Ehrenfeucht principles in set theory

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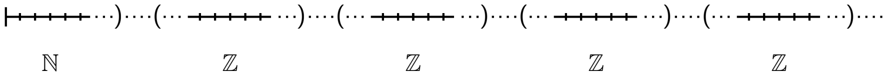
Models of arithmetic

Peano Arithmetic

- The first-order language of arithmetic is $\mathcal{L}_A = \{+, \cdot, <, 0, 1\}$.
- Peano Arithmetic (PA) is a collection of statements in $\mathcal{L}_A$ codifying the fundamental properties of the natural numbers.
  - Commutativity and associativity of addition and multiplication, distributive law, ordering is discrete with least element 0, 0 is the additive identity, etc.
  - Induction scheme: for every $\mathcal{L}_A$-formula $\varphi(x, \vec{y})$,

$$\forall \vec{y}[(\varphi(0, \vec{y}) \wedge (\forall x\, \varphi(x, \vec{y}) \to \varphi(x+1, \vec{y}))) \to \forall x \varphi(x, \vec{y})].$$

- PA proves the least-number principle:
  every definable (with parameters) set has a $<$-least element.
- The standard model of PA is $\mathbb{N}$.
- A countable nonstandard model of PA looks like:

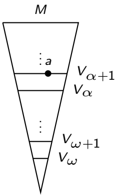


densely many copies of $\mathbb{Z}$

Models of set theory

Zermelo-Fraenkel set theory (with choice)

- The first-order language of set theory is $\mathcal{L}_S = \{\in\}$.
- Zermelo-Fraenkel set theory ($\mathrm{ZF(C)}$) is a collection of statements in $\mathcal{L}_S$ codifying the fundamental properties of sets.
- A model $M \models \mathrm{ZF}$ is the union of the von Neumann hierarchy $M = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha^M$:
  - $V_0^M = \emptyset$,
  - $V_{\alpha+1}^M = P^M(V_\alpha)$ (Powerset of V_α^M),
  - $V_\lambda^M = \bigcup_{\alpha<\lambda} V_\alpha^M$ for limit ordinals λ.
  - If $a \in M$, then $\mathrm{rank}(a) = \alpha$ is least such that $a \in V_{\alpha+1}^M$.



- **Theorem**: (Lévy-Montague reflection) For every n, there are unboundedly many ordinals α in M such that $V_\alpha^M \prec_{\Sigma_n} M$.

There is an active exchange of concepts, methods and techniques between model theory of models of PA and model theory of models of ZF.

Some general model theory

Suppose M is a first-order structure and $P \subseteq M$.

Definition:

- $a \neq b$ in M are indiscernible if they have the same type: for every $\varphi(x)$, $M \models \varphi(a) \leftrightarrow \varphi(b)$.
- $a \neq b$ in M are P-indiscernible if they have the same type with parameters from P.
- Otherwise they are (P)-discernible.

It is common for models to have indiscernible elements:

- A first-order structure (in a countable language) of size greater than 2^ω has indiscernible elements.
- A non-rigid first-order structure has indiscernible elements.
- If T is a first-order theory with an infinite model, then there are models of T with sets of indiscernible elements of any cardinality (by compactness).

Ehrenfeucht's lemma in arithmetic

A powerful tool in model theory of models of PA is

Ehrenfeucht's lemma: (Ehrenfeucht, '73) If $a \neq b$ in a model $M \models \mathrm{PA}$ and b is definable from a in M, then a and b are discernible in M.

Here is an application.

Theorem: If $M \models \mathrm{PA}$ is the Skolem closure of a single element $a \in M$, then it has no non-trivial automorphisms.

Proof:

- a is the only element of its type by Ehrenfeucht's lemma.
- An automorphism of M must take a to a and therefore fixes everything. $\square$

Note: Given any first-order structure, we can ask if Ehrenfeucht's lemma holds there.

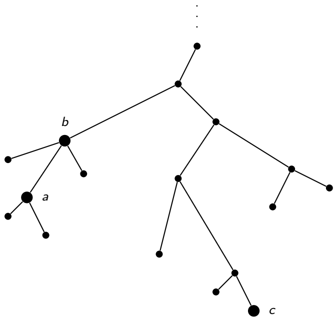
Question: Does Ehrenfeucht's lemma hold for models of ZF?

Proof of Ehrenfeucht's lemma

Proof: Suppose $a \neq b$ and b is definable from a.
Fix a definable function $f(x)$ such that $f(a) = b$.

Case 1: $a < f(a) = b$

- G is the graph whose edges are pairs $\{x, f(x)\}$ such that $x < f(x)$.
- G has an edge between a and b.
- G is loop-free.
- $d(x, y)$ is the length of the shortest path between x and y in G, if connected.
- c is $<$-least in the connected component of a and b (exists by the least-number principle).
- $d(c, a)$ and $d(c, b)$ differ by 1.
- $d(c, a)$ is even iff $d(c, b)$ is odd.

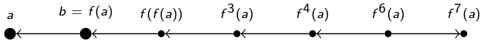


Proof of Ehrenfeucht's lemma (continued)

$f(x)$ is a definable function such that $f(a) = b$.

Case 2: $a > f(a) = b$

- $d(x)$ is the number of times $f(x)$ can be iterated before the values stop decreasing.
- $d(x)$ exists by the least-number principle.
- $d(a)$ and $d(b)$ differ by 1.



$\square$

Ordinal definable sets

Suppose $M \models \mathrm{ZF}$ and fix a definable bijection $F : \mathrm{ORD}^{<\omega^M} \xrightarrow[\text{onto}]{1-1} \mathrm{ORD}$.

Definition:

- A set is ordinal definable if it is definable with ordinal parameters (using F, we can always assume that there is a single ordinal parameter).
- OD is the collection of all ordinal definable sets.
- HOD is the collection of all hereditarily ordinal definable sets.

Lemma: A set $a \in \mathrm{OD}$ iff M satisfies that a is ordinal definable in some V_α.

Proof:
($\Rightarrow$): If $a \in \mathrm{OD}$, then a is defined by the same formula in some V_α (by reflection).

($\Leftarrow$): Suppose M satisfies that a is defined by $\ulcorner\varphi(x,\beta)\urcorner$ in V_α.
Note that φ might be nonstandard. But

$$\psi(x, \ulcorner\varphi\urcorner, \beta, \alpha) := \exists y\, y = V_\alpha \wedge y \models \ulcorner\varphi(x,\beta)\urcorner$$

defines a in M. $\square$

Corollary: The collection OD is first-order definable in M.

A definable well-ordering of OD

Lemma: M has a definable well-ordering $<$ of OD.

Proof: Fix $x, y \in \mathrm{OD}$.

- x is definable in V_α by $\ulcorner\varphi(x,\xi)\urcorner$, where α and $F(\ulcorner\varphi\urcorner,\xi)$ are least.
- y is definable in V_β by $\ulcorner\psi(x,\mu)\urcorner$, where β and $F(\ulcorner\psi\urcorner,\mu)$ are least.

$x < y$ if $\alpha < \beta$ or $\alpha = \beta$ and $F(\ulcorner\varphi\urcorner,\xi) < F(\ulcorner\psi\urcorner,\mu)$. $\square$

Lemma: HOD is a transitive model of ZFC.

Proof:

- Transitivity is by definition.
- ZF a is direct verification.
- If $a \in \mathrm{OD}$, then the well-ordering of OD restricted to a is ordinal definable.

The axiom: $V = \mathrm{HOD}$

Definition: The axiom $V = \mathrm{HOD}$ states that every set is hereditarily ordinal definable.

Corollary: If $M \models \mathrm{ZF}$, then M is a model of $V = \mathrm{HOD}$ iff M has a definable well-ordering.

Proof:
($\Rightarrow$): M has a definable well-ordering of ordinal definable sets.

($\Leftarrow$): Assume (wlog) order-type of well-ordering is ORD. Each $a \in M$ is the α^{th}-element for some ordinal α. $\square$

Ehrenfeucht's lemma and ordinal definable sets

Theorem: If $a \neq b$ in a model $M \models \mathrm{ZF}$, a is ordinal definable and b is definable from a in M, then a and b are discernible in M.

"Ehrenfeucht's lemma holds for ordinal definable sets."

Proof:

- b is ordinal definable.
- Use same argument as for models of PA with the well-ordering $<$ of OD-sets. $\square$

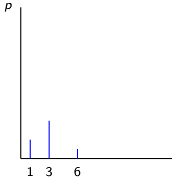
Corollary: Ehrenfeucht's lemma holds for every model of $\mathrm{ZF} + V = \mathrm{HOD}$.

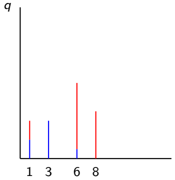
A counterexample to Ehrenfeucht's lemma

Theorem: (Fuchs, G., Hamkins) If $M \models \mathrm{ZFC}$ and $M[c]$ is the Cohen forcing extension, then there are $a \neq b$ in $M[c]$ such that a and b are inter-definable, but a and b are M-indiscernible.

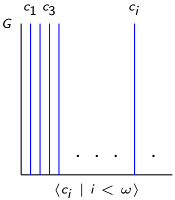
Proof:

- Poset $\mathbb{P}$ to add a Cohen real is isomorphic to finite-support product $\prod_{i<\omega} \mathbb{P}$.
- The poset $\prod_{i<\omega} \mathbb{P}$ adds ω-many Cohen reals:
  - conditions: finite functions $p : \mathrm{dom}(p) \to {}^{<\omega}2$ on ω,
  - order: $q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all $n \in \mathrm{dom}(p)$, $q(n)$ extends $p(n)$.





A counterexample to Ehrenfeucht's lemma (continued)



- $G \subseteq \prod_{i<\omega} \mathbb{P}$ is M-generic.
- $C = \{c_i \mid i < \omega\}$.
- $E = \{c_{2i} \mid i < \omega\}$ and $O = \{c_{2i+1} \mid i < \omega\}$.
- $\langle C, E \rangle$ and $\langle C, O \rangle$ are M-indiscernible.

- $\dot{C}$, $\dot{E}$, and $\dot{O}$ are $\mathbb{P}$-names for C, E, and O.
- Suppose $M[G] \models \varphi(\langle C, E \rangle, d)$ with $d \in M$. Then there is $q \Vdash \varphi(\langle \dot{C}, \dot{E} \rangle, \check{d})$ in G.
- If $i \in \text{dom}(q)$ is even/odd, then there is odd/even m_i such that $q(i) \subseteq c_{m_i}$.
- $\pi : \mathbb{P} \to \mathbb{P}$ is an automorphism such that
  - π switches even and odd coordinates.
  - $\pi(i) = m_i$ for all $i \in \text{dom}(q)$.
- $H = \pi \text{ " } G$ is M-generic for $\prod_{i<\omega} \mathbb{P}$ and $M[G] = M[H]$.
- $q \in H$.
- $\dot{C}_H = C$, $\dot{E}_H = O$, and $\dot{O}_H = E$.
- $M[H] = M[G] \models \varphi(\langle C, O \rangle, d)$. $\square$

Is Ehrenfeucht's lemma equivalent to $V = \mathrm{HOD}$?

Question: If Ehrenfeucht's lemma holds in $M \models \mathrm{ZF(C)}$, is M a model of $V = \mathrm{HOD}$?

Ehrenfeucht Principles

Parametric generalizations of Ehrenfeucht's lemma.

Definition: (Fuchs, G., Hamkins) Let $M \models \mathrm{ZF}$ and let $A, P, Q \subseteq M$. The principle $\mathrm{EL}(A, P, Q)$ for M asserts that if $a \in A$, $a \neq b$, and b is definable in M from a with parameters from P, then a and b are Q-discernible.

"P-definability from A implies Q-discernibility."

Observation:

- $\mathrm{EL}(M, \emptyset, \emptyset)$ is Ehrenfeucht's lemma.
- $\mathrm{EL}(\mathrm{OD}, \emptyset, \emptyset)$ holds for every $M \models \mathrm{ZF}$.
- $\mathrm{EL}(M[c], \emptyset, M)$ fails in every Cohen forcing extension $M[c]$.
- $\mathrm{EL}(M, \emptyset, \emptyset)$ implies $\mathrm{EL}(M, P, P)$ for every P.
- $\mathrm{EL}(A, P, Q)$ gets stronger if A, P are enlarged or Q is diminished.

Interesting Ehrenfeucht principles

Suppose $M \models \mathrm{ZF}$.

$\mathrm{EL}(M, \mathrm{ORD}, \mathrm{ORD})$

- If $a \neq b$ and b is definable from a with ordinal parameters, then a and b are ORD-discernible.
- First-order expressible (similar argument to definability of OD).

$\mathrm{EL}(M, M, \emptyset)$

- M contains no indiscernibles.
- (Leibniz) **Identity of Indiscernibles**: any two objects differ on some property.
- (Enayat) M is Leibnizian.
- **Theorem**: (Enayat, '04) There are uncountable Leibnizian models of set theory.
- **Theorem**: (Mycielski, '95) A theory T extending ZF has a Leibnizian model iff T proves the Leibniz-Mycielski axiom (next slide).

$\mathrm{EL}(M, M, \mathrm{ORD})$

- Any $a \neq b$ are ORD-discernible.
- First-order expressible.

Leibniz-Mycielski axiom

Leibniz-Mycielski axiom (LM): (Mycielski, '95) If $a \neq b$ in $M \models \mathrm{ZF}$, then a and b are discernible in some V_α.

Observation: Suppose $M \models \mathrm{ZF}$.

- LM is equivalent to $\mathrm{EL}(M, M, \mathrm{ORD})$.

  Proof: Fix $a \neq b$.

  $(\Rightarrow)$: Suppose LM holds.
  - $V_\alpha \models \varphi(a) \wedge \neg\varphi(b)$.
  - $\psi(x, \alpha) := \exists y\, y = V_\alpha \wedge y \models \varphi(x)$.
  - $M \models \psi(a, \alpha) \wedge \neg\psi(b, \alpha)$.

  $(\Leftarrow)$: Suppose $\mathrm{EL}(M, M, \mathrm{ORD})$ holds.
  - $M \models \varphi(a, \beta) \wedge \neg\varphi(b, \beta)$ for some ordinal β.
  - $V_\gamma \models \varphi(a, \beta) \wedge \neg\varphi(b, \beta)$ (by reflection).
  - δ codes $\langle \beta, \gamma \rangle$.
  - γ, β are definable in $V_{\delta+1}$ without parameters.
  - a and b are discernible in $V_{\delta+1}$. $\square$

Leibniz-Mycielski axiom (continued)

Corollary:

- LM is first-order expressible.
- LM is equivalent to LM^*: If $a \neq b$ in $M \models \mathrm{ZF}$, then M satisfies that a and b are discernible in some V_α.

Observation: $V = \mathrm{HOD} \to \mathrm{LM}$ over ZF.

Question: Is LM equivalent to $V = \mathrm{HOD}$ over $\mathrm{ZF(C)}$?

Leibniz-Mycielski axiom (continued)

Theorem: (Enayat, '04) $M \models \mathrm{ZF} + \mathrm{LM}$ iff M has a definable injection from M into subsets of ordinals

$$F : M \xrightarrow{1-1} {}^{<\mathrm{ORD}}2.$$

Proof:

($\Rightarrow$): Suppose LM holds.

- Fix a set a. Let β_a be least above rank(a) such that $V_{\beta_a} \prec_{\Sigma_2} M$.
- T_a consists of pairs $\langle \ulcorner\varphi\urcorner, \alpha\rangle$ such that $V_\alpha \models \ulcorner\varphi(a)\urcorner$ for some $\alpha < \beta_a$.
- Suppose $a \neq b$ and (wlog) rank(a) $\leq$ rank(b).
- $V_\gamma \models \varphi(a) \wedge \neg\varphi(b)$ for some $\gamma < \beta_b$.
- $\langle \ulcorner\neg\varphi\urcorner, \gamma\rangle$ is in T_b but not in T_a.
- $T_a \neq T_b$.
- $F(a) = T_a$ (viewed as subset of ordinals via coding).

($\Leftarrow$): Suppose $F : M \xrightarrow{1-1} {}^{<\mathrm{ORD}}2$ and $a \neq b$. Then (wlog) $\alpha \in F(a)$, but $\alpha \notin F(b)$. $\square$

LM and choice principles

Corollary: If $M \models \mathrm{ZF} + \mathrm{LM}$, then M has a definable linear ordering.

Proof: ${}^{<\mathrm{ORD}}2$ is linearly ordered lexicographically. $\square$

The existence of a definable linear ordering is a weak global choice principle.

Theorem: (Easton, '64) There are models of ZFC without a definable linear ordering.

Proof: Consider the class forcing extension $V[G]$, where a Cohen subset is added to every regular cardinal. $\square$.

Question: Are there models of ZFC having a definable linear ordering, but no definable well-ordering ($V = \mathrm{HOD}$ fails)?

Question: Is LM equivalent over $\mathrm{ZF(C)}$ to the existence of a definable linear ordering?

LM and choice principles (continued)

Theorem: (Solovay, '04) LM does not imply (even countable) AC over ZF.

Proof: Similar to Cohen's argument that countable AC fails in $L(\{c_i \mid i < \omega\})$, where c_i are the Cohen reals explicitly added by $\prod_{i<\omega} \mathbb{P}$. But uses Jensen reals.
Work in L.

- Jensen forcing $\mathbb{Q}$:
  - A subposet of Sacks forcing which has the ccc in L (constructed using $\diamondsuit$).
  - Adds a unique $\mathbb{Q}$-generic real over L.
  - The collection of all Jensen reals in any V is Π^1_2-definable.
- $\mathbb{Q}^* = \prod_{i<\omega} \mathbb{Q}$ (finite-support).
- **Theorem:** (Kanovei, '14) If $G \subseteq \mathbb{Q}^*$ is L-generic, then the Jensen reals $\{c_i \mid i < \omega\}$ added explicitly by G are the only Jensen reals in $L[G]$.
- Consider $L(C)$, where $C = \{c_i \mid i < \omega\}$.
  - countable AC fails in $L(C)$.
  - C is definable in $L(C)$ (as the collection of all Jensen reals).
  - Every set in $L(C)$ is ordinal definable from a unique minimal finite subset of C.
  - $L(C)$ has a definable injection into subsets of ordinals.
  - $L(C)$ is a model of LM. $\square$

Questions

Question: Suppose $V \models \mathrm{ZFC}$. Which arrows reverse?

$$\begin{tikzcd} V = \mathrm{HOD} \arrow[d] \\ \text{Ehrenfeucht's lemma holds for } V \end{tikzcd}$$

$$\begin{tikzcd} V = \mathrm{HOD} \arrow[d] \\ \text{Leibniz-Mycielski axiom} \arrow[d] \\ V \text{ has a definable linear order} \end{tikzcd}$$

Thank you!