FORCING AND GAPS IN 2^ω

1. Introduction

The material in these notes draws mainly on Teruyuki Yorioka's thesis [Yor04] and Marion Scheepers' survey paper [Sch93].

For $a, b \in 2^\omega$, we say that a is *eventually dominated* by b, denoted by $a \leq^* b$, if $a(n) \leq b(n)$ for all but finitely many n. Let $\mathcal{A} = \langle a_\alpha \mid \alpha < \kappa \rangle$ and $\mathcal{B} = \langle b_\beta \mid \beta < \lambda \rangle$, where κ and λ are infinite regular cardinals, be a pair of sequences in 2^ω. The pair $(\mathcal{A}, \mathcal{B})$ is called a (κ, λ)-*pregap* if $a_{\alpha_1} \leq^* a_{\alpha_2} \leq^* b_{\beta_2} \leq^* b_{\beta_1}$ for all $\alpha_1 < \alpha_2 < \kappa$ and $\beta_1 < \beta_2 < \lambda$. That is, we have:

$$a_0 \leq^* a_1 \leq^* \cdots \leq^* a_\alpha \leq^* \cdots \leq^* b_\beta \leq^* \cdots \leq^* b_1 \leq^* b_0$$

We say that a set $c \in 2^\omega$ *separates* the pregap $(\mathcal{A}, \mathcal{B})$ if $a_\alpha \leq^* c \leq^* b_\beta$ for all $\alpha < \kappa$ and $\beta < \lambda$. That is, we have:

$$a_0 \leq^* a_1 \leq^* \cdots \leq^* a_\alpha \leq^* \cdots \leq^* c \leq^* \cdots \leq^* b_\beta \leq^* \cdots \leq^* b_1 \leq^* b_0$$

If there is no such set c, then we say that the pregap $(\mathcal{A}, \mathcal{B})$ is a (κ, λ)-*gap*.

Much of the literature on gaps also studies gaps in ω^ω under the eventual domination ordering and there similar results are obtained as the ones we discuss in this talk. In what follows we tacitly associate elements of 2^ω with subsets of ω.

Theorem 1.1 (Hadamard, 1894)**.** *There are no (ω, ω)-gaps.*

Proof. Consider a pregap $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A} = \langle a_n \mid n < \omega \rangle$ and $\mathcal{B} = \langle b_m \mid m < \omega \rangle$. Let $\overline{b}_m$ denote the complement of b_m and define $c_n = a_n \setminus (\bigcup_{m \leq n} \overline{b}_m)$. The set $c = \bigcup_{n < \omega} c_n$ separates $(\mathcal{A}, \mathcal{B})$. $\square$

Theorem 1.2 (Hausdorff, 1909)**.** *There is an (ω_1, ω_1)-gap.*

For a proof see [Jec03] (Section 29).

In these notes, we focus on the interaction between (ω_1, ω_1)-gaps and forcing. In particular, we are interested in the following questions:

Question 1.3. Can we force to add an (ω_1, ω_1)-gap?

Let us call an (ω_1, ω_1)-gap *destructible*, if there is an ω_1-preserving forcing which adds a set separating it. Note that every (ω_1, ω_1)-gap is trivially destructible, if we remove the requirement that the forcing is ω_1-preserving, by collapsing ω_1 to ω. We call an (ω_1, ω_1)-gap *indestructible* if it is not destructible. Kunen showed (1976) that Hausdorff's gap is indestructible.

Question 1.4. Are there destructible (ω_1, ω_1)-gaps?

Question 1.5. Can we force to make an (ω_1, ω_1)-gap indestructible?

There is an "equivalent" way of defining (ω_1, ω_1)-gaps in 2^ω that makes the presentation of the concepts involved easier. Given a pair of sequences $(\mathcal{A}, \mathcal{B})$ in 2^ω, where $\mathcal{A} = \langle a_\alpha \mid \alpha < \omega_1 \rangle$ and $\mathcal{B} = \langle b_\beta \mid \beta < \omega_1 \rangle$, consider the corresponding pair $(\mathcal{A}, \overline{\mathcal{B}})$, where $\overline{\mathcal{B}} = \langle \overline{b}_\beta \mid \beta < \omega_1 \rangle$, where $\overline{b}_\beta$ denotes the complement of b_β.

Now observe that $(\mathcal{A},\mathcal{B})$ is a pregap if and only if $a_{\alpha_1} \leq^* a_{\alpha_2}$, $\bar{b}_{\alpha_1} \leq^* \bar{b}_{\alpha_2}$ for all $\alpha_1 < \alpha_2 < \omega_1$ and $a_\alpha \cap \bar{b}_\beta$ is finite for all $\alpha, \beta < \omega_1$. Observe also that a set c separates $(\mathcal{A},\mathcal{B})$ if and only if $a_\alpha \leq^* c$ and $\bar{b}_\alpha \cap c$ is finite for all $\alpha < \omega_1$. Next, we consider the sequence $\mathcal{A}^* = \langle a^*_\alpha \mid \alpha < \omega_1 \rangle$, where a^*_α differs on finitely many coordinates from a_α in such a way that $a^*_\alpha \cap \bar{b}_\alpha = \emptyset$. Again, we have that $(\mathcal{A},\mathcal{B})$ is a pregap if and only if $a^*_{\alpha_1} \leq^* a^*_{\alpha_2}$, $\bar{b}_{\alpha_1} \leq^* \bar{b}_{\alpha_2}$ for all $\alpha_1 < \alpha_2 < \omega_1$ and $a^*_\alpha \cap \bar{b}_\beta$ is finite for all $\alpha, \beta < \omega_1$. Also, again, a set c separates $(\mathcal{A},\mathcal{B})$ if and only if $a^*_\alpha \leq^* c$ and $\bar{b}_\alpha \cap c$ is finite for all $\alpha < \omega_1$. This analysis shows that we can redefine an (ω_1,ω_1)-pregap in 2^ω as a pair of sequences $(\mathcal{A},\mathcal{B})$, where $\mathcal{A} = \langle a_\alpha \mid \alpha < \omega_1 \rangle$ and $\mathcal{B} = \langle b_\beta \mid \beta < \omega_1 \rangle$, such that:

(1) $a_{\alpha_1} \leq^* a_{\alpha_2}$ and $b_{\alpha_1} \leq^* b_{\alpha_2}$ for all $\alpha_1 < \alpha_2 < \omega_1$,
(2) $a_\alpha \cap b_\beta$ is finite for all $\alpha, \beta < \omega_1$,
(3) $a_\alpha \cap b_\alpha = \emptyset$ for all $\alpha < \omega_1$.

We further redefine that a set c separates the pregap $(\mathcal{A},\mathcal{B})$ if $a_\alpha \leq^* c$ and $b_\alpha \cap c$ is finite for all $\alpha < \omega_1$. We shall use the redefined terminology for the remainder of the notes.

We now introduce a Ramsey-theoretic characterization of when an (ω_1,ω_1)-pregap $(\mathcal{A},\mathcal{B})$ is a gap.

Lemma 1.6 (Folklore). *An (ω_1,ω_1)-pregap $(\mathcal{A},\mathcal{B})$ is a gap if and only if for every uncountable $X \subseteq \omega_1$, there are $\alpha, \beta \in X$ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset$.*

Proof. Suppose that $(\mathcal{A},\mathcal{B})$, where $\mathcal{A} = \langle a_\alpha \mid \alpha < \omega_1 \rangle$ and $\mathcal{B} = \langle b_\beta \mid \beta < \omega_1 \rangle$, is not a gap and fix a separating set c such that $a_\alpha \leq^* c$ and $b_\alpha \cap c$ is finite for all $\alpha < \omega_1$. We argue that there is an uncountable $X \subseteq \omega_1$ and associated $n \in \omega$ and $s, t \in 2^n$ such that for all $\alpha \in X$:

(1) $a_\alpha \cap n = s$ and $b_\alpha \cap n = t$,
(2) $a_\alpha \setminus n \subseteq c$ and $b_\alpha \cap c \subseteq n$.

To see why, note that

$$\omega_1 = \bigcup_{n \in \omega, s, t \in 2^n} X_{(n,s,t)}$$

where

$$X_{(n,s,t)} = \{\alpha < \omega_1 \mid a_\alpha \cap n = s, b_\alpha \cap n = t, a_\alpha \setminus n \subseteq c, b_\alpha \cap c \subseteq n\}.$$

Fix $\alpha, \beta \in X$. It is clear that $a_\alpha \setminus n \cap b_\beta \setminus n = \emptyset$. By our redefinition of a pregap, we have that $a_\alpha \cap b_\alpha = \emptyset$, from which it follows that $s \cap t = \emptyset$. Hence $a_\alpha \cap b_\beta \cap n = \emptyset$ as well. Thus, we have found an uncountable $X \subseteq \omega_1$ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$ for all $\alpha, \beta \in X$.

Conversely suppose that there is an uncountable $X \subseteq \omega_1$ such that for all $\alpha, \beta \in X$, we have $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$. Since X is cofinal in ω_1, it is clear that $c = \bigcup_{\alpha \in X} a_\alpha$ separates $(\mathcal{A},\mathcal{B})$, and so it is not a gap. □

For a pregap $(\mathcal{A},\mathcal{B})$, suppose that $f_{(\mathcal{A},\mathcal{B})} : [\omega_1]^2 \to 2$ is defined by $f_{(\mathcal{A},\mathcal{B})}(\alpha,\beta) = 0$ if $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$ and otherwise $f_{(\mathcal{A},\mathcal{B})}(\alpha,\beta) = 1$. Lemma 1.6 states that $(\mathcal{A},\mathcal{B})$ is a gap if and only if $f_{(\mathcal{A},\mathcal{B})}$ cannot have an uncountable homogeneous set with value 0. If $(\mathcal{A},\mathcal{B})$ is a gap, can $f_{(\mathcal{A},\mathcal{B})}$ have an uncountable homogeneous set with value 1? The answer depends on whether the gap is destructible.

Lemma 1.7 (Kunen, 1976?). *An (ω_1,ω_1)-gap $(\mathcal{A},\mathcal{B})$ is destructible if and only if for every uncountable $X \subseteq \omega_1$, there are $\alpha \neq \beta \in X$ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.*

Thus, $(\mathcal{A},\mathcal{B})$ is a destructible gap if and only if $f_{(\mathcal{A},\mathcal{B})}$ does not have an uncountable homogeneous set. We shall prove Lemma 1.7 in Section 3.

2. Creating destructible gaps by forcing

In this section, we shall show that there is a destructible (ω_1,ω_1)-gap in the Cohen forcing extension, by constructing it from an (ω_1,ω)-gap of the ground model together with the Cohen real.

Theorem 2.1 (Todorčević, 1984). *There is a destructible (ω_1,ω_1)-gap in the Cohen forcing extension.*

Proof. Fix an (ω_1,ω_1)-gap in the ground model, which exists by Hausdorff's result. Let $\mathbb{C}$ be the Cohen poset and $C \subseteq \mathbb{C}$ be V-generic. Also, let c be the Cohen real constructed from C. In $V[C]$, we let $\mathcal{A} \cap c = \langle a_\alpha \cap c \mid \alpha < \omega_1\rangle$ and $\mathcal{B} \cap c = \langle b_\beta \cap c \mid \beta < \omega_1\rangle$. We shall argue that $(\mathcal{A} \cap c, \mathcal{B} \cap c)$ is a destructible gap by verifying the Ramsey-theoretic characterizations using density arguments. We shall show that every uncountable $X \subseteq \omega_1$ in $V[c]$ has ordinals α, β such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset$ (gap) and also has ordinals $\gamma \neq \delta$ such that $(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset$ (destructible).

Fix an uncountable $X \subseteq \omega_1$ in $V[C]$. First, we claim that there is an uncountable $Y \subseteq \omega_1$ in V such that $Y \subseteq X$. Fix a $\mathbb{C}$-name $\dot{X}$ such that $(\dot{X})_C = X$ and a condition $p \in C$ such that $p \Vdash$ "$\dot{X} \subseteq \check{\omega}_1$ is uncountable". To verify the claim, we shall argue that it is dense below p to have a condition q and an uncountable $Y \subseteq X$ such that $q \Vdash \check{Y} \subseteq \dot{X}$. Fix $q' \leq p$. Since q' forces that $\dot{X}$ is an uncountable subset of ω_1, there must be a condition $q_0 \leq q$ and an ordinal α_0 such that $q_0 \Vdash \check{\alpha}_0 \in \dot{X}$. Inductively, suppose that we have constructed a sequence of conditions $\langle q_\xi \mid \xi < \delta\rangle$ below q' for some $\delta < \omega_1$ and a corresponding increasing sequence of ordinals $\alpha_0 < \alpha_1 < \cdots < \alpha_\xi \cdots$ such that $q_\xi \Vdash \check{\alpha}_\xi \in \dot{X}$. Since q' forces that $\dot{X}$ is an uncountable subset of ω_1, there must be a condition q_δ and an ordinal $\alpha_\delta > \alpha_\xi$ for all $\xi < \delta$ such that $q_\delta \Vdash \check{\alpha}_\delta \in \dot{X}$. In this manner, we construct a sequence of conditions $\langle q_\xi \mid \xi < \omega_1\rangle$ below q' and a corresponding increasing sequence of ordinals $\alpha_0 < \alpha_1 < \cdots < \alpha_\xi < \cdots$ such that $q_\xi \Vdash \check{\alpha}_\xi \in \dot{X}$. Since $\mathbb{C}$ is countable there must a condition $q \in \mathbb{C}$ such that $q_\xi = q$ for uncountably many ξ. Let $Y = \{\alpha < \omega_1 \mid q \Vdash \check{\alpha} \in \dot{X}\}$ and observe that clearly $q \Vdash \check{Y} \subseteq \dot{X}$.

Now we fix an uncountable $Y \subseteq X$ and a condition $q \in C$ such that $q \Vdash \check{Y} \subseteq \dot{X}$. We claim that below any condition $q' \leq q$, there is a condition r and ordinals $\alpha, \beta \in Y$ such that

$$(a_\alpha \cap b_\beta \cap r) \cup (a_\alpha \cap b_\beta \cap r) \neq \emptyset,$$

as well as a condition r' and ordinals $\gamma \neq \delta$ in Y such that

$$(a_\gamma \cap b_\delta \cap r') \cup (a_\delta \cap b_\gamma \cap r') = \emptyset.$$

It follows immediately from the claim that $(\mathcal{A}\cap c, \mathcal{B}\cap c)$ is a destructible gap in $V[C]$. To verify the claim, we fix a condition $q' \leq q$ and let $n = \mathrm{dom}(q)$. Consider the pair of sequences $(\mathcal{A}^*, \mathcal{B}^*)$, where $\mathcal{A}^* = \langle a_\alpha \setminus n \mid \alpha < \omega_1\rangle$ and $\mathcal{B}^* = \langle b_\beta \setminus n \mid \beta < \omega_1\rangle$, and note that it remains a gap in V. It follows that there exist $\alpha, \beta \in Y$ such that

$$S = (a_\alpha \setminus n \cap b_\beta \setminus n) \cup (a_\beta \setminus n \cap b_\alpha \setminus n) \neq \emptyset.$$

Thus, we may choose $m \in S$ and extend q' to a condition r with $r(m) = 1$. Next, we observe that there is an an uncountable $Z \subseteq Y$ and associated sequences $s, t \in 2^n$ such that for all $\alpha, \beta \in Z$, $a_\alpha \cap n = s$ and $b_\beta \cap n = t$. It follows that $a_\alpha \cap b_\beta \cap n = \emptyset$ for all $\alpha, \beta \in Z$, as $a_\alpha \cap b_\alpha = \emptyset$ for any α by assumption. Choose any two ordinals $\gamma \neq \delta$ in Z. Since $T = (a_\gamma \setminus n \cap b_\delta \setminus n) \cup (a_\delta \setminus n \cap b_\gamma \setminus n)$ is finite, we may extend q' to a condition r' with $r'(m) = 0$ for all $m \in T$. □

Corollary 2.2. *It is relatively consistent that there are destructible (ω_1, ω_1)-gaps.*

3. Forcing to separate a gap

In this section, we study a forcing notion due to Laver (1979) that adds a set separating a destructible gap. We use this forcing to prove Lemma 1.7 and argue that under MA all (ω_1, ω_1)-gaps are destructible.

Fix an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A} = \langle a_\alpha \mid \alpha < \omega_1 \rangle$ and $\mathcal{B} = \langle b_\beta \mid \beta < \omega_1 \rangle$. The forcing $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ consists of conditions $\langle L, R, s \rangle$ such that

1. L, R are finite subsets of ω_1,
2. $s \in 2^n$ for some $n < \omega$,
3. for all $\alpha \in L$, $\beta \in R$, $a_\alpha \cap b_\beta \subseteq n$.

Let $\langle L, R, s \rangle$ and $\langle L', R', s' \rangle$ be two conditions in $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ with $s \in 2^n$ and $s' \in 2^{n'}$. Then $\langle L', R', s' \rangle \leq \langle L, R, s \rangle$ if

1. $L \subseteq L'$ and $R \subseteq R'$,
2. s' end-extends s,
3. for all $\alpha \in L$, $\beta \in R$, $a_\alpha \cap n' \setminus n \subseteq s'$ and $b_\beta \cap (n' \setminus n) \cap s' = \emptyset$.

The subsets L and R act as promises that s will grow into a separating set for $(\mathcal{A}, \mathcal{B})$. Let us argue that if $G \subseteq \mathbb{P}_{(\mathcal{A},\mathcal{B})}$ is V-generic, then the union $c = \bigcup_{\langle L,R,s \rangle \in G} s$ separates $(\mathcal{A}, \mathcal{B})$ in $V[G]$. It suffices to show that for every $\alpha, \beta < \omega_1$, the set $\mathcal{D}_{\alpha,\beta} = \{\langle L, R, s \rangle \in \mathbb{P}_{(\mathcal{A},\mathcal{B})} \mid \alpha \in L, \beta \in R\}$ is dense in $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$. Given a condition $\langle L, R, s \rangle$, where $s \in 2^n$, we choose n' such that $a_\alpha \cap b_\beta \subseteq n'$ for all $\beta \in R$ and extend s to $s' \in 2^{n'}$ with $s'(m) = 1$ exactly when $a_\delta(m) = 1$ for some $\delta \in L$ and $m \geq n$. The condition $\langle L \cup \{\alpha\}, R, s' \rangle$ is below $\langle L, R, s \rangle$ since $a_\delta \cap b_\gamma \subseteq n$ for all $\delta \in L, \gamma \in R$. Similarly, we construct a condition $\langle L \cup \{\alpha\}, R \cup \{\beta\}, s'' \rangle$ below $\langle L \cup \{\alpha\}, R, s' \rangle$.

Observe that two conditions $\langle L, R, s \rangle$ and $\langle L', R', s \rangle$ with the same sequence $s \in 2^n$ are incompatible precisely when there is $k \geq n$ and $\alpha \in L \cup L'$ and $\beta \in R \cup R'$ such that $\kappa \in a_\alpha \cap b_\beta$.

Lemma 3.1. *If a pregap $(\mathcal{A}, \mathcal{B})$ is not a gap, then $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ has the ccc.*

Proof. Suppose that a pregap $(\mathcal{A}, \mathcal{B})$ is not a gap and fix a separating set c such that $a_\alpha \leq^* c$ and $b_\beta \cap c$ is finite for all $\alpha < \omega_1$. For a sequence $s \in 2^n$, we let $C(s) = \{\langle L, R, s \rangle \in \mathbb{P}_{(\mathcal{A},\mathcal{B})} \mid \forall \alpha \in L, \beta \in R \; a_\alpha \setminus c \subseteq n \text{ and } b_\beta \cap c \subseteq n\}$. It follows that for any two conditions $\langle L, R, s \rangle$ and $\langle L', R', s \rangle$ in $C(s)$, if $\alpha \in L$ and $\beta \in R'$, then $a_\alpha \cap b_\beta \subseteq n$. Thus, any two conditions in $C(s)$ are compatible. Next, we argue that $\mathcal{D} = \bigcup_{s \in 2^{<\omega}} C(s)$ is dense in $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$. Given a condition $\langle L, R, s \rangle \in \mathbb{P}_{(\mathcal{A},\mathcal{B})}$ with $s \in 2^n$, we choose n' such that $a_\alpha \setminus n' \subseteq c$ and $b_\beta \setminus n' \cap c = \emptyset$ for all $\alpha \in L$, $\beta \in R$, and extend s to $s' \in 2^{n'}$ with $s'(m) = 1$ exactly when $a_\alpha(m) = 1$ for some $\alpha \in L$ and $m \geq n$. Since for all $\alpha \in R$, $\beta \in L$, $a_\alpha \cap b_\beta \subseteq n$, it follows that for $\beta \in R$, $b_\beta \cap (n' \setminus n) \cap s' = \emptyset$. Thus, $\langle L, R, s' \rangle$ is a condition in $C(s')$ below $\langle L, R, s \rangle$. Now we suppose that P is an uncountable subset of $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$. For every $p \in P$, fix a condition

$q_p \leq p$ with q_p in $\mathcal{D}$. Since there are only countably many $s \in 2^{<\omega}$, there is an s such that uncountably many of the q_p are in $C(s)$. Choose any two conditions p and p' in P with q_p and $q_{p'}$ in $C(s)$, and observe that p and p' are compatible. Thus, $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ cannot have uncountable antichains. □

Theorem 3.2 (Kunen, Woodin?)**.** *Suppose $(\mathcal{A},\mathcal{B})$ is an (ω_1,ω_1)-gap, then the following are equivalent:*

(1) *$(\mathcal{A},\mathcal{B})$ is destructible,*
(2) *for every uncountable $X \subseteq \omega_1$, there are $\alpha \neq \beta$ in X such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$,*
(3) *$\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ has the ccc.*

Proof.
(1) $\Rightarrow$ (3): Suppose that $(\mathcal{A},\mathcal{B})$ is a destructible (ω_1,ω_1)-gap. Then there is some ω_1-preserving forcing $\mathbb{P}$ and a V-generic $G \subseteq \mathbb{P}$ such that $(\mathcal{A},\mathcal{B})$ is no longer a gap in the forcing extension $V[G]$. Note that the definition of $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ produces the same poset whether applied in V or in $V[G]$. It follows by Lemma 3.1 that $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ has the ccc in $V[G]$. But if $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ had an uncountable antichain in V, it would remain an uncountable antichain in $V[G]$ since $\mathbb{P}$ is ω_1-preserving. Thus, $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ has the ccc in V.
(3) $\Rightarrow$ (1): Suppose $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ has the ccc, then it is an ω_1-preserving forcing that destroys the gap $(\mathcal{A},\mathcal{B})$.
(3) $\Rightarrow$ (2): Suppose that $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ has the ccc. For an ordinal $\alpha < \omega_1$, let $p_\alpha = \langle L,R,s\rangle$ where $L = \{\alpha\}$, $R = \{\alpha\}$, and $s = \emptyset$. Note that each p_α is a condition in $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ since by our assumption $a_\alpha \cap b_\alpha = \emptyset$ for any α. Fix an uncountable $X \subseteq \omega_1$ and consider the corresponding subset $P = \{p_\alpha \in \mathbb{P}_{(\mathcal{A},\mathcal{B})} \mid \alpha \in X\}$. Since $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ has the ccc, there must be two conditions compatible conditions p_α and p_β in P. Let $q = \langle L,R,s\rangle$, where $s \in 2^n$, be a condition below both p_α and p_β. Note that $\alpha,\beta \in L \cap R$, and so $a_\alpha \cap b_\beta \subseteq n$ and $a_\beta \cap b_\alpha \subseteq n$. Also, $a_\alpha \cap n \subseteq s$ and $b_\alpha \cap n \cap s = \emptyset$ since $q \leq p_\alpha$ and $a_\beta \cap n \subseteq s$ and $b_\alpha \cap n \cap s = \emptyset$ since $q \leq p_\beta$. It follows that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.
$\neg$(3) $\Rightarrow$ $\neg$(2): Suppose that $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ does not have the ccc and fix an uncountable antichain $\{\langle L_\alpha, R_\alpha, s_\alpha\rangle \mid \alpha < \omega_1\}$ in $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$. Let $\{\xi_\alpha \mid \alpha < \omega_1\}$ be an increasing sequence of ordinals such that ξ_α is larger than all ordinals in $L_\alpha \cup R_\alpha$. By thinning out, we may make the following list of assumptions.
(1) There is $s \in 2^n$ such that all $s_\alpha = s$.
(2) There are $k,m \in \omega$ such that for all $\alpha < \omega_1$,

$$|L_\alpha| = k \text{ and } |R_\alpha| = m.$$

(3) There is a fixed $l > n$ such that for all $\alpha < \omega_1$,

$$\forall \delta \in L_\alpha \ a_\delta \setminus l \subseteq a_{\xi_\alpha} \text{ and } \forall \delta \in R_\alpha \ b_\delta \setminus l \subseteq b_{\xi_\alpha}.$$

(4) There are sequences $s_i, t_j \in 2^l$ for $i < k$ and $j < m$ such that for all $\alpha < \omega_1$,

$$\{a_\delta \cap l \mid \delta \in L_\alpha\} = \{s_i \mid i < m\} \text{ and } \{b_\delta \cap l \mid \delta \in R_\alpha\} = \{t_j \mid j < k\}.$$

Let $X = \{\xi_\alpha \mid \alpha < \omega_1\}$. We shall argue that for all $\xi_{\alpha_1} \neq \xi_{\alpha_2} \in X$,

$$(a_{\xi_{\alpha_1}} \cap b_{\xi_{\alpha_2}}) \cup (a_{\xi_{\alpha_2}} \cap b_{\xi_{\alpha_1}}) \neq \emptyset.$$

Fix $\alpha_1 \neq \alpha_2$ in ω_1. Since conditions $\langle L_{\alpha_1}, R_{\alpha_1}, s\rangle$ and $\langle L_{\alpha_2}, R_{\alpha_2}, s\rangle$ are incompatible, by our earlier observation, there must be $n' \geq n$ and $\alpha \in L_{\alpha_1} \cup L_{\alpha_2}$ and

$\beta \in R_{\alpha_1} \cup R_{\alpha_2}$ with $n' \in a_\alpha \cap b_\beta$. Indeed, it must be that $n' \geq l$ by assumption (4). Now it follows using assumption (3) that $n' \in (a_{\xi_{\alpha_1}} \cap b_{\xi_{\alpha_2}}) \cup (a_{\xi_{\alpha_2}} \cap b_{\xi_{\alpha_1}})$. □

Corollary 3.3. *Under* MA*, every* (ω_1, ω_1)*-gap is indestructible.*

Proof. Suppose to the contrary that MA holds and $(\mathcal{A}, \mathcal{B})$ is a destructible (ω_1, ω_1)-gap. It follows by Theorem 3.2 that $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ has the ccc. Notice that if a filter for the poset $\mathbb{P}_{(\mathcal{A},\mathcal{B})}$ meets ω_1-many dense sets, namely $\mathcal{D}_{\alpha,\beta} = \{\langle L, R, s\rangle \in \mathbb{P}_{(\mathcal{A},\mathcal{B})} \mid \alpha \in L, \beta \in R\}$ for $\alpha, \beta < \omega_1$, then it may be used to construct a separating set for $(\mathcal{A}, \mathcal{B})$. Thus, we have obtained a contradiction, showing that $(\mathcal{A}, \mathcal{B})$ cannot be destructible. □

4. Forcing to make a gap indestructible

If $(\mathcal{A}, \mathcal{B})$ is an (ω_1, ω_1)-gap, then the forcing to make it indestructible adds an uncountable subset X of ω_1 such that for all $\alpha, \beta \in X$, $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$ with conditions that are finite subsets of ω_1.

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