}$, we can find $p \in \mathbb{P}$ such that $p \leq p_\xi$ for all $\xi < \beta$. Let $p_\beta \leq p$ and $a_\beta \in \mathbb{Q}$ such that $p_\beta \Vdash \check{a}_\beta \in \dot{A}$ and $a_\beta \neq a_\xi$ for all $\xi < \beta$. Such a_β must exist since we assumed $r \Vdash$ "$\dot{A}$ is a maximal antichain of $\check{\mathbb{Q}}$ of size ω_1" and $p \leq r$. Thus, we can build a descending sequence $\langle p_\xi \mid \xi < \omega_1 \rangle$ of elements of $\mathbb{P}$ and a

corresponding sequence $\langle a_\xi \mid \xi < \omega_1 \rangle$ of elements of $\mathbb{Q}$ such that $p_\xi \Vdash \check{a}_\xi \in \dot{A}$. But clearly $\langle a_\xi \mid \xi < \omega_1 \rangle$ is an antichain in V of size ω_1, which contradicts the assumption that $\mathbb{Q}$ was c.c.c.. □

Theorem 1.49. *If $\mathbb{P}$ is a poset in $M \prec H_\lambda$, then a V-generic filter $G \subseteq \mathbb{P}$ is M-generic if and only if $M \cap Ord = M[G] \cap Ord$.* (See, for example, [20], p. 105)

Proof. ($\Longrightarrow$): Suppose that a V-generic filter $G \subseteq \mathbb{P}$ is M-generic. Recall that $M[G] = \{\dot{a}_G \mid \dot{a} \in M \text{ is a } \mathbb{P}\text{-name}\}$. Let α be an ordinal of $M[G]$, then $\alpha = \dot{a}_G$ for some $\mathbb{P}$-name $\dot{a} \in M$. Let $D = \{p \in \mathbb{P} \mid \exists \beta \text{ ordinal such that } p \Vdash \dot{a} = \check{\beta} \text{ or } p \Vdash \text{“}\dot{a} \text{ is not an ordinal”}\}$. Clearly D is dense in $\mathbb{P}$ and $D \in M$ since it is definable from $\dot{a}$ and $\mathbb{P}$. Since G is M-generic, $D \cap M \cap G \neq \emptyset$. So let $q \in D \cap M \cap G$. Observe that q must force $\dot{a} = \check{\alpha}$, and hence α is definable from q and $\dot{a}$. Since both of these are in M, it follows that $\alpha \in M$. ($\Longleftarrow$): Suppose that $M \cap \text{Ord} = M[G] \cap \text{Ord}$. Fix $A \in M$ a maximal antichain of $\mathbb{P}$. We need to show that $A \cap M \cap G \neq \emptyset$. In H_λ, enumerate $A = \{q_\xi \mid \xi < \delta\}$. Since $M \prec H_\lambda$, we can assume without loss of generality that this enumeration is in M. Also in H_λ, we can build by mixing a $\mathbb{P}$-name $\dot{a}$ such that $q_\xi \Vdash \dot{a} = \check{\xi}$. Again, without loss of generality, we can assume $\dot{a} \in M$. Let $\dot{a}_G = \alpha$, then α is in $M[G]$, and hence in M by our assumption.

It also follows that $q_\alpha \in G$. But since $\alpha \in M$, the element $q_\alpha \in M$ as well. Thus, $q_\alpha \in A \cap M \cap G$, and consequently $A \cap M \cap G \neq \emptyset$. □

Theorem 1.50. *Suppose $\mathbb{Q}$ is a c.c.c. poset and $g \subseteq \mathbb{Q}$ is a V-generic filter. Also suppose $V[g]$ has the property that any $M \in V[g]$ such that $M \prec H_\lambda^V$ Mostowski collapses to an element of V. Then $\mathcal{P}(\mathbb{N})^V$ is proper in $V[g]$.*

Proof. I will prove that generic conditions exist for countable elementary substructures $\langle M[g], M, \mathbb{Q}, g\rangle \prec \langle H_\lambda[g], H_\lambda, \mathbb{Q}, g\rangle$ in $V[g]$. Here we view H_λ as a predicate and $\mathbb{Q}$ and g as constants. Since for large enough λ, we have $H_\lambda[g] = H_\lambda^{V[g]}$ (see Theorem 2.59) and the substructures $M[g]$ form a club, this suffices for properness. We need to prove that for every $A \in M[g] \cap \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$, there exists $B \in \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ such that $B \subseteq_{\mathrm{Fin}} A$ and every $V[g]$-generic filter $G \subseteq \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ containing B is also $M[g]$-generic. We will first consider M. Observe that $M \prec H_\lambda$, and so we can use the hypothesis that the Mostowski collapse of M must be in V. This is important since M itself may not be in V. Let $\pi : M \to \overline{M}$ be the Mostowski collapse, then $\overline{M} \in V$. Observe that $\pi(\mathcal{P}(\mathbb{N})^V) = \mathcal{P}(\mathbb{N})^V \cap M = \mathcal{P}(\mathbb{N})^V \cap \overline{M}$ since $\pi(A) = A$ for all $A \in \mathcal{P}(\mathbb{N})^V \cap M$. Also $\overline{M}$ thinks that $\mathscr{D}$ is a dense subset of $\pi(\mathcal{P}(\mathbb{N})^V/\mathrm{Fin}) = \mathcal{P}(\mathbb{N})^V/\mathrm{Fin} \cap M$ if and only if $\mathscr{D} = \pi(\mathscr{D}') = \mathscr{D}' \cap M$ and M thinks $\mathscr{D}'$ is dense in $\mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$. Since $\overline{M}$ is countable and in V, we can

enumerate the dense subsets of $\mathcal{P}(\mathbb{N})^V/\mathrm{Fin} \cap M$ that are in $\overline{M}$ in a countable sequence inside V. By the countable closure of $\mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ in V, we can find $B \in \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ such that every dense set in $\overline{M}$ contains something above B. Clearly B is $\overline{M}$-generic. I will end up proving that B is, in fact, $M[g]$-generic.

The first step is to show that B is M-generic. Suppose $G \subseteq \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ is $V[g]$-generic and $B \in G$. To see that G is M-generic, let $\mathscr{D}' \in M$ be a dense subset of $\mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$. Observe that $\pi(\mathscr{D}') = \mathscr{D}$ is dense in $\overline{M}$, and thus $G \cap \mathscr{D} \neq \emptyset$. But $\mathscr{D} = \mathscr{D}' \cap M$, and so $\mathscr{D}' \cap M \cap G \neq \emptyset$. This completes the argument that B is M-generic.

To see that G is $M[g]$-generic, I will show that $M[g] \cap Ord = M[g][G] \cap Ord$. We begin by observing that g is M-generic. To see this, suppose that $A \in M$ is a maximal antichain of $\mathbb{Q}$ in $V[g]$. Since $M \prec H_\lambda$, we have $A \in H_\lambda$, and hence A is countable in H_λ. Thus, $A \subseteq M$ and since g is V-generic, it must be that $A \cap g = A \cap M \cap g \neq \emptyset$. By the proof of Theorem 1.49, we conclude that $M[g] \cap Ord = M \cap Ord$. Observe also that $M[g][G] = M[G][g]$ since we are forcing with the product $\mathbb{Q} \times \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$. Thus, it suffices to show that $M[G][g] \cap Ord = M \cap Ord$. Since G is M-generic and clearly V-generic, we have, by the proof of Theorem 1.49, that $M \cap Ord = M[G] \cap Ord$. By Lemma 1.48, $\mathbb{Q}$ remains c.c.c. in $V[G]$. Since $M \prec H_\lambda$, it follows that $M[G] \prec H_\lambda[G] = H_\lambda^{V[G]}$. If $A \in M[G]$ is a max-

imal antichain of $\mathbb{P}$, then A is countable in $H_\lambda[G]$, and hence $A \subseteq M[G]$. Finally, since g is $V[G]$-generic, we have that g is $M[G]$-generic. Therefore $M[G] \cap Ord = M[G][g] \cap Ord$. □

Corollary 1.51. *Suppose $V = L$ or $V = L[A]$ for some $A \subseteq \omega_1$. Let $\mathbb{Q}$ be the forcing to add a Cohen real and let $g \subseteq \mathbb{Q}$ be V-generic. Then $\mathcal{P}(\mathbb{N})^V$ is an uncountable proper Scott set in $V[g]$ that is not the whole powerset of $\mathbb{N}$.*

Proof. Suppose $M \prec H_\lambda^V$ in $V[g]$. If $V = L$, then $H_\lambda^V = L_\lambda$. Therefore the Mostowski collapse of M is some L_α by condensation, and hence in L. The argument for $V = L[A]$ where $A \subseteq \omega_1$ is identical. So we have satisfied the hypothesis of Theorem 1.50, and hence $\mathcal{P}(\mathbb{N})^V$ is proper in $V[g]$. Since $\mathbb{Q}$ does not collapse cardinals, $\mathcal{P}(\mathbb{N})^V$ still has size ω_1 in $V[g]$. However, since $\mathbb{Q}$ adds new reals, $\mathcal{P}(\mathbb{N})^V$ is no longer the whole powerset of $\mathbb{N}$ in $V[g]$. □

This shows that there can be proper Scott sets that are not countable or $\mathcal{P}(\mathbb{N})$. But it does not give us a new instance of Scott's Problem since we already know that there exist models of PA whose standard system is $\mathcal{P}^L(\mathbb{N})$ or $\mathcal{P}^{L[A]}(\mathbb{N})$ by compactness applied in the respective models.

Next, I will show how to force the existence of proper Scott sets. I will begin by looking at what properness means in the specific context of Scott sets.

Proposition 1.52. *Suppose $\mathfrak{X}$ is a Scott set and $\mathscr{A}$ is a countable antichain of $\mathfrak{X}/\mathrm{Fin}$. Then for $B \in \mathfrak{X}$:*

(1) *Every V-generic filter $G \subseteq \mathfrak{X}/\mathrm{Fin}$ containing B meets $\mathscr{A}$.*

(2) *There exists a finite list $A_0, \ldots, A_n \in \mathscr{A}$ such that $B \subseteq_{\mathrm{Fin}} A_0 \cup \ldots \cup A_n$.*

Proof.

(2)$\Longrightarrow$(1): Suppose $B \subseteq_{\mathrm{Fin}} A_0 \cup \ldots \cup A_n$ for some $A_0, \ldots, A_n \in \mathscr{A}$. Since a V-generic filter G is an ultrafilter, one of the A_i must be in G.

(1)$\Longrightarrow$(2): Assume that every V-generic filter G containing B meets $\mathscr{A}$ and suppose toward a contradiction that (2) does not hold. Enumerate $\mathscr{A} = \{A_0, A_1, \ldots, A_n, \ldots\}$. It follows that for all $n \in \mathbb{N}$, the intersection $B \cap (\mathbb{N} - A_0) \cap \cdots \cap (\mathbb{N} - A_n)$ is infinite. Define $C = \{c_n \mid n \in \mathbb{N}\}$ such that c_0 is the least element of $B \cap (\mathbb{N} - A_0)$ and c_{n+1} is the least element of $B \cap (\mathbb{N} - A_0) \cap \cdots \cap (\mathbb{N} - A_{n+1})$ greater than c_n. Clearly $C \subseteq B$ and $C \subseteq_{\mathrm{Fin}} (\mathbb{N} - A_n)$ for all $n \in \mathbb{N}$. Let G be a V-generic filter containing C, then $B \in G$ and $(\mathbb{N} - A_n) \in G$ for all $n \in \mathbb{N}$. But this contradicts our assumption that G meets $\mathscr{A}$. □

Corollary 1.53. *Let $\mathfrak{X}$ be a Scott set. Then $\mathfrak{X}$ is proper if and only if there exists $\lambda > 2^{|\mathfrak{X}|}$ such that for every countable $M \prec H_\lambda$ containing $\mathfrak{X}$, whenever $C \in M \cap \mathfrak{X}/\mathrm{Fin}$, then there is $B \subseteq_{\mathrm{Fin}} C$ in $\mathfrak{X}/\mathrm{Fin}$ such that for*

every maximal antichain $\mathscr{A} \in M$ *of* $\mathfrak{X}/\mathrm{Fin}$*, there are* $A_0, \ldots, A_n \in \mathscr{A} \cap M$ *with* $B \subseteq_{\mathrm{Fin}} A_0 \cup \cdots \cup A_n$.

Proof.

($\Longrightarrow$): Suppose $\mathfrak{X}$ is proper. Then there is $\lambda > 2^{|\mathfrak{X}|}$ such that for every countable $M \prec H_\lambda$ containing $\mathfrak{X}$ and every $C \in M \cap \mathfrak{X}/\mathrm{Fin}$, there is an M-generic $B \subseteq_{\mathrm{Fin}} C$ in $\mathfrak{X}/\mathrm{Fin}$. Fix a countable $M \prec H_\lambda$ containing $\mathfrak{X}$ and $C \in M \cap \mathfrak{X}/\mathrm{Fin}$. Let $B \subseteq_{\mathrm{Fin}} C$ be M-generic. Thus, every V-generic filter containing B must meet $\mathscr{A} \cap M$ for every maximal antichain $\mathscr{A} \in M$ of $\mathfrak{X}/\mathrm{Fin}$. But since $\mathscr{A} \cap M$ is countable, by Proposition 1.52, there exist $A_0, \ldots, A_n \in \mathscr{A} \cap M$ such that $B \subseteq_{\mathrm{Fin}} A_0 \cup \cdots \cup A_n$.

($\Longleftarrow$): Suppose that there is $\lambda > 2^{|\mathfrak{X}|}$ such that for every countable $M \prec H_\lambda$ containing $\mathfrak{X}$, whenever $C \in M \cap \mathfrak{X}/\mathrm{Fin}$, then there is $B \subseteq_{\mathrm{Fin}} C$ in $\mathfrak{X}/\mathrm{Fin}$ such that for every maximal antichain $\mathscr{A} \in M$ of $\mathfrak{X}/\mathrm{Fin}$, there are $A_0, \ldots, A_n \in \mathscr{A} \cap M$ with $B \subseteq_{\mathrm{Fin}} A_0 \cup \cdots \cup A_n$. Fix a countable $M \prec H_\lambda$ with $\mathfrak{X} \in M$ and $C \in M \cap \mathfrak{X}/\mathrm{Fin}$. Let $B \subseteq_{\mathrm{Fin}} C$ be as above. By Proposition 1.52, every V-generic filter G containing B must meet $\mathscr{A} \cap M$ for every maximal antichain $\mathscr{A} \in M$. Thus, B is M-generic. Since M was arbitrary, we can conclude that $\mathfrak{X}$ is proper. □

The hypothesis of Corollary 1.53 can be weakened to finding for some H_λ,

only a club of countable M having the desired property since, as was pointed out earlier, this is equivalent for properness.

Definition 1.54. Let $\mathfrak{X}$ be a countable Scott set, let $\mathcal{D}$ be some collection of dense subsets of $\mathfrak{X}/\mathrm{Fin}$, and let $B \in \mathfrak{X}$. We say that an infinite set $A \subseteq \mathbb{N}$ is $\langle \mathfrak{X}, \mathcal{D} \rangle$-*generic below* B if $A \subseteq_{\mathrm{Fin}} B$ and for every $\mathscr{D} \in \mathcal{D}$, there is $C \in \mathscr{D}$ such that $A \subseteq_{\mathrm{Fin}} C$.

Here one should think of the context of having some big Scott set $\mathfrak{Y} \in M \prec H_\lambda$ such that M is countable, $\mathfrak{X} = \mathfrak{Y} \cap M$, and $\mathcal{D} = \{\mathscr{D} \cap M \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ is dense in } \mathfrak{Y}/\mathrm{Fin}\}$. We think of A as coming from the bigger Scott set $\mathfrak{Y}$ and the requirement for A to be $\langle \mathfrak{X}, \mathcal{D} \rangle$-generic is a strengthening of the requirement to be M-generic.

Lemma 1.55. *Let $\mathfrak{X}$ be a countable Scott set. Assume that $B \in \mathfrak{X}/\mathrm{Fin}$ and $G \subseteq \mathfrak{X}/\mathrm{Fin}$ is a V-generic filter containing B. Then in $V[G]$, there is an infinite $A \subseteq \mathbb{N}$ such that $A \subseteq_{\mathrm{Fin}} C$ for all $C \in G$. Furthermore, if $\mathcal{D}$ is the collection of dense subsets of $\mathfrak{X}/\mathrm{Fin}$ of V, then such an A is $\langle \mathfrak{X}, \mathcal{D} \rangle$-generic below B.*

Proof. Since G is countable and directed in $V[G]$, there exists an infinite $A \subseteq \mathbb{N}$ such that $A \subseteq_{\mathrm{Fin}} C$ for all $C \in G$. For the "furthermore" part, fix a

dense subset $\mathscr{D}$ of $\mathfrak{X}/\mathrm{Fin}$ in V. Since there is $C \in G \cap \mathscr{D}$, we have $A \subseteq_{\mathrm{Fin}} C$. It is clear that $A \subseteq_{\mathrm{Fin}} B$ since $B \in G$. □

Lemma 1.56. *Let $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \cdots \subseteq \mathfrak{X}_\xi \subseteq \cdots$ for $\xi < \omega_1$ be a continuous chain of countable Scott sets and let $\mathfrak{X} = \cup_{\xi<\omega_1} \mathfrak{X}_\xi$. Assume that for every $\xi < \omega_1$, if $B \in \mathfrak{X}_\xi$ and $\mathcal{D}$ is a countable collection of dense subsets of $\mathfrak{X}_\xi$, there is $A \in \mathfrak{X}/\mathrm{Fin}$ that is $\langle \mathfrak{X}_\xi, \mathcal{D}\rangle$-generic below B. Then $\mathfrak{X}$ is proper.*

Proof. Fix a countable $M \prec H_\lambda$ such that $\langle \mathfrak{X}_\xi \mid \xi < \omega_1\rangle \in M$. It suffices to show that generic conditions exist for such M since these form a club. I claim that $\mathfrak{X} \cap M = \mathfrak{X}_\alpha$ where $\alpha = \mathrm{Ord}^M \cap \omega_1$. Suppose $\xi \in \alpha$, then $\xi \in M$, and hence $\mathfrak{X}_\xi \in M$. Since $\mathfrak{X}_\xi$ is countable, it follows that $\mathfrak{X}_\xi \subseteq M$. Thus, $\mathfrak{X}_\alpha \subseteq \mathfrak{X} \cap M$. Now suppose $A \in \mathfrak{X} \cap M$, then the least ξ such that $A \in \mathfrak{X}_\xi$ is definable in H_λ. It follows that $\xi \in M$, and hence $\xi \in \alpha$. Thus, $\mathfrak{X} \cap M \subseteq \mathfrak{X}_\alpha$. This establishes that $\mathfrak{X} \cap M = \mathfrak{X}_\alpha$. Fix $B \in \mathfrak{X}_\alpha$ and let $\mathcal{D} = \{\mathscr{D} \cap M \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathfrak{X}/\mathrm{Fin}\}$. By hypothesis, there is $A \in \mathfrak{X}/\mathrm{Fin}$ that is $\langle \mathfrak{X}_\alpha, \mathcal{D}\rangle$-generic below B. Clearly A is M-generic. Thus, we were able to find an M-generic element below every $B \in M \cap \mathfrak{X}/\mathrm{Fin}$. □

Theorem 1.57. *There is a generic extension of V by a c.c.c. poset, which satisfies* ¬CH *and contains a proper arithmetically closed Scott set of size ω_1.*

Proof. First, note that we can assume without loss of generality that

$V \models \neg\text{CH}$ since this is forceable by a c.c.c. forcing.

The forcing to add an arithmetically closed proper Scott set will be a c.c.c. iteration $\mathbb{P}$ of length ω_1. The iteration $\mathbb{P}$ will add, step-by-step, a continuous chain $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \cdots \subseteq \mathfrak{X}_\xi \subseteq \cdots$ for $\xi < \omega_1$ of countable arithmetically closed Scott sets such that $\cup_{\xi<\omega_1}\mathfrak{X}_\xi$ will have the property of Lemma 1.56. The idea will be to obtain generic elements for $\mathfrak{X}_\xi$, as in Lemma 1.55, by adding generic filters. Once $\mathfrak{X}_\xi$ has been constructed, I will force over $\mathfrak{X}_\xi/\text{Fin}$ below every one of its elements cofinally often before the iteration is over. Every time such a forcing is done, I will obtain a generic element for a new collections of dense sets. This element will be added to $\mathfrak{X}_{\delta+1}$ where δ is the stage at which the forcing was done.

Fix a bookkeeping function f mapping ω_1 onto $\omega_1 \times \omega$, having the properties that any pair $\langle\alpha, n\rangle$ appears cofinally often in the range and if $f(\xi) = \langle\alpha, n\rangle$, then $\alpha \leq \xi$. Let $\mathfrak{X}_0$ be any countable arithmetically closed Scott set and fix an enumeration $\mathfrak{X}_0 = \{B_0^0, B_1^0, \ldots, B_n^0 \ldots\}$. Each subsequent $\mathfrak{X}_\xi$ will be created in $V^{\mathbb{P}_\xi}$. Suppose λ is a limit and G_λ is generic for $\mathbb{P}_\lambda$. In $V[G_\lambda]$, define $\mathfrak{X}_\lambda = \cup_{\xi<\lambda}\mathfrak{X}_\xi$ and fix an enumeration $\mathfrak{X}_\lambda = \{B_0^\lambda, B_1^\lambda, \ldots, B_n^\lambda, \ldots\}$. Consult $f(\lambda) = \langle\xi, n\rangle$ and define $\dot{\mathbb{Q}}_\lambda = \mathfrak{X}_\xi/\text{Fin}$ below B_n^ξ. Suppose $\delta = \beta + 1$, then $\mathbb{P}_\delta = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$ where $\dot{\mathbb{Q}}_\beta$ is $\mathfrak{X}_\xi/\text{Fin}$ for some $\xi \leq \beta$ below one of its elements. In $V[G_\delta] = V[G_\beta][H]$, let $A \subseteq_{\text{Fin}} B$ for all $B \in H$ and define $\mathfrak{X}_\delta$

to be the arithmetic closure of $\mathfrak{X}_\beta$ and A. Also in $V[G_\delta]$, fix an enumeration $\mathfrak{X}_\delta = \{B_0^\delta, B_1^\delta, \ldots, B_n^\delta, \ldots\}$. Consult $f(\delta) = \langle \xi, n \rangle$ and define $\dot{\mathbb{Q}}_\delta = \mathfrak{X}_\xi/\text{Fin}$ below B_n^ξ. At limits, use finite support.

The poset $\mathbb{P}$ is c.c.c. since it is a finite support iteration of c.c.c. posets (see [9], p.271). Let G be V-generic for $\mathbb{P}$. It should be clear that we can use G in $V[G]$ to construct an arithmetically closed Scott set $\mathfrak{X} = \cup_{\xi<\omega_1}\mathfrak{X}_\xi$. A standard nice name counting argument shows that $(2^\omega)^V = (2^\omega)^{V[G]}$. Since we assumed at the beginning that $V \models \neg\text{CH}$, it follows that $V[G] \models \neg\text{CH}$.

Finally, we must see that $\mathfrak{X}$ satisfies the hypothesis of Lemma 1.56 in $V[G]$. Fix $\mathfrak{X}_\xi$, a set $B \in X_\xi$, and a countable collection $\mathcal{D}$ of dense subsets of $\mathfrak{X}_\xi/\text{Fin}$. Since the poset $\mathbb{P}$ is a finite support c.c.c. iteration and all elements of $\mathcal{D}$ are countable, they must appear at some stage α below ω_1. Since we force with $\mathfrak{X}_\xi/\text{Fin}$ below B cofinally often, we have added a $\langle \mathfrak{X}_\xi, \mathcal{D} \rangle$-generic condition below B at some stage above α. □

Corollary 1.58. *There is a generic extension of V in which* CH *holds and which contains a proper arithmetically closed Scott set of size ω_1 that is not the whole powerset of* $\mathbb{N}$.

Proof. As before, we can assume without loss of generality that $V \models \neg\text{CH}$. Force with $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P}$ is the forcing iteration from Theorem 1.57 and $\mathbb{Q}$

is the poset which adds a subset to ω_1 with countable conditions. Let $G * H$ be V-generic for $\mathbb{P} * \dot{\mathbb{Q}}$, then clearly CH holds in $V[G][H]$. Also the Scott set $\mathfrak{X}$ created from G remains proper in $V[G][H]$ since $\mathbb{Q}$ is a countably closed forcing, and therefore cannot affect the properness of a Scott set. □

We can push this argument further to show that it is consistent with ZFC that there are continuum many uncountable arithmetically closed proper Scott sets.

Theorem 1.59. *There is a generic extension of V by a c.c.c. poset, which satisfies* ¬CH *and contains continuum many uncountable proper arithmetically closed Scott sets.*

Proof. We start by forcing MA + ¬CH. Since this can be done by a c.c.c. forcing notion ([9], p. 272), we can assume without loss of generality that $V \models \mathrm{MA} + \neg\mathrm{CH}$.

Define a finite support product $\mathbb{Q} = \Pi_{\xi<2^\omega}\mathbb{P}^\xi$ where every $\mathbb{P}^\xi$ is an iteration of length ω_1 as described in Theorem 1.57. Since Martin's Axiom implies that finite support products of c.c.c. posets are c.c.c. (see [9], p. 277), the product poset $\mathbb{Q}$ is c.c.c.. Let $G \subseteq \mathbb{Q}$ be V-generic, then each $G^\xi = G \restriction \mathbb{P}^\xi$ together with $\mathbb{P}^\xi$ can be used to build an arithmetically closed Scott set $\mathfrak{X}^\xi$ as described in Theorem 1.57. Each such $\mathfrak{X}^\xi$ will be the union of an increasing

chain of countable arithmetically closed Scott sets $\mathfrak{X}^\xi_\gamma$ for $\gamma < \omega_1$. First, I claim that all $\mathfrak{X}^\xi$ are distinct. Fixing $\alpha < \beta$, I will show that $\mathfrak{X}^\alpha \neq \mathfrak{X}^\beta$. Consider $V[G \restriction \beta + 1] = V[G \restriction \beta][G^\beta]$ a generic extension by $(\mathbb{Q} \restriction \beta) \times \mathbb{P}^\beta$. Observe that $\mathfrak{X}^\alpha$ already exists in $V[G \restriction \beta]$. Recall that to build $\mathfrak{X}^\beta$, we start with an arithmetically closed countable Scott set $\mathfrak{X}^\beta_0$ and let the first poset in the iteration $\mathbb{P}^\beta$ be $\mathfrak{X}^\beta_0/\mathrm{Fin}$. Let g be the generic filter for $\mathfrak{X}^\beta_0/\mathrm{Fin}$ definable from G^β. The next step in constructing $\mathfrak{X}^\beta$ is to pick $A \subseteq \mathbb{N}$ such that $A \subseteq_{\mathrm{Fin}} B$ for all $B \in g$ and define $\mathfrak{X}^\beta_1$ to be the arithmetic closure of $\mathfrak{X}^\beta_0$ and A. It should be clear that g is definable from A and $\mathfrak{X}^\beta_0$. Since g is $V[G \restriction \beta]$-generic, it follows that $g \notin V[G \restriction \beta]$. Thus, $A \notin V[G \restriction \beta]$, and hence $\mathfrak{X}^\beta \neq \mathfrak{X}^\alpha$. It remains to show that each $\mathfrak{X}^\alpha$ is proper in $V[G]$. Fix $\alpha < 2^\omega$ and let $V[G] = V[G \restriction \alpha][G^\alpha][G_{\mathrm{tail}}]$ where G_{tail} is the generic for $\mathbb{Q}$ above α. By the commutativity of products, $V[G \restriction \alpha][G^\alpha][G_{\mathrm{tail}}] = V[G \restriction \alpha][G_{\mathrm{tail}}][G^\alpha]$ and G^α is $V[G \restriction \alpha][G_{\mathrm{tail}}]$-generic. Fix a countable $M \prec H^{V[G]}_\lambda$ containing the sequence $\langle \mathfrak{X}^\alpha_\xi \mid \xi < \omega_1 \rangle$ as an element. By Lemma 1.56, $M \cap \mathfrak{X}^\alpha$ is some $\mathfrak{X}^\alpha_\gamma$. This is the key step of the proof since it allows us to know exactly what $M \cap \mathfrak{X}^\alpha$ is, even though we know nothing about M. Let $G^\alpha_\xi = G^\alpha \restriction \mathbb{P}^\alpha_\xi$ for $\xi < \omega_1$. Let $\mathcal{D} = \{\mathscr{D} \cap M \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathfrak{X}^\alpha/\mathrm{Fin}\}$. There must be some $\beta < \omega_1$ such that $\mathcal{D} \in V[G \restriction \alpha][G_{\mathrm{tail}}][G^\alpha_\beta]$. By construction, there must be some stage $\delta > \beta$ at which we forced with $\mathfrak{X}^\alpha_\gamma/\mathrm{Fin}$ and added a set

A such that $A \subseteq_{\text{Fin}} B$ for all $B \in H$ where $G^\alpha_{\delta+1} = G^\alpha_\delta * H$. Now observe that H is $V[G \restriction \alpha][G_{\text{tail}}][G^\alpha_\delta]$-generic for $\mathfrak{X}^\alpha_\gamma/\text{Fin}$. Therefore H meets all the sets in $\mathcal{D}$. So we can conclude that A is M-generic.

A standard nice name counting argument will again show that $(2^\omega)^V = (2^\omega)^{V[G]}$. Thus, $V[G]$ contains continuum many arithmetically closed proper Scott sets of size ω_1. □

Question 1.60. Is it consistent that there exist proper Scott sets of size ω_2 that are not the whole powerset of $\mathbb{N}$?

There appears to be a construction for building proper Scott sets under PFA. The idea is, in some sense, to mimic the forcing iteration like that of Theorem 1.57 in the ground model. Unfortunately, the main problem with the construction is that it is not clear whether we are getting the whole $\mathcal{P}(\mathbb{N})$. This problem never arose in the forcing construction since we were building Scott sets of size ω_1 and knew that the continuum was larger than ω_1. I will describe the construction and a possible way of ensuring that the resulting Scott set is not $\mathcal{P}(\mathbb{N})$.

Fix an enumeration $\{\langle A_\xi, B_\xi\rangle \mid \xi < \omega_2\}$ of $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$. Also fix a bookkeeping function f from ω_2 onto ω_2 such that each element appears cofinally in the range. I will build an arithmetically closed Scott set $\mathfrak{X}$ of

size ω_2 as the union of an increasing chain of arithmetically closed Scott sets $\mathfrak{X}_\xi$ for $\xi < \omega_2$. Start with any arithmetically closed Scott set $\mathfrak{X}_0$ of size ω_1. Suppose we have constructed $\mathfrak{X}_\beta$ for $\beta \leq \alpha$ and we need to construct $\mathfrak{X}_{\alpha+1}$. Consult $f(\alpha) = \gamma$ and consider the pair $\langle A_\gamma, B_\gamma \rangle$ in the enumeration of $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$. First, suppose that A_γ codes a countable Scott set $\mathfrak{Y} \subseteq \mathfrak{X}_\alpha$ and B_γ codes a countable collection $\mathcal{D}$ of dense subsets of $\mathfrak{Y}$. Let G be some filter on $\mathfrak{Y}$ meeting all sets in $\mathcal{D}$ and let $A \subseteq_{\text{Fin}} C$ for all $C \in G$. Define $\mathfrak{X}_{\alpha+1}$ to be the arithmetic closure of $\mathfrak{X}_\alpha$ and A. If the pair $\langle A_\gamma, B_\gamma \rangle$ does not code such information, let $\mathfrak{X}_{\alpha+1} = \mathfrak{X}_\alpha$. At limit stages take unions.

I claim that $\mathfrak{X}$ is proper. Fix some countable $M \prec H_\lambda$ containing $\mathfrak{X}$. Let $\mathfrak{Y} = M \cap \mathfrak{X}$ and let $\mathcal{D} = \{\mathscr{D} \cap M \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathfrak{X}/\text{Fin}\}$. There must be some γ such that $\langle A_\gamma, B_\gamma \rangle$ codes $\mathfrak{Y}$ and $\mathcal{D}$. Let δ such that $M \cap \mathfrak{X}$ is contained in $\mathfrak{X}_\delta$, then there must be some $\alpha > \delta$ such that $f(\alpha) = \gamma$. Thus, at stage α in the construction we considered the pair $\langle A_\gamma, B_\gamma \rangle$. Since $\alpha > \delta$, we have $M \cap \mathfrak{X} = M \cap \mathfrak{X}_\alpha$. It follows that at stage α we added an M-generic set A to $\mathfrak{X}$.

A way to prove that $\mathfrak{X} \neq \mathcal{P}(\mathbb{N})$ would be to show that some fixed set C is not in $\mathfrak{X}$. Suppose the following question had a positive answer:

Question 1.61. Let $\mathfrak{X}$ be an arithmetically closed Scott set such that $C \notin \mathfrak{X}$ and $\mathfrak{Y} \subseteq \mathfrak{X}$ be a countable Scott set. Is there a $\mathfrak{Y}/\text{Fin}$-name $\dot{A}$

such that $1_{\mathfrak{Y}/\mathrm{Fin}} \Vdash$ "$\dot{A} \subseteq_{\mathrm{Fin}} B$ for all $B \in \dot{G}$ and $\check{C}$ is not in the arithmetic closure of $\dot{A}$ and $\check{\mathfrak{X}}$"?

Assuming that the answer to Question 1.61 is positive, let us construct a proper Scott set $\mathfrak{X}$ in such a way that C is not in $\mathfrak{X}$. We will carry out the above construction being careful in our choice of the filters G and elements A. Start with $\mathfrak{X}_0$ that does not contain C and assume that $C \notin \mathfrak{X}_\alpha$. Suppose the pair $\langle A_\gamma, B_\gamma \rangle$ considered at stage α codes meaningful information. It follows that A_γ codes a countable Scott set $\mathfrak{Y} \subseteq \mathfrak{X}_\alpha$ and B_γ codes a countable collection $\mathcal{D}$ of dense subsets of $\mathfrak{Y}$. Choose some transitive $N \prec H_{\omega_2}$ of size ω_1 such that $\mathfrak{X}_\alpha$, $\mathfrak{Y}$, and $\mathcal{D}$ are elements of N. Since we assumed a positive answer to Question 1.61, H_{ω_2} satisfies that there exists a $\mathfrak{Y}/\mathrm{Fin}$-name $\dot{A}$ such that $1_{\mathfrak{Y}/\mathrm{Fin}} \Vdash$ "$\dot{A} \subseteq_{\mathrm{Fin}} B$ for all $B \in \dot{G}$ and $\check{C}$ is not in the arithmetic closure of $\dot{A}$ and $\check{\mathfrak{X}}_\alpha$". But then N satisfies the same statement by elementarity. Hence there is $\dot{A} \in N$ such that N satisfies $1_{\mathfrak{Y}/\mathrm{Fin}} \Vdash$ "$\dot{A} \subseteq_{\mathrm{Fin}} B$ for all $B \in \dot{G}$ and $\check{C}$ is not in the arithmetic closure of $\dot{A}$ and $\check{\mathfrak{X}}_\alpha$". Now use PFA to find an N-generic filter G for $\mathfrak{Y}/\mathrm{Fin}$. Since G is fully generic for the model N, the model $N[G]$ will satisfy that C is not in the arithmetic closure of $\mathfrak{X}_\alpha$ and $A = \dot{A}_G$. Thus, it is really true that C is not in the arithmetic closure of $\mathfrak{X}_\alpha$ and A. Since G also met all the dense sets in $\mathcal{D}$ and $A \subseteq_{\mathrm{Fin}} B$ for all $B \in G$, we can let $\mathfrak{X}_{\alpha+1}$ be the arithmetic closure of $\mathfrak{X}_\alpha$ and A. Thus, $C \notin \mathfrak{X}_{\alpha+1}$. We

can conclude that $C \notin \mathfrak{X}$.

Finally, is there always a Scott set $\mathfrak{X}$ which is *not* proper?

Theorem 1.62 (Enayat, 2006)**.** *There is an arithmetically closed Scott set* $\mathfrak{X}$ *such that* $\mathfrak{X}$/Fin *collapses* ω_1*. Hence* $\mathfrak{X}$ *is not proper.* [4]

Clearly $\mathfrak{X}$/Fin cannot be proper since all proper posets preserve ω_1.

1.9 Weakening the Hypothesis

There are several ways in which the hypothesis of Theorem 1.7 can be modified. PFA is a very strong set theoretic axiom, and therefore it is important to see whether this assumption can be weakened to something that is lower in consistency strength. In fact, there are weaker versions of PFA that will still work with Theorem 1.7. It is also possible to make slightly different assumptions on $\mathfrak{X}$. Instead of assuming that $\mathfrak{X}$ is proper, it is sufficient to assume that $\mathfrak{X}$ is the union of a chain of proper Scott sets.

The definition of properness refers to countable structures $M \prec H_\lambda$ and the existence of M-generic elements for them. If we fix a cardinal κ and modify the definition to consider M of size κ instead, we will get the notion of κ-properness. In this extended definition, the notion of properness we considered up to this point becomes $\aleph_0$-properness. For example, the κ-c.c.

and $< \kappa$-closed posets are κ-proper. Hamkins and Johnstone [10] recently proposed a new axiom PFA($\mathfrak{c}$-proper) which states that for every poset $\mathbb{P}$ that is proper and 2^ω-proper and every collection $\mathcal{D}$ of ω_1 many dense subsets of $\mathbb{P}$, there is a filter on $\mathbb{P}$ that meets all of them. PFA($\mathfrak{c}$-proper) is much weaker in consistency strength than PFA. While the best large cardinal upper bound on the consistency strength of PFA is a supercompact cardinal, an upper bound for PFA($\mathfrak{c}$-proper) is an unfoldable cardinal. Unfoldable cardinals were defined by Villaveces [23] and are much weaker than measurable cardinals. In fact, unfoldable cardinals are consistent with $V = L$. The axiom PFA($\mathfrak{c}$-proper) also decides the size of the continuum is ω_2 [10]. It is enough for Theorem 1.7 to assume that PFA($\mathfrak{c}$-proper) holds:

Theorem 1.63. *Assuming* PFA($\mathfrak{c}$*-proper*)*, every arithmetically closed proper Scott set is the standard system of a model of* PA*.*

Proof. Every κ^+-c.c. poset is κ-proper. It is clear that every Scott set $\mathfrak{X}$ is $(2^\omega)^+$-c.c.. It follows that every Scott set is 2^ω-proper. Thus, PFA($\mathfrak{c}$-proper) applies to proper Scott sets. □

It is also easy to see that we do not need the whole Scott set $\mathfrak{X}$ to be proper. For the construction, it would suffice if $\mathfrak{X}$ was a union of a chain of arithmetically closed proper Scott sets. Call a Scott set *piecewise proper* if

it is the union of a chain of arithmetically closed proper Scott sets of size $\leq \omega_1$. Under this definition, any arithmetically closed Scott set of size $\leq \omega_1$ is trivially piecewise proper since it is the union of a chain of countable Scott sets. The modified construction using piecewise proper Scott sets does not require all of PFA but only a much weaker version known as PFA$^-$. The axiom PFA$^-$ is the assertion that for every proper poset $\mathbb{P}$ of size ω_1 and every collection $\mathcal{D}$ of ω_1 many dense subsets of $\mathbb{P}$, there is a filter on $\mathbb{P}$ that meets all of them. PFA$^-$ has no large cardinal strength. The axiom is equiconsistent with ZFC [20] (p. 122). This leads to the following modified version of Theorem 1.7:

Theorem 1.64. *Assuming* PFA$^-$*, every arithmetically closed piecewise proper Scott set of size* $\leq \omega_2$ *is the standard system of a model of* PA*.*

Proof. It suffices to show that the ω_2-Ehrenfeucht Principle holds for arithmetically closed piecewise proper Scott sets of size ω_2. So suppose M is a model of PA of size ω_1 and $\mathfrak{X}$ is a piecewise proper Scott set of size ω_2 such that $\mathrm{SSy}(M) \subseteq \mathfrak{X}$. Since $\mathfrak{X}$ is piecewise proper, it is the union of a chain of arithmetically closed proper Scott sets $\mathfrak{X}_\xi$ for $\xi < \omega_2$. Fix any $A \in \mathfrak{X}$, then there is an ordinal $\alpha < \omega_2$ such that $\mathrm{SSy}(M)$ and A are contained in $\mathfrak{X}_\alpha$. Since $\mathfrak{X}_\alpha$ is proper, the ω_2-Ehrenfeucht Principle holds for $\mathfrak{X}_\alpha$ by Theorem

1.26. Thus, there is $M \prec N$ such that $A \in \mathrm{SSy}(N) \subseteq \mathfrak{X}_\alpha \subseteq \mathfrak{X}$. □

Question 1.65. Is it consistent that there exist piecewise proper Scott sets of size ω_2?

Chapter 2

Ramsey-like Embedding Properties

2.1 Introduction

Most large cardinals are defined in terms of the existence of certain elementary embeddings with that cardinal as the critical point. For example, a cardinal κ is measurable if and only if there exists an inner model M and an elementary embedding $j : V \to M$ with critical point κ; a cardinal κ is weakly compact if and only if for every transitive set M of size κ with $\kappa \in M$, there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ. In contrast, for several large cardinals that are weaker than measurable cardinals, such as ineffable cardinals and subtle cardinals, there is no known characterization in terms of elementary embeddings. I will study the seldom used elementary embedding property of Ramsey cardinals, which

I resurrected from several references (see, for example, [17] and [3]). I will analyze and generalize this property to define new large cardinals with "nicer" and more workable elementary embedding characterizations. My aim will be to use these new embedding properties to obtain indestructibility results for Ramsey cardinals. I will also attempt to place these new large cardinal axioms into the existing hierarchy. I hope that this project will motivate set theorists who work with smaller large cardinals to focus on investigating their elementary embedding properties. I will begin with some basic definitions and terminology.

Let ZFC^- denote the fragment of ZFC consisting of ZF without the powerset axiom and a form of choice which states that every set is bijective with some ordinal. I will call a transitive set $M \models \mathrm{ZFC}^-$ of size κ with $\kappa \in M$ a *weak κ-model*. I will say that a weak κ-model M is a *κ-model* if additionally $M^{<\kappa} \subseteq M$. Observe that for any cardinal κ, if $M \prec H_{\kappa^+}$ has size κ with $\kappa \subseteq M$, then M a weak κ-model. Similarly, if $\lambda > \kappa$ is a regular cardinal and $X \prec H_\lambda$ has size κ with $\kappa + 1 \subseteq X$, then the Mostowski collapse of X is a weak κ-model. So there are always many weak κ-models for any cardinal κ. If additionally $\kappa^{<\kappa} = \kappa$, we can use a Skolem-Lowenheim type construction to build κ-models $M \prec H_{\kappa^+}$ and substructures $X \prec H_\lambda$ whose collapse will be a κ-model. Unless specifically stated otherwise, the sources and targets of

elementary embeddings are assumed to be transitive. I will call an elementary embedding $j : M \to N$ *κ-powerset preserving* if it has critical point κ and M and N have the *same* subsets of κ. For example, it is trivially true that any elementary embedding $j : V \to M$ with critical point κ such that $M \subseteq V$ has to be κ-powerset preserving.

I will define the Ramsey-like embedding properties shortly. To provide a motivation for this collection of definitions, I will begin by recalling the various equivalent definitions of weakly compact cardinals.

Theorem 2.1. *If $\kappa^{<\kappa} = \kappa$, then the following are equivalent.*

1. *κ is weakly compact. That is, κ is uncountable and every κ-satisfiable theory in a $L_{\kappa,\kappa}$ language of size at most κ is satisfiable.*

2. *For every $A \subseteq \kappa$, there is a transitive structure W properly extending V_κ and $A^* \subseteq W$ such that $\langle V_\kappa, \in, A\rangle \prec \langle W, \in, A^*\rangle$.*

3. *κ is inaccessible and every κ-tree has a cofinal branch.*

4. *Every $A \subseteq \kappa$ is contained in weak κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ.*

5. *Every $A \subseteq \kappa$ is contained in a κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ.*

6. *Every $A \subseteq \kappa$ is contained in a κ-model $M \prec H_{\kappa^+}$ for which there exists an elementary embedding $j : M \to N$ with critical point κ.*

7. *For every κ-model M, there exists an elementary embedding $j : M \to N$ with critical point κ.*

8. *For every κ-model M, there exists an elementary embedding $j : M \to N$ with critical point κ such that j and M are elements of N.*

For a proof of these equivalences see [8] (ch. 6). Now we are ready to state the large cardinal notions that will be the focus of this chapter. The general idea is to consider the various equivalent elementary embedding properties of weakly compact cardinals with the added assumption the the embeddings have to be *κ-powerset preserving*. We will soon see that this additional assumption destroys the equivalence in the strongest possible sense.

Definition 2.2 (Weak Ramsey Embedding Property)**.** A cardinal κ has the *weak Ramsey embedding property* if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a κ-powerset preserving elementary embedding $j : M \to N$. We say that a cardinal is *weakly Ramsey* if it has the weak Ramsey embedding property.

Definition 2.3 (Ramsey Embedding Property)**.** A cardinal κ has the *Ramsey embedding property* if every $A \subseteq \kappa$ is contained in a weak κ-model M for

which there exists a κ-powerset preserving elementary embedding $j : M \to N$ satisfying the property that whenever $\langle A_n \mid n \in \omega \rangle$ is a sequence of subsets of κ such that each $A_n \in M$ and $\kappa \in j(A_n)$, then $\cap_{n\in\omega} A_n \neq \emptyset$.

For $\langle A_n \mid n \in \omega \rangle \in M$, of course, the conclusion follows trivially. So the content here is for sequences *not* in M. I will show later that a cardinal is Ramsey if and only if it has the Ramsey embedding property.

Definition 2.4 (Strong Ramsey Embedding Property)**.** A cardinal κ has the *strong Ramsey embedding property* if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a κ-powerset preserving elementary embedding $j : M \to N$. We say that a cardinal is *strongly Ramsey* if it has the strong Ramsey embedding property.

Definition 2.5 (Super Ramsey Embedding Property)**.** A cardinal κ has the *super Ramsey embedding property* if every $A \subseteq \kappa$ is contained in a κ-model $M \prec H_{\kappa^+}$ for which there exists a κ-powerset preserving elementary embedding $j : M \to N$. We say that a cardinal is *super Ramsey* if it has the super Ramsey embedding property.

Definition 2.6 (Total Ramsey Embedding Property)**.** A cardinal κ has the *total Ramsey embedding property* if for every κ-model $M \prec H_{\kappa^+}$, there exists

a κ-powerset preserving elementary embedding $j : M \to N$. We say that a cardinal is *totally Ramsey* if it has the total Ramsey embedding property.

The following theorem, which will be proved in parts in the subsequent sections, summarizes what is known about the relationships among the large cardinal notions defined above and other well-established large cardinals. Also see the diagram following the theorem.

Theorem 2.7.

1. *If κ is a measurable cardinal, then κ is the κ^{th} super Ramsey cardinal.*

2. *If κ is a super Ramsey cardinal, then κ is the κ^{th} strongly Ramsey cardinal.*

3. *If κ is a strongly Ramsey cardinal, then κ is the κ^{th} Ramsey cardinal.*

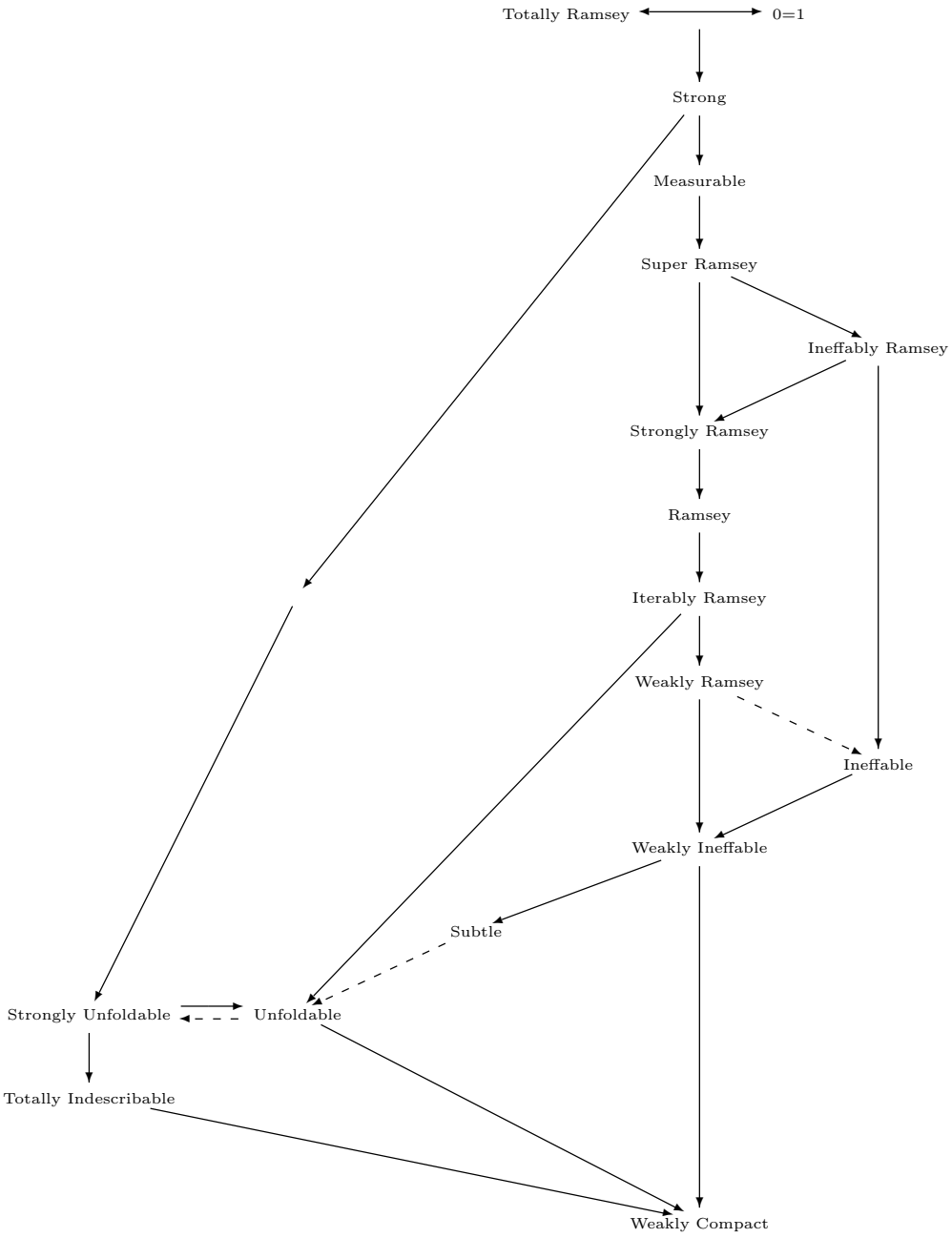
4. *κ has the Ramsey embedding property if and only if κ is Ramsey.*

5. *If κ is a super Ramsey cardinal, then κ is the κ^{th} ineffably Ramsey cardinal.*

6. *If κ is a weakly Ramsey cardinal, then κ is a weakly ineffable limit of ineffable cardinals, but not necessarily ineffable.*

7. *There are no totally Ramsey cardinals.*

The solid arrows indicate direct implications and the dashed arrows indicate relative consistency.



This shows, surprisingly, that the various embedding properties that define weakly compact cardinals differ in strength when you add the powerset preservation property and one such property is even inconsistent!

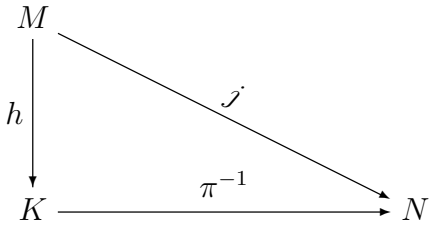
Some straightforward observations, which will be useful later on, are in order. First, note that since a weak κ-model can take the Mostowski collapse of any of its elements, the definitions above hold not just for any $A \subseteq \kappa$, but more generally for any $A \in H_{\kappa^+}$. Second, observe that the "total Ramsey embedding property" $\rightarrow$ "super Ramsey embedding property" $\rightarrow$ "strong Ramsey embedding property" $\rightarrow$ "Ramsey embedding property" $\rightarrow$ "weak Ramsey embedding property". The only implication that requires some explanation is that "strong Ramsey embedding property" $\rightarrow$ "Ramsey embedding property". If κ has the strong Ramsey embedding property, then the sequence $\langle A_n \mid n \in \omega \rangle$ is an element of M. By elementarity, we have $j(\langle A_n \mid n \in \omega \rangle) = \langle j(A_n) \mid n \in \omega \rangle$, and hence $\kappa \in \cap j(\langle A_n \mid n \in \omega \rangle)$. Again by elementarity, we get $\cap \langle A_n \mid n \in \omega \rangle \neq \emptyset$.

Definition 2.8. Suppose M is a transitive model of ZFC^- and λ is a cardinal in M. Then $U \subseteq \mathcal{P}(\lambda) \cap M$ is an *M-ultrafilter* if $\langle M, U \rangle \models U$ is an ultrafilter. We say that an M-ultrafilter U is *normal* if every regressive function $f : A \rightarrow \lambda$ in M with $A \in U$ is constant on some $B \in U$.

Proposition 2.9. *Suppose M is a weak κ-model such that $M^\alpha \subseteq M$ for some $\alpha < \kappa$ and $j : M \to N$ is a κ-powerset preserving embedding. Let $X = \{j(f)(\kappa) \mid f : \kappa \to M \text{ and } f \in M\} \subseteq N$ and let $\pi : X \to K$ be the Mostowski collapse. Then $h = \pi \circ j : M \to K$ is an elementary embedding with critical point κ having the properties:*

1. *h is κ-powerset preserving,*
2. *h is an ultrapower by a normal M-ultrafilter on κ,*
3. *$\kappa \in j(A)$ if and only if $\kappa \in h(A)$ for all $A \subseteq \kappa$ in M,*
4. *K has size κ,*
5. *$K^\alpha \subseteq K$,*

and we get the commutative diagram:



Proof. Clearly the size of K is κ since M has size κ. Define $U = \{A \subseteq \kappa \mid A \in M \text{ and } \kappa \in j(A)\}$. It follows by standard seed theory arguments that U is a normal M-ultrafilter, the embedding h is an ultrapower by U, and $K^\alpha \subseteq K$ [8] (ch. 0). For (1), observe that $\pi(\beta) = \beta$ for

all $\beta \leq \kappa$, and hence the critical point of π^{-1} is above κ. This implies that N and K have the same subsets of κ. Finally, for (3), observe that $\kappa \in j(A) \leftrightarrow \pi(\kappa) \in \pi \circ j(A) \leftrightarrow \kappa \in \pi \circ j(A) \leftrightarrow \kappa \in h(A)$. □

By Proposition 2.9, we can always assume in the definitions of Ramsey-like embeddings that $j : M \to N$ is an ultrapower by a normal M-ultrafilter on κ, the target N has size κ, and if $M^\alpha \subseteq M$, then $N^\alpha \subseteq N$. Also from condition (3), it follows that the Ramsey embedding property can be restated in terms of having a κ-powerset preserving elementary embedding $j : M \to N$ by a normal M-ultrafilter U on κ, with the property that whenever $\langle A_n \mid n \in \omega \rangle$ is a sequence of elements of U, then $\cap_{n\in\omega} A_n \neq \emptyset$.

2.2 How Large Are These Cardinals?

In this section, I will show that the definitions above do define large cardinal notions and discuss where these large cardinals fit into the large cardinal hierarchy.

Proposition 2.10. *If κ is a weakly Ramsey cardinal, then κ is the κ^{th} weakly compact cardinal.*

Proof. Since κ satisfies the Ramsey embedding property, to show that it is weakly compact, it suffices to verify that $\kappa^{<\kappa} = \kappa$. I will show that κ is

inaccessible. First, suppose to the contrary that κ is not regular. Then there exists a cofinal map $f : \alpha \to \kappa$ for some ordinal $\alpha < \kappa$. Choose a weak κ-model M containing f for which there exists a κ-powerset preserving embedding $j : M \to N$. Since $M \models$ "f is cofinal in κ", it follows by elementarity that $N \models$ "$j(f)$ is cofinal in $j(\kappa)$". But $j(f) = f : \alpha \to \kappa$ since the critical point of j is κ. Thus, it is impossible that $j(f)$ is cofinal in $j(\kappa)$, which is a contradiction. So κ is regular. Next, we show that κ is a strong limit. Again, suppose to the contrary that κ is not a strong limit. Then $\mathcal{P}(\alpha) \geq \kappa$ for some $\alpha < \kappa$. Fix an injective function $f : \kappa \to \mathcal{P}(\alpha)$. Choose a weak κ-model M containing f for which there exists a κ-powerset preserving embedding $j : M \to N$. Since $M \models \forall \xi\, f(\xi) \subseteq \alpha$, it follows by elementarity that $N \models \forall \xi\, j(f)(\xi) \subseteq \alpha$. Let $A = j(f)(\kappa)$ and note that $A \subseteq \alpha$. Since M and N have the same subsets of κ, the set A is in M and $j(A) = A$. It follows that $N \models \exists \xi\, j(f)(\xi) = j(A)$, and so by elementarity $M \models \exists \xi f(\xi) = A$. But now we have a contradiction since $f \subseteq j(f)$ and $N \models$ "$j(f)$ is injective" by elementarity. So κ is a strong limit. This concludes the argument that κ is weakly compact.

It remains to show that κ is a limit of weakly compact cardinals. We start by choosing a weak κ-model M containing V_κ for which there exists a κ-powerset preserving $j : M \to N$. Observe that it suffices to show that

κ is weakly compact in N. If κ is weakly compact in N, then for every $\beta < \kappa$, we have $N \models$ "$\exists \alpha < j(k)\ \alpha > \beta$ and α is weakly compact". It follows that $M \models$ "$\exists \alpha < \kappa\ \alpha > \beta$ and α is weakly compact". But since $V_\kappa \in M$, the model M is correct about α being weakly compact. Thus, it remains to show that $N \models$ "κ is weakly compact". I will show that $N \models$ "Every κ-tree has a cofinal branch". Suppose $T \subseteq \kappa$ is a κ-tree in N. Since M and N have the same subsets of κ, the tree $T \in M$. Consider $j(T) \in N$. Since κ is the critical point of j and levels of T have size less than κ, the tree $j(T)$ restricted to the first κ many levels is exactly T. Since the height of $j(T)$ is $j(\kappa)$, it must have some element on the κ^{th} level. The predecessors of that element will be a cofinal branch of T in N. □

Definition 2.11. An uncountable regular cardinal κ is *ineffable* if for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ with $A_\alpha \subseteq \alpha$, there exists $A \subseteq \kappa$ such that the set $S = \{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$ is stationary. An uncountable regular cardinal κ is *weakly ineffable* if such an A can be found for which the corresponding set S has size κ. A cardinal κ is *ineffably Ramsey* if κ is both ineffable and Ramsey.

Ineffable cardinals are weakly compact limits of weakly compact cardinals and weakly ineffable limits of weakly ineffable cardinals. This is true since

ineffable cardinals are Π^1_2-indescribable [2] (p. 315) and being weakly ineffable or weakly compact is a Π^1_2- statement satisfied by ineffable cardinals. We will see below that a Ramsey cardinal is a limit of ineffable cardinals. However, since being Ramsey is a Π^1_2-statement, the least Ramsey cardinal cannot be ineffable.

Theorem 2.12. *If κ is a weakly Ramsey cardinal, then κ is a weakly ineffable limit of ineffable cardinals.*

Proof. Fix $\vec{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ with each $A_\alpha \subseteq \alpha$. Choose a weak κ-model M containing $\vec{A}$ and V_κ for which there exists a κ-powerset preserving embedding $j : M \to N$. Consider $j(\vec{A})$ and, in particular, $A = j(\vec{A})(\kappa)$. Since $A \subseteq \kappa$, by the powerset preservation property, $A \in M$. I will show that the set $S = \{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$ is stationary in M. Observe that $j(\vec{A})(\kappa) = j(A) \cap \kappa$, and hence $\kappa \in j(S)$. Let C be any club in M, then clearly $\kappa \in j(C)$. Thus, $\kappa \in j(C) \cap j(S)$. It follows that $C \cap S \neq \emptyset$. Since S is stationary in M, it must have size κ. So we have shown that κ is weakly ineffable. As in the proof of Theorem 2.10, to show that κ is a limit of ineffable cardinals, we show that $N \models$ "κ is ineffable". The argument above actually shows that κ is ineffable in M. Since M and N have the same subsets of κ, they have the same clubs on κ. Hence they agree on stationarity, and thus, κ is ineffable

in N. This completes the proof that κ is a limit of ineffable cardinals. $\square$

The conclusion of Theorem 2.12 cannot be improved to say that κ is ineffable since a Ramsey cardinal is always weakly Ramsey and the least Ramsey cardinal is not ineffable. In fact, I will show later on that even the strongly Ramsey cardinals are not necessarily ineffable (Corollary 2.58).

Definition 2.13. An uncountable regular cardinal κ is *subtle* if for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ with $A_\alpha \subseteq \alpha$ and every club C on κ, there exist ordinals $\alpha < \beta$ in C such that $A_\beta \cap \alpha = A_\alpha$.

Proposition 2.14. *If κ is a weakly Ramsey cardinal, then κ is subtle.*

It is known that weakly ineffable cardinals are subtle, but I will give a direct proof here.

Proof. Fix any $\vec{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ with $A_\alpha \subseteq \alpha$ and a club C on κ. Choose a weak κ-model M containing $\vec{A}$ and C for which there exists a κ-powerset preserving $j : M \to N$. Define S and A as in the proof of Theorem 2.12 and recall that M thinks that S is stationary. Thus, there are $\alpha < \beta$ in $S \cap C$. Clearly for such $\alpha < \beta$, we have $A_\beta \cap \alpha = A \cap \beta \cap \alpha = A_\alpha$. $\square$

Corollary 2.15. *If κ is a weakly Ramsey cardinal, then $\diamondsuit_\kappa$ holds.*

Proof. If κ is subtle, then $\diamondsuit_\kappa$ holds. [1] □

Next, we examine the strength of the other Ramsey-like embedding properties.

Theorem 2.16. *If κ is a super Ramsey cardinal, then κ is the κ^{th} ineffably Ramsey cardinal.*

Proof. If κ has the super Ramsey embedding property, then we can choose $M \prec H_{\kappa^+}$ in the proof of Theorem 2.12. This insures that if M thinks that a set is stationary, then M is correct about this. Therefore the proof of Theorem 2.12 shows in this case that κ is ineffable. Once we show that having the Ramsey embedding property is equivalent to being Ramsey (Theorem 2.35), it will follow that κ is also Ramsey. Thus, κ is ineffably Ramsey. We already saw that if $j : M \to N$ is κ-powerset preserving and M is a κ-model, then $N \models$ "κ is ineffable". I will also show later that if $j : M \to N$ is κ-powerset preserving and M is a κ-model, then $N \models$ "κ is Ramsey" (Corollary 2.36). This shows that κ is a limit of ineffably Ramsey cardinals. □

Theorem 2.17. *If κ is a super Ramsey cardinal, then κ is the κ^{th} strongly Ramsey cardinal.*

[1] See [2], p. 315 for a proof that ineffable cardinals have diamond and observe that only subtleness is needed.

Proof. Choose a κ-model $M \prec H_{\kappa^+}$ for which there exists a κ-powerset preserving embedding $j : M \to N$. Note that $V_\kappa \in M$ since M is a κ-model (use the Replacement axiom in M). As usual, it suffices to show that κ is a strongly Ramsey cardinal in N. By Proposition 2.9, there is always a κ-powerset preserving embedding witnessing the strong Ramsey embedding property whose target has size κ. It follows that $H_{\kappa^+} \models$ "κ is a strongly Ramsey cardinal". Therefore $M \models$ "κ is a strongly Ramsey cardinal" by elementarity and N agrees about this since M and N have the same elements with transitive closure of size $\leq \kappa$. □

The most surprising result is the following:

Theorem 2.18. *There are no totally Ramsey cardinals.*

Proof. Suppose that there exists a totally Ramsey cardinal and let κ be the *least* totally Ramsey cardinal. Choose any κ-model $M \prec H_{\kappa^+}$ and a κ-powerset preserving embedding $j : M \to N$. The strategy will be to show that κ is totally Ramsey in N. Observe, first, that $H^N_{\kappa^+} = M$. If $B \in H^N_{\kappa^+}$, then $B \in M$ by the powerset preservation property. Conversely, since $M \prec H_{\kappa^+}$, if $B \in M$, then M thinks $|Trcl(B)| \leq \kappa$, and hence $B \in H^N_{\kappa^+}$. Thus, to show that κ is totally Ramsey in N, we need to verify in N that every κ-model $m \prec M$ has a κ-powerset preserving embedding. So let $m \in N$

be a κ-model such that $m \prec M$. Observe that $m \in M$ and $m \prec H_{\kappa^+}$. By Proposition 2.9, H_{κ^+} contains a κ-powerset preserving embedding for m. By elementarity, M contains some κ-powerset preserving embedding $h : m \to n$. Since M thinks that $|Trcl(h)| \leq \kappa$, it must be in N as well. Thus, κ is totally Ramsey in N. It follows that there is a totally Ramsey cardinal α below κ. This is, of course, impossible since we assumed that κ was the least totally Ramsey cardinal. Thus, there cannot be any totally Ramsey cardinals. $\square$

This is surprising since, as was pointed out earlier, these embedding properties without powerset preservation are equivalent modulo the assumption that $\kappa^{<\kappa} = \kappa$. Once we add the powerset condition on the embeddings, the equivalence is strongly violated. Of course, now the question arises whether there can be any super Ramsey cardinals.

Theorem 2.19. *If κ is a measurable cardinal, then κ is the κ^{th} super Ramsey cardinal.*

Proof. Fix $A \subseteq \kappa$. Let $j : V \to M$ be an ultrapower by a measure on κ, then j is κ-powerset preserving. First, we reduce j to a κ-powerset preserving embedding of sets by considering the restriction $j : H_{\kappa^+} \to H^M_{j(k)^+}$. The problem is that the set H_{κ^+} is still too big. So second, in some sense, we would like to take an elementary substructure of size κ of the embedding we currently

have. To make this precise, consider the structure $\langle H^M_{j(\kappa)^+}, H_{\kappa^+}, j\rangle$. This is a structure whose universe is $H^M_{j(\kappa)^+}$ with a unary relation H_{κ^+} and a binary relation j. Observe that $\langle H^M_{j(\kappa)^+}, H_{\kappa^+}, j\rangle \models$ "j is a κ-powerset preserving embedding from H_{κ^+} to $H^M_{j(\kappa)^+}$". Now we can take an elementary substructure $\langle N', K, h'\rangle$ of size κ such that $A \in N'$, $\kappa \subseteq N'$, and $N'^{<\kappa} \subseteq N'$. The last is possible, by the usual Skolem-Lowenheim type construction, since κ is inaccessible. By elementarity, we have that $A \in K$, $\kappa \subseteq K$, and $K^{<\kappa} \subseteq K$. Also we have $K \prec H_{\kappa^+}$, and since $\kappa \subseteq K$, we have that K is transitive. Thus, K is a κ-model containing A and elementary in H_{κ^+}. Let $\pi : N' \to N$ be the Mostowski collapse and observe that $\pi \restriction K = id$. Finally, let $h = \pi'' h'$. Elementarity and the fact that π is an isomorphism imply that $h : K \to N$ is a κ-powerset preserving embedding. This concludes the proof that κ is super Ramsey. It remains to show that κ is a limit of super Ramsey cardinals. It will suffice to show that $M \models$ "κ is super Ramsey". But this follows since the h we built has transitive closure of size κ and is therefore in M. □

Corollary 2.20. *Con(*ZFC + ∃ *measurable cardinal)* $\implies$ *Con(*ZFC + ∃ *proper class of super Ramsey cardinals)*

Next, observe that we cannot hope for any of the cardinals with the Ramsey-like embeddings to have property (8) of weakly compact cardinals

from Theorem 2.1. In fact, the following is true:

Proposition 2.21. *If κ has the property that every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a κ-powerset preserving embedding $j : M \to N$ such that $j''\{B \subseteq \kappa \mid B \in M\}$ is an element of N, then κ is a limit of measurable cardinals.*

Proof. Fix a weak κ-model M containing V_κ for which there exists a κ-powerset preserving $j : M \to N$ such that $X = j''\{B \subseteq \kappa \mid B \in M\}$ is an element of N. Define $U = \{B \subseteq \kappa \mid B \in M \text{ and } \kappa \in j(B)\}$ and observe that U is a normal M-ultrafilter. Since $X \in N$, we can define $\{C \cap \kappa \mid C \in X \text{ and } \kappa \in C\} = \{B \subseteq \kappa \mid \kappa \in j(B)\} = U$ in N. Therefore U is an element of N. But, by the powerset preservation property, U is also a normal N-ultrafilter, and hence N thinks that κ is measurable. It follows that κ must be a limit of measurable cardinals. □

For example, if κ is 2^κ-supercompact, then κ will have the above property. To see this, fix a 2^κ-supercompact embedding $j : V \to M$ and $A \subseteq \kappa$. Choose some cardinal λ such that $j(\lambda) = \lambda$ and $j''2^\kappa \in H^M_{\lambda^+}$. As before, we first restrict j to a set embedding $j : H_{\lambda^+} \to H^M_{\lambda^+}$. Observe that $H^M_{\lambda^+} \subseteq H_{\lambda^+}$. Thus, it makes sense to consider the structure $\langle H_{\lambda^+}, H^M_{\lambda^+}, j\rangle$. Take an elementary substructure $\langle K', N', h'\rangle$ of $\langle H_{\lambda^+}, H^M_{\lambda^+}, j\rangle$ of size κ such that $A \in K'$,

$j''2^\kappa \in K'$, $\kappa + 1 \subseteq K'$, and $K'^{<\kappa} \subseteq K'$. Let $\pi : K' \to K$ be the Mostowski collapse, $\pi \restriction N' = N$, and $h = \pi'' h'$. Observe that N is the Mostowski collapse of N'. It is easy to see that K is a κ-model containing A, the map $h : K \to N$ is a κ-powerset preserving embedding, and $h''\{B \subseteq \kappa \mid B \in K\}$ is an element of N.

Property (2) of weakly compact cardinals from Theorem 2.1 is called the *Extension Property*. Finally, I will show that the weak Ramsey cardinals also have an extension-like property. Suppose $\mathfrak{X} \subseteq \mathcal{P}(\kappa)$. The structure $\langle V_\kappa, B\rangle_{B\in\mathfrak{X}}$ will be a structure in the language consisting of $\in$ and unary predicate symbols for every element of $\mathfrak{X}$ with the natural interpretation.

Theorem 2.22. *A cardinal κ is weakly Ramsey if and only if every $A \subseteq \kappa$ belongs to a collection $\mathfrak{X} \subseteq \mathcal{P}(\kappa)$ such that the structure $\langle V_\kappa, B\rangle_{B\in\mathfrak{X}}$ has a proper transitive elementary extension $\langle W, B^*\rangle_{B\in\mathfrak{X}}$ with $\mathcal{P}(\kappa)^W = \mathfrak{X}$.*

Proof. ($\Longrightarrow$): Suppose that κ is a weakly Ramsey cardinal and $A \subseteq \kappa$. Fix a weak κ-model M containing A and V_κ for which there exists a κ-powerset preserving embedding $j : M \to N$. Let $\mathfrak{X} = \{B \subseteq \kappa \mid B \in M\}$. It is easy to verify that $\langle V_\kappa, B\rangle_{B\in\mathfrak{X}} \prec \langle V_{j(\kappa)}, j(B)\rangle_{B\in\mathfrak{X}}$.

($\Longleftarrow$): Fix $A \subseteq \kappa$. The set A belongs to a collection $\mathfrak{X} \subseteq \mathcal{P}(\kappa)$ such that the structure $\langle V_\kappa, B\rangle_{B\in\mathfrak{X}}$ has a proper transitive elementary extension $\langle W, B^*\rangle_{B\in\mathfrak{X}}$

with $\mathcal{P}(\kappa)^W = \mathfrak{X}$. Since V_κ satisfies that H_{α^+} exists for every $\alpha < \kappa$, it follows by elementarity that H_{κ^+} exists in W. Let $M = H^W_{\kappa^+}$ and observe that M is a weak κ-model containing A. Define $U = \{B \in \mathfrak{X} \mid k \in B^*\}$. I claim that U is a weakly amenable normal M-ultrafilter. See ahead to Definition 2.26 and Proposition 2.27 for an explanation of weakly amenable ultrafilters. It should be clear that U is an ultrafilter. To show that U is normal, fix a regressive $f : B \to \kappa$ in M for some $B \in U$. Since we can code f as a subset of κ and $\mathcal{P}(\kappa)^W = \mathfrak{X}$, we can think of f being in $\mathfrak{X}$. Now we can consider the regressive $f^* : B^* \to \kappa^*$ and let $f^*(\kappa) = \alpha < \kappa$. The set $C = \{\xi \in \kappa \mid f(\xi) = \alpha\}$ is in W. Since it is clear that $\kappa \in C^*$, we have $C \in U$. Thus, U is normal. To show that U is weakly amenable, suppose $B \subseteq \kappa \times \kappa$ is in M. We need to see that the set $C = \{\alpha \in \kappa \mid B_\alpha \in U\}$ is in M. Again, since we can code B as a subset of κ, we think of B as being in $\mathfrak{X}$. In W, we can define the set $\{\alpha \in \kappa \mid \kappa \in B^*_\alpha\}$ and it is clear that this set is exactly C. This completes the argument that $C \in M$, and hence U is weakly amenable.

Next, I will show that the ultrapower of M by U is well-founded. It will help first to verify that if $C \in \mathfrak{X}$ codes a well-founded relation on κ, then C^* codes a well-founded relation on Ord^W. If $C \in \mathfrak{X}$ codes a well-founded relation, $\langle V_\kappa, B\rangle_{B\in\mathfrak{X}}$ satisfies that $C \restriction \alpha$ has a rank function for all $\alpha < \kappa$. It follows that $\langle W, B^*\rangle_{B\in\mathfrak{X}}$ satisfies that $C^* \restriction \alpha$ has a rank function

for all $\alpha < Ord^W$. We can assume that W has size κ since if this is not the case, we can take an elementary substructure of size κ which contains V_κ as a subset and collapse it. Since κ is weakly compact and we assumed that W has size κ, we can find a proper well-founded elementary extension $\langle X, E, B^{**}\rangle_{B\in\mathfrak{X}}$ for the structure $\langle W, \in, B^*\rangle_{B\in\mathfrak{X}}$ (Theorem 2.1 (1)). There is no reason to expect that X is an end-extension or that E is the true membership relation, but that is not important for us. We only care that E is well-founded. By elementarity, it follows that $\langle X, E, B^{**}\rangle_{B\in\mathfrak{X}}$ satisfies that $C^{**} \restriction \alpha$ has a rank function for all $\alpha < Ord^X$. In particular, if $\alpha > Ord^W$ in X, then $\langle X, E, B^{**}\rangle_{B\in\mathfrak{X}}$ satisfies that $C^{**} \restriction \alpha$ has a rank function. Since the structure $\langle X, E, B^{**}\rangle_{B\in\mathfrak{X}}$ is well-founded and can only add new elements to C^*, if C^* was not well-founded to begin with, X would detect this. Hence C^* is really well-founded. Now, we go back to proving that the ultrapower of M by U is well-founded. Suppose towards a contradiction that there exists a membership descending sequence $\ldots E\,[f_n]\,E \ldots E\,[f_1]\,E\,[f_0]$ of elements of the ultrapower. Each $f_n : \kappa \to M$ is an element of M and for every $n \in \omega$, the set $A_n = \{\alpha \in \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in U$. In M, fix some $F_n \subseteq \kappa$ coding the function f_n. We can use the codes F_0 and F_1 and the set A_0 to define $B_0 \subseteq \kappa$, which codes for every $\alpha \in A_0$, a membership isomorphism from the transitive closure of $F_1(\alpha)$ to the subset of the transitive closure of $F_0(\alpha)$ that

corresponds to the transitive closure of $f_1(\alpha)$. In this way, we define B_n for every $n \in \omega$. Observe that $\langle V_\kappa, B\rangle_{B\in\mathfrak{X}}$ satisfies that for every $\alpha \in A_0$, the set B_0 codes a membership isomorphism from the transitive closure of $F_1(\alpha)$ to a subset of the transitive closure of $F_0(\alpha)$ that corresponds to the transitive closure of an element of $F_0(\alpha)$. Since $A_0 \in U$, we know that $\kappa \in A_0^*$. Hence $\langle W, B^*\rangle_{B\in\mathfrak{X}}$ satisfies that the set B_0^* codes a membership isomorphism from the transitive closure of $F_1^*(\kappa)$ to a subset of the transitive closure of $F_0^*(\kappa)$ that corresponds to the transitive closure of an element of $F_0^*(\kappa)$. The same statement holds of course for all $n \in \omega$. Since each F_n was well-founded, then so is each F_n^* by the above argument. Hence we can Mostowski collapse each F_n^* to obtain some function $g_n : Ord^W \to Ord$. Finally, observe that the $g_n(\kappa)$ form a descending $\in$-sequence. This follows since the transitive closure of each $F_{n+1}^*(\kappa)$ was membership isomorphic to the transitive closure of an element of $F_n^*(\kappa)$. Thus, we reached a contradiction showing that the the ultrapower of M by U is well-founded. Let N be the Mostowski collapse of M/U and observe finally that since U was weakly amenable, the ultrapower embedding $j : M \to N$ is κ-powerset preserving (Proposition 2.27). □

Question 2.23. If κ is Ramsey, does it follow that κ is the κ^{th} weakly Ramsey cardinal?

This question has been answered affirmatively by Ian Sharpe [24] (see Section 2.5 for details).

Question 2.24. Are weakly Ramsey cardinals consistent with $V = L$?

2.3 Ramsey Cardinals

In this section, I will give a proof that Ramsey cardinals are exactly the cardinals with the Ramsey embedding property. Before that we need to verify some facts about product M-ultrafilters and κ-powerset preserving embeddings.

Definition 2.25. A cardinal κ is *Ramsey* if every coloring $F : [\kappa]^{<\omega} \to 2$ has a homogenous set of size κ.

Definition 2.26. Suppose M is a weak κ-model and U is a normal M-ultrafilter on κ. Then U is *weakly amenable* if for every $A \subseteq \kappa \times \kappa$ in M, the set $\{\alpha \in \kappa \mid A_\alpha \in U\} \in M$.

It is easy to see that if U is weakly amenable, then for every $A \subseteq \kappa^n \times \kappa$ in M, the set $\{\vec{\alpha} \in \kappa^n \mid A_{\vec{\alpha}} \in U\} \in M$. If U is a weakly amenable normal M-ultrafilter on κ, we can define *product* ultrafilters U^n on $\mathcal{P}(k^n) \cap M$ for every $n \in \omega$. We define U^n by induction on n such that $A \subseteq \kappa^n \times \kappa$ is in $U^{n+1} = U^n \times U$ if and only if $A \in M$ and $\{\vec{\alpha} \in \kappa^n \mid$

$A_{\vec{\alpha}} \in U\} \in U^n$. Note that this definition makes sense only in the presence of weak amenability. It turns out that weakly amenable normal M-ultrafilters on κ are exactly the ones that give rise to κ-powerset preserving embeddings.

Proposition 2.27. *If M is a weak κ-model and $j : M \to N$ is the ultrapower by a normal M-ultrafilter U on κ, then U is weakly amenable if and only if j is κ-powerset preserving.* ([11], p. 246)

Proof. ($\Longrightarrow$): Suppose $j : M \to N$ is the ultrapower by a weakly amenable normal M-ultrafilter U on κ. Fix $A \subseteq \kappa$ in N and let $A = [f]_U$. Observe that for all $\alpha < \kappa$, we have $\alpha \in A \leftrightarrow [c_\alpha]_U \in [f]_U \leftrightarrow \{\xi \in \kappa \mid \alpha \in f(\xi)\} \in U$. In M, we can define $B = \{\langle \alpha, \xi \rangle \in \kappa \times \kappa \mid \alpha \in f(\xi)\}$. It follows, by the weak amenability of U, that $C = \{\alpha \in \kappa \mid B_\alpha \in U\} \in M$. But clearly $C = A$, and hence $A \in M$.

($\Longleftarrow$): Suppose $j : M \to N$ is the ultrapower by a normal M-ultrafilter U on κ and j is κ-powerset preserving. Fix $A \subseteq \kappa \times \kappa$ in M and let $B = \{\alpha \in \kappa \mid A_\alpha \in U\}$. We need to show that $B \in M$. Observe that $A_\alpha \in U$ if and only if $\kappa \in j(A_\alpha) = j(A)_\alpha$. Thus, $B = \{\alpha \in \kappa \mid \kappa \in j(A)_\alpha\}$ is in N. By the powerset preservation property, $B \in M$. Thus, U is weakly amenable. □

Notice that with the definition of weakly amenable ultrafilters we can

restate the Ramsey embedding property. We say that an ultrafilter U is *ω-closed* if whenever $\langle A_n \mid n \in \omega \rangle$ is a sequence of elements of U, the intersection $\cap_{n\in\omega} A_n \neq \emptyset$.

Proposition 2.28. *A cardinal κ has the Ramsey embedding property if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an ω-closed weakly amenable normal M-ultrafilter on κ.*

This holds since an ω-closed ultrafilter must give rise to a well-founded ultrapower. The next couple of propositions address the issue of iterating ultrapowers.

Lemma 2.29. *Suppose M is a weak κ-model, U is a weakly amenable normal M-ultrafilter on κ, and $j : M \to N$ is the well-founded ultrapower by U^n with critical point κ. Define $j(U) = \{A \subseteq j(\kappa) \mid A = [f]_{U^n}$ and $\{\vec{\alpha} \in \kappa^n \mid f(\vec{\alpha}) \in U\} \in U^n\}$. Then $j(U)$ is well-defined and $j(U)$ is a weakly amenable normal N-ultrafilter on $j(\kappa)$ such that $A \in U$ implies $j(A) \in j(U)$.* ([11], p. 246)

Proof. First, let us verify that $j(U)$ is well-defined. Suppose $A \subseteq j(\kappa)$ in N and $A = [f]_{U^n} = [g]_{U^n}$. Then $X = \{\vec{\alpha} \in \kappa^n \mid f(\vec{\alpha}) = g(\vec{\alpha})\} \in U^n$. Let $X_1 = \{\vec{\alpha} \in \kappa^n \mid f(\vec{\alpha}) \in U\}$ and $X_2 = \{\vec{\alpha} \in \kappa^n \mid g(\vec{\alpha}) \in U\}$. Observe that $X_1 \in U^n$ implies that $X_1 \cap X \in U^n$, which, in turn, implies that $X_2 \in U^n$.

Next, we check that $j(U)$ is an ultrafilter. Let $A = [f_A]_{U^n}$ and $B = [f_B]_{U^n}$ be elements of N and let A^c denote the complement of A. If $E = [f_E]_{U^n}$ is any set, define $X_E = \{\vec{\alpha} \in \kappa^n \mid f_E(\vec{\alpha}) \in U\}$. Suppose $A \subseteq B \subseteq j(\kappa)$ and $A \in j(U)$. Then $X_A \in U^n$ and $Y = \{\vec{\alpha} \in \kappa^n \mid f(\vec{\alpha}) \subseteq g(\vec{\alpha})\} \in U^n$. Therefore $X_A \cap Y \subseteq X_B$, and hence $X_B \in U^n$. This shows that $B \in j(U)$. Suppose A and B are both in $j(U)$. Let $C = [f_C]_{U^n} = A \cap B$ and $Z = \{\vec{\alpha} \in \kappa^n \mid f_A(\vec{\alpha}) \cap f_B(\vec{\alpha}) = f_C(\vec{\alpha})\} \in U^n$. Therefore $X_A \cap X_B \cap Z \subseteq X_C$, and hence $X_C \in U^n$. This shows that $C \in j(U)$. Suppose $A \subseteq j(\kappa)$ is not in $j(U)$. We can assume without loss of generality that $f_A(\vec{\alpha}) \subseteq \kappa$ for all $\vec{\alpha} \in \kappa^n$. Define $f_{A_c} : \kappa^n \to M$ so that $f_{A_c}(\vec{\alpha})$ is the complement of $f_A(\vec{\alpha})$ in κ, then clearly $A^c = [f_{A_c}]_{U^n}$. Since A is not in $j(U)$, the set X_A is not in U^n. It follows that the complement of X_A is in U^n, but the complement of X_A is precisely X_{A_c}. Thus, $X_{A_c} \in U^n$, and hence $A_c \in j(U)$. Finally, it is clear that $j(\kappa) \in j(U)$. This completes the proof that $j(U)$ is an ultrafilter.

Next, we check that $j(U)$ is normal. Fix $A = [f_A]_{U^n} \in j(U)$ and a regressive $[f_F]_{U^n} = F : A \to j(\kappa)$. First, we can assume without loss of generality that $f_A(\vec{\alpha}) \in U$ for all $\vec{\alpha} \in \kappa^n$. This follows by the weak amenability of U since it allows us to tell which $f(\vec{\alpha})$ are in U, and once we know that we can make $f(\vec{\alpha}) = \kappa$ on the rest. Second, we can assume without loss of generality that $f_F(\vec{\alpha}) : f_A(\vec{\alpha}) \to \kappa$ is regressive for all $\vec{\alpha} \in \kappa^n$. Consider

the set $C = \{\langle \vec{\alpha}, \beta, \gamma\rangle \mid f_F(\vec{\alpha})(\gamma) = \beta\}$ and let $D = \{\langle \vec{\alpha}, \beta\rangle \mid C_{\langle \vec{\alpha},\beta\rangle} \in U\}$. Then $D \in M$ by weak amenability. In M, define $h : \kappa^n \to M$ by letting $h(\vec{\alpha})$ be the least β such that $\langle \vec{\alpha}, \beta\rangle \in D$. Again in M, define $l : \kappa^n \to M$ by $l(\vec{\alpha}) = \{\gamma \in \kappa \mid f_F(\vec{\alpha})(\gamma) = h(\vec{\alpha})\}$. Observe that for all $\vec{\alpha} \in \kappa^n$, we have $l(\vec{\alpha}) \in U$ and $f_F(\vec{\alpha})$ is constant on $l(\vec{\alpha})$. Thus, F is constant on $[l]_{U^n} \in j(U)$. This completes the proof that $j(U)$ is normal.

Next, we show that $j(U)$ is weakly amenable. Fix $A = [f]_{U^n} \subseteq j(\kappa) \times j(\kappa)$ in N. We need to show that $B = \{\xi \in j(\kappa) \mid A_\xi \in j(U)\} \in N$. We can assume without loss of generality that $f(\vec{\alpha}) \subseteq \kappa \times \kappa$ for all $\vec{\alpha} \in \kappa^n$. Define $g : \kappa^n \to M$ by $g(\vec{\alpha}) = \{\xi \in \kappa \mid f(\vec{\alpha})_\xi \in U\}$. Let us verify that $g \in M$. Define $C = \{\langle \vec{\alpha}, \xi, \beta\rangle \mid \langle \xi, \beta\rangle \in f(\vec{\alpha})\}$ and define $D = \{\langle \vec{\alpha}, \xi\rangle \mid C_{\langle \vec{\alpha},\xi\rangle} \in U\}$. Observe that $D \in M$ by weak amenability. We see that $g(\vec{\alpha}) = \{\xi \in \kappa \mid \langle \vec{\alpha}, \xi\rangle \in D\}$, and hence $g \in M$. It remain to show that $[g]_{U^n} = B$. Let $[h]_{U^n} \in [g]_{U^n}$ and assume without loss of generality that $h(\vec{\alpha}) \in g(\vec{\alpha})$ for all $\vec{\alpha} \in \kappa^n$. Thus, by definition of g, we have $f_A(\vec{\alpha})_{h(\vec{\alpha})} \in U$ for all $\vec{\alpha} \in \kappa^n$. It follows that $E = \{\vec{\alpha} \in \kappa^n \mid f(\vec{\alpha})_{h(\vec{\alpha})} \in U\} \in U^n$, and hence $[f]_{U^n[h]_{U^n}} \in j(U)$. Thus, $[g]_{U^n} \subseteq B$ has been established. Now suppose $[h]_{U^n} \in B$, then $[f]_{U^n[h]_{U^n}} \in j(U)$, and therefore $E \in U^n$, which, in turn, implies that $[h]_{U^n} \in [g]_{U^n}$. Thus, $B \subseteq [g]_{U^n}$ has been established.

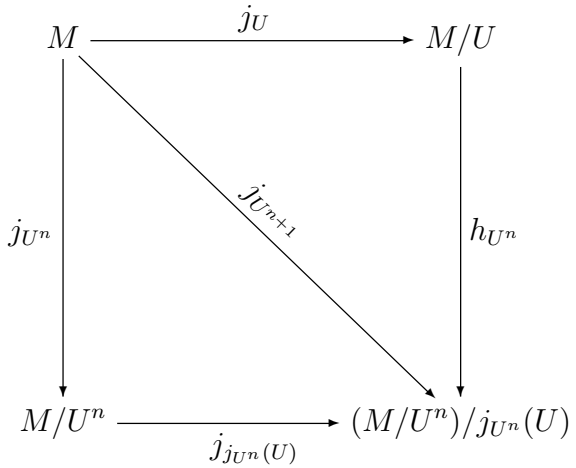
Finally, we check that $A \in U$ implies $j(A) \in j(U)$. We need to show that

$[c_A]_{U^n} \in j(U)$, which is true if and only if $\{\vec{\alpha} \in \kappa^n \mid A \in U\} = \kappa \in U^n$. $\square$

Question 2.30. If the ultrapower by U is well-founded, is the ultrapower by $j(U)$ necessarily well-founded?

The next lemma is a standard fact from iterating ultrapowers (see, for example, [8], ch. 0) that must be checked here since we are restricted to only using functions that are elements of the respective models.

Lemma 2.31. *Suppose M is a weak κ-model, U is a weakly amenable normal M-ultrafilter on κ, and $j_U : M \to M/U$ is the well-founded ultrapower by U. Suppose further that $j_{U^n} : M \to M/U^n$ and $h_{U^n} : M/U \to (M/U)/U^n$ are the well-founded ultrapowers by U^n. Then the ultrapower $j_{j_{U^n}(U)} : M/U^n \to (M/U^n)/j_{U^n}(U)$ by $j_{U^n}(U)$ and the ultrapower $j_{U^{n+1}} : M \to M/U^{n+1}$ by U^{n+1} are also well-founded. Moreover, $(M/U^n)/j_{U^n}(U) = (M/U)/U^n = M/U^{n+1}$ and the following diagram commutes:*



Proof. We will define an isomorphism Φ between $(M/U^n)/j_{U^n}(U)$ and M/U^{n+1}. Fix $F : \kappa^n \times \kappa \to M$ with $F \in M$. Define $f_{\vec{\alpha}} : \kappa \to M$ by $f_{\vec{\alpha}}(\beta) = F(\vec{\alpha}, \beta)$ and define $G : \kappa^n \to M$ by $G(\vec{\alpha}) = f_{\vec{\alpha}}$. Observe that $G \in M$ by Replacement and consider $f = [G]_{U^n} \in M/U^n$. We have that $f : j_{U^n}(\kappa) \to M/U^n$, and hence $[f]_{j_{U^n}(U)}$ is in $(M/U^n)/j_{U^n}(U)$. Define $\Phi([F]_{U^{n+1}}) = [f]_{j_{U^n}(U)}$. Standard arguments show that Φ is well-defined and preserves membership (see, for example, [8], ch. 0). I will show that Φ is onto. Fix $[f]_{j_{U^n}(U)} \in (M/U^n)/j_{U^n}(U)$, then $f : j_{U^n}(\kappa) \to M/U^n$ and $f = [G]_{U^n}$ where $G : \kappa^n \to M$ is an element of M. We can assume without loss of generality that $G(\vec{\alpha}) : \kappa \to M$ for all $\vec{\alpha} \in \kappa^n$. Use G to define $F : \kappa^n \times \kappa \to M$ reversing the earlier procedure. Clearly such $F \in M$ and $\Phi([F]_{U^{n+1}}) = [f]_{j_{U^n}(U)}$.

Next, we define an isomorphism Ψ between $(M/U)/U^n$ and M/U^{n+1}. Fix $f : \kappa^n \to M/U$ with $f \in M/U$, then $f = [g]_U$ where $g : \kappa \to M$ and $g \in M$. Let $f(\vec{\alpha}) = [f_{\vec{\alpha}}]_U$ where $f_{\vec{\alpha}} : \kappa \to M$ and $f_{\vec{\alpha}} \in M$. Observe that $[f_{\vec{\alpha}}]_U = [g]_U([c_{\vec{\alpha}}]_U)$, which implies that $\{\xi \in \kappa \mid f_{\vec{\alpha}}(\xi) = g(\xi)(\vec{\alpha})\} \in U$. Define $g_{\vec{\alpha}} : \kappa \to M$ by $g_{\vec{\alpha}}(\xi) = g(\xi)(\vec{\alpha})$ if $\vec{\alpha} \in \mathrm{dom}(g(\xi))$ and $\emptyset$ otherwise. The sequence $\langle g_{\vec{\alpha}} \mid \vec{\alpha} \in \kappa^n \rangle \in M$. It should be clear that $[f_{\vec{\alpha}}]_U = [g_{\vec{\alpha}}]_U$ for all $\vec{\alpha} \in \kappa^n$. Define $F : \kappa^n \times \kappa \to M$ by $F(\vec{\alpha}, \xi) = g_{\vec{\alpha}}(\xi)$ and observe that $F \in M$. Define $\Psi : (M/U)/U^n \to M/U^{n+1}$ by $\Psi([f]_{U^n}) = [F]_{U^{n+1}}$.

Again, standard arguments shows that Ψ is well-defined and membership preserving (see [8], ch. 0). So we need only to verify that Ψ is onto. Fix $F' : \kappa^n \times \kappa \to M$ with $F' \in M$. Define $\langle F'_{\vec{\alpha}} \mid \vec{\alpha} \in \kappa^n \rangle \in M$ such that $F'_{\vec{\alpha}}(\xi) = F(\vec{\alpha}, \xi)$. Define $g : \kappa \to M$ such that each $g(\beta)$ is a function with domain β^n and $g(\xi)(\vec{\alpha}) = F'_{\vec{\alpha}}(\xi)$. Observe that $\mathrm{dom}([g]_U) = k^n = [id]^n_U$ since $\{\xi \in \kappa \mid \mathrm{dom}(g(\xi)) = \xi^n\} = \kappa \in U$. I claim $\Psi([[g]_U]_{U^n}) = [F']_{U^{n+1}}$. First, we argue that $[g]_U(\vec{\alpha}) = [F'_{\vec{\alpha}}]_U$. As before, $[g]_U(\vec{\alpha}) = [g]_U([c_{\vec{\alpha}}]_U)$, and so we need to show that $\{\xi \in \kappa \mid g(\xi)(\vec{\alpha}) = F'_{\vec{\alpha}}(\xi)\} \in U$. But the set $\{\xi \in \kappa \mid g(\xi)(\vec{\alpha}) = F'_{\vec{\alpha}}(\xi)\} \supseteq \{\xi \in \kappa \mid \vec{\alpha} \in \mathrm{dom}(g(\xi))\}$, and the later set is clearly in U. Let $\Psi([[g]_U]_{U^n}) = [F]_{U^{n+1}}$ and $g_{\vec{\alpha}}$ be defined from g by the above construction. It follows that for all $\vec{\alpha} \in \kappa^n$, we have $[g_{\vec{\alpha}}]_U = [F'_{\vec{\alpha}}]_U$, and therefore $\{\xi \in \kappa \mid g_{\vec{\alpha}}(\xi) = F'_{\vec{\alpha}}(\xi)\} \in U$. Finally, this shows that $\{\vec{\alpha} \in \kappa^n \mid \{\xi \in \kappa \mid F(\vec{\alpha}, \xi) = F'(\vec{\alpha}, \xi)\} \in U\} = \kappa^n \in U^n$. Thus, $[F]_{U_{n+1}} = [F']_{U_{n+1}}$ as promised.

It is a standard argument to verify that the diagram commutes. □

Proposition 2.32. *If M is a weak κ-model and U is an ω-closed weakly amenable normal M-ultrafilter on κ, then the ultrapowers of M by U^n are well-founded for all $n \in \omega$.*

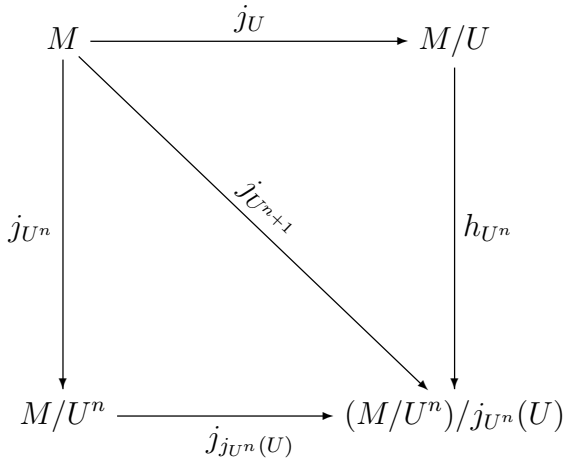
Proof. We argue by induction on n that for every weak κ-model M and every

ω-closed weakly amenable normal M-ultrafilter U on κ, the ultrapower of M by U^n is well-founded. We already noted earlier that if U is an ω-closed M-ultrafilter, the ultrapower of M by U is well-founded. So suppose that for every M and every ω-closed weakly amenable normal M-ultrafilter U, the ultrapowers of M by U^m are well-founded for every $m \leq n$. From Lemma 2.31, we know that if the ultrapower of M by U and the ultrapower of M/U by U^n are well-founded, then the ultrapower of M by U^{n+1} is well-founded as well. This completes the induction step. □

Lemma 2.33. *Suppose M is a weak κ-model and U is a weakly amenable normal M-ultrafilter on κ such the embeddings $j_{U^n} : M \to M/U^n$ are well-founded for all $n \in \omega$. Then $A \in U^n$ if and only if $\langle \kappa, j_U(\kappa), \ldots, j_{U^{n-1}}(\kappa) \rangle \in j_{U^n}(A)$.*

Proof. The proof is by induction on n. Since $j : M \to M/U$ is a normal ultrapower, $A \in U$ if and only if $\kappa \in j_U(A)$. Assume that the statement holds for n. To prove the statement for $n+1$, consider the commutative

diagram below:



Suppose $\langle \kappa, j_U(\kappa), \ldots, j_{U^n}(\kappa)\rangle \in j_{U^{n+1}}(A)$, then $\langle \kappa, j_U(\kappa), \ldots, j_{U^n}(\kappa)\rangle \in j_{j_{U^n}(U)}(j_{U^n}(A))$, and so $j_{U^n}(\kappa) \in j_{j_{U^n}(U)}(j_{U^n}(A))_{\langle \kappa, j_U(\kappa), \ldots, j_{U^{n-1}}(\kappa)\rangle}$. It follows that $j_{U^n}(\kappa) \in j_{j_{U^n}(U)}(j_{U^n}(A)_{\langle \kappa, j_U(\kappa), \ldots, j_{U^{n-1}}(\kappa)\rangle})$ since the critical point of $j_{j_{U^n}(U)}$ is $j_{U^n}(\kappa)$. Since $j_{U^n}(U)$ is a normal ultrafilter on $j_{U^n}(\kappa)$, we have that $j_{U^n}(A)_{\langle \kappa, j_U(\kappa), \ldots, j_{U^{n-1}}(\kappa)\rangle} \in j_{U^n}(U)$. By the inductive assumption, we know that $[id]_{U^n} = \langle \kappa, j_U(\kappa), \ldots, j_{U^{n-1}}(\kappa)\rangle$, and therefore we have $j_{U^n}(A)_{[id]_{U^n}} \in j_{U^n}(U)$. But now this implies that $\{\vec{\alpha} \in \kappa^n \mid A_{\vec{\alpha}} \in U\} \in U^n$. So $A \in U^{n+1}$ as desired. For the forward implication, just reverse the steps above. □

Lemma 2.34. *Suppose M is a weak κ-model and U is a weakly amenable normal M-ultrafilter on κ such that the ultrapowers by U^n are well-founded for all $n \in \omega$. If $A \in U^n$, then there exists $B \in U$ such that for all $\alpha_1 < \cdots < \alpha_n \in B$, we have $\langle \alpha_1, \ldots, \alpha_n\rangle \in A$.*

Proof. I will argue by induction on n. Throughout the proof, I will be using the machinery of diagonal intersections in disguise. It is easier to see what is going on in this instance without using the terminology of diagonal intersections. The base case $n = 1$ is clearly trivial. So suppose the statement holds for n. Let $A \in U^{n+1}$, then $Z = \{\vec{\alpha} \in \kappa^n \mid A_{\vec{\alpha}} \in U\} \in U^n$. By the inductive assumption, there is a set $Y \in U$ such that for all $\beta_1 < \cdots < \beta_n \in Y$, we have $\langle \beta_1, \ldots, \beta_n \rangle \in Z$. Define $B = \{\xi \in Y \mid \forall \beta_1 < \cdots < \beta_n \in Y$ if $\beta_n < \xi$, then $\xi \in A_{\langle \beta_1, \ldots, \beta_n \rangle}\}$, then $j(B) = \{\xi \in j(Y) \mid \forall \beta_1 < \cdots < \beta_n \in j(Y)$ if $\beta_n < \xi$, then $\xi \in j(A)_{\langle \beta_1, \ldots, \beta_n \rangle}\}$. Recall that $B \in U$ if and only if $\kappa \in j(B)$. It is clear that $\kappa \in j(Y)$. If $\beta_1 < \cdots < \beta_n \in j(Y)$ and $\beta_n < \kappa$, then $\beta_1, \ldots, \beta_n \in Y$. It follows that $\kappa \in j(A_{\langle \beta_1, \ldots, \beta_n \rangle}) = j(A)_{\langle \beta_1, \ldots, \beta_n \rangle}$ since $A_{\langle \beta_1, \ldots, \beta_n \rangle} \in U$. So $\kappa \in j(B)$, and hence $B \in U$. Also clearly if $\beta_1 < \cdots < \beta_n < \xi$ are in B, then $\langle \beta_1, \ldots, \beta_n, \xi \rangle \in A$. □

We are now ready to prove that cardinals with the Ramsey embedding property are Ramsey. The other direction will involve a much more complicated argument starting with Definition 2.37.

Theorem 2.35. *A cardinal κ has the Ramsey embedding property if and only if κ is Ramsey.*

Proof of the forward direction. Fix a coloring $F : [\kappa]^{<\omega} \to 2$ and choose a

weak κ-model M containing F for which there exists an ω-closed weakly amenable normal M-ultrafilter U on κ. Let $j : M \to N$ be the ultrapower by U. We need to show that F has a homogenous set H of size κ. First, observe that if $\langle A_n \mid n \in \omega \rangle$ is a sequence such that each $A_n \in U$, then $\cap_{n\in\omega} A_n$ has size κ. To see this, suppose $\cap_{n\in\omega} A_n$ has size less than κ. Choose $\alpha < \kappa$ such that $\cap_{n\in\omega} A_n$ is contained in α and let $B = \{\xi \in \kappa \mid \xi > \alpha\} \in U$. But then $B \cap (\cap_{n\in\omega} A_n) = \emptyset$, which contradicts that U is ω-closed. Define $f_n : [\kappa]^n \to 2$ by $f_n = F \restriction [\kappa]^n$ and observe that $\langle f_n \mid n \in \omega \rangle \in M$. The strategy will be to find for every $n \in \omega$, a set $H_n \in U$ homogenous for f_n. Then $\cap_{n\in\omega} H_n$ will have size κ and be homogenous for F. Let $j_{U^n}(f_n)(\kappa, j_U(\kappa), \ldots, j_{U^{n-1}}(\kappa)) = i \in 2$ and consider $A = \{\vec{\alpha} \in \kappa^n \mid f_n(\vec{\alpha}) = i\}$. Clearly $A \in U^n$ since $\langle \kappa, j_U(\kappa), \ldots, j_{U^{n-1}}(\kappa) \rangle \in j_{U^n}(A)$. By Proposition 2.34, there exists $B \in U$ such that for all $\beta_1 < \cdots < \beta_n \in B$, we have $\langle \beta_1, \ldots, \beta_n \rangle \in A$. Letting $B = H_n$, we get the desired result. □

Corollary 2.36. *If κ is a strongly Ramsey cardinal, then κ is the κ^{th} Ramsey cardinal.*

Proof. Clearly κ is Ramsey. So we need to show that κ is a limit of Ramsey cardinals. Fix a κ-model M for which there exists a κ-powerset preserving $j : M \to N$. As usual, we show that $N \models$ "κ is Ramsey". Fix a coloring

$F : [\kappa]^{<\omega} \to 2$ in N. By the powerset preservation property, $F \in M$. We can argue, as in Theorem 2.35, to construct $H_n \in U$ homogenous for $F \restriction [k]^n$. Since $M^{<\kappa} \subseteq M$, the sequence $\langle H_n \mid n \in \omega \rangle \in M$, and therefore $H = \cap_{n\in\omega} H_n$ is in M as well. Hence $H \in N$, and so $N \models$ "κ is Ramsey". □

Now we begin to build up the machinery required to show that if κ is Ramsey, then κ has the Ramsey embedding property. Recall that a partition property $\kappa \to (\alpha)^n_\gamma$ means that every coloring $F : [\kappa]^n \to \gamma$ has a homogenous set of order-type α.

Definition 2.37. A cardinal κ is *α-Erdős* if α is a limit ordinal and κ is the least cardinal such that $\kappa \to (\alpha)^{<\omega}_2$.

Observe that κ is Ramsey if and only if κ is κ-Erdős. Also recall that α-Erdős cardinals are inaccessible.

Theorem 2.38. *If κ is α-Erdős, then $\kappa \to (\alpha)^{<\omega}_\beta$ for every $\beta < \kappa$.*

Lemma 2.39. *Suppose κ is α-Erdős and $\mathcal{A}$ is a structure in some countable language such that $\kappa \subseteq \mathcal{A}$. Then there is $I \subseteq \kappa$ of order-type α such that I are indiscernibles for $\mathcal{A}$.*

See [9] (p. 298) for a discussion of the above concepts. The results below are from [3] (p. 128).

Definition 2.40. Suppose κ is a cardinal. A function $f : [\kappa]^{<\omega} \to \kappa$ is said to be *regressive* if $f(\alpha_1, \ldots, \alpha_n) < \alpha_1$ for every $\alpha_1 > \emptyset$.

Definition 2.41. Suppose κ is a cardinal and $\mathcal{A} = \langle L_\kappa[A], A\rangle$ where $A \subseteq \kappa$. Then $I \subseteq \kappa$ is a set of *good indiscernibles* for $\mathcal{A}$ if for all $\gamma \in I$:

1. $\langle L_\gamma[A], A\rangle \prec \langle L_\kappa[A], A\rangle$.

2. γ is a cardinal.

3. $I - \gamma$ is a set of indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$.

Lemma 2.42. *If κ is α-Erdős and C is a club on κ, then every regressive $f : [C]^{<\omega} \to \kappa$ has a homogenous set of order-type α.*

Proof. Fix a regressive $f : [C]^{<\omega} \to \kappa$ where C is a club on κ. Since κ is α-Erdős, for every $\xi < \kappa$, we have $\xi \nrightarrow (\alpha)^{<\omega}_2$. Therefore for every $\xi < \kappa$, there is a counterexample $h_\xi : [\xi]^{<\omega} \to 2$ for which there is no homogenous set of order-type α. Let $f_n = f \restriction [C]^n$ and define $g_n : [C]^n \to 2$ by $g_n(\xi_1, \ldots, \xi_n) = h_{\xi_n}(\xi_1, \ldots, \xi_{n-1})$. We consider the structure $\mathcal{A} = \langle \kappa, \in, C, f_1, \ldots, f_n, \ldots, g_1, \ldots, g_n, \ldots\rangle$. Since this is a structure in a countable language and C has size κ, Lemma 2.39 tells us that there is a set $I \subseteq C$ of indiscernibles for $\mathcal{A}$ of order-type α. Choose such an I with the *least* first element and let $I = \{i_\xi \mid \xi < \alpha\}$. I claim that for every $\xi < \alpha$, the

ordinal i_ξ is a limit point of C. Suppose to the contrary that this is not the case, then since i_ξ are indiscernibles, we have $\mathcal{A} \models$ "i_ξ is not a limit point of C" for all $\xi < \alpha$. Thus, each i_ξ is a successor element of C. Define j_ξ to be the predecessor of i_ξ in C. It follows that $\{j_\xi \mid \xi < \alpha\}$ are indiscernibles for $\mathcal{A}$. But this is impossible since I had the least first element by assumption. So every i_ξ is a limit point of C.

I will now show that the set I is homogenous for f. Suppose to the contrary that this is not the case, then there exist two sequences $i_{\xi_1} < \cdots < i_{\xi_n}$ and $i_{\alpha_1} < \cdots < i_{\alpha_n}$ such that $f(i_{\xi_1}, \ldots, i_{\xi_n}) \neq f(i_{\alpha_1}, \ldots, i_{\alpha_n})$. We can assume without loss of generality that $\xi_n < \alpha_1$. Define $u_0 = \langle i_0, \ldots, i_{n-1} \rangle$ and $u_1 = \langle i_n, \ldots, i_{2n-1} \rangle$. Since α is a limit ordinal, it is possible, continuing in this manner, to define u_ξ for $\xi < \alpha$. By indiscernibility, it must be that either $f(u_0) < f(u_1)$ or $f(u_0) > f(u_1)$. Suppose $f(u_0) > f(u_1)$, then by indiscernibility, we have an infinite descending sequence in the ordinals, which is impossible. It follows that $f(u_0) < f(u_1)$, and hence $f(u_\xi) < f(u_\beta)$ for all $\xi < \beta < \alpha$. Define $d_\xi = f(u_\xi)$ and let c_ξ be the least element of C above d_ξ. Since $d_0 < d_1$, it follows that $c_0 \leq c_1$. So either $c_0 < c_1$ or $c_0 = c_1$. First, suppose $c_0 < c_1$, then by indiscernibility, $c_\xi < c_\beta$ for all $\xi < \beta < \alpha$. Therefore $\{c_\xi \mid \xi < \alpha\}$ are indiscernibles for $\mathcal{A}$. Recall that $d_0 = f(u_0) < i_0$ since f is regressive. But since i_0 is a limit point of C, it

must be that $c_0 < i_0$. Again, this is impossible since we chose I to have the least first element. So $c_0 = c_1$, and hence all the c_ξ are equal to each other by indiscernibility. Thus, $d_\xi < c_0$ for all $\xi < \alpha$. I claim that this implies $\{d_\xi \mid \xi < \alpha\}$ are homogenous for h_{c_0}, which will be a contradiction. By indiscernibility, it suffices to show that $h_{c_0}(d_0, \ldots, d_{n-1}) = h_{c_0}(d_n, \ldots, d_{2n-1})$. Suppose to the contrary that $h_{c_0}(d_0, \ldots, d_{n-1}) \neq h_{c_0}(d_n, \ldots, d_{2n-1})$, then $g_{n+1}(d_0, \ldots, d_{n-1}, c_0) \neq g_{n+1}(d_n, \ldots, d_{2n-1}, c_0)$ by definition. But by indiscernibility:

1. $g_{n+1}(d_0, \ldots, d_{n-1}, c_0) \neq g_{n+1}(d_{2n}, \ldots, d_{3n-1}, c_0)$

2. $g_{n+1}(d_n, \ldots, d_{2n-1}, c_0) \neq g_{n+1}(d_{2n}, \ldots, d_{3n-1}, c_0)$

This situation is impossible since g_{n+1} has only two colors. So we reached a contradiction showing that h_{c_0} has a homogenous set. This, in turn, contradicts the fact that h_{c_0} was a counterexample. Thus, finally it follows that I is homogenous for f. □

Lemma 2.43. *If κ is Ramsey and $A \subseteq \kappa$, then $\langle L_\kappa[A], A\rangle$ has a collection I of good indiscernibles of size κ.*

Proof. Define $C' = \{\alpha \in \kappa \mid \alpha$ is a cardinal and $\langle L_\alpha[A], A\rangle \prec \langle L_\kappa[A], A\rangle\}$, then C' is a club. Fix any bijection $f : \kappa \times \kappa \to \kappa - \{\emptyset\}$. Use f to inductively

define a pairing function $g : \kappa^{<\omega} \to \kappa$ such that $id = g \restriction \kappa : \kappa \to \kappa$, and given $g \restriction [\kappa]^n$, we define $g \restriction [\kappa]^{n+1} : [\kappa]^{n+1} \to \kappa$ by $g(\alpha_1, \alpha_2 \ldots, \alpha_{n+1}) = f(\alpha_1, g(\alpha_2, \ldots, \alpha_{n+1}))$. Let $C'' = \{\alpha \in \kappa \mid \alpha \times \alpha \text{ is closed under } f\}$, then C'' is a club. Observe that for all $\alpha \in C''$, the set $[\alpha]^{<\omega}$ is closed under g. Let $C = C' \cap C'' \cap \{\xi \in \kappa \mid \xi > \omega\}$, then C is a club. Fix an enumeration $\langle \varphi_m \mid m \in \omega \rangle$ of formulas in the language of $\langle L_\kappa[A], A \rangle$. Consider the following condition on an ordered tuple $\langle \alpha_1, \ldots, \alpha_{2n} \rangle$ of length $2n$ of elements of κ:

(*) there exist ordinals $\delta_1, \ldots, \delta_k$ with $\delta_k < \alpha_1$ and $m \in \omega$ such that $\langle L_\kappa[A], A \rangle \not\models \varphi_m(\vec{\delta}, \alpha_1, \ldots, \alpha_n) \leftrightarrow \varphi_m(\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n})$

If $\langle \alpha_1, \ldots, \alpha_{2n} \rangle$ satisfies (*), let $w(\vec{\alpha})$ be the least $\lambda = g(m, \vec{\delta})$ where $\vec{\delta}$ and m witness (*), and $\emptyset$ otherwise. Define $h : [C]^{<\omega} \to \kappa$ by:

$$h(\alpha_1, \ldots, \alpha_{2n+1}) = \emptyset,$$

$$h(\alpha_1, \ldots, \alpha_{2n}) = w(\vec{\alpha}).$$

It follows that the value of h is either $\emptyset$ or $h(\alpha_1, \ldots, \alpha_{2n}) = g(m, \delta_1, \ldots, \delta_k)$ where $\delta_k < \alpha_1$ and $m < \omega < \alpha_1$. Since C is closed under g, the function h is regressive. By Lemma 2.42, the function h is homogenous on a set $I \subseteq C$ of size κ. I claim that the value of h on I is $\emptyset$ for every $n \in \omega$. First, let us argue that if this is the case, then I are good indiscernibles for $\langle L_\kappa[A], A \rangle$.

Observe that since the range of f excludes $\emptyset$, the function $h(\alpha_1, \ldots, \alpha_{2n}) = \emptyset$ if and only if $\vec{\alpha}$ does not satisfy (*). Fix $\gamma \in I$. Since $\gamma \in C$, it follows that γ is a cardinal and $\langle L_\gamma[A], A\rangle \prec \langle L_\kappa[A], A\rangle$. So it remains to show that $I - \gamma$ are indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$. Suppose to the contrary that this is not the case. Then there exists a formula φ_m, ordinals $\delta_1, \ldots, \delta_k < \gamma$, and two ordered n-tuples $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ in I such that $\langle L_\kappa[A], A\rangle \models \varphi_m(\vec{\delta}, \vec{\alpha})$ but $\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \vec{\beta})$. I claim that we can assume without loss of generality that $\beta_n < \alpha_1$. We can always choose $\gamma_1 < \cdots < \gamma_n \in I$ such that $\gamma_1 > \alpha_n, \beta_n$. If $\langle L_\kappa[A], A\rangle \models \varphi_m(\vec{\delta}, \vec{\gamma})$, we can take the $\vec{\gamma}$ and $\vec{\beta}$ sequences. If $\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \vec{\gamma})$, we can take the $\vec{\gamma}$ and $\vec{\alpha}$ sequences. This shows that the assumption is really without loss of generality. Finally, we can conclude that it is impossible that $h(\vec{\beta}, \vec{\alpha}) = \emptyset$ since $\vec{\delta}$ and m witness otherwise. Therefore I are good indiscernibles for $\langle L_\kappa[A], A\rangle$.

It remains to prove that the value of h on I is always $\emptyset$. Suppose to the contrary that this is not the case. Then there is n such that for all $\alpha_1 < \cdots < \alpha_{2n}$ in I, the function $h(\vec{\alpha}) = \lambda = g(m, \delta_1, \ldots, \delta_k)$ since I is homogenous for h. Fix some $\alpha_1 < \cdots < \alpha_n < \alpha_{n+1} < \cdots < \alpha_{2n} < \alpha_{2n+1} < \cdots < \alpha_{3n}$ in I. It follows that:

1. $h(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}) = \lambda$

2. $h(\alpha_1, \ldots, \alpha_n, \alpha_{2n+1}, \ldots, \alpha_{3n}) = \lambda$

3. $h(\alpha_{n+1}, \ldots, \alpha_{2n}, \alpha_{2n+1}, \ldots, \alpha_{3n}) = \lambda$

From this we conclude that:

1. $\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \alpha_1, \ldots, \alpha_n) \leftrightarrow \varphi_m(\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n})$

2. $\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \alpha_1, \ldots, \alpha_n) \leftrightarrow \varphi_m(\vec{\delta}, \alpha_{2n+1}, \ldots, \alpha_{3n})$

3. $\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}) \leftrightarrow \varphi_m(\vec{\delta}, \alpha_{2n+1}, \ldots, \alpha_{3n})$

To see that this is impossible, suppose, for example, that $\langle L_\kappa[A], A\rangle \models \varphi_m(\vec{\delta}, \alpha_1, \ldots, \alpha_n)$. Then $\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n})$ by (1). This implies that $\langle L_\kappa[A], A\rangle \models \varphi_m(\vec{\delta}, \alpha_{2n+1}, \ldots, \alpha_{3n})$ by (3). Finally, this last statement implies that $\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \alpha_1, \ldots, \alpha_n)$. Thus, we have reached a contradiction. The argument is similar if you suppose that $\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \alpha_1, \ldots, \alpha_n)$. □

Lemma 2.44. *If κ is inaccessible and $A \subseteq \kappa$, then $\langle L_\kappa[A], A\rangle \models$ ZFC in the extended language.*

Proof.

Pairing: Let a and b be in $L_\kappa[A]$, then there is $\alpha < \kappa$ such that a and b are already in $L_\alpha[A]$. Thus, clearly $\langle a, b\rangle \in L_\kappa[A]$.

Separation: Let X and $\vec{b}$ be in $L_\kappa[A]$ and $\varphi(x, \vec{y})$ be a formula in the language of $\langle L_\kappa[A], A\rangle$. Define $Y = \{a \in X \mid \langle L_\kappa[A], A\rangle \models \varphi(a, \vec{b})\}$. Let $\alpha < \kappa$ such that X and $\vec{b}$ are in $L_\alpha[A]$ and $\langle L_\alpha[A], A\rangle \prec \langle L_\kappa[A], A\rangle$, then clearly Y is definable over $\langle L_\alpha[A], A\rangle$, and therefore $Y \in L_{\alpha+1}[A]$.

Union: Let $X \in L_\kappa[A]$, then there is $\alpha < \kappa$ such that $X \in L_\alpha[A]$. It follows that $\cup X \in L_{\alpha+1}[A]$.

Powerset: Let $X \in L_\kappa[A]$, then there is $\alpha < \kappa$ such that $X \in L_\alpha[A]$. Since $|L_\beta[A]| = |\beta|$ for all β, it follows that $|X| \leq |\alpha| < \kappa$. Since κ is inaccessible, $|\mathcal{P}(X)| < \kappa$. Again by the inaccessibility of κ, there must be $\delta < \kappa$ such that $\mathcal{P}(X) \cap L_\kappa[A] = \mathcal{P}(X) \cap L_\delta[A]$. Let $B = \mathcal{P}(X) \cap L_\delta[A]$, then $B \in L_{\delta+1}[A]$ and $L_\kappa[A] \models B = \mathcal{P}(X)$.

Infinity: $\omega \in L_\kappa[A]$.

Replacement: Let X and $\vec{b}$ be in $L_\kappa[A]$ and $\varphi(x, y, \vec{z})$ be a formula in the language of $\langle L_\kappa[A], A\rangle$ such that $\langle L_\kappa[A], A\rangle \models \forall x \in X \, \exists! \, y \, \varphi(x, y, \vec{b})$. Then there exists $\alpha < \kappa$ such that X and $\vec{b}$ are in $L_\alpha[A]$ and $\langle L_\alpha[A], A\rangle \prec \langle L_\kappa[A], A\rangle$. Define $Y = \{y \in L_\kappa[A] \mid \exists x \in X \, \langle L_\kappa[A], A\rangle \models \varphi(x, y, \vec{b})\}$. Since $X \in L_\alpha[A]$, it follows that for every $x \in X$, there is a unique $y \in L_\alpha[A]$ such that $\langle L_\kappa[A], A\rangle \models \varphi(x, y, \vec{b})$. Thus, Y is definable over $\langle L_\alpha[A], A\rangle$, and hence $Y \in L_{\alpha+1}[A]$.

Choice: In $\langle L_\kappa[A], A\rangle$, use the usual method of defining a well-ordering in

$L[A]$. □

We are now ready to prove that Ramsey cardinals have the Ramsey embedding property.

Proof of the backward direction of Theorem 2.35. Fix $A \subseteq \kappa$ and consider the structure $\mathcal{A} = \langle L_\kappa[A], A\rangle$. By Lemma 2.44, $\mathcal{A} \models \text{ZFC}$. As was noted above, $\mathcal{A}$ has a definable well-ordering, and therefore definable Skolem functions. By Lemma 2.43, the structure $\mathcal{A}$ has a collection I of good indiscernibles of size κ. For every $\gamma \in I$ and $n \in \omega$, let $\vec{\gamma} = \{\gamma_1, \ldots, \gamma_n\}$ where $\gamma_1 < \cdots < \gamma_n$ are the first n elements in I above γ. Given $\gamma \in I$ and $n \in \omega$, define $\widetilde{\mathcal{A}}^n_\gamma = Scl_{\mathcal{A}}(\gamma + 1 \cup \{\vec{\gamma}\})$, the Skolem closure using the definable Skolem functions of $\langle L_\kappa[A], A\rangle$. Since $\langle L_\kappa[A], A\rangle \models \text{ZFC}$, it follows that $\langle L_\kappa[A], A\rangle$ satisfies that H_λ exists for every λ. Also it holds in $\langle L_\kappa[A], A\rangle$ that for every cardinal λ, the structure $\langle H^{\mathcal{A}}_{\lambda^+}, A\rangle \models \text{ZFC}^-$. Thus, it is really true that $\langle H^{\mathcal{A}}_{\lambda^+}, A\rangle \models \text{ZFC}^-$ by the absoluteness of satisfaction. Since $\langle \widetilde{\mathcal{A}}^n_\gamma, A\rangle \prec \langle L_\kappa[A], A\rangle$ and $\gamma \in \widetilde{\mathcal{A}}^n_\gamma$, we have $H^{\mathcal{A}}_{\gamma^+} \in \widetilde{\mathcal{A}}^n_\gamma$. Next, define $\mathcal{A}^n_\gamma = \widetilde{\mathcal{A}}^n_\gamma \cap H^{\mathcal{A}}_{\gamma^+}$. From now on, to simplify notation, I will write H_{γ^+} instead of $H^{\mathcal{A}}_{\gamma^+}$, but I will always mean the H_{γ^+} of $\mathcal{A}$.

Lemma 2.44.1. *$\mathcal{A}^n_\gamma$ is transitive and $\langle \mathcal{A}^n_\gamma, A\rangle \models \text{ZFC}^-$.*

Proof. First, we show that $\mathcal{A}^n_\gamma$ is transitive. Fix $a \in \mathcal{A}^n_\gamma$ and $b \in a$. The set a is coded by a subset of $\gamma \times \gamma$ in $\langle L_\kappa[A], A\rangle$. By elementarity, $\widetilde{\mathcal{A}}^n_\gamma$ contains a set $E \subseteq \gamma \times \gamma$ coding a and the Mostowski collapse $\pi : \langle \gamma, E\rangle \to Trcl(a)$. Let $\alpha \in \gamma$ such that $\langle L_\kappa[A], A\rangle \models \pi(\alpha) = b$. Since $\gamma \subseteq \widetilde{\mathcal{A}}^n_\gamma$, the ordinal $\alpha \in \widetilde{\mathcal{A}}^n_\gamma$, and so by elementarity, $b \in \widetilde{\mathcal{A}}^n_\gamma$. It is clear that $b \in \mathcal{A}^n_\gamma$ also. Thus, we have shown that $\mathcal{A}^n_\gamma$ is transitive.

Next, we show that $\langle \mathcal{A}^n_\gamma, A\rangle \models \mathrm{ZFC}^-$. Let $\pi : \widetilde{\mathcal{A}}^n_\gamma \to N$ be the Mostowski collapse. Since $\mathcal{A}^n_\gamma \subseteq \widetilde{\mathcal{A}}^n_\gamma$ is transitive, it follows that $\pi(x) = x$ for all $x \in \mathcal{A}^n_\gamma$. Therefore $\pi(H_{\gamma^+}) = \{\pi(x) \mid x \in H_{\gamma^+} \cap \widetilde{\mathcal{A}}^n_\gamma = \mathcal{A}^n_\gamma\} = \{x \mid x \in \mathcal{A}^n_\gamma\} = \mathcal{A}^n_\gamma$. Also observe that $H_{\gamma^+} \cap A \in L_\kappa[A]$, and therefore $H_{\gamma^+} \cap A \in \widetilde{\mathcal{A}}^n_\gamma$. By the transitivity of $\mathcal{A}^n_\gamma$, the ordinals of $\mathcal{A}^n_\gamma$ is some ordinal $\alpha \subseteq \widetilde{\mathcal{A}}^n_\gamma$. Therefore $\pi(\xi) = (\xi)$ for all $\xi \in \alpha$ and the image $\pi(H_{\gamma^+} \cap A) = (H_{\gamma^+} \cap A) \cap \widetilde{\mathcal{A}}^n_\gamma = \mathcal{A}^n_\gamma \cap A$. We conclude that $\langle \mathcal{A}^n_\gamma, A\rangle$ is an element of the Mostowski collapse N. Since π is an isomorphism, it follows that N thinks $\mathcal{A}^n_\gamma$ is its H_{γ^+}. From this we conclude that $N \models$ "$\langle \mathcal{A}^n_\gamma, A\rangle \models \mathrm{ZFC}^-$". But satisfaction is absolute, and so it is really true that $\langle \mathcal{A}^n_\gamma, A\rangle \models \mathrm{ZFC}^-$. □

Lemma 2.44.2. *For every $\gamma \in I$ and $n \in \omega$, we have $\mathcal{A}^n_\gamma \prec \mathcal{A}^{n+1}_\gamma$.*

Proof. It is certainly clear that $\widetilde{\mathcal{A}}^n_\gamma \prec \widetilde{\mathcal{A}^{n+1}_\gamma}$ since these are Skolem closures of sets that extend each other. Let $\rho : \widetilde{\mathcal{A}^{n+1}_\gamma} \to N'$ be the Mostowski collapse.

It remains to observe that:

$$\begin{aligned}
\langle \mathcal{A}^n_\gamma, A\rangle \models \varphi(a) \leftrightarrow \\
\langle N, \pi'' A\rangle \models \text{“}\langle \mathcal{A}^n_\gamma, A\rangle \models \varphi(a)\text{”} \leftrightarrow \\
\langle \widetilde{\mathcal{A}}^n_\gamma, A\rangle \models \text{“}\langle H_{\gamma^+}, A\rangle \models \varphi(a)\text{”} \leftrightarrow (\pi(a) = a \text{ and } \pi(\langle H_{\gamma^+}, A\rangle) = \langle \mathcal{A}^n_\gamma, A\rangle) \\
\langle \widetilde{\mathcal{A}^{n+1}_\gamma}, A\rangle \models \text{“}\langle H_{\gamma^+}, A\rangle \models \varphi(a)\text{”} \leftrightarrow \\
\langle N', \rho'' A\rangle \models \text{“}\langle \mathcal{A}^{n+1}_\gamma, A\rangle \models \varphi(a)\text{”} \leftrightarrow \\
\langle \mathcal{A}^{n+1}_\gamma, A\rangle \models \varphi(a).
\end{aligned}$$

□

Recall that if $a \in \widetilde{\mathcal{A}}^n_\gamma$, then $a = h(\vec{b}, \gamma, \vec{\gamma})$ where h is a definable Skolem function, the sequence $\vec{b} \subseteq \gamma$, and $\vec{\gamma} = \{\gamma_1, \ldots, \gamma_n\}$ are the first n elements above γ in I. Given $\gamma < \delta \in I$, define $f^n_{\gamma\delta} : \widetilde{\mathcal{A}}^n_\gamma \to \widetilde{\mathcal{A}}^n_\delta$ by $f^n_{\gamma\delta}(a) = h(\vec{b}, \delta, \vec{\delta})$ where $a = h(\vec{b}, \gamma, \vec{\gamma})$ is as above. Observe that since $I - \gamma$ are indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$, the map $f^n_{\gamma\delta}$ is clearly well-defined and elementary. Also $f^n_{\gamma\delta}(\gamma) = \delta$ and $f^n_{\gamma\delta}(\xi) = \xi$ for all $\xi < \gamma$. So the critical point of $f^n_{\gamma\delta}$ is γ. Finally, note that for all $\gamma < \delta < \beta \in I$, we have $f^n_{\gamma\beta} \circ f^n_{\beta\delta} = f^n_{\gamma\delta}$.

Lemma 2.44.3. *The map $f^n_{\gamma\delta} \restriction \mathcal{A}^n_\gamma : \mathcal{A}^n_\gamma \to \mathcal{A}^n_\delta$ is elementary.*

Proof. Fix $a \in \mathcal{A}^n_\gamma$ and recall that $\widetilde{\mathcal{A}}^n_\gamma$ thinks $a \in H_{\gamma^+}$. By elementarity of $f^n_{\gamma\delta}$, it follows that $\widetilde{\mathcal{A}}^n_\delta$ thinks $f^n_{\gamma\delta}(a) \in H_{\delta^+}$. Therefore $f^n_{\gamma\delta} : \mathcal{A}^n_\gamma \to \mathcal{A}^n_\delta$. So it

remains to check elementarity:

$$\langle \mathcal{A}^n_\gamma, A\rangle \models \varphi(a) \leftrightarrow$$
$$\langle \widetilde{\mathcal{A}}^n_\gamma, A\rangle \models \text{“}\langle H_{\gamma^+}, A\rangle \models \varphi(a)\text{”} \leftrightarrow$$
$$\langle \widetilde{\mathcal{A}}^n_\delta, A\rangle \models \text{“}\langle H_{\delta^+}, A\rangle \models \varphi(f^n_{\gamma\delta}(a))\text{”} \leftrightarrow$$
$$\langle \mathcal{A}^n_\delta, A\rangle \models \varphi(f^n_{\gamma\delta}(a)).$$

□

For $\gamma \in I$, define $U^n_\gamma = \{X \in \mathcal{P}(\gamma) \cap \mathcal{A}^n_\gamma \mid \gamma \in f^n_{\gamma\delta}(X) \text{ for some } \delta > \gamma\}$. Observe that we could have equivalently used “for *all* $\delta > \gamma$” in the definition.

Lemma 2.44.4. *U^n_γ is a normal $\mathcal{A}^n_\gamma$-ultrafilter on γ.*

Proof. First, let us verify that U^n_γ is an ultrafilter. Clearly $\gamma \in U^n_\gamma$ since $\gamma \in f^n_{\gamma\gamma_1}(\gamma) = \gamma_1$ where γ_1 is the least element in I above γ. Let $X \in U^n_\gamma$ and suppose $X \subseteq Y \subseteq \gamma$ where $Y \in \mathcal{A}^n_\gamma$. Since $X \in U^n_\gamma$, we have $\gamma \in f^n_{\gamma\gamma_1}(X)$. Since $X \subseteq Y$, it follows that $f^n_{\gamma\gamma_1}(X) \subseteq f^n_{\gamma\gamma_1}(Y)$, and hence $\gamma \in f^n_{\gamma\gamma_1}(Y)$. Thus, $Y \in U^n_\gamma$. Suppose X and Y are in U^n_γ, then $\gamma \in f^n_{\gamma\gamma_1}(X)$ and $\gamma \in f^n_{\gamma\gamma_1}(Y)$. It clearly follows that $\gamma \in f^n_{\gamma\gamma_1}(X \cap Y) = f^n_{\gamma\gamma_1}(X) \cap f^n_{\gamma\gamma_1}(Y)$. Thus, $X \cap Y \in U^n_\gamma$. Finally, suppose X is not in U^n_γ and let X^c be the complement of X in γ. Then $\gamma \notin f^n_{\gamma\gamma_1}(X)$, and so clearly $\gamma \in f^n_{\gamma\gamma_1}(X^c)$. Thus, $X^c \in U^n_\gamma$.

Next, we check that U^n_γ is normal. Fix $X \in U^n_\gamma$ and a regressive

$F : X \to \gamma$ in $\mathcal{A}^n_\gamma$. Then $\gamma \in f^n_{\gamma\gamma_1}(X)$ and $f^n_{\gamma\gamma_1}(F) : f^n_{\gamma\gamma_1}(X) \to \gamma_1$ is regressive. Let $f^n_{\gamma\gamma_1}(F)(\gamma) = \alpha < \gamma$ and define $Y = \{\xi \in \gamma \mid F(\xi) = \alpha\}$. The set $Y \in \mathcal{A}^n_\gamma$ since Y is definable from F and α and $Y \in H_{\gamma^+}$. Also clearly $Y \in U^n_\gamma$ since $\gamma \in f^n_{\gamma\gamma_1}(Y)$. □

Observe that if $a_0, \ldots, a_n \in L_\gamma[A]$ for some $\gamma \in I$, then for every formula $\varphi(\vec{x})$, we have $\langle L_\kappa[A], A\rangle \models \varphi(\vec{a}) \leftrightarrow \langle L_\gamma[A], A\rangle \models \varphi(\vec{a}) \leftrightarrow \langle L_\kappa[A], A\rangle \models$ "$\langle L_\gamma[A], A\rangle \models \varphi(\vec{a})$". It follows that for every $\gamma \in I$, the model $\langle L_\kappa[A], A\rangle$ has a truth predicate definable from γ for formulas with parameters from $L_\gamma[A]$. For example, this implies that $\widetilde{\mathcal{A}}^n_\gamma$ is definable in $\widetilde{\mathcal{A}}^{n+1}_\gamma$. To see this, recall that $x \in \widetilde{\mathcal{A}}^n_\gamma$ if and only if "$\exists h\, \exists \vec{v} < \gamma$ h is a Skolem term and $x = h(\vec{v}, \gamma, \gamma_1, \ldots, \gamma_n)$". This is a definition for $\widetilde{\mathcal{A}}^n_\gamma$ in $\widetilde{\mathcal{A}}^{n+1}_\gamma$ since $\widetilde{\mathcal{A}}^{n+1}_\gamma$ has γ_{n+1}, from which it can define a truth predicate for $L_{\gamma_{n+1}}[A]$.

Lemma 2.44.5. *The ultrafilter $U^n_\gamma \in \mathcal{A}^{n+2}_\gamma$.*

Proof. In $\widetilde{\mathcal{A}}^{n+2}_\gamma$, we have $U^n_\gamma = \{x \in \mathcal{P}(\gamma) \mid \exists h\, \exists \vec{v} < \gamma$ h is a Skolem term and $h = (\vec{v}, \gamma, \gamma_1, \ldots, \gamma_n) \wedge \gamma \in h(\vec{v}, \gamma_1, \gamma_2, \ldots, \gamma_{n+1})\}$. This follows since $\gamma_{n+2} \in \widetilde{\mathcal{A}}^{n+2}_\gamma$, and therefore we can define a truth predicate for $L_{\gamma_{n+2}}[A]$, which is good enough for the definition above. So far we have shown that U^n_γ is in $\widetilde{\mathcal{A}}^{n+2}_\gamma$. To finish the argument, observe that $\langle L_\kappa[A], A\rangle \models U^n_\gamma \in H_{\gamma^+}$, and therefore $U^n_\gamma \in \mathcal{A}^{n+2}_\gamma$. □

It should be clear that $U_\gamma^n \subseteq U_\gamma^{n+1}$ and $f_{\gamma\delta}^n \subseteq f_{\gamma\delta}^{n+1}$. Define $\mathcal{A}_\gamma = \cup_{n\in\omega}\mathcal{A}_\gamma^n$ and $U_\gamma = \cup_{n\in\omega}U_\gamma^n$. Also define $f_{\gamma\delta} = \cup_{n\in\omega}f_{\gamma\delta}^n : \mathcal{A}_\gamma \to \mathcal{A}_\delta$ and observe that it is elementary in the language with a predicate for A. Since each U_γ^n was a normal $\mathcal{A}_\gamma^n$-ultrafilter on γ, it is easy to see that U_γ is a normal $\mathcal{A}_\gamma$-ultrafilter on γ.

Lemma 2.44.6. *The normal $\mathcal{A}_\gamma$-ultrafilter U_γ is weakly amenable.*

Proof. Consider $B \subseteq \gamma \times \gamma$ with $B \in \mathcal{A}_\gamma$. Define $C = \{\xi \in \gamma \mid B_\xi \in U_\gamma\}$. We need to show that $C \in \mathcal{A}_\gamma$. Since $B \in \mathcal{A}_\gamma$, it follows that $B \in \mathcal{A}_\gamma^n$ for some $n \in \omega$. But then $C = \{\xi \in \gamma \mid B_\xi \in U_\gamma^n\}$ and $U_\gamma^n \in \mathcal{A}_\gamma^{n+2} \subseteq \mathcal{A}_\gamma$. Thus, $C \in \mathcal{A}_\gamma$. □

Now for every $\gamma \in I$, we have an associated structure $\langle \mathcal{A}_\gamma, \in, A, U_\gamma\rangle$. Also if $\gamma < \delta$ in I, we have an elementary embedding $f_{\gamma\delta} : \mathcal{A}_\gamma \to \mathcal{A}_\delta$ with critical point γ between the structures $\langle \mathcal{A}_\gamma, A\rangle$ and $\langle \mathcal{A}_\delta, A\rangle$. Observe also that $X \in U_\gamma$ if and only if $f_{\gamma\delta}(X) \in U_\delta$. This is a directed system of models, and so we can take the direct limit of this directed system. Define $\langle B, E, A', V\rangle = \lim_{\gamma\in I}\langle \mathcal{A}_\gamma, \in, A, U_\gamma\rangle$.

Lemma 2.44.7. *The relation E on B is well-founded.*

Proof. The elements of B are functions t with domains $\{\xi \in I \mid \xi \geq \alpha\}$ for

some $\alpha \in I$ satisfying the properties:

1. $t(\gamma) \in \mathcal{A}_\gamma$,

2. for $\gamma < \delta$ in domain of t, we have $t(\delta) = f_{\gamma\delta}(t(\gamma))$,

3. there is no $\xi \in I \cap \alpha$ for which there is $a \in \mathcal{A}_\xi$ such that $f_{\xi\alpha}(a) = t(a)$.

Note that each t is determined once you know any $t(\xi)$ by extending uniquely forward and backward. It follows, by standard arguments (for example, [9], p. 157), that $B \models \varphi(t_1, \ldots, t_n)$ if and only if $\exists\gamma\, \mathcal{A}_\gamma \models \varphi(t_1(\gamma), \ldots, t_n(\gamma))$ if and only if for all γ in the intersection of the domains of the t_i, the structure $\mathcal{A}_\gamma \models \varphi(t_1(\gamma), \ldots, t_n(\gamma))$. It is good to keep in mind that this truth definition holds only of atomic formulas in the case where the formulas involve the predicate for the ultrafilter.

Suppose to the contrary that E is not well-founded, then there is a descending E-sequence $\cdots E\, t_n\, E \cdots E\, t_1\, E\, t_0$. Find γ_0 such that $\mathcal{A}_{\gamma_0} \models t_1(\gamma_0) \in t_0(\gamma_0)$. Next, find $\gamma_1 > \gamma_0$ such that $\mathcal{A}_{\gamma_1} \models t_2(\gamma_1) \in t_1(\gamma_1)$. In this fashion, define an increasing sequence $\gamma_0 < \gamma_1 < \cdots < \gamma_n < \cdots$ such that $\mathcal{A}_{\gamma_n} \models t_{n+1}(\gamma_n) \in t_n(\gamma_n)$. Let $\gamma \in I$ such that $\gamma > \sup_{n\in\omega}\gamma_n$. It follows that for all $n \in \omega$, the structure $\mathcal{A}_\gamma \models f_{\gamma_n\gamma}(t_{n+1}(\gamma_n)) \in f_{\gamma_n\gamma}(t_n(\gamma_n))$, and therefore $\mathcal{A}_\gamma \models t_{n+1}(\gamma) \in t_n(\gamma)$. But, of course, this is impossible. Thus, E is well-founded. □

Let $\langle \mathcal{A}_\kappa, \in, A^*, U_\kappa \rangle$ be the Mostowski collapse of $\langle B, E, A', V \rangle$.

Lemma 2.44.8. *The cardinal $\kappa \in \mathcal{A}_\kappa$.*

Proof. Fix $\alpha \in \kappa$ and let $\gamma \in I$ be the least ordinal greater than α, then $f_{\gamma\delta}(\alpha) = \alpha$ for all $\delta > \gamma$. Define t_α to have domain $\{\xi \in I \mid \xi \geq \gamma\}$ such that $t_\alpha(\xi) = \alpha$. I claim that t_α is an element of B. We need to show that it cannot be extended further backward (condition (3) in proof of Lemma 2.44.7). Suppose $\beta < \gamma$ is in I, then $\beta \leq \alpha$. If $\xi < \beta$, then $f_{\beta\gamma}(\xi) = \xi$. The value of $f_{\beta\gamma}(\beta) = \gamma$. Thus, if $\xi > \beta$, then $f_{\beta\gamma}(\xi) > \gamma$. Hence the value of $f_{\beta\gamma}(\xi)$ can never be α. This shows that t_α is in B. Next, observe that t_α has exactly α predecessors in B, namely t_ξ for $\xi < \alpha$. Hence t_α collapses to α in $\mathcal{A}_\kappa$. This shows that $\kappa \subseteq \mathcal{A}_\kappa$. Define t_κ to have domain I such that $t_\kappa(\gamma) = \gamma$. Clearly $t_\alpha \, E \, t_\kappa$ for all $\alpha \in \kappa$. Suppose $s \, E \, t_\kappa$, then there is γ in domain of s such that $s(\gamma) \in t_\kappa(\gamma) = \gamma$. Let $s(\gamma) = \alpha < \gamma$. Since each t is determined by a single coordinate, this clearly implies that $s = t_\alpha$. Thus, t_κ collapses to κ, and so $\kappa \in \mathcal{A}_\kappa$. □

Define $j_\gamma : \mathcal{A}_\gamma \to \mathcal{A}_\kappa$ such that $j_\gamma(a)$ is the collapse of the function t for which $t(\gamma) = a$. This makes sense since, by the above remark, specifying the value of t on a coordinate completely determines t. The maps j_γ are fully elementary in the language of $\langle A_\gamma, A \rangle$ and elementary for atomic formulas in

the language with a predicate for the ultrafilter. Observe that $j_\gamma(\xi) = \xi$ for all $\xi < \gamma$ since if $t(\gamma) = \xi$, then $t = t_\xi$. Also $j_\gamma(\gamma) = \kappa$ since if $t(\gamma) = \gamma$, then $t = t_\kappa$. So the critical point of each j_γ is κ. Finally, if $\gamma < \delta$ in I, then $j_\delta \circ f_{\gamma\delta} = j_\gamma$. For every element a of $\mathcal{A}_\kappa$, there is $\gamma \in I$ such that for all $\delta \geq \gamma$, the element a is the image of something in $\mathcal{A}_\delta$ under j_δ. For example, this shows that if a and b are two elements of $\mathcal{A}_\kappa$, then there is $\gamma \in I$ and $a', b' \in \mathcal{A}_\gamma$ such that $a = j_\gamma(a')$ and $b = j_\gamma(b')$.

Lemma 2.44.9. *The set U_κ consists of subsets of κ.*

Proof. Fix $X \in U_\kappa$, then $X = j_\gamma(Y)$ for some $\gamma \in I$ and $Y \in \mathcal{A}_\gamma$. By the elementarity of j_γ for atomic formulas, it follows that $Y \in U_\gamma$, and hence $Y \subseteq \gamma$. Thus, $X = j_\gamma(Y) \subseteq j_\gamma(\gamma) = \kappa$. □

Lemma 2.44.10. *The set U_κ is a normal $\mathcal{A}_\kappa$-ultrafilter on κ.*

Proof. The set κ is in U_κ since $\gamma \in U_\gamma$ and $j_\gamma(\gamma) = \kappa$ for any $\gamma \in I$. Suppose $X \in U_\kappa$ and $X \subseteq Y \subseteq \kappa$ with $Y \in \mathcal{A}_\kappa$. There is $\gamma \in I$ such that $X = j_\gamma(X')$ and $Y = j_\gamma(Y')$. It follows that $X' \in U_\gamma$ and $X' \subseteq Y'$. Therefore $Y' \in U_\gamma$, and hence $j_\gamma(Y') = Y \in U_\kappa$ by elementarity. The argument for intersections and complements is similar. For normality, fix $X \in U_\kappa$ and a regressive $F : X \to \kappa$ in $\mathcal{A}_\kappa$. Let $F = j_\gamma(F')$ and $X = j_\gamma(X')$. It follows that

$F' : X' \to \gamma$ is regressive on $X' \in U_\gamma$. Thus, F' is constant on some $Y' \subseteq X'$ in U_γ, and hence F is constant on $j_\gamma(Y') = Y \in U_\kappa$. □

Lemma 2.44.11. *The $\mathcal{A}_\kappa$-ultrafilter U_κ is weakly amenable.*

Proof. Fix $Y \subseteq \kappa \times \kappa$ and define $X = \{\alpha \in \kappa \mid Y_\alpha \in U_\kappa\}$. We need to show that $X \in \mathcal{A}_\kappa$. Let $Y = j_\gamma(Y')$, then $Y' \subseteq \gamma \times \gamma$ and $X' = \{\alpha \in \gamma \mid Y'_\alpha \in U_\gamma\}$ is in $\mathcal{A}_\gamma$ by the weak amenability of U_γ. It is easy to see that $j_\gamma(X') = X$. □

Lemma 2.44.12. *A set $X \in U_\kappa$ if and only if there exists $\alpha \in I$ such that $\{\xi \in I \mid \xi > \alpha\} \subseteq X$.*

Proof. Fix $X \subseteq \kappa$ in $\mathcal{A}_\kappa$ and $\alpha \in I$ such that for all $\gamma > \alpha$, there is $X' \in \mathcal{A}_\gamma$ with $j_\gamma(X') = X$. Consider $\gamma > \alpha$, then:

$$\begin{aligned} X \in U_\kappa &\leftrightarrow \\ X' \in U_\gamma &\leftrightarrow \\ \gamma \in f_{\gamma\gamma_1}(X') &\leftrightarrow \\ j_{\gamma_1}(\gamma) \in j_{\gamma_1} \circ f_{\gamma\gamma_1}(X') = j_\gamma(X') &\leftrightarrow \\ \gamma \in j_\gamma(X') = X. & \end{aligned}$$

Thus, if $X \in U_\kappa$, then $\{\xi \in I \mid \xi > \alpha\} \subseteq X$. Now suppose for some α, the set $\{\xi \in I \mid \xi > \alpha\} \subseteq X$. Let $\beta \in I$ such that for all $\delta > \beta$ in I, there is

$X' \in \mathcal{A}_\delta$ such that $j_\delta(X') = X$. Choose any γ larger than β and α, then $\gamma \in j_\gamma(X') = X$. Follow the backward arrows to conclude that $X \in U_\kappa$. □

Lemma 2.44.13. *The $\mathcal{A}_\kappa$-ultrafilter U_κ is ω-closed.*

Proof. Fix $\langle A_n \mid n \in \omega \rangle$ a sequence of elements of U_κ. We need to show that $\cap_{n\in\omega} A_n \neq \emptyset$. For each A_n, there exists $\gamma_n \in I$ such that $X_n = \{\xi \in I \mid \xi > \gamma_n\} \subseteq A_n$. Thus, $\cap_{n\in\omega} X_n \subseteq \cap_{n\in\omega} A_n$ and clearly $\cap_{n\in\omega} X_n$ has size κ. □

It remains to show that $A \in \mathcal{A}_\kappa$.

Lemma 2.44.14. *The set $A^* \restriction \kappa = A$, and hence $A \in \mathcal{A}_\kappa$.*

Proof. Fix $\alpha \in A$ and let $\gamma \in I$ such that $\gamma > \alpha$, then $\langle \mathcal{A}_\gamma, A \rangle \models \alpha \in A$. It follows that $\langle \mathcal{A}_\kappa, A \rangle \models j_\gamma(\alpha) \in A^*$, but $j_\gamma(\alpha) = \alpha$, and so $\alpha \in A^*$. Thus, $A \subseteq A^*$. Now fix $\alpha \in A^* \restriction \kappa$ and let $\gamma \in I$ such that $\gamma > \alpha$, then $j_\gamma(\alpha) = \alpha$, and so $j_\gamma(\alpha) \in A^*$. It follows that $\alpha \in A$. Thus, $A^* \restriction \kappa \subseteq A$. We conclude that $A = A^* \restriction \kappa$. □

We have found a weak κ-model M containing A for which there exists an ω-closed weakly amenable normal M-ultrafilter on κ, namely $M = \mathcal{A}_\kappa$. This concludes the proof that if κ is Ramsey, then κ has the Ramsey embedding property. □

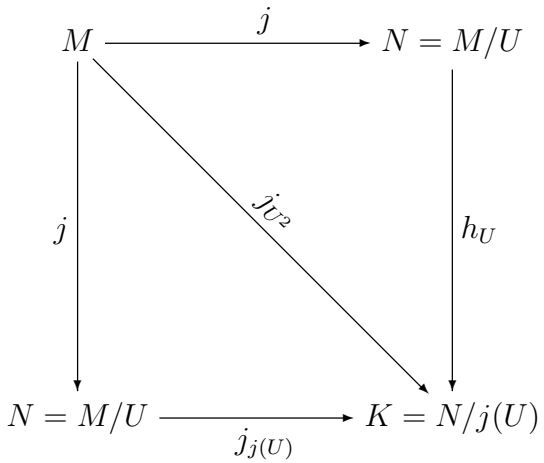
I will end this section by giving an interesting reformulation of the Ramsey embedding property.

Proposition 2.45. *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model $M \models \text{ZFC}$ for which there exists $j : M \to N$ an ultrapower by an ω-closed normal M-ultrafilter such that $M \prec N$.*

The difference from the earlier definition of the Ramsey embedding property is that now for every $A \subseteq \kappa$, we have $A \in M$ where M is a model of all of ZFC and $j : M \to N$ an ultrapower by an ω-closed normal M-ultrafilter such that not only do M and N have the same subsets of κ, but actually $M \prec N$.

Proof. Fix $A \subseteq \kappa$ and choose a weak κ-model M containing A and V_κ for which there exists an ω-closed weakly amenable normal M-ultrafilter U on κ. Let $j : M \to N$ be the ultrapower by U. I will refer to the commutative diagram from Lemma 2.31. For the case $n = 1$, the diagram becomes the

following:



Let $M' = V^N_{j(\kappa)}$, then M' is a transitive model of ZFC since $V_\kappa \models$ ZFC. Let $K' = V^K_{j_{j(U)}(j(\kappa))} = V^K_{h_U(j(\kappa))}$. Observe that since M' is transitive, $h_U \restriction M' : M' \to K'$ is the same as the ultrapower of M' by U. It remains to show that $M' \prec K'$, but this follows easily from the $j_{j(U)}$ side of the commutative diagram since $j_{j(U)} \restriction M' : M' \to N'$ and $j_{j(U)}$ is identity on M'. □

Corollary 2.46. *A cardinal κ is strongly Ramsey if and only if every $A \subseteq \kappa$ is contained in a κ-model $M \models$ ZFC for which there exists an elementary embedding $j : M \to N$ with critical point κ such that $M \prec N$.*

Proof. Using the previous proof it suffices to show that M' is closed under $< \kappa$-sequences. We can assume without loss of generality that N is closed under $< \kappa$-sequences (Proposition 2.9). Therefore $V^N_{j(\kappa)}$ must be closed under $< \kappa$-sequences since N thinks that $j(\kappa)$ is inaccessible. □

Question 2.47. If κ is a weakly Ramsey cardinal, does there exist for every $A \subseteq \kappa$, a weak κ-model $M \models \text{ZFC}$ and an elementary embedding $j : M \to N$ with critical point κ such that $M \prec N$?

In the above proofs, we get elementary embeddings $j : M \to N$ such that $M = V^N_{j(\kappa)}$. It is also worth noting that, on the opposite side of the spectrum, it is always possible to get κ-powerset preserving embeddings where $M = H^N_{\kappa^+}$.

Proposition 2.48. *If $j : M \to N$ is a κ-powerset preserving embedding of weak κ-models and $A \subseteq \kappa$ is in M, then there is another κ-powerset preserving embedding of weak κ-models $h : M' \to N'$ such that $A \in M'$ and $M' = H^{N'}_{\kappa^+}$.*

Proof. Define $M' = \{x \in M \mid M \models |Trcl(x)| \leq \kappa\}$ and $N' = \{x \in N \mid N \models |Trcl(x)| \leq j(\kappa)\}$. Clearly M' is a transitive subclass of M containing A. It is easy to check that $j \restriction M' : M' \to N'$ is a κ-powerset preserving embedding of κ-models and $M' = H^{N'}_{\kappa^+}$. □

2.4 Indestructibility for Ramsey Cardinals

In this section, I will prove some basic indestructibility results for strongly Ramsey cardinals. These results will be obtained with the usual techniques

for lifting embeddings. That is, I will start with a ground model having the embedding property, force with a certain poset and show that the forcing extension still satisfies the embedding property by lifting the ground model embedding to the forcing extension. For details about lifting arguments see [8] (ch. 1). I will make use of the following key lemmas from [8] (ch. 1).

Lemma 2.49 (The Lifting Criterion)**.** *Suppose that $j : M \to N$ is an elementary embedding of two models of* ZFC$^-$ *having forcing extensions $M[G]$ and $N[H]$ by poset $\mathbb{P}$ and $j(\mathbb{P})$ respectively. The embedding j lifts to an embedding $j : M[G] \to N[H]$ with $j(G) = H$ if and only if $j''G \subseteq H$.*

Thus, whenever we want to lift an embedding $j : M \to N$, we are always looking for an N-generic filter H such that $j''G \subseteq H$.

Lemma 2.50 (The Diagonalization Criterion)**.** *If $\mathbb{P}$ is a poset in a model M of* ZFC$^-$ *and for some cardinal δ the following criterion are satisfied,*

1. *$M^{<\delta} \subseteq M$,*

2. *$\mathbb{P}$ is $\leq \delta$-closed in M,*

3. *M has at most δ many antichains of $\mathbb{P}$,*

then there is an M-generic filter for $\mathbb{P}$.

Lemma 2.51 (Ground Closure Criterion)**.** *Suppose that $M \subseteq V$ is a model of* ZFC$^-$*, $M^{<\delta} \subseteq M$, and there is in V, an M-generic filter $H \subseteq \mathbb{P}$ for poset $\mathbb{P} \in M$. Then $M[H]^{<\delta} \subseteq M[H]$. The same statement holds for $M^\delta \subseteq M$.*

Lemma 2.52 (Generic Closure Criterion)**.** *Suppose that $M \subseteq V$ is a model of* ZFC$^-$*, $M^{<\delta} \subseteq M$, and $\mathbb{P} \in M$ has δ-c.c.. If $G \subseteq \mathbb{P}$ is V-generic, then $M[G]^{<\delta} \subseteq M[G]$ in $V[G]$. The same statement holds for $M^\delta \subseteq M$ under the assumption that $\mathbb{P}$ has δ^+-c.c..*

A forcing iteration $\mathbb{P}$ is said to have *Easton support* if direct limits are taken at inaccessible cardinals and inverse limits are taken everywhere else. If δ and γ are cardinals, I will call $Add(\delta, \gamma)$ the poset that adds γ many subsets to δ with conditions of size less than δ.

Theorem 2.53. *If κ is a strongly Ramsey cardinal, then this is preserved in a forcing extension by the class forcing of the* GCH*.*

Proof. Let $\mathbb{P}$ be the Ord-length Easton support iteration where at stage α if α is an infinite cardinal in $V^{\mathbb{P}_\alpha}$, we force with $Add(\alpha^+, 1)$, and with the trivial poset otherwise. The poset $\mathbb{P}$ forces ZFC + GCH [8] (ch. 2). It is clearly enough in our case to consider $\mathbb{P}_\kappa$, the iteration up to κ, since the forcing above κ is $\leq \kappa$-closed, and therefore cannot destroy the strong Ramsey embedding property. Let $G \subseteq \mathbb{P}_\kappa$ be V-generic. We need to show

that κ has the strong Ramsey embedding property in $V[G]$. Fix $A \subseteq \kappa$ in $V[G]$ and let $\dot{A}$ be a nice $\mathbb{P}_\kappa$-name such that $\dot{A}_G = A$. Observe that since we take direct limits at inaccessible stages, we can regard $\mathbb{P}_\kappa$ as being a subset of V_κ, and therefore $\dot{A} \in H_{\kappa^+}$. Fix a κ-model M containing $\dot{A}$ and $\mathbb{P}_\kappa$ for which there exists a κ-powerset preserving $j : M \to N$. By Proposition 2.9, we can assume that N is also a κ-model. The strategy will be to lift j to $M[G]$ in $V[G]$. Since $\dot{A} \in M$, clearly A will be in $M[G]$. Also since $\mathbb{P}_\kappa$ has κ-c.c. (see [8], ch. 2 for chain conditions of Easton support iterations), by the generic closure criterion, we will have $M[G]^{<\kappa} \subseteq M[G]$ in $V[G]$. Thus, we will be done if we can lift the embedding j to $M[G]$ and show that the lift is still κ-powerset preserving in $V[G]$. To do this we first need to find an N-generic filter H for $j(\mathbb{P}_\kappa)$ containing $j''G$. Elements of $\mathbb{P}_\kappa$ have bounded support in κ, and so elements of $j''G$ are simply elements of G with trivial coordinates stretching until $j(\kappa)$. Since M and N agree on the definition of $\mathbb{P}_\kappa$, we can factor $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{P}}_{\text{tail}}$. Observe that if we can find an $N[G]$-generic filter G_{tail} for $(\dot{\mathbb{P}}_{\text{tail}})_G = \mathbb{P}_{\text{tail}}$, the filter $G * G_{\text{tail}}$ will contain $j''G$ and we can lift j by the lifting criterion. I will show that G_{tail} exists by showing that $N[G]$ and $\mathbb{P}_{\text{tail}}$ satisfy the requirements of the diagonalization criterion in $V[G]$. By the generic closure criterion, we have $N[G]^{<\kappa} \subseteq N[G]$. We assumed N to have size κ, and therefore $N[G]$ has size κ as well. So $N[G]$ contains at most

κ many antichains of $\mathbb{P}_{\text{tail}}$. Finally, by analyzing the GCH forcing iteration, it is clear that $N \models$ "$\mathbb{P}_{\text{tail}}$ is $\leq \kappa$-closed" since the first nontrivial forcing in $\mathbb{P}_{\text{tail}}$ is $Add(\kappa^+, 1)$. So in $V[G]$, we have $j : M[G] \to N[G][G_{\text{tail}}]$. The final step is to show that the lift of j is still κ-powerset preserving. The model $N[G]$ satisfies that $\mathbb{P}_{\text{tail}}$ is $\leq \kappa$-closed, and therefore the subsets of κ are the same for $N[G]$ and $N[G][G_{\text{tail}}]$. It remains to show that $M[G]$ and $N[G]$ have the same subsets of κ. Let $B \in N[G]$ be a subset of κ. Let $\dot{B} \in N$ be a nice $\mathbb{P}_\kappa$-name for B, then $|Trcl(\dot{B})| \leq \kappa$, and therefore by the powerset preservation property, $\dot{B} \in M$. It follows that $B \in M[G]$, completing the argument. □

Next, I will show that strongly Ramsey cardinals can have fast functions. A *fast function* is a generic function that acts like the Laver function on a supercompact cardinal. It is also similar to a diamond sequence in that it exhibits all possible behavior below κ. For a cardinal κ, the fast function forcing $\mathbb{F}_\kappa$ consists of conditions that are partial functions $p : \kappa \to \kappa$ such that $\gamma \in \text{dom}(p)$ implies $p''\gamma \subseteq \gamma$, and for every inaccessible cardinal $\gamma \leq \kappa$, we have $|\text{dom}(p \restriction \gamma)| < \gamma$. The ordering is inclusion. The union $f : \kappa \to \kappa$ of the generic filter for $\mathbb{F}_\kappa$ is called a fast function. Observe that if κ is inaccessible, the forcing $\mathbb{F}_\kappa \subseteq V_\kappa$.

For any $\gamma < \kappa$, define $\mathbb{F}_{[\gamma,\kappa)}$ to be the set of all $p \in \mathbb{F}_\kappa$ whose domain is contained in $[\gamma, \kappa)$. If $p \in \mathbb{F}_\kappa$, define the poset $\mathbb{F}_\kappa \restriction p$ to be $\mathbb{F}_\kappa$ below p. A useful fact about the fast function forcing is that for any $p \in \mathbb{F}_\kappa$ and $\gamma \in \mathrm{dom}(p)$, the poset $\mathbb{F}_\kappa \restriction p$ is isomorphic to $\mathbb{F}_\gamma \restriction (p \restriction \gamma) \times \mathbb{F}_{[\gamma,\kappa)} \restriction (p \restriction [\gamma, \kappa))$. This basically tells us that the poset $\mathbb{F}_\kappa$ can be viewed as a product. For example, if $\gamma \leq \alpha$ and $p = \{\langle \gamma, \alpha \rangle\}$, then $\mathbb{F}_\kappa$ factors as $\mathbb{F}_\gamma \times \mathbb{F}_{(\alpha,\kappa)}$. Observe that in this case the second factor $\mathbb{F}_{(\alpha,\kappa)}$ is clearly $\leq \alpha$-closed. For a detailed analysis of the fast function forcing, see [8] (ch. 2).

Theorem 2.54. *If κ is a strongly Ramsey cardinal, then this is preserved in a forcing extension $V[f]$ by the fast function forcing. Moreover, if $j : M \to N$ is any κ-powerset preserving embedding of κ-models and $\theta < j(\kappa)$, then there is a κ-powerset preserving lift of j to $j : M[f] \to N[j(f)]$ such that $j(f)(\kappa) = \theta$.*

Proof. Let f be a V-generic fast function. Fix any κ-powerset preserving embedding of κ-models $j : M \to N$ in V and $\theta < j(\kappa)$. Since $\mathbb{F}_\kappa$ is a definable subset of V_κ and $V_\kappa \in M$, we get that $\mathbb{F}_\kappa \in M$ by Separation. First, we need to verify that $M[f]^{<\kappa} \subseteq M[f]$ in $V[f]$. I will show that for arbitrarily large inaccessible cardinals $\alpha < \kappa$, we have $M[f]^\alpha \subseteq M[f]$ in $V[f]$. It will be useful to observe that the sets $D_\beta = \{p \in \mathbb{F}_\kappa \mid \exists \alpha > \beta \ \alpha \text{ is inaccessible and}$

$\alpha \notin \mathrm{dom}(f)\}$ are dense in $\mathbb{F}_\kappa$. To see this, fix $q \in \mathbb{F}_\kappa$ and choose some $\beta < \alpha < \gamma$ above the domain of q such that α is inaccessible. Let $p = q \cup \{\langle \beta, \gamma \rangle\}$, then $p \leq q$ and $\alpha \notin \mathrm{dom}(p)$. Now given $\beta < \kappa$, let $\alpha > \beta$ be some inaccessible not in the domain of f. By density, this implies that there is $\gamma < \alpha$ in the domain of f such that $f(\gamma) = \delta > \alpha$. Let p be a condition in the generic containing $\langle \gamma, \delta \rangle$. Below p, the poset $\mathbb{F}_\kappa$ factors as $\mathbb{F}_\gamma \times \mathbb{F}_{(\delta,\kappa)}$. Let $f = f_\gamma \times f_{(\delta,\kappa)}$. We need to show that $M[f_\gamma][f_{(\delta,\kappa)}]^\alpha \subseteq M[f_\gamma][f_{(\delta,\kappa)}]$ in $V[f]$. By the generic closure criterion, $M[f_\gamma]^\alpha \subseteq M[f_\gamma]$ in $V[f_\gamma]$ since $\mathbb{F}_\gamma$ is clearly α^+-c.c.. Also since $\mathbb{F}_{(\delta,\kappa)}$ is $\leq \alpha$-closed, $M[f_\gamma]^\alpha \subseteq M[f_\gamma]$ in $V[f]$. Finally, by the ground closure criterion, $M[f_\gamma][f_{(\delta,\kappa)}]^\alpha \subseteq M[f_\gamma][f_{(\delta,\kappa)}]$ in $V[f]$. Next, we lift the embedding j. According to N, the poset $j(\mathbb{F}_\kappa) = \mathbb{F}_{j(\kappa)}$ is the poset to add a fast function on $j(\kappa)$. The condition $p = \langle \kappa, \theta \rangle$ is clearly an element of $\mathbb{F}_{j(\kappa)}$, and so we can factor $\mathbb{F}_{j(\kappa)}$ below p to get $\mathbb{F}_\kappa \times \mathbb{F}_{(\kappa,j(\kappa))} \restriction p$. Let $\mathbb{F}_{(\kappa,j(\kappa))} \restriction p = \mathbb{F}_{\mathrm{tail}}$. Our strategy will be to find an N-generic filter for $\mathbb{F}_\kappa \times \mathbb{F}_{\mathrm{tail}}$ containing $j''f = f$. The condition p will then be in its upward closure, guaranteeing that $j(f)(k) = \theta$. Since $\mathbb{F}_\kappa \times \mathbb{F}_{\mathrm{tail}}$ is a product, the order in which we force does not matter. It suffices to find an N-generic filter for $\mathbb{F}_{\mathrm{tail}}$ in V since we already have f being V-generic for the $\mathbb{F}_\kappa$ part of the forcing. We verify the diagonalization criterion for N and $\mathbb{F}_{\mathrm{tail}}$ in V. Recall that N is a κ-model by assumption. It remains to note that in N, the poset

$\mathbb{F}_{\text{tail}}$ is $\leq \kappa$-closed by our observations above. So in V, there is an N-generic filter f_{tail} for $\mathbb{F}_{\text{tail}}$. Therefore we can lift $j : M \to N$ to $j : M[f] \to N[f][f_{\text{tail}}]$ in $V[f]$. Since $f \times f_{\text{tail}} = j(f)$, we have $j(f)(\kappa) = \theta$. It remains to check that the lift of j is still κ-powerset preserving. Since $\mathbb{F}_{\text{tail}}$ is $\leq \kappa$-closed, there are no new subsets of κ in $N[f_{\text{tail}}]$, and so M and $N[f_{\text{tail}}]$ have the same subsets of κ. Fix $B \subseteq \kappa$ in $N[f_{\text{tail}}][f]$. Let $\dot{B}$ be a nice $\mathbb{F}_\kappa$-name for B in $N[f_{\text{tail}}]$, then $|Trcl(\dot{B})| \leq \kappa$. By the previous observation, $\dot{B} \in M$. It follows that $B \in M[f]$, completing the argument. □

It follows from the method of iterating ultrapowers (see Section 2.5) that if κ is strongly Ramsey, then for every $A \subseteq \kappa$, there exists a κ-powerset preserving $j : M \to N$ with N arbitrarily large.

Question 2.55. Theorem 2.54 shows that we can lift every κ-powerset preserving embedding of κ-*models* in the forcing extension $V[f]$ by $\mathbb{F}_\kappa$. Can we lift a κ-powerset preserving $j : M \to N$ where M is a κ-model but N has size *larger* than κ?

Let κ be a cardinal. A *slim κ-Kurepa tree* is a κ-tree with at least κ^+ many branches in which every level $\alpha \geq \omega$ has size at most $|\alpha|$. For every inaccessible cardinal, the tree $2^{<\kappa}$ is a κ-Kurepa tree but not slim. So the addition of the slimness requirement is an attempt to make the existence

of Kurepa trees a meaningful question for large cardinals. We know that if κ is a measurable cardinal, then there cannot exist a slim κ-Kurepa tree. Ineffable cardinals also cannot have slim Kurepa trees (See [2], p. 317).

Theorem 2.56 (with T. Johnstone and J. Reitz)**.** *It is relatively consistent with* ZFC *that κ is strongly Ramsey and there exists a slim κ-Kurepa tree.*

Proof. For an inaccessible κ, I will define a poset $\mathbb{S}$ which adds a slim κ-Kurepa tree. The elements of $\mathbb{S}$ are ordered triples $\langle T, f, g\rangle$ having the properties:

1. $T \subseteq \kappa$ is a normal slim β-tree for some ordinal $\beta < \kappa$.

2. $f : T \to [T]$ such that $\alpha \in f(\alpha)$ for all $\alpha \in T$.

3. $g : \kappa^+ \to [T]$ has domain of size at most $|\beta|$.

We define $\langle T', f', g'\rangle < \langle T, f, g\rangle$ in $\mathbb{S}$ if:

1. T' end-extends T and $T' \neq T$.

2. For all $\alpha \in T$, the branch $f'(\alpha)$ extends $f(\alpha)$.

3. For all $\alpha \in \mathrm{dom}(g)$, the branch $g'(\alpha)$ extends $g(\alpha)$.

Lemma 2.56.1. *The poset $\mathbb{S}$ has size κ^+, it is $< \kappa$-closed, and has κ^+-c.c..*

Proof. Fixing $\beta < \kappa$, we see that there are at most $\kappa^{|\beta|} = \kappa$ many β-trees whose universe is a subset of κ. This is true since κ was assumed to be inaccessible. Also there are κ many f and κ^+ many g for every T. It follows that $\mathbb{S}$ has size κ^+.

Next, we check that $\mathbb{S}$ is $< \kappa$-closed. Let $\delta < \kappa$ and $\langle \langle T_\alpha, f_\alpha, g_\alpha \rangle \mid \alpha < \delta \rangle$ be a strictly decreasing sequence of conditions in $\mathbb{S}$. Suppose that each T_α is a γ_α-tree. We need to find a lower bound for this sequence. Define $T = \cup_{\alpha<\delta} T_\alpha$ and let γ be the height of T. Observe that $\gamma = \sup_{\alpha<\delta}\gamma_\alpha$. Clearly T is a slim γ-tree and $\gamma < \kappa$ since κ was assumed to be inaccessible. Define $f : T \to [T]$ by $f(\xi) = \cup_{\beta\leq\alpha<\delta} f_\alpha(\xi)$ where β is the least ordinal such that $\xi \in T_\beta$. Define $g : \kappa^+ \to [T]$ such that $\mathrm{dom}(g) = \cup_{\alpha<\delta}\mathrm{dom}(g_\alpha)$ and for every $\xi \in \mathrm{dom}(g)$, we have $g(\xi) = \cup_{\beta\leq\alpha<\delta} g_\alpha(\xi)$ where β is the least ordinal such that $\xi \in \mathrm{dom}(g_\beta)$. To see that $|\mathrm{dom}(g)| \leq \gamma$, observe that $|\mathrm{dom}(g_\alpha)| \leq |\gamma_\alpha|$, and therefore $|\mathrm{dom}(g)| \leq \sup_{\alpha<\delta}|\gamma_\alpha|$. Since we assumed that the sequence of conditions is strictly decreasing, we know each tree is strictly taller than the previous one, and therefore $\delta \leq \gamma$. It follows that $\sup_{\alpha<\delta}|\gamma_\alpha| \leq |\gamma|$. Here is precisely where we need the requirement that strictly stronger conditions have strictly taller trees. So $\langle T, f, g \rangle$ is a lower bound for our sequence.

Finally, let us show that $\mathbb{S}$ has κ^+-c.c.. Suppose $A \subseteq \mathbb{S}$ has size κ^+. We need to show that A cannot be an antichain. Since there are only κ many

T and κ many f for every fixed T, we can assume without loss of generality that all elements of A have the same tree T and the same f. Thus, only g varies. Define $X = \{\mathrm{dom}(g) \mid g \text{ appears in } A\}$ and observe that X must have size κ^+ since there are less than κ many g with a fixed domain. Applying a Δ-system argument, we see that, without loss of generality, we can assume X has a root r. Clearly any two g that agree on r must be compatible and there are less than κ many ways of defining g on r. We conclude that there are κ^+ many g with the same r, and therefore there are κ^+ many compatible elements in A. Hence A cannot be an antichain. □

Thus, the forcing $\mathbb{S}$ does not collapse any cardinals. Suppose $G \subseteq \mathbb{S}$ is V-generic. Define $\mathcal{T}$ to be the union of T that appear in elements of G. Since G is a filter, it is clear that $\mathcal{T}$ is a slim tree.

Lemma 2.56.2. *The tree $\mathcal{T}$ is a slim κ-Kurepa tree in $V[G]$.*

Proof. For every $\alpha < \kappa$, define $D_\alpha = \{\langle T, f, g\rangle \in \mathbb{S} \mid \mathrm{ht}(T) \geq \alpha\}$. I claim that D_α is dense in $\mathbb{S}$. If this is so, then $\mathcal{T}$ will have height κ. Fix $\langle T, f, g\rangle \in \mathbb{S}$. If T has height $\geq \alpha$, we are done. So suppose the height of T is less than α. We will build up T to height α recursively. Suppose $\langle T, f, g\rangle$ has been extended to $\langle T_\beta, f_\beta, g_\beta\rangle \in \mathbb{S}$ with T_β of height β. If $\beta = \gamma + 1$, we will extend T_β to $T_{\beta+1}$ of height $\beta + 1$ by adding two successor nodes to every node in

T_β. Extend f_β and g_β in the obvious fashion to obtain $\langle T_{\beta+1}, f_{\beta+1}, g_{\beta+1}\rangle$. If β is a limit ordinal, we build $T_{\beta+1}$ from T_β by adding a node on level β to every branch mentioned in f_β and g_β. The function f_β guarantees that the tree T_β has many branches, and therefore can be extended to a tree of height $\beta+1$. Since $\langle T_\beta, f_\beta, g_\beta\rangle$ was an element of $\mathbb{S}$, it follows that the β^{th} level of $T_{\beta+1}$ has size at most $|\beta|$, and hence $T_{\beta+1}$ is a slim $(\beta+1)$-tree. Extend f_β and g_β in the obvious fashion to obtain $\langle T_{\beta+1}, f_{\beta+1}, g_{\beta+1}\rangle$. At limit stages use the $<\kappa$-closure of $\mathbb{S}$. This concludes the proof that $\mathcal{T}$ is a slim κ-tree and it remains to show that $\mathcal{T}$ has κ^+ many branches. Observe that the sets $E_\alpha = \{\langle T, f, g\rangle \in \mathbb{S} \mid \alpha \in \mathrm{dom}(g)\}$ for $\alpha < \kappa^+$ are dense in $\mathbb{S}$. This is clear since we do not require g to be one-to-one. For every $\alpha < \kappa^+$, we can union up the $g(\alpha)$ for g appearing in elements of G to get a branch of $\mathcal{T}$. If we can show that any such two branches are distinct, we will be done. This uses the fact that $\mathbb{S}$ does not collapse any cardinals. Fixing $\alpha \neq \beta \in \kappa^+$, observe that it is dense that α and β appear in the domain of g and $g(\alpha) \neq g(\beta)$. This is true since we can extend any tree to a successor level and separate the two branches. Therefore the κ^+ many branches obtained from the generic filter are distinct. □

Notice that it is easy to modify $\mathbb{S}$ to a poset $\mathbb{Q}$ which adds a slim κ-tree

with κ many branches. It is just necessary to change the definition of g to $g : \kappa \to [T]$. The poset $\mathbb{Q}$ has size κ.

Next, I define a κ-length Easton support iteration $\mathbb{P}_\kappa$ where at stage α if α is an inaccessible cardinal in $V^{\mathbb{P}_\alpha}$, we force with the poset $\mathbb{Q}_\alpha$ which adds a slim α-tree with α-many branches, and with the trivial poset otherwise. The iteration $\mathbb{P}_\kappa$ has size κ and κ-c.c. (see [8] for a discussion of chain conditions of Easton support iterations). Observe that if $\alpha < \kappa$ is inaccessible, then we can factor $\mathbb{P}_\kappa = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{P}}_{(\alpha,\kappa)}$ where $\mathbb{Q}_\alpha$ is the poset which adds a slim α-tree with α many branches. There is a nontrivial forcing at stage α since forcing with $\mathbb{P}_\alpha$ preserves inaccessible cardinals. The poset $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ has α^+-c.c.. If $H \subseteq \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ is V-generic, then the poset $\mathbb{P}_{(\alpha,\kappa)} = (\dot{\mathbb{P}}_{(\alpha,\kappa)})_H$ will be $\leq \alpha$-closed in $V[H]$.

Let $G_\kappa * G$ be V-generic for $\mathbb{P}_\kappa * \dot{\mathbb{S}}$. I claim that κ has the strong Ramsey embedding property in $V[G_\kappa][G]$ and $V[G_\kappa][G]$ has a slim κ-Kurepa tree. It is easy to see that $\mathbb{P}_\kappa$ preserves inaccessible cardinals, and therefore κ remains inaccessible in $V[G_\kappa]$. It follows, by Lemma 2.56.2, that there is a slim κ-Kurepa tree in $V[G_\kappa][G]$.

Fix $A \subseteq \kappa$ in $V[G_\kappa][G]$ and let $\dot{A}$ be a nice $\mathbb{P}_\kappa * \dot{\mathbb{S}}$-name such that $\dot{A}_{G_\kappa * G} = A$. Since $\mathbb{P}_\kappa * \dot{\mathbb{S}}$ has κ^+-c.c., every antichain in $\dot{A}$ has size at most κ, and therefore $\dot{A} \in H_{\kappa^+}$. To make all the arguments go through nicely, it will help

to think of elements of $\mathbb{S}$ as coded by subsets of κ^+, and therefore names for elements of $\dot{\mathbb{S}}$ simply become nice names for subsets of κ^+. But now we cannot use the same techniques we have used before because the poset $\mathbb{P}_\kappa * \dot{\mathbb{S}}$ does not have size κ, and therefore cannot be put into a κ-model. This is the extra difficulty we must overcome. I show how to do that below. Let $\gamma < \kappa^+$ such that γ contains the domain of every g mentioned in $\dot{A}$. Observe that there is an obvious automorphism of $\mathbb{S}$ under which elements with g whose domain is contained in γ get moved to elements with g whose domain is contained in κ. The image of $\dot{A}$ under this automorphism when interpreted by the image of the generic filter under the same automorphism will still yield A. Thus, we can assume without loss of generality that the domain of every g mentioned in $\dot{A}$ is contained in κ. Next, consider $\{\langle T, f, g\rangle \in \mathbb{S} \mid \mathrm{dom}(g) \subseteq \kappa\}$ and observe that this is precisely the poset $\mathbb{Q}$ defined above which adds a slim κ-tree with κ many branches. I will argue that it suffices to force with $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ over M, which has size κ as needed (to see that $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ has size κ use the fact that $\mathbb{P}_\kappa$ has κ-c.c.). The following lemma is key to this argument:

Lemma 2.56.3. *If $G \subseteq \mathbb{S}$ is V-generic, then the restriction of G to $\mathbb{Q}$ is V-generic for $\mathbb{Q}$.*

Proof. We need to show that for every dense set D of $\mathbb{Q}$, the intersection

$D \cap (G \restriction \mathbb{Q}) \neq \emptyset$. Given a dense subset D of $\mathbb{Q}$, define $D' = \{p \in \mathbb{S} \mid \exists q \in D\ p \leq q\}$. I claim that D' is dense in $\mathbb{S}$. If it is so, then we are clearly done. Fixing $p \in \mathbb{S}$, we need to find $r \in D'$ such that $r \leq p$. First, we will build $p' \in \mathbb{Q}$ such that for all $q \in \mathbb{Q}$ below p', the condition p will be compatible with q. Once we have such p', it is easy to find r. To see this, let $q \in D$ below p'. By assumption, q is compatible with p, and hence there is r below both p and q. But then $r \in D'$. This completes the argument for the denseness of D'. It remains to construct p'. Let $p = \langle T, f, g\rangle$. Define $g' : \kappa^+ \to [T]$ such that for $\alpha < \kappa$ in the domain of g, we have $g'(\alpha) = g(\alpha)$ and for $\alpha \geq \kappa$ in the domain of g, there is $\xi < \kappa$ in the domain of g' such that $g'(\xi) = g(\alpha)$. That is, we "copied over" branches from g that were on coordinates greater than κ to coordinates less than κ. Define $p' = \langle T, f, g'\rangle$. To see that p' works, fix $q = \langle T_q, f_q, g_q\rangle \leq p'$ in $\mathbb{Q}$. We need to show that q and p are compatible. Define $g'' : \kappa^+ \to [T_q]$ by $g''(\xi) = g_q(\xi)$ for all $\xi < \kappa$ and $g''(\xi) = g(\xi)$ for all $\xi \geq \kappa$. Observe that each $g''(\xi)$ is a branch through T_q since q was below p' and p' had all the branches from p "copied over". Finally, let S be any proper extension of T_q with g'' extended to a function $h : \kappa^+ \to [S]$ and f_q extended to a function $l : S \to [S]$ in the obvious fashion. It should now be clear that $\langle S, l, h\rangle$ is below both q and p. $\square$

Let H be the restriction of G to $\mathbb{Q}$. The strategy, as hinted above, will be to find a κ-model M containing $\dot{A}$ and $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ for which there exists a κ-powerset preserving embedding $j : M \to N$ of κ-models. We will lift j to $M[G_\kappa][H]$ in $V[G_\kappa][G]$. This suffices since clearly $\dot{A}_{G_\kappa * G} = \dot{A}_{G_\kappa * H}$.

Fix a κ-model M containing $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ and $\dot{A}$ for which there exists a κ-powerset preserving embedding $j : M \to N$ of κ-models. First, we will lift j to $M[G_\kappa]$ in $V[G_\kappa][G]$. To do this, we need to find an N-generic filter for $j(\mathbb{P}_\kappa)$ containing $j''G_\kappa$ as a subset. Observe that $j(\mathbb{P}_\kappa)$ factors as $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\text{tail}}$. It is key here that at stage κ, in the iteration $j(\mathbb{P}_\kappa)$, we force with $\dot{\mathbb{Q}}$. We already have a generic filter $G_\kappa * H$ for $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ and clearly $j''G \subseteq G_\kappa * H$. So, as usual, we only need an $N[G_\kappa][H]$-generic filter for $\mathbb{P}_{\text{tail}} = (\dot{\mathbb{P}}_{\text{tail}})_{G_\kappa * H}$. Let us verify the diagonalization criterion for $N[G_\kappa][H]$ and $\mathbb{P}_{\text{tail}}$ in $V[G_\kappa][G]$. Since N has size κ, the model $N[G_\kappa][H]$ has size κ as well, and therefore can contain at most κ many antichains of $\mathbb{P}_{\text{tail}}$. Also clearly $N[G_\kappa][H]$ satisfies that $\mathbb{P}_{\text{tail}}$ is $\leq \kappa$-closed. It only requires some work to show that $N[G_\kappa][H]^{<\kappa} \subseteq N[G_\kappa][H]$ in $V[G_\kappa][G]$. Fix an inaccessible $\alpha < \kappa$ and factor $\mathbb{P}_\kappa = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{P}}_{(\alpha,\kappa)}$. Since $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ has α^+-c.c., by the generic closure criterion, $N[G \restriction (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)]^\alpha \subseteq N[G \restriction (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)]$ in $V[G \restriction (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)]$. The remaining part of the forcing is $\leq \alpha$-closed, and so $N[G \restriction (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)]^\alpha \subseteq N[G \restriction (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)]$ in $V[G_\kappa]$. Thus, $N[G \restriction (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)]^{<\kappa} \subseteq N[G \restriction (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)]$ in

$V[G_\kappa]$. By the ground closure criterion, it follows that $N[G_\kappa]^{<\kappa} \subseteq N[G_\kappa]$ in $V[G_\kappa]$. The poset $\mathbb{S}$ is $< \kappa$-closed, and hence $N[G_\kappa]^{<\kappa} \subseteq N[G_\kappa]$ in $V[G_\kappa][G]$. Finally, by the ground closure criterion again, $N[G_\kappa][H]^{<\kappa} \subseteq N[G_\kappa][H]$ in $V[G_\kappa][G]$. Since we verified the diagonalization criterion, we can conclude that $V[G_\kappa][G]$ has an $N[G_\kappa][H]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$. So letting $j(G_\kappa) = G_\kappa * H * G_{\text{tail}}$, we are able to lift j to $j : M[G_\kappa] \to N[j(G_\kappa)]$ in $V[G_\kappa][G]$.

Next, we must lift j to $M[G_\kappa][H]$. For this step, we need to find an $N[j(G_\kappa)]$-generic filter for the poset $j(\mathbb{Q})$ containing $j''H = H$ as a subset. By elementarity, $N[j(G_\kappa)]$ thinks that $j(\mathbb{Q})$ is the poset to add a slim $j(\kappa)$-tree with $j(\kappa)$ many branches. Observe that by our choice of a generic filter for $j(\mathbb{P}_\kappa)$, the set $H \in N[j(G_\kappa)]$. Using H, we can define $\langle \mathcal{T}, \mathcal{F}, \mathcal{G} \rangle$ where $\mathcal{T}$ is the slim κ-tree that is the union of the trees in H, the function $\mathcal{F}$ is the union of the f that appear in H, and the function $\mathcal{G}$ is the union of the g that appear in H. Clearly $\langle \mathcal{T}, \mathcal{F}, \mathcal{G} \rangle$ is an element of $j(\mathbb{Q})$ and a *master condition* for our lift. That is, we will build a generic filter for $j(\mathbb{Q})$ by diagonalization below the condition $\langle \mathcal{T}, \mathcal{F}, \mathcal{G} \rangle$ to ensure that $j''H = H$ is contained in it. So we have to verify the diagonalization criterion for $j(\mathbb{Q})$ and $N[j(G_\kappa)]$ in $V[G_\kappa][G]$. Once again, the model $N[j(G_\kappa)]$ has size κ and thinks that $j(\mathbb{Q})$ is $< j(\kappa)$-closed. We only need to verify the closure. By the previous argument, $N[G_\kappa][H]^{<\kappa} \subseteq N[G_\kappa][H]$ in $V[G_\kappa][G]$. By the ground closure criterion, we

can extend this to $N[j(G_\kappa)]^{<\kappa} \subseteq N[j(G_\kappa)]$ in $V[G_\kappa][G]$. Since we verified the diagonalization criterion, we can conclude that $V[G_\kappa][G]$ has an $N[j(G_\kappa)]$-generic filter for $j(\mathbb{Q})$. So we can lift j to $j : M[G_\kappa][H] \to N[j(G_\kappa)][j(H)]$. By the arguments already used, it is easy to see that $M[G_\kappa][H]$ is a κ-model.

To finish the argument, it only remains to check that the lift of j is κ-powerset preserving. The model $N[j(G_\kappa)]$ satisfies that $j(\mathbb{Q})$ is $< j(\kappa)$-closed, and therefore the subsets of κ are the same for $N[j(G_\kappa)]$ and $N[j(G_\kappa)][j(H)]$. Also the last factor $\mathbb{P}_{\text{tail}}$ of $j(\mathbb{P}_\kappa)$ is $\leq \kappa$-closed, and therefore the subsets of κ are the same for $N[j(G_\kappa)]$ and $N[G_\kappa][H]$. We are left with showing that $M[G_\kappa][H]$ and $N[G_\kappa][H]$ have the same subsets of κ. Fix $B \subseteq \kappa$ in $N[G_\kappa][H]$ and let $\dot{B}$ be a nice $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$-name for B in N. Clearly $|Trcl(\dot{B})| \leq \kappa$ in N, and therefore $\dot{B} \in M$ as well. It follows that $B \in M[G_\kappa][H]$, which completes the argument that the lift of j is a κ-powerset preserving embedding. □

Corollary 2.57. *It is relatively consistent with* ZFC *that κ is Ramsey and there exists a slim κ-Kurepa tree.*

Corollary 2.58. *It is relatively consistent with* ZFC *that there is a strongly Ramsey cardinal that is not ineffable.*

Proof. An ineffable cardinal cannot have a slim Kurepa tree [2]. But we just showed that it is relatively consistent that there is a strongly Ramsey cardinal

κ having a slim κ-Kurepa tree. This cardinal cannot be ineffable. $\square$

The first two results about the GCH and the generic fast function extend to cardinals with the super Ramsey embedding property as well. This is easy to see once we have the lemmas below.

Lemma 2.59. *For every poset $\mathbb{P} \in H_\lambda$ and every V-generic $G \subseteq \mathbb{P}$, we have $H_\lambda^{V[G]} = H_\lambda[G]$.*

Proof. I will begin by showing that the statement holds for regular λ. So assume that λ is regular. I will show that if $p \in \mathbb{P}$ and $p \Vdash |Trcl(\tau)| < \lambda$, then there exists $\sigma \in H_\lambda$ such that $p \Vdash \tau = \sigma$. I will argue by induction on the rank of τ. So suppose that the statement holds for all $\mathbb{P}$-names τ with rank $\alpha < \beta$. Fix $p \in \mathbb{P}$ and τ of rank β such that $p \Vdash |Trcl(\tau)| < \lambda$. Since $|\mathbb{P}| < \lambda$ and λ is regular, there is $\gamma < \lambda$ such that $p \Vdash |Trcl(\tau)| \leq \gamma$. It follows that there is a nice $\mathbb{P}$-name $\dot{A} \in H_\lambda$ for a subset of $\gamma \times \gamma$ such that $p \Vdash$ "$\dot{A}$ is well-founded and $\dot{\pi} : \langle \gamma, \dot{A} \rangle \to Trcl(\tau)$ is the Mostowski collapse". We will construct a $\mathbb{P}$-name $\sigma \in H_\lambda$ such that $p \Vdash \tau = \sigma$. For every $q \leq p$ and every $\xi \in \gamma$, check if $q \Vdash \dot{\pi}(\xi) = \rho$ for some ρ in the domain of τ. If this is the case, we choose one such ρ. By the inductive assumption, there is a $\mu \in H_\lambda$ such that $q \Vdash \mu = \rho$ since ρ has rank less than β and $q \Vdash \rho \in \tau$. Choose one such μ and put the pair $\langle q, \mu \rangle$ into σ. Since $\mathbb{P} \in H_\lambda$ and each $\mu \in H_\lambda$, it is clear that

the name σ will be in H_λ. I will show that if $H \subseteq \mathbb{P}$ is V-generic and $p \in H$, then $\tau_H = \sigma_H$, from which it will follow that $p \Vdash \tau = \sigma$. Let $x \in \sigma_H$, then there is $\langle q, \nu \rangle \in \sigma$ such that $x = \nu_H$ and $q \in H$. It follows, by construction, that there is some $\xi \in \gamma$ such that $q \Vdash \dot{\pi}(\xi) = \nu$. But since $q \in H$, it really is true that $\dot{\pi}_H(\xi) = \nu_H$, and hence $\nu_H \in \tau_H$. So $\sigma_H \subseteq \tau_H$. Suppose that $x \in \tau_H$, then there is $\langle r, \nu \rangle \in \tau$ such that $x = \nu_H$ and $r \in H$. Find q in H below both r and p such that $q \Vdash \dot{\pi}(\xi) = \nu$ for some $\xi \in \gamma$. Since ν is in the domain of τ, there is another name $\zeta \in H_\lambda$ such that $q \Vdash \dot{\pi}(\xi) = \zeta$ and $\langle q, \zeta \rangle \in \sigma$. But then clearly $\zeta_H = \nu_H = x \in \sigma$. So $\tau_H \subseteq \sigma_H$. This completes the inductive step. It follows that $H_\lambda^{V[G]} \subseteq H_\lambda[G]$. We still need to show the inclusion in the other direction. Here we need to show that if $\tau \in H_\lambda$ is a $\mathbb{P}$-name, then $|Trcl(\tau_H)| < \lambda$ in any generic extension. Again, we argue by induction on the rank of τ. So suppose that the statement is true for τ of rank $\alpha < \beta$. Fix some $\tau \in H_\lambda$ of rank exactly β and a V-generic $H \subseteq \mathbb{P}$. In $V[H]$, the element $\tau_H = \{\mu_H \mid \langle \mu, p \rangle \in \tau \text{ and } p \in H\}$. Since clearly each $\mu \in H_\lambda$ and rank of μ is less than β, by the inductive assumption, we conclude that $|Trcl(\mu_H)| < \lambda$. Now since $|H| < \lambda$ as well, we see that $|Trcl(\tau_H)|$ has size less than λ. This completes the inductive step. It follows that $H_\lambda[G] \subseteq H_\lambda^{V[G]}$. This finishes the argument for regular λ.

To complete the argument, suppose that λ is singular. Since $\mathbb{P} \in H_\lambda$,

there is a cardinal $\gamma < \lambda$ such that $\mathbb{P} \in H_{\gamma^+}$. Now observe that $H_\lambda^{V[G]} = \cup_{\gamma<\delta<\lambda} H_{\delta^+}^{V[G]}$. But $\cup_{\gamma<\delta<\lambda} H_{\delta^+}^{V[G]} = \cup_{\gamma<\delta<\lambda} H_{\delta^+}[G] = H_\lambda[G]$. □

Lemma 2.60. *Suppose $\mathbb{P}$ is a poset, $G \subseteq \mathbb{P}$ is V-generic, and $M \prec H_\lambda$ is transitive and contains $\mathbb{P}$. Then $M[G] \prec H_\lambda[G]$.*

Proof. By the Tarski-Vaught test, it suffices to verify that if $H_\lambda[G] \models \exists x\, \varphi(x, a)$ for some $a \in M$, then M has a witness for $\varphi(x, a)$ as well. Suppose $H_\lambda[G] \models \exists x\, \varphi(x, a)$ for some $a \in M$. Fix a $\mathbb{P}$-name $\dot{a}$ such that $\dot{a}_G = a$ and $p \in G$ such that $H_\lambda \models$ "$p \Vdash \exists x\, \varphi(x, \dot{a})$". Since $\mathbb{P} \in M$ and M is transitive, it follows that $p \in M$ as well. But then $M[G] \models \exists x\, \varphi(x, a)$ by the usual theorems about forcing. □

Now the next two theorems follow easily.

Theorem 2.61. *If κ is a super Ramsey cardinal, then this is preserved in a forcing extension by the class forcing of the* GCH.

Theorem 2.62. *If κ is a super Ramsey cardinal, then this is preserved in a forcing extension $V[f]$ by the fast function forcing. Moreover, if $j : M \to N$ is any κ-powerset preserving embedding of κ-models and $\theta < j(\kappa)$, then there is a κ-powerset preserving lift of j to $j : M[f] \to N[j(f)]$ such that $j(f)(\kappa) = \theta$.*

2.5 Iterably Ramsey Cardinals

There are other definitions of large cardinal notions that would fit in well with the series of definitions in this chapter. Lemma 2.29 shows that given an ultrapower $j : M_0 \to M_1$ by a weakly amenable normal M_0-ultrafilter on κ, we can construct a weakly amenable normal M_1-ultrafilter on $j(\kappa)$ and let M_2 be the corresponding ultrapower of M_1. If M_2 is well-founded, we repeat the process. We can continue this process for ω many steps, resulting in the sequence $\langle M_n, U_n \rangle$ of models with their ultrafilters. At the ω^{th} step, we can take the direct limit $\langle M_\omega, U_\omega \rangle$ of these structures. It can be shown that U_ω is a weakly amenable normal M_ω-ultrafilter. If M_ω is well-founded, we can collapse it and proceed to iterate further. For the proofs involved in this construction, see [11] (p. 244). We can continue this process as long as the iterates are well-founded. It is shown in [11] (p. 244) that if U_0 is ω-closed, then every model in this iteration will, in fact, be well-founded. It is also known that this assumption is not necessary [11] (p. 256). If M is a transitive model of ZFC^-, I will call an M-ultrafilter U *iterable* if it is normal weakly amenable and the construction described above can be iterated through all ordinals. It was just stated above that every ω-closed weakly amenable normal M-ultrafilter is iterable. This naturally leads to the

following large cardinal notion:

Definition 2.63. A cardinal κ is *iterably Ramsey* if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an iterable M-ultrafilter on κ.

Ramsey cardinals would be an example of iterably Ramsey cardinals. It also easily follows that iterable cardinals imply $0^{\#}$. It has recently been shown by Ian Sharpe [24] that iterably Ramsey cardinals are strictly weaker than Ramsey cardinals. In fact, he showed that a Ramsey cardinal κ is the κ^{th} iterably Ramsey cardinal. Observe that this answers Question 2.23 affirmatively. Another notion that is closely related to iterably Ramsey cardinals is the following:

Definition 2.64. A cardinal κ has the *High Ramsey Embedding Property* if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exist κ-powerset preserving elementary embeddings $j_\alpha : M \to N_\alpha$ with $Ord^{N_\alpha} > \alpha$ for all ordinals α. We say that a cardinal is *high Ramsey* if it has the high Ramsey embedding property.

It should be clear that iterably Ramsey cardinals are high Ramsey and high Ramsey cardinals imply $0^{\#}$.

Question 2.65. Are weakly Ramsey cardinals iterably Ramsey?

Question 2.66. Are high Ramsey cardinals weaker than iterably Ramsey cardinals?

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