JENSEN FORCING AT AN INACCESSIBLE AND A MODEL OF KELLEY-MORSE SATISFYING CC BUT NOT DC_ω

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Abstract. Jensen used $\diamondsuit$ to construct a forcing notion $\mathbb{J}$ of perfect trees which has the ccc and adds a unique generic real. Given an inaccessible κ, we generalize Jensen's construction, using $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$, to obtain a forcing notion $\mathbb{J}(\kappa)$ of perfect κ-trees that is $<\kappa$-closed, has the κ^+-cc, and adds a unique generic subset of κ. As is the case with the poset $\mathbb{J}$, we show that products and iterations of $\mathbb{J}(\kappa)$ have "unique generics" properties. We use a non-linear iteration of $\mathbb{J}(\kappa)$ along a tree to produce a model of Kelley-Morse with the Choice Scheme in which the second-order dependent choice principle DC_ω fails.

1. Introduction

Second-order set theory has two types of objects: sets and classes. Unlike in first-order set theory, where classes are relegated to the meta-theory, in second-order set theory we are able to formally study properties of classes. Second-order set theory is formalized in a two-sorted logic with separate sorts (variables and quantifiers) for sets and classes. To distinguish between the two types of objects, we will use lower case letters for sets and upper case letters for classes. Complexity of formulas in second-order set theory is determined by the number of alternations of class quantifiers. A second-order formula is Σ^1_n (or Π^1_n) if it has the form an alternation of n class quantifiers followed by a first-order formula. A model of second-order set theory is a triple $\mathscr{V} = (V, \in, \mathcal{C})$, where V consists of the sets, $\mathcal{C}$ consists of the classes, and $\in$ is the membership relation between sets and between sets and classes. Each element of $\mathcal{C}$ is viewed as a sub-collection of V consisting of those elements which are $\in$-related to it. An axiomatization for second-order set theory consists of axioms for sets, classes, and the interactions between the two sorts. A typical axiomatization consists of the basic axioms: ZFC axioms for sets, extensionality for classes, the class replacement axiom asserting that every class function restricted to a set is a set, the global well-order axiom asserting that there is a class well-ordering of all sets, together with some class existence axioms. The most common class existence axioms take the form of comprehension axioms, asserting that collections defined by formulas of some complexity are classes. One of the weakest second-order axiomatizations is the theory GBC, which consists of the above mentioned basic axioms together with comprehension for first-order formulas. The theory GBC is equiconsistent with ZFC because every model of ZFC that has a definable global well-order (such as the constructible universe L), when taken together with its definable collections, is a model of GBC. We get increasingly stronger second-order theories by extending the amount of available comprehension to formulas of higher complexity. This hierarchy of theories culminates in the theory Kelley-Morse KM, which consists of the basic axioms together with comprehension for all second-order formulas. The theory KM has consistency strength greater than ZFC, but less than ZFC with the existence of an inaccessible cardinal.

Although, in terms of comprehension, KM is the strongest possible second-order theory, it can be made much more robust by adding choice axioms for classes. Analogously to how sets have much more structure in models of ZFC than in models of ZF, classes have much more structure in models of KM together with class choice axioms than they do in models of just KM. The two most common choice principles used in the second-order context are the choice scheme CC and the dependent choice schemes DC_α (for a regular cardinal α or $\alpha=\mathrm{Ord}$). The *choice scheme* asserts for every second-order formula $\varphi(x,X,A)$, with parameter A, that if for every set x, there is a class X such that $\varphi(x,X,A)$ holds, then there is a single class Y collecting on its slices witnesses for every set x, namely, for every set x, $\varphi(x,Y_x,A)$ holds, where $Y_x=\{y\mid \langle x,y\rangle\in Y\}$ is the x's slice of Y. Given a regular cardinal α or $\alpha=\mathrm{Ord}$, the *dependent choice scheme* DC_α asserts for every second-order formula $\psi(X,Y,A)$, with parameter A, that if ψ defines a relation without terminal nodes (for every class X, there is a class Y such that $\psi(X,Y,A)$ holds), then we can make α-many dependent choices along ψ, namely, there is a class F such that for every $\xi<\alpha$, $\psi(F\upharpoonright\xi,F_\xi,A)$ holds, where F_ξ is the ξ-th slice of F and $F\upharpoonright\xi$ is the restriction of the class F to slices indexed by ordinals $\eta<\xi$ ($F\upharpoonright\xi=\{\langle\eta,y\rangle\in F\mid\eta<\xi\}$). The choice scheme is viewed as the analogue of AC for classes and the DC_α-scheme is viewed as the analogue of dependent choice DC_α. The second author and Hamkins showed that the theory KM is not strong enough to prove even the weakest instances of the choice scheme, those where we make only ω-many choices for a first-order formula. That is, it is consistent that there is a model of KM and a first-order assertion $\varphi(x,X)$ such that in the model, for every $n\in\omega$, there is a class X such that $\varphi(n,X)$ holds, but there is no class Y such that $\varphi(n,Y_n)$ holds for every $n\in\omega$ [GH]. Given this situation, the next natural question which arises is whether the theory Kelley-Morse together with the choice scheme $\mathrm{KM}+\mathrm{CC}$ proves DC_ω, the weakest of the dependent choice principles. The axiom of choice AC, of course, implies all the dependent choice axioms DC_α over ZF, but results from second-order arithmetic suggested that $\mathrm{KM}+\mathrm{CC}$ may not be able to prove at least some of the dependent choice principles DC_α. The study of the two fields has shown that there are many parallels as well as some differences between these mathematical domains, with second-order arithmetic being the more well-developed of the two because of its connections to analysis.

Second-order arithmetic has two types of objects: numbers and sets of numbers (which we think of as the reals). A typical axiomatization of second-order arithmetic consists of the basic axioms: the PA axioms for numbers and the set induction axiom asserting that induction holds for every set, together with some set existence axioms. The primary set existence axioms are again comprehension axioms and choice principles for sets. The choice scheme CC, in this context, is the exact analogue of the choice scheme in second-order set theory and the dependent choice scheme DC is the exact analogue of DC_ω. The analogue of GBC is the second-order arithmetic theory ACA_0, which is equiconsistent with PA, and the analogue of KM is the theory Z_2, which has comprehension for all second-order formulas. An argument, using Shoenfield absoluteness, shows that Z_2 implies CC for Σ^1_2-formulas (note the contrast with the KM situation, where KM does not prove CC even for first-order formulas) and that $\mathrm{Z}_2+\mathrm{CC}$ implies DC for Σ^1_2-formulas [Sim09].

Feferman and Levy constructed a model of ZF in which ω_1 is a countable union of countable sets as a symmetric submodel of a forcing extension by the finite-support product $\Pi_{n<\omega}\operatorname{Coll}(\omega,\aleph_n)$ of collapse forcings [Lév70]. The reals of this classical choiceless model form a model of Z_2 in which CC optimally fails for a Π^1_2-assertion. The authors of the present article, together with Kanovei, showed that it is consistent that there is a model of $Z_2+\mathrm{CC}$ in which DC fails for a Π^1_2-assertion. The model was obtained as the reals of a symmetric submodel of a forcing extension by a tree iteration of Jensen's forcing [FGK19].

Jensen's forcing $\mathbb{J}$ is a subposet of Sacks forcing that was constructed by Jensen in L using the guessing principle $\diamondsuit$. The forcing $\mathbb{J}$ has two key properties: it has the ccc and it adds a unique (Π^1_2-singleton) generic real over L. Jensen used the forcing $\mathbb{J}$ to show that it is consistent to have a Π^1_2-singleton non-constructible real (every Σ^1_2-singleton is in L by Shoenfield's absoluteness) [Jen70]. Since then his forcing has found a number of other applications. While $\diamondsuit$ is crucial to the construction of a poset with the two key properties of $\mathbb{J}$, the constructible universe is not. A subposet of Sacks forcing with the ccc that adds a unique generic real can be constructed in any universe with $\diamondsuit$, with only the complexity of the generic real possibly being lost (see Section 6 for details).

Lyubetsky and Kanovei extended the "unique generics" property of Jensen's forcing to a finite-support ω-length product $\mathbb{J}^{<\omega}$ of $\mathbb{J}$. They showed that in any forcing extension $L[G]$ by $\mathbb{J}^{<\omega}$, the only generic reals for $\mathbb{J}$ are those added by the slices of G [KL17]. Abraham showed that, for $n<\omega$, a finite n-length iteration $\mathbb{J}_n$ of $\mathbb{J}$ adds a unique n-length generic sequence of reals [Abr84]. The authors of the current article and Kanovei showed that certain tree iterations of $\mathbb{J}$ have a "unique generics" property, and this property was instrumental in obtaining the model of second-order arithmetic satisfying AC but not DC.

In this paper, we generalize Jensen's construction from ω to an inaccessible cardinal κ to define an analogue of Jensen's forcing with perfect κ-trees. We show that if κ is an inaccessible cardinal and the guessing principle $\diamondsuit_{\kappa^+}(\operatorname{Cof}(\kappa))$ holds, then there is a poset $\mathbb{J}(\kappa)$, whose elements are perfect κ-trees ordered by the subset relation, with the three key properties: it is $<\kappa$-closed, it has the κ^+-cc, and it adds a unique generic subset of κ. A number of obstacles needed to be overcome to make Jensen's construction work for perfect κ-trees. Perfect trees have several nice properties properties, like the existence of greatest lower bounds for compatible conditions, which fail for perfect κ-trees, and these properties played a significant role in the construction of Jensen's forcing. Another problem which did not arise in the original construction was presented by the $<\kappa$-closure requirement. The poset $\mathbb{J}$ is built up as the union of a continuous chain of posets of length ω_1 with maximal antichains being sealed at stages given by $\diamondsuit$ along the way. In the case of the poset $\mathbb{J}(\kappa)$, we couldn't simply take unions at the limit stages of building up the analogous chain because then we would lose $<\kappa$-closure, but closing the union under $<\kappa$-sequences could potentially unseal maximal antichains. We shall argue that this doesn't happen. We also define finite iterations and tree iterations of the poset $\mathbb{J}(\kappa)$ partially analogously to how finite iterations $\mathbb{J}_n$ and tree iterations of $\mathbb{J}$ were defined in [FGK19].

We show that bounded-support κ-length products of $\mathbb{J}(\kappa)$, finite iterations of $\mathbb{J}(\kappa)$, and certain tree iterations of $\mathbb{J}(\kappa)$ have "unique generics" properties analogous to those of Jensen's forcing.

Theorem 1.1. *Suppose that κ is an inaccessible cardinal and $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$ holds. There is a poset $\mathbb{J}(\kappa)$ whose elements are perfect κ-trees ordered by the subset relation with the following three properties:*

1. *$\mathbb{J}(\kappa)$ is $<\kappa$-closed.*
2. *$\mathbb{J}(\kappa)$ has the κ^+-cc.*
3. *In any forcing extension by $\mathbb{J}(\kappa)$, there is a unique generic subset of κ. If the starting universe is L, then the unique generic subset is a Π^1_1-singleton.*

Let $\mathbb{J}(\kappa)^{<\kappa}$ denote the bounded-support κ-length product of the poset $\mathbb{J}(\kappa)$ and let $\mathbb{J}(\kappa)_n$, for $n<\omega$, denote the n-length iteration of $\mathbb{J}(\kappa)$ (defined in Section 11).

Theorem 1.2. *Suppose that κ is an inaccessible cardinal and $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$ holds. In a forcing extension $V[G]$ by $\mathbb{J}(\kappa)^{<\kappa}$, the only generic subsets of κ for $\mathbb{J}(\kappa)$ are those added by the slices of G.*

Theorem 1.3. *Suppose that κ is an inaccessible cardinal, $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$ holds, and $n\in\omega$. The iteration $\mathbb{J}(\kappa)_n$ of the poset $\mathbb{J}(\kappa)$ of length n adds a unique n-length generic sequence of subsets of κ.*

Theorem 1.4. *Suppose that κ is an inaccessible cardinal and $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$ holds. In any forcing extension by the tree iteration of $\mathbb{J}(\kappa)$ along the tree $(\kappa^+)^{<\omega}$, for every $n\in\omega$, the only n-length generic sequences of subsets of κ for the iteration $\mathbb{J}(\kappa)_n$ are those arising from the n-length nodes of the generic tree.*

For details on the definitions of the iterations $\mathbb{J}(\kappa)_n$ and the tree iterations, see Section 11.

We construct a symmetric submodel of a forcing extension by the tree iteration of $\mathbb{J}(\kappa)$ along the tree $(\kappa^+)^{<\omega}$ such that in this model $\mathscr{V}=(V_\kappa,\in,V_{\kappa+1})$ is a model of $\mathrm{KM}+\mathrm{CC}$ in which the scheme DC_ω fails.

Theorem 1.5. *It is consistent that there is a model of* $\mathrm{KM}+\mathrm{CC}+\neg\mathrm{DC}_\omega$.

2. κ-perfect posets

Throughout what follows we assume that κ is an inaccessible cardinal. We will say that $T\subseteq 2^{<\kappa}$ is a *tree* if $T\neq\emptyset$ and whenever $s\in T$ and $t\leq s$ (in the sequence end-extension order), then $t\in T$. Given a tree T and a node $t\in T$, we let $T_t=\{s\in T\mid s\geq t \text{ or } s\leq t\}$ be the subtree of T consisting of the stem up to t together with everything above t. We will say that a tree $T\subseteq 2^{<\kappa}$ is a *κ-tree* if T has height κ. Next, we define the notion of a perfect κ-tree T following [Kan80].

Definition 2.1. A tree $T\subseteq 2^{<\kappa}$ is a *perfect κ-tree* if it satisfies the following:

1. Every node of T has a splitting node above it (T is splitting).
2. For every limit ordinal $\alpha<\kappa$, if $s\in 2^\alpha$ and $s\restriction\beta\in T$ for every $\beta<\alpha$, then $s\in T$ (T is closed).
3. For every limit ordinal $\alpha<\kappa$ if $s\in 2^\alpha$ and for cofinally many $\beta<\alpha$, $s\restriction\beta$ splits, then s splits (the splitting nodes of T are closed).

We define a partial order on perfect κ-trees by asserting that if T and S are perfect κ-trees, then $T\leq S$ if $T\subseteq S$.

The following propositions from [Kan80] are not difficult to prove.

Proposition 2.2 ([Kan80])**.** *If a tree $T\subseteq 2^{<\kappa}$ satisfies conditions* (1) *and* (2) *above, then condition* (3) *is equivalent to the assertion that for every branch $f\in 2^\kappa$ of T, the set $C_f=\{\alpha<\kappa\mid f\restriction\alpha \text{ splits}\}$ is a club.*

In fact, even though we won't make use of it, we note that given a perfect κ-tree T, the subset of κ consisting of all limit levels of T at which every node splits is a club.

Proposition 2.3. *If T is a perfect κ-tree, then*

$$C = \{\alpha < \kappa \mid \textit{every node in } T \cap 2^\alpha \textit{ splits}\}$$

is a club.

Proof. Clearly C is closed, so it remains to check that it is unbounded. Fix $\alpha_0 < \kappa$. Let α_1 be a level above α_0 such that for every node $s \in T \cap 2^{\alpha_0}$, s splits below level α_1. Similarly, given α_n for some $n < \omega$, let α_{n+1} be a level above α_n such that for every node $s \in T \cap 2^{\alpha_n}$, s splits below level α_{n+1}. Let $\alpha = \sup_{n<\omega} \alpha_n$. Fix a node $t \in T \cap 2^\alpha$. We will show that t has cofinally many splitting nodes below it. Fix $\beta < \alpha$, and let $n < \omega$ be least such that $\alpha_n > \beta$. Let $s \leq t$ be such that $s \in 2^{\alpha_n} \cap T$. Let $\gamma < \alpha_{n+1}$ be the least level having a node $s' \geq s$ which splits. By the leastness of γ, $s' \leq t$. Since t was arbitrary, it follows that every node on level α of T splits. □

Proposition 2.4 ([Kan80])**.** *Suppose that $\beta < \kappa$ and $\{T_\xi \mid \xi < \beta\}$ is a $\leq$-decreasing sequence of perfect κ-trees. Then $T = \bigcap_{\xi<\beta} T_\xi$ is a perfect κ-tree.*

Because of the presence of limit levels, the perfect κ-trees do not behave as nicely as perfect trees (subtrees of $2^{<\omega}$ in which every node has a splitting node above it). For instance, consider the standard fact that if T and S are perfect trees whose intersection contains a perfect tree, then there is a maximal perfect tree $T \wedge S \subseteq T \cap S$, the *meet* of T and S. Let's see that this maximality property fails for perfect κ-trees.

Proposition 2.5. *There are perfect κ-trees S and T whose intersection contains a perfect κ-tree but no maximal one.*

Proof. Fix some $t \in 2^\omega$. Let T be the tree $2^{<\kappa}$ with all nodes $\geq t^\frown 00$ and $\geq t^\frown 10$ removed. Let S be the tree $2^{<\kappa}$ with all nodes $\geq t^\frown 01$ and $\geq t^\frown 11$ removed. Clearly, S and T are perfect κ-trees, and there are many different perfect κ-trees contained in $S \cap T$. Note next that a perfect κ-tree in the intersection of S and T cannot contain the node t and therefore has to be bounded below t. Suppose that $R \subseteq S \cap T$ is a perfect κ-tree and n is largest such that $t \restriction n \in R$. Let $t(n+1) = i$. Then $R \cup (2^{<\kappa})_{t \restriction n+1^\frown(1-i)}$ is a perfect κ-tree contained in $S \cap T$. Thus, there cannot be a maximal perfect κ-tree contained in $S \cap T$. □

Another example comes from considering closure under unions. While perfect trees are closed under finite ($<\omega$-sized) unions, and we would like, by analogy, for perfect κ-trees to be closed under $<\kappa$-sized unions, they are not closed even under ω-sized unions.

Proposition 2.6. *There are ω-many perfect κ-trees $\{T_n \mid n < \omega\}$ whose union is not a perfect κ-tree.*

Proof. Let $\{r_n \mid n < \omega\}$ be a sequence of distinct elements of 2^ω such that for every $m < \omega$ and $s \in 2^m$, there is some r_n with $s \subseteq r_n$. For each $n < \omega$, let $T_n = (2^{<\kappa})_{r_n}$, which is clearly a perfect κ-tree. Now observe that $\bigcup_{n<\omega} T_n$ includes $2^{<\omega}$, but does not include every node in 2^ω, and hence cannot be a perfect κ-tree because it fails the closure requirement (2) from Definition 2.1. □

Next, we will define the notion of a κ-perfect poset, whose elements are perfect κ-trees ordered by the subset relation.

Definition 2.7. A collection $\mathbb{P}$ of perfect κ-trees ordered by the subset relation is a *κ-perfect poset* if it satisfies the following conditions:

(1) $2^{<\kappa} \in \mathbb{P}$.
(2) If $T \in \mathbb{P}$ and $t \in T$, then $T_t \in \mathbb{P}$.
(3) If $\{T_\xi \mid \xi < \beta\}$, with $\beta < \kappa$, is a $\subseteq$-decreasing sequence of trees in $\mathbb{P}$, then $T = \bigcap_{\xi<\beta} T_\xi \in \mathbb{P}$.
(4) Suppose $T \in \mathbb{P}$, $\alpha < \kappa$ is a successor ordinal, and $\{T^{(s)} \mid s \in T \cap 2^\alpha\}$ is a collection of elements of $\mathbb{P}$ such that $T^{(s)} \subseteq T_s$. Then $\bigcup_{s\in 2^\alpha} T^{(s)} \in \mathbb{P}$.

We will refer to property (3) as the *$<\kappa$-intersection property* and to property (4) as the *weak union property*.

Note that the union $\bigcup_{s\in 2^\alpha} T^{(s)}$ in the definition of the weak union property must be a perfect κ-tree because we chose α to be a successor ordinal. Note also that a κ-perfect poset is $<\kappa$-closed by the $<\kappa$-intersection property.

We can actually strengthen the weak union property as follows. Suppose T is a perfect κ-tree, X is an antichain of nodes on successor levels of T, with $|X| < \kappa$, and

$$\vec{T} = \{T^{(s)} \mid s \in X\}$$

is a collection of perfect κ-trees such that $T^{(s)} \subseteq T_s$ for every $s \in X$. We say that a tree S is the tree T *slimmed down by* $\vec{T}$ if S is the result of replacing T_s by $T^{(s)}$ in T for every $s \in X$.

Proposition 2.8. *Suppose $\mathbb{P}$ is a κ-perfect poset, $T \in \mathbb{P}$, X is an antichain of nodes on successor levels of T, with $|X| < \kappa$, and $\vec{T} = \{T^{(s)} \mid s \in X\}$ is a collection of perfect κ-trees such that $T^{(s)} \subseteq T_s$ for every $s \in X$. Then the tree S, which is the tree T slimmed down by $\vec{T}$, is in $\mathbb{P}$.*

Proof. Let $\beta < \kappa$ be a limit ordinal such that all nodes in X appear on levels below β. For every successor ordinal $\alpha < \beta$, let $X_\alpha = X \cap 2^\alpha$. Enumerate the successor ordinals below β as $\{\alpha_\xi \mid \xi < \beta\}$. Let $S_0 = T$. Suppose we have defined a $\subseteq$-decreasing sequence $\{S_\xi \mid \xi < \gamma\} \subseteq \mathbb{P}$ for some $\gamma < \beta$. Let $S'_\gamma = \bigcap_{\xi<\gamma} S_\xi$, which is in $\mathbb{P}$ by the $<\kappa$-intersection property. Let S_γ be S'_γ slimmed down by $\vec{T}_\gamma = \{T^{(s)} \mid s \in X_{\alpha_\gamma}\}$, which is in $\mathbb{P}$ by the weak union property. Clearly $S_\gamma \subseteq S_\xi$ for all $\xi < \gamma$. Now it is easy to see that $S = \bigcap_{\xi<\beta} S_\xi$, which must be in $\mathbb{P}$ by the $<\kappa$-intersection property. □

The following proposition is standard. Given a perfect κ-tree T, we will denote by $[T]$ the collection of all cofinal branches through T.

Proposition 2.9. *Suppose that $\mathbb{P}$ is a κ-perfect poset and $G \subseteq \mathbb{P}$ is V-generic. Then $A = \bigcap_{T\in G} T$ is a cofinal branch of every $T \in G$ and for every $T \in \mathbb{P}$, if $A \in [T]$, then $T \in G$. Thus, A determines G.*

Since A is a characteristic function of a subset of κ, we can view a generic filter for a κ-perfect poset as a subset of κ.

Recall that perfect posets, whose elements are perfect trees ordered by subset, are required to be closed under meets: if T and S are elements of a perfect poset, then so is $T \wedge S$ (see [FGK19] or [Abr84]). Closure under meets is an extremely useful property. In particular, it follows from it that if perfect trees T and S are compatible (have a perfect tree in their intersection), then they are compatible in any perfect poset to which they belong. This suggests that a useful property for κ-perfect posets might be to insist that they respect compatibility of perfect κ-trees in the following strong sense.

Definition 2.10. Let us say that a κ-perfect poset $\mathbb{P}$ has the *$<\kappa$-compatibility property* if whenever $X \subseteq \mathbb{P}$ has size less than κ and $\bigcap_{T\in X} T$ contains a perfect κ-tree, then there is a tree $S \in \mathbb{P}$ such that $S \subseteq \bigcap_{T\in X} T$.

If a κ-tree doesn't have a subtree that is a perfect κ-tree, then a κ-perfect forcing cannot add such a subtree (see Proposition 2.12 below). Suppose that λ is an inaccessible cardinal. Let us first see that it is consistent to have a λ-tree T that does not contain a subtree that is a perfect λ-tree and a forcing notion preserving the inaccessibility of λ such that in the forcing extension, T has a perfect λ-tree as a subset. Recall that $S \subseteq \lambda$ is called *fat stationary* if for every club $C \subseteq \lambda$ and $\alpha < \lambda$, the intersection $S \cap C$ contains a closed set of order-type α. A fat stationary set is said to be *non-trivial* if it doesn't contain a club. If λ is at least Mahlo, then the complement of the regular cardinals in λ is a non-trivial fat stationary set. Also, we can force with $\mathrm{Add}(\lambda, 1)$ to add a Cohen subset to λ and use the characteristic function f of the Cohen subset to define fat stationary and co-fat stationary sets $S = f^{-1}(0)$ and $\bar{S} = f^{-1}(1)$. If S is fat stationary, then the club shooting forcing $\mathbb{P}_S$ for S, whose conditions are closed bounded subsets of S ordered by end-extension, is $<\lambda$-distributive and adds a club subset to S [AS83].

Proposition 2.11. *It is consistent that λ is an inaccessible cardinal and there is a λ-tree T without a subtree that is a perfect λ-tree and a poset $\mathbb{P}$ such that in any forcing extension by $\mathbb{P}$, λ remains inaccessible and T has a perfect λ-tree as a subset.*

Proof. Fix a stationary set S in λ. Define a κ-tree T^S by $s \in T^S$ whenever $s(\alpha) = 0$ for every $\alpha \notin S$. Note that T^S splits precisely at levels $\alpha \in S$. Let's argue that T contains a subtree that is a perfect κ-subtree if and only if S contains a club.

Suppose, first, that S contains a club C. Consider the tree T^C consisting of $s \in 2^{<\kappa}$ such that $s(\alpha) = 0$ for every $\alpha \notin C$. Clearly T^C is a subtree of T^S. The tree T^C has a splitting node above every node because C is unbounded, and it is obviously closed. Fix any branch $f \in 2^\kappa$ through T^C, and observe that

$$C_f = \{\alpha < \kappa \mid f \restriction \alpha \text{ splits}\} = C,$$

and hence is a club. So by Proposition 2.2, T^C is a perfect κ-tree. Next, suppose that S does not contain a club. Suppose towards a contradiction that $T \subseteq T^S$ is a perfect κ-tree. Fix a branch $f \in 2^\kappa$ through T, and observe that C_f is a club by Proposition 2.2. But then $C_f \subseteq S$, contradicting our assumption that S does not contain a club.

Now suppose further that S is a non-trivial fat stationary set and force with the poset $\mathbb{P}_S$. Since $\mathbb{P}_S$ is $<\kappa$-distributive, κ remains inaccessible in the forcing extension by $\mathbb{P}_S$. By the above argument, the κ-tree T^S does not have a subtree that is a perfect κ-tree in V, but will have one in any forcing extension by $\mathbb{P}_S$. $\square$

Proposition 2.12. *Suppose $\mathbb{P}$ is a $<\kappa$-closed poset and T is a κ-tree. If T does not have a subtree that is a perfect κ-tree, then it won't have one in any forcing* *extension by $\mathbb{P}$. In particular, a κ-perfect poset cannot add a subtree that is a perfect κ-tree to a κ-tree without one.*

Proof. Suppose that T is a κ-tree that doesn't have any subtrees that are perfect κ-trees. Suppose towards a contradiction that there is a forcing extension $V[G]$ by $\mathbb{P}$ in which T gets a perfect κ-tree as a subset, and fix a condition $p\in\mathbb{P}$ forcing this. Let $\theta\gg\kappa$ be a regular cardinal and let $X\prec H_\theta$ be an elementary substructure of size κ such that $p,\mathbb{P},T\in X$ and $X^{<\kappa}\subseteq X$. Let $\pi:X\to M$ be the Mostowski collapse map, and note that $M^{<\kappa}\subseteq M$. Let $\mathbb{Q}=\pi(\mathbb{P})$, $q=\pi(p)$. Observe that we have $\pi(T)=T$ since $\kappa\subseteq X$ (follows from closure). By elementarity, in M, q forces that T contains a perfect κ-tree as a subset. Since, by elementarity, $\mathbb{Q}$ is $<\kappa$-closed in M and $M^{<\kappa}\subseteq M$, we can build in V an M-generic filter G for M with $q\in M$ (see Proposition 4.2). But then T has a perfect κ-tree as a subset in $M[G]\subseteq V$, which contradicts our assumption. □

Let's describe the smallest κ-perfect poset $\mathbb{P}_{\min}$, which we will construct in κ-many steps as follows. Let

$$\mathbb{P}_0=\{(2^{<\kappa})_t\mid t\in 2^{<\kappa}\}.$$

Suppose inductively that $\mathbb{P}_\xi$ has been defined. Let $\mathbb{P}'_{\xi+1}$ be the collection of all trees $T\in\mathbb{P}_\xi$ slimmed down by some $\vec{T}=\{T^{(s)}\mid s\in 2^\beta\cap T\}\subseteq\mathbb{P}_\xi$, with β a successor ordinal. Let $\mathbb{P}_{\xi+1}$ be the collection of all trees $T=\bigcap_{\alpha<\beta}T_\alpha$ for some $\subseteq$-decreasing sequence $\{T_\alpha\mid\alpha<\beta\}$, with $\beta<\kappa$, of trees in $\mathbb{P}'_{\xi+1}$. At limit stages, we take unions. The poset

$$\mathbb{P}_{\min}=\bigcup_{\xi<\kappa}\mathbb{P}_\xi.$$

The poset $\mathbb{P}_{\min}$ is κ-perfect by construction. Note also that it has size κ.

Given a tree $T\in\mathbb{P}_{\min}$, let us say that α is a *nice level* of T if for every node t on level α of T, $T_t=(2^{<\kappa})_t$. Clearly, if α is a nice level of T and $\bar{\alpha}>\alpha$, then $\bar{\alpha}$ is also a nice level of T.

Proposition 2.13. *Every tree $T\in\mathbb{P}_{\min}$ has a nice level.*

Proof. We will argue by induction on $\xi<\kappa$ that every tree in $\mathbb{P}_\xi$ has a nice level. Clearly, every tree in $\mathbb{P}_0$ has a nice level. So suppose inductively that every tree in $\mathbb{P}_\xi$ has a nice level. Let $S\in\mathbb{P}'_{\xi+1}$. Then S is a tree $T\in\mathbb{P}_\xi$ slimmed down by some $\vec{T}=\{T^{(s)}\mid s\in 2^\beta\cap T\}\subseteq\mathbb{P}_\xi$, with β a successor ordinal. For each tree $T^{(s)}$, let α_s be a nice level and let γ be a nice level of T. Let α be above γ and α_s for every $s\in T\cap 2^\beta$. Then clearly α is a nice level for S. Next, suppose that $\{T_\eta\mid\eta<\beta\}\subseteq\mathbb{P}'_{\xi+1}$, with $\beta<\kappa$, is a $\subseteq$-decreasing sequence of trees. For each $\eta<\beta$, let α_η be a nice level of T_η. Let $\alpha>\alpha_\eta$ for every $\eta<\beta$ and let $T=\bigcap_{\eta<\beta}T_\eta$. Then clearly α is a nice level for T. □

Proposition 2.14. *The poset $\mathbb{P}_{\min}$ has the $<\kappa$-compatibility property.*

Proof. Let $X\subseteq\mathbb{P}_{\min}$ be such that $|X|<\kappa$ and $\bigcap_{T\in X}T$ contains a perfect κ-tree S. Since X has size less than κ, there is a level α which is nice for every $T\in X$. Thus, for every $T\in X$ and $t\in T\cap 2^\alpha$, $T_t=(2^{<\kappa})_t$. Since $S\subseteq\bigcap_{T\in X}T$ is a perfect κ-tree, there is some $t\in S$ on level α. So $t\in T$ for every $T\in X$. But then $(2^{<\kappa})_t\subseteq T$ for every $T\in X$, and so $(2^{<\kappa})_t\subseteq\bigcap_{T\in X}T$. □

Definition 2.15. Let $\mathbb{P}^{<\kappa}$ denote the bounded support κ-length product of a κ-perfect poset $\mathbb{P}$. Conditions in $\mathbb{P}^{<\kappa}$ are functions $p : \kappa \to \mathbb{P}$ for which there is a $\beta < \kappa$ such that for every $\gamma \geq \beta$, $p(\gamma) = 2^{<\kappa}$. We call the *domain* of a condition $p \in \mathbb{P}^{<\kappa}$ the collection of all non-trivial coordinates of p.

3. Growing κ-perfect posets

In this section, we will show how to grow a κ-perfect poset to a larger κ-perfect poset in a specially chosen forcing extension of V.

Definition 3.1. We associate to each κ-perfect poset $\mathbb{P}$, the poset $\mathbb{Q}(\mathbb{P})$ whose conditions are pairs (T, α) such that $T \in \mathbb{P}$ and α is a successor ordinal ordered so that $(T_2, \alpha_2) \leq (T_1, \alpha_1)$ whenever $\alpha_2 \geq \alpha_1$, $T_2 \subseteq T_1$, and $T_1 \cap 2^{\alpha_1} = T_2 \cap 2^{\alpha_1}$. We call $\mathbb{Q}(\mathbb{P})$ the *fusion poset* for $\mathbb{P}$ because fusion arguments with perfect κ-trees in $\mathbb{P}$ can be expressed by meeting well-chosen dense sets in $\mathbb{Q}(\mathbb{P})$.

The poset $\mathbb{Q}(\mathbb{P})$ is $<\kappa$-closed and adds a generic perfect κ-tree.

Proposition 3.2. *The poset $\mathbb{Q}(\mathbb{P})$ is $<\kappa$-closed.*

Proof. Suppose that $\{(T_\xi, \alpha_\xi) \mid \xi < \beta\}$, with $\beta < \kappa$, is a $\leq$-descending sequence of conditions in $\mathbb{Q}(\mathbb{P})$. Let $\alpha = \bigcup_{\xi<\beta} \alpha_\xi$. Let $T = \bigcap_{\xi<\beta} T_\xi$, which is in $\mathbb{P}$ by the $<\kappa$-intersection property. Let's verify that $(T, \alpha + 1) \leq (T_\xi, \alpha_\xi)$ for every $\xi < \beta$. Since the sequence is descending we have that $T_\eta \cap 2^{\alpha_\xi} = T_\xi \cap 2^{\alpha_\xi}$ for all $\xi < \eta < \beta$. Thus, $T \cap 2^{\alpha_\xi} = T_\xi \cap 2^{\alpha_\xi}$. □

Let $G \subseteq \mathbb{Q}(\mathbb{P})$ be V-generic. Note that by the $<\kappa$-closure of $\mathbb{Q}(\mathbb{P})$, κ remains inaccessible in $V[G]$. Let $\mathcal{T} = \bigcup_{(T,\alpha)\in G} T \cap 2^{\leq\alpha}$.

Proposition 3.3.

1. *$\mathcal{T}$ is a perfect κ-tree in $V[G]$.*
2. *$\mathcal{T} \leq T$ for every condition $(T, \alpha) \in G$.*

Proof. First, let's prove (2). Fix a condition $(T, \alpha) \in G$ and a node $t \in \mathcal{T}$, with $t \in 2^\beta$. By density, there is $(S, \gamma) \in G$ with $\gamma \geq \alpha, \beta$. Let $(\bar{T}, \bar{\alpha}) \in G$ be below both (T, α) and (S, γ). In particular, $\bar{T} \subseteq T$ and $\bar{\alpha} \geq \beta$. Next, let $(R, \delta) \in G$ be such that $t \in R \cap 2^{\leq\delta}$. Let $(\bar{R}, \bar{\delta}) \in G$ be below both $(\bar{T}, \bar{\alpha})$ and (R, δ). Then $\bar{R} \subseteq \bar{T}, R$ and $\bar{R} \cap 2^{\min\{\bar{\alpha},\delta\}} = R \cap 2^{\min\{\bar{\alpha},\delta\}} = \bar{T} \cap 2^{\min\{\bar{\alpha},\delta\}}$. In particular, $t \in \bar{R}$. Thus, since $\bar{R} \subseteq \bar{T}$, $t \in \bar{T} \subseteq T$.

Next, let's prove (1). Suppose that $t \in 2^\gamma$ is such that $t \restriction \eta \in \mathcal{T}$ for every $\eta < \gamma$. Choose some $(T, \alpha) \in G$ with $\alpha > \gamma$, which exists by density. Then, by (2), $t \restriction \eta \in T$ for every $\eta < \gamma$. Hence $t \in T$ by closure, and since $\alpha > \gamma$, $t \in \mathcal{T}$. Closure of splitting nodes is shown similarly. So it remains to show that above every node in $\mathcal{T}$, there is a splitting node in $\mathcal{T}$.

Fix a node $t \in \mathcal{T}$. There is some condition $(T, \alpha) \in G$ such that $t \in T \cap 2^{\leq\alpha}$. We will show that conditions $(S, \beta) \leq (T, \alpha)$ such that t splits in S below level β are dense below (T, α). Choose any condition $(S, \bar{\beta}) \leq (T, \alpha)$, and note that $t \in S$. Since S is a perfect κ-tree, there must be some successor level $\beta \geq \bar{\beta}$ such that t splits in S below level β. Then the condition (S, β) works. This completes the argument that $\mathcal{T}$ is a perfect κ-tree in $V[G]$. □

Definition 3.4. Let $\mathbb{Q}(\mathbb{P})^{<\kappa}$ denote the bounded support κ-length product of the poset $\mathbb{Q}(\mathbb{P})$. Conditions in $\mathbb{Q}(\mathbb{P})^{<\kappa}$ are functions $p : \kappa \to \mathbb{Q}(\mathbb{P})$ for which there is

$\beta < \kappa$ such that for all $\xi \geq \beta$ we have that $f(\xi) = (2^{<\kappa}, 0)$ is trivial. We call the *domain* of a condition $p \in \mathbb{Q}(\mathbb{P})^{<\kappa}$ the collection of all non-trivial coordinates of p.

Clearly, $\mathbb{Q}(\mathbb{P})^{<\kappa}$ is $<\kappa$-closed, and hence preserves the inaccessibility of κ.

For the remainder of this section, suppose that $G \subseteq \mathbb{Q}(\mathbb{P})^{<\kappa}$ is V-generic. Let $\{\mathcal{T}_\xi \mid \xi < \kappa\}$ be the κ-length sequence of generic perfect κ-trees derived from G, and let $\dot{\mathcal{T}}_\xi$ for $\xi < \kappa$ be the canonical $\mathbb{Q}(\mathbb{P})^{<\kappa}$-names for the trees $\mathcal{T}_\xi$.

First, we observe that any two distinct trees $\mathcal{T}_\xi$ and $\mathcal{T}_\eta$ are going to have a bounded intersection.

Proposition 3.5. *For any $\xi \neq \eta < \kappa$, the trees $\mathcal{T}_\xi$ and $\mathcal{T}_\eta$ have a bounded intersection. Hence, $\{\mathcal{T}_\xi \mid \xi < \kappa\}$ is an antichain in the $\subseteq$-order.*

Proof. Fix $\xi \neq \eta < \kappa$. We will show that the intersection $\mathcal{T}_\xi \cap \mathcal{T}_\eta$ has size less than κ. Fix any condition $p \in G$. Let $p(\xi) = (T_\xi, \alpha_\xi)$ and $p(\eta) = (T_\eta, \alpha_\eta)$. Pick a successor ordinal α above both α_ξ and α_η, and strengthen p to the condition q such that $q(\xi) = (T_\xi, \alpha)$, $q(\eta) = (T_\eta, \alpha)$, and $q(\beta) = p(\beta)$ on the rest of the coordinates. For each node $t \in 2^\alpha \cap T_\xi \cap T_\eta$ choose two incompatible nodes t_0 and t_1 above t such that $t_0 \in T_\xi$ and $t_1 \in T_\eta$. Let $\bar{T}_\xi$ be the tree we get by replacing each $(T_\xi)_t$ with $(T_\xi)_{t_0}$ in T_ξ, which is in $\mathbb{P}$ by the weak union property, and let $\bar{T}_\eta$ be the tree we get by replacing each $(T_\eta)_t$ with $(T_\eta)_{t_1}$ in T_η. Let $\bar{q}$ be the condition such that $\bar{q}(\xi) = (\bar{T}_\xi, \alpha)$, $\bar{q}(\eta) = (\bar{T}_\eta, \alpha)$, and $\bar{q}(\beta) = q(\beta)$ on the rest of the coordinates. Since $\bar{q} \leq p$, we have argued that it is dense below p to have conditions where the trees on coordinates ξ and η have intersection of size less than κ. □

Now working in $V[G]$, we will define a κ-perfect poset $\mathbb{P}^*$ extending $\mathbb{P}$. The poset $\mathbb{P}^*$ is going to be generated by $\mathbb{P}$ and

$$\mathbb{U} = \{(\mathcal{T}_\xi)_t \mid \xi < \kappa, t \in \mathcal{T}_\xi\}$$

by closing to obtain the weak union property and the $<\kappa$-intersection property, analogously to how the poset $\mathbb{P}_{\min}$ was generated by the trees $\{(2^{<\kappa})_t \mid t \in 2^{<\kappa}\}$. Let $\mathbb{P}^*_0 = \mathbb{P} \cup \mathbb{U}$. Suppose inductively that $\mathbb{P}^*_\xi$ has been defined. Let $\mathbb{P}'_{\xi+1}$ be the collection of all trees $T \in \mathbb{P}^*_\xi$ slimmed down by some $\vec{T} = \{T^{(s)} \mid s \in T \cap 2^\beta\} \subseteq \mathbb{P}^*_\xi$, with β a successor ordinal. Let $\mathbb{P}^*_{\xi+1}$ be the collection of all trees $T = \bigcap_{\alpha<\beta} T_\alpha$ for a $\subseteq$-decreasing sequence $\{T_\alpha \mid \alpha < \beta\}$, with $\beta < \kappa$, of trees in $\mathbb{P}'_{\xi+1}$. At limit stages, we take unions. The poset $\mathbb{P}^* = \bigcup_{\xi<\kappa} \mathbb{P}^*_\xi$. Clearly $\mathbb{P}^*$ is a κ-perfect poset in $V[G]$. Note that if the original poset $\mathbb{P}$ had size κ, then so does the poset $\mathbb{P}^*$.

The next proposition will be used to show that if $\mathbb{P}$ has the $<\kappa$-compatibility property, then so does $\mathbb{P}^*$.

Proposition 3.6. *Suppose $\{S_\xi \mid \xi < \rho\} \subseteq \mathbb{P}$, with $\rho < \kappa$, and for some $\alpha < \kappa$, $(\mathcal{T}_\alpha)_t \cap (\bigcap_{\xi<\rho} S_\xi)$ contains a perfect κ-tree as a subset. Then there is a node $s \in \mathcal{T}_\alpha$, with $s \geq t$, such that $(\mathcal{T}_\alpha)_s \subseteq \bigcap_{\xi<\rho} S_\xi$.*

Proof. Fix a condition $p \in \mathbb{Q}(\mathbb{P})^{<\kappa}$, and let $p(\alpha) = (T, \gamma)$. If $t \notin T$, let $\bar{p} = p$. So suppose that $t \in T$. By strengthening p, if necessary, we can assume that γ is above the level of t. Fix a node $s \geq t$ on level γ of T. If $T_s \subseteq \bigcap_{\xi<\rho} S_\xi$, then let $U^{(s)} = T_s$. Otherwise, there is some node $\bar{s} \geq s$ such that $\bar{s} \notin \bigcap_{\xi<\rho} S_\xi$. In this case, let $U^{(s)} = T_{\bar{s}}$. Let $\bar{T}$ be the tree T with each T_s replaced by $U^{(s)}$. Let $\bar{p}$ be the condition with $\bar{p}(\alpha) = (\bar{T}, \gamma)$ and $\bar{p}(\beta) = p(\beta)$ for every $\beta \neq \alpha$. Clearly, $\bar{p} \leq p$. Thus, conditions of the form $\bar{p}$ are dense in $\mathbb{Q}(\mathbb{P})^{<\kappa}$. Let $\bar{p} \in G$ be some such condition. First, observe that, since $t \in \mathcal{T}_\alpha$, $\bar{p}(\alpha) = (\bar{T}, \gamma)$ as defined above. Since $(\mathcal{T}_\alpha)_t \cap (\bigcap_{\xi<\rho} S_\xi)$ contains a perfect κ-tree, there must be some node $s \in 2^\gamma$ such that $(\mathcal{T}_\alpha)_s \cap (\bigcap_{\xi<\rho} (S_\xi)_s)$ contains a perfect κ-tree. Thus, it must have been the case that $\bar{T}_s = U^{(s)} \subseteq \bigcap_{\xi<\rho} S_\xi$. Hence, since $\bar{p} \in G$, $(\mathcal{T}_\alpha)_s \subseteq \bar{T}_s \subseteq \bigcap_{\xi<\rho} S_\xi$ by Proposition 3.3 (2).

□

Given a tree $T \in \mathbb{P}^*$, let us say that a level α of T is *nice* if for every $t \in T \cap 2^\alpha$, either $T_t \in \mathbb{P}$ or $T_t = (\mathcal{T}_\xi)_t$ for some $\xi < \kappa$. Clearly, if α is a nice level of T, then every level $\beta > \alpha$ is also nice.

Proposition 3.7. *Every $T \in \mathbb{P}^*$ has a nice level.*

Proof. We will argue by induction on $\xi < \kappa$ that every tree in $\mathbb{P}^*_\xi$ has a nice level. Clearly, every tree in $\mathbb{P}^*_0$ has a nice level. So suppose inductively that every tree in $\mathbb{P}^*_\xi$ has a nice level. Let $S \in \mathbb{P}'_{\xi+1}$. Then S is a tree $T \in \mathbb{P}^*_\xi$ slimmed down by some $\vec{T} = \{T^{(s)} \mid s \in 2^\beta \cap T\} \subseteq \mathbb{P}_\xi$, with β a successor ordinal. For each tree $T^{(s)}$, let $\alpha_s > \beta$ be a nice level and let $\gamma > \beta$ be a nice level for T. Let α be above γ and α_s for every $s \in T \cap 2^\beta$. Then clearly α is a nice level for S. Next, suppose that $\{T_\eta \mid \eta < \beta\} \subseteq \mathbb{P}'_{\xi+1}$, with $\beta < \kappa$, is a $\subseteq$-decreasing sequence of trees. For every $\eta < \beta$, let α_η be a nice level for T_η. Let $\alpha > \alpha_\eta$ for every $\eta < \beta$, and let $T = \bigcap_{\eta<\beta} T_\eta$. Fix $t \in T \cap 2^\alpha$. If cofinally many $(T_\eta)_t \in \mathbb{P}$, then this sequence of tree is in V by closure, and so $T_t = \bigcap_{\eta<\beta} (T_\eta)_t$ is in $\mathbb{P}$ by the $<\kappa$-intersection property. Otherwise, there is $\eta < \beta$ such that for every $\eta < \nu < \beta$, $(T_\nu)_t = (\mathcal{T}_{\rho_\nu})_t$ for some ρ_ν. But then by Proposition 3.5, all $\rho_\nu = \rho$ for some single ρ. Therefore, $T_t = (\mathcal{T}_\rho)_t$.

□

Proposition 3.8. *Suppose that $\mathbb{P}$ has the $<\kappa$-compatibility property. Then $\mathbb{P}^*$ has the $<\kappa$-compatibility property.*

Proof. Suppose that $\{T_\eta \mid \eta < \beta\}$, with $\beta < \kappa$, is some sequence of trees in $\mathbb{P}^*$ such that there is a perfect κ-tree $R \subseteq \bigcap_{\eta<\beta} T_\eta$. We need to argue that there is $T \in \mathbb{P}^*$ with $T \subseteq \bigcap_{\eta<\beta} T_\eta$. For each $\eta < \beta$, let α_η be a nice level for the tree T_η, and fix α such that $\alpha > \alpha_\eta$ for every $\eta < \beta$. Fix a node t on level α of R. Then $R_t \subseteq \bigcap_{\eta<\beta} (T_\eta)_t$. If every $(T_\eta)_t \in \mathbb{P}$, then the entire sequence is in V by closure, and so $\mathbb{P}$ has a perfect κ-tree $T \subseteq \bigcap_{\eta<\beta} (T_\eta)_t$ because we assumed that $\mathbb{P}$ has the $<\kappa$-compatibility property. Otherwise, $\bigcap_{\eta<\beta} (T_\eta)_t = (\mathcal{T}_\rho)_t \cap (\bigcap_{\xi<\delta} S_\xi)$ for some collection of trees $S_\xi \in \mathbb{P}$ (there can only be one $(\mathcal{T}_\rho)_t$ by Proposition 3.5). Now by Proposition 3.6, there is a node $s \geq t$ in $\mathcal{T}_\rho$ such that $(\mathcal{T}_\rho)_s \subseteq \bigcap_{\xi<\delta} S_\xi$. Hence, $T = (\mathcal{T}_\rho)_s \subseteq \bigcap_{\eta<\beta} (T_\eta)_t$. In either case, we have found a tree $T \in \mathbb{P}^*$ with $T \subseteq \bigcap_{\eta<\beta} T_\eta$.

□

Proposition 3.9.

(1) *The antichain $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$ is maximal in $\mathbb{P}^*$.*
(2) *$\mathbb{U}$ is dense in $\mathbb{P}^*$.*
(3) *Every maximal antichain of $\mathbb{P}$ from V remains maximal in $\mathbb{P}^*$.*

Proof. Fix $T \in \mathbb{P}^*$. Choose a nice level β of T and some $t \in T \cap 2^\beta$. If $T_t \in \mathbb{P}$, then by density, there is a condition $q \in G$ and ordinals α, γ such that $q(\alpha) = (T_t, \gamma)$.

By Proposition 3.3 (2), we then have $\mathcal{T}_\alpha \leq T_t$. If $T_t \notin \mathbb{P}$, then $T_t = (\mathcal{T}_\alpha)_s$ for some s. Thus, $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$ is a maximal antichain in $\mathbb{P}^*$ and $\mathbb{U}$ is dense in $\mathbb{P}^*$.

Fix a maximal antichain $\mathcal{A} \in V$ of $\mathbb{P}$. By the density of $\mathbb{U}$, it suffices to argue that every element of $\mathbb{U}$ meets $\mathcal{A}$. So fix some tree $(\mathcal{T}_\alpha)_t \in \mathbb{U}$. Fix a condition $p \in \mathbb{Q}(\mathbb{P})^{<\kappa}$, and let $p(\alpha) = (T, \gamma)$. If $t \notin T$, let $\bar{p} = p$. So suppose that $t \in T$. By strengthening p, if necessary, we can assume that γ is above the level of t. Fix a node $s \geq t$ on level γ of T. Since $\mathcal{A}$ is maximal in $\mathbb{P}$, there is $A \in \mathcal{A}$ compatible with T_s. Let $U \in \mathbb{P}$ such that $U \subseteq T_s, A$. Let $\bar{T}$ be the tree we get by replacing T_s with U in T. Let $\bar{p}$ be the condition with $\bar{p}(\alpha) = (\bar{T}, \gamma)$ and $\bar{p}(\beta) = p(\beta)$ for all $\beta \neq \alpha$. Clearly, $\bar{p} \leq p$. Thus, conditions of the form $\bar{p}$ are dense in $\mathbb{Q}(\mathbb{P})^{<\kappa}$. Let $\bar{p} \in G$ be some such condition. Since $t \in \mathcal{T}_\alpha$, $\bar{p}(\alpha) = (\bar{T}, \gamma)$ as constructed above. Then we have $(\mathcal{T}_\alpha)_s \subseteq \bar{T}_s \subseteq A \in \mathcal{A}$, verifying that $(\mathcal{T}_\alpha)_t$ is compatible with an element of $\mathcal{A}$. □

Property (3) of $\mathbb{P}^*$ above generalizes to κ-length bounded support products $\mathbb{P}^{*<\kappa}$. Let $\mathbb{U}^{<\kappa}$ consist of conditions in $\mathbb{P}^{*<\kappa}$ such that for every α in the domain of p, $p(\alpha) \in \mathbb{U}$. Clearly, $\mathbb{U}^{<\kappa}$ is dense in $\mathbb{P}^{*<\kappa}$.

Proposition 3.10. *Every maximal antichain of $\mathbb{P}^{<\kappa}$ from V remains maximal in $\mathbb{P}^{*<\kappa}$.*

Proof. Fix a maximal antichain $\mathcal{A} \in V$ of $\mathbb{P}^{<\kappa}$. By density, it suffices to argue that every condition $q \in \mathbb{U}^{<\kappa}$ meets $\mathcal{A}$. So fix a condition $q \in \mathbb{U}^{<\kappa}$. Let D_q be the domain of q and let $q(\alpha) = (\mathcal{T}_{\xi_\alpha})_{t_\alpha}$ for every $\alpha \in D_q$. Note that we can have $\alpha \neq \beta \in D_q$ such that $\xi_\alpha = \xi_\beta$. By thinning out the trees $q(\alpha)$, for $\alpha \in D_q$, we can assume without loss of generality that if $\xi_\alpha = \xi_\beta$, then t_α and t_β are incompatible nodes.

Fix $p \in \mathbb{Q}(\mathbb{P})^{<\kappa}$, and let $p(\xi) = (T_\xi, \gamma_\xi)$ for every $\xi < \kappa$. If for some $\alpha \in D_q$, $t_\alpha \notin T_{\xi_\alpha}$, let $\bar{p} = p$. So suppose that for all $\alpha \in D_q$, $t_\alpha \in T_{\xi_\alpha}$. By strengthening p, if necessary, we can assume that there is a single γ such that $\gamma = \gamma_{\xi_\alpha}$ and it is above the level of t_α for every $\alpha \in D_q$. For every $\alpha \in D_q$, fix a node $s_\alpha \geq t_\alpha$ on level γ of T_{ξ_α}, and note that by our assumption on q, if $\alpha \neq \beta$, but $\xi_\alpha = \xi_\beta$, then $s_\alpha \neq s_\beta$. Let $r \in \mathbb{P}^{<\kappa}$ be the condition defined by $r(\alpha) = (T_{\xi_\alpha})_{s_\alpha}$ for every $\alpha \in D_q$ and $r(\alpha)$ is trivial otherwise. Since $\mathcal{A}$ is maximal in $\mathbb{P}^{<\kappa}$, there is a condition $a \in \mathcal{A}$ compatible with r. Fix $\bar{r} \leq a, r$. Given $\eta < \kappa$, if $\eta = \xi_\alpha$ for some $\alpha \in D_q$, let $\bar{T}_\eta$ be the tree we get by replacing $(T_\eta)_{s_\beta}$ with $\bar{r}(\beta)$ for every β with $\eta = \xi_\beta$. Let $\bar{p}$ be the condition such that $\bar{p}(\eta) = (\bar{T}_\eta, \gamma)$ for every $\eta = \xi_\alpha$ for some $\alpha \in D_q$, and $\bar{p}(\eta) = p(\eta)$ otherwise. Clearly, $\bar{p} \leq p$. Thus, conditions of the form $\bar{p}$ are dense in $\mathbb{Q}(\mathbb{P})^{<\kappa}$. Let $\bar{p} \in G$ be some such condition. Since $t_\alpha \in T_{\xi_\alpha}$ for every $\alpha \in D_q$, it follows that $\bar{p}(\eta) = (\bar{T}_\eta, \gamma)$ for every $\eta = \xi_\alpha$ for some $\alpha \in D_q$ as constructed above. Then we have for every $\alpha \in D_q$, $(\mathcal{T}_{\xi_\alpha})_{s_\alpha} \subseteq (\bar{T}_{\xi_\alpha})_{s_\alpha} \subseteq a(\alpha)$. Thus, q and a are compatible. □

Finally, we will argue that generic filters for $\mathbb{P}^*$ restrict to generic filters for $\mathbb{P}$. This property will be used extensively in later arguments.

Proposition 3.11. *Suppose $H^* \subseteq \mathbb{P}^*$ is $V[G]$-generic. Then $H = H^* \cap \mathbb{P}$ is V-generic for $\mathbb{P}$.*

Proof. By Proposition 3.9 (3), H meets every maximal antichain of $\mathbb{P}$ from V. So it remains to check that H is a filter. Clearly, H is upward closed. So it remains to verify that any two conditions in H are compatible in H. Fix $T_1, T_2 \in H$. Since $T_1, T_2 \in H^*$, there is $S \in H^*$ such that $S \subseteq T_1, T_2$. Let β be a nice level of S and let $t \in S \cap 2^\beta$ be such that $S_t \in H^*$. If $S_t \in \mathbb{P}$, then we are done. Otherwise, $S_t = (\mathcal{T}_\alpha)_t$ for some $\alpha < \kappa$.

Fix a condition $p \in \mathbb{Q}(\mathbb{P})^{<\kappa}$, and let $p(\alpha) = (T, \gamma)$. If $t \notin T$, let $\bar{p} = p$. So suppose that $t \in T$. By strengthening p, if necessary, we can assume that $\gamma > \beta$. Fix a node $s \geq t$ on level γ of T. If $T_s \subseteq T_1, T_2$, let $U^{(s)} = T_s$. Otherwise, there is some node $\bar{s} \geq s$ such that $\bar{s} \notin T_1$ or $\bar{s} \notin T_2$. In this case, let $U^{(s)} = T_{\bar{s}}$. Let $\bar{T}$ be the tree T with each T_s replaced by $U^{(s)}$. Let $\bar{p}$ be the condition with $\bar{p}(\alpha) = (\bar{T}, \gamma)$ and $\bar{p}(\beta) = p(\beta)$ for every $\beta \neq \alpha$. Clearly, $\bar{p} \leq p$. Thus, conditions of the form $\bar{p}$ are dense in $\mathbb{Q}(\mathbb{P})^{<\kappa}$. Let $\bar{p} \in G$ be some such condition. Since $t \in T_\alpha$, $\bar{p}(\alpha) = (\bar{T}, \gamma)$ as constructed above. Since $(\mathcal{T}_\alpha)_t \subseteq T_1, T_2$ by assumption, it must have been the case that $(\mathcal{T}_\alpha)_s \subseteq T_s \subseteq T_1, T_2$. Thus, we have $T_s \subseteq T_1, T_2$ and $T_s \in H$.

□

Proposition 3.12. *Suppose $H^* \subseteq \mathbb{P}^{*<\kappa}$ is $V[G]$-generic. Then $H = H^* \cap \mathbb{P}$ is V-generic for $\mathbb{P}^{<\kappa}$.*

Proof. By Proposition 3.10, H meets every maximal antichain of $\mathbb{P}$ from V. Clearly, H is upward closed. So it remains to verify that any two conditions in H are compatible in H. For every $\alpha < \kappa$, let

$$H^*_\alpha = \{p(\alpha) \mid p \in H^*\}.$$

Then H^*_α is $V[G]$-generic for $\mathbb{P}^*$. Let H_α be the restriction of H^*_α to $\mathbb{P}$, which is V-generic by Proposition 3.11. Fix $p_1, p_2 \in H$, and observe that for every $\alpha < \kappa$, $p_1(\alpha), p_2(\alpha) \in H_\alpha$. For every $\alpha < \kappa$, let $p_\alpha \leq p_1(\alpha), p_2(\alpha)$ with $p_\alpha \in H_\alpha$, making sure to choose p_α to be $\mathbb{1}_\mathbb{P}$ whenever both $p_1(\alpha) = p_2(\alpha) = \mathbb{1}_\mathbb{P}$. Let $p \in \mathbb{P}$ be the condition with $p(\alpha) = p_\alpha$ (note that $p \in V$ by closure). We will argue that $p \in H^*$. It suffices to show that p is compatible with every condition $q \in H^*$. So fix $q \in H^*$. Suppose towards a contradiction that p is not compatible with q. Then there must be a coordinate α such that $p(\alpha) = p_\alpha$ is not compatible with $q(\alpha)$. Since $p_\alpha \in H_\alpha$, there is a condition $\bar{p} \in H^*$ such that $\bar{p}(\alpha) = p_\alpha$. But then $\bar{p}$ and q would be incompatible, which is the desired contradiction. □

4. κ-Jensen forcing at an inaccessible κ

In this section, we will construct a poset generalizing the Jensen poset $\mathbb{J}$ at an inaccessible cardinal κ. We will construct the κ-Jensen poset $\mathbb{J}(\kappa)$ in the constructible universe L (see Section 6 for how to generalize the construction beyond L). Following the construction in [Abr84], the Jensen poset $\mathbb{J}$ is built up as the union of an ω_1-length chain of countable perfect posets, where unions are taken at limit stages and maximal antichains are sealed at successor stages using a $\diamondsuit$-sequence. The κ-Jensen poset $\mathbb{J}(\kappa)$ will be analogously built up as the union of a κ^+-length chain of κ-perfect posets of size κ. The construction will use a $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$-sequence. The successor stages of the construction will be completely analogous to the construction of $\mathbb{J}$ from [Abr84]: based on the information provided by a $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$-sequence, we will either do nothing or grow our current poset $\mathbb{P}$ to $\mathbb{P}^*$ as constructed in a forcing extension by $\mathbb{Q}(\mathbb{P})^{<\kappa}$ of a carefully chosen transitive model M. At limit stages, we will take unions, but then close (if necessary) to obtain the weak union property and the $<\kappa$-intersection property.

For the duration of the construction, we let

$$\vec{D} = \langle D_\alpha \mid \alpha \in \mathrm{Cof}(\kappa) \cap \kappa^+ \rangle$$

be the canonical[1] $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$-sequence of L. At successor stages dictated by $\vec{D}$ (as explained below), we will force over sufficiently nice transitive models of size κ to obtain the next poset.

Definition 4.1. A *κ-suitable model* is a transitive model $M \models \mathrm{ZFC}^- +$"$\mathcal{P}(\kappa)$ exists" such that:

(1) $|M| = \kappa$,
(2) $M^{<\kappa} \subseteq M$,
(3) $M = L_\alpha$ for some α.

Natural examples of κ-suitable models M arise as Mostowski collapses of elementary substructures $\bar{M} \prec L_{\kappa^{++}}$ with $|\bar{M}| = \kappa$ and $\bar{M}^{<\kappa} \subseteq \bar{M}$. Note that if M is a κ-suitable model and $\delta = (\kappa^+)^M$, then $\vec{D} \restriction \delta = \langle D_\alpha \mid \alpha \in \mathrm{Cof}(\kappa) \cap \delta \rangle$ is an element of M because we chose to use the canonical sequence (note that M is correct about $\mathrm{Cof}(\kappa) \cap \delta$ since it is closed under sequences of length less than κ). Note, also, that any transitive model M that is closed under $<\kappa$-sequences must be correct about whether a poset $\mathbb{P} \in M$ is κ-perfect, and it must also correctly construct $\mathbb{Q}(\mathbb{P})$ and $\mathbb{Q}(\mathbb{P})^{<\kappa}$.

Next, we need the following two standard propositions, which are usually used for lifting elementary embeddings arising from large cardinals to forcing extensions.

Proposition 4.2. *Suppose that $M \models \mathrm{ZFC}^-$ is a transitive model of size κ such that $M^{<\kappa} \subseteq M$ and $\mathbb{P} \in M$ is a $<\kappa$-closed poset. Then there is an M-generic filter for $\mathbb{P}$.*

The proof is a generalization of the standard diagonalization argument to show that generic filters exist for countable models. The closure of M and $\mathbb{P}$ are used to get through limit stages.

Proposition 4.3. *Suppose that $M \models \mathrm{ZFC}^-$ is a transitive (set or class) model such that $M^{<\kappa} \subseteq M$ and $\mathbb{P} \in M$ is a poset. If $G \subseteq \mathbb{P}$ is M-generic, then $M[G]^{<\kappa} \subseteq M[G]$.*

The proof can be found, for example, in [GJ22] (Lemma 3.7).

Let M be a κ-suitable model and suppose that $\mathbb{P} \in M$ is κ-perfect. By Proposition 4.2, we can fix an M-generic filter $G \subseteq \mathbb{Q}(\mathbb{P})^{<\kappa}$. In $M[G]$, we can construct the poset $\mathbb{P}^*$. Then, by Proposition 3.9 (3) and Proposition 3.10, every maximal antichain $\mathcal{A}$ of $\mathbb{P}$ or $\mathbb{P}^{<\kappa}$ from M remains maximal in $\mathbb{P}^*$ or $\mathbb{P}^{*<\kappa}$ respectively. Since by Proposition 4.3, $M[G]^{<\kappa} \subseteq M[G]$, $\mathbb{P}^*$ is a κ-perfect poset extending $\mathbb{P}$. Finally, the proof of Proposition 3.8 shows that if $\mathbb{P}$ has the $<\kappa$-compatibility property, then the poset $\mathbb{P}^*$ has the $<\kappa$-compatibility property not just in $M[G]$, but fully in V.

The κ-Jensen poset $\mathbb{J}(\kappa)$ will be the union of an increasing chain $\langle \mathbb{P}_\alpha \mid \alpha < \kappa^+ \rangle$ of κ-perfect posets constructed as follows. Let $\mathbb{P}_0 = \mathbb{P}_{\min}$. Suppose that $\mathbb{P}_\alpha$ has been defined. We let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha$ unless the following happens. Suppose that $\alpha \in \mathrm{Cof}(\kappa) \cap \kappa^+$ and D_α codes an extensional and well-founded binary relation

[1] $\vec{D}$ is the sequence defined by letting $D_\alpha = D$ for the L-least pair (D, C) such that $D \subseteq \alpha$, C is a club in α and D is not guessed on any $\xi \in C \cap \mathrm{Cof}(\kappa)$ by $\langle D_\xi \mid \xi \in \mathrm{Cof}(\kappa) \cap \alpha \rangle$ if such a pair exists, or letting $D_\alpha = \alpha$ otherwise.

$E \subseteq \alpha \times \alpha$ such that the collapse of E is a κ-suitable model M_α with $\mathbb{P}_\alpha \in M_\alpha$ and $\alpha = (\kappa^+)^{M_\alpha}$. In this case, we take the L-least M_α-generic filter $G_\alpha \subseteq \mathbb{Q}(\mathbb{P}_\alpha)^{<\kappa}$ and let $\mathbb{P}_{\alpha+1} = \mathbb{P}^*_\alpha$ as constructed in $M[G_\alpha]$. At limit stages of cofinality κ, we simply take unions. So suppose that λ is a limit stage of cofinality $\gamma < \kappa$. We will construct the λ-stage poset $\mathbb{P}_\lambda$ in κ-many steps by closing under the $<\kappa$-intersection property and the weak union property. Let $\mathbb{P}_\lambda^{(0)} = \bigcup_{\xi<\lambda} \mathbb{P}_\xi$. Suppose inductively that $\mathbb{P}_\lambda^{(\xi)}$ has been defined. Let $\mathbb{P}_\lambda^{\prime(\xi+1)}$ be the collection of all trees $T \in \mathbb{P}_\lambda^{(\xi)}$ slimmed down by some $\vec{T} = \{T^{(s)} \mid s \in T \cap 2^\beta\} \subseteq \mathbb{P}_\lambda^{(\xi)}$, with β a successor ordinal. Let $\mathbb{P}_\lambda^{(\xi+1)}$ be the collection of all trees $T = \bigcap_{\alpha<\beta} T_\alpha$ for some $\subseteq$-decreasing sequence $\{T_\alpha \mid \alpha < \beta\} \subseteq \mathbb{P}_\lambda^{\prime(\xi+1)}$, with $\beta < \kappa$. At limit stages, we take unions. Let $\mathbb{P}_\lambda = \bigcup_{\xi<\kappa} \mathbb{P}_\lambda^{(\xi)}$. Clearly, $\mathbb{P}_\lambda$ is a κ-perfect poset extending $\bigcup_{\xi<\lambda} \mathbb{P}_\xi$. Thus, every $\mathbb{P}_\xi$ for $\xi < \kappa^+$ is a κ-perfect poset by construction. Let

$$\mathbb{J}(\kappa) = \mathbb{P}_{\kappa^+} = \bigcup_{\xi<\kappa^+} \mathbb{P}_\xi.$$

Thus, the following is clear.

Proposition 4.4. *The poset $\mathbb{J}(\kappa)$ is κ-perfect.*

Proposition 4.5. *For every $\beta < \alpha < \kappa^+$, $M_\beta[G_\beta] \in M_\alpha$.*

Proof. Fix $\beta < \alpha < \kappa^+$. Since $\alpha = (\kappa^+)^{M_\alpha}$, it follows that $\langle D_\xi \mid \xi \in \mathrm{Cof}(\kappa) \cap \alpha\rangle$ is an element of M_α, and hence $M_\beta \in M_\alpha$. Also, since G_β was chosen to be the L-least M_β-generic, it is definable from M_β, and hence must be in M_α as well. Thus, $M_\beta[G_\beta] \in M_\alpha$. □

Let $\mathcal{T}_\nu^{(\xi)}$, where $\xi < \kappa^+$, $\xi+1$ is a non-trivial stage, and $\nu < \kappa$, be the M_ξ-generic perfect κ-trees added in $M_\xi[G_\xi]$. Given a tree $T \in \mathbb{J}(\kappa)$, let us say that a level α of T is *nice* if for every $t \in T \cap 2^\alpha$, one of the following holds:

(1) There is a $\subseteq$-decreasing sequence $\{(\mathcal{T}_{\rho_\xi}^{(\mu_\xi)})_t \mid \xi < \nu\}$ for some $\nu < \kappa$ such that $T_t = \bigcap_{\xi<\nu} (\mathcal{T}_{\rho_\xi}^{(\mu_\xi)})_t$.
(2) There are $\mu < \kappa^+$ and $\rho < \kappa$ such that $T_t = (\mathcal{T}_\rho^{(\mu)})_t$.
(3) $T_t = (2^{<\kappa})_t$.

Note that in a $\subseteq$-decreasing sequence $\{(\mathcal{T}_{\rho_\xi}^{(\mu_\xi)})_t \mid \xi < \nu\}$, the indices μ_ξ must be weakly increasing because if $\mathcal{T}$ is the generic perfect κ-tree added by $\mathbb{Q}(\mathbb{P})$, then by density, we cannot have $T \subseteq \mathcal{T}$ for any $T \in \mathbb{P}$. Note also that if α is a nice level of a T, then every $\bar{\alpha} > \alpha$ is also a nice level of T.

Proposition 4.6. *Suppose that $\alpha < \kappa^+$ is such that $\alpha + 1$ is non-trivial successor stage and $\mathcal{A} \in M_\alpha$ is a maximal antichain of $\mathbb{P}_\alpha$. Then for every $\xi < \kappa$, there is a level β_ξ of $\mathcal{T}_\xi^{(\alpha)}$ such that for every node t on level β_ξ, there is $A_t \in \mathcal{A}$ such that $(\mathcal{T}_\xi^{(\alpha)})_t \subseteq A_t$.*

The proof is an easy modification of the proof of Proposition 3.9(3).

Lemma 4.7. *Every $T \in \mathbb{J}(\kappa)$ has a nice level.*

Lemma 4.7 will be proved simultaneously with Lemma 4.8 below by induction on $\xi < \kappa^+$ to show that every tree in $\mathbb{P}_\xi$ has a nice level.

Proof. By Proposition 2.13, every tree in $\mathbb{P}_{\min}$ has a nice level (according to our latest definition of "nice"). Suppose inductively that every tree in $\mathbb{P}_\xi$ has a nice level and $\xi+1$ is a non-trivial successor stage. Let T be a tree in $\mathbb{P}_{\xi+1}$. Let α' be a nice level of T. Fix a node t on level α'. Then either $T_t = (\mathcal{T}^{(\xi)}_\rho)_t$ or $T_t \in \mathbb{P}_\xi$ by Proposition 3.7. In the second case, let α_t be a nice level for T_t. Let $\alpha > \alpha'$ and $\alpha > \alpha_t$ for all t on level α' of T. Then clearly α is a nice level for T. Next, suppose that λ is a limit stage and for every $\xi < \lambda$, every tree in $\mathbb{P}_\xi$ has a nice level. If λ has cofinality κ, then clearly every tree in $\mathbb{P}_\lambda = \bigcup_{\xi<\lambda}\mathbb{P}_\xi$ has a nice level.

So suppose that λ has cofinality less than κ. By our inductive assumption, every tree in $\mathbb{P}^{(0)}_\lambda = \bigcup_{\xi<\lambda}\mathbb{P}_\xi$ has a nice level. Now suppose inductively that every tree in $\mathbb{P}^{(\xi)}_\lambda$ has a nice level. Let $S \in \mathbb{P}'^{(\xi+1)}_\lambda$. Then S is a tree $T \in \mathbb{P}^{(\xi)}_\lambda$ slimmed down by some $\vec{T} = \{T^{(s)} \mid s \in T\cap 2^\beta\} \subseteq \mathbb{P}^{(\xi)}_\lambda$, with β a some successor ordinal. For each tree $T^{(s)}$, let $\alpha_s > \beta$ be a nice level and let $\gamma > \beta$ be a nice level for T. Let α be above γ and α_s for every $s \in T\cap 2^\beta$. Then clearly α is a nice level for S. Next, suppose that $\{T_\eta \mid \eta < \beta\} \subseteq \mathbb{P}'^{(\xi+1)}_\lambda$, with $\beta < \kappa$, is a $\subseteq$-decreasing sequence of trees. Let α'_η be a nice level for T_η, and let $\alpha' > \alpha'_\eta$ for all $\eta < \beta$. Suppose that $\mu < \bar{\mu} < \lambda$ and $\mu+1, \bar{\mu}+1$ are non-trivial successor stages. By Lemma 4.8 applied below λ,

$$\mathcal{A} = \{\mathcal{T}^{(\mu)}_\rho \mid \rho < \kappa\} \in M_\mu[G_\mu] \subseteq M_{\bar{\mu}}$$

(Proposition 4.5) is a maximal antichain in $\mathbb{P}_{\bar{\mu}}$. Thus, by Proposition 4.6, for every $\bar{\rho} < \kappa$, there is a level $\beta_{\bar{\rho}}$ of $\mathcal{T}^{(\bar{\mu})}_{\bar{\rho}}$ such that for every node t on level $\beta_{\bar{\rho}}$, $(\mathcal{T}^{(\bar{\mu})}_{\bar{\rho}})_t \subseteq (\mathcal{T}^{(\mu)}_\rho)_t$ for some $\rho < \kappa$. Now consider the collection of all trees $\mathcal{T}^{(\mu)}_\rho$ such that for some $\eta < \beta$ and node t on level α' of T_η, $(T_\eta)_t = \bigcap_{\xi<\nu}(\mathcal{T}^{(\mu_\xi)}_{\rho_\xi})_t$ and $\mu = \mu_\xi$, $\rho = \rho_\xi$ for some $\xi < \nu$. Note that this collection must have size less than κ. So we can choose a large enough level $\alpha > \alpha'$ such that if $\mu < \bar{\mu}$, then for every node t on level α, it is decided whether $(\mathcal{T}^{(\bar{\mu})}_{\bar{\rho}})_t \subseteq (\mathcal{T}^{(\mu)}_\rho)_t$.

Let's argue that α is a nice level for T. Fix $t \in T\cap 2^\alpha$. Note that $T_t = \bigcap_{\eta<\beta}(T_\eta)_t$. By our choice of level α, each $(T_\eta)_t$ fits one of the criteria (1), (2), or (3) for a nice level. First, suppose that there are cofinally many η such that $(T_\eta)_t$ satisfies (1). By our assumption on the level α, we can take each of the decreasing sequences from the cofinally many η and intertwine them into a single decreasing sequence, resulting in T_t satisfying condition (1). If not, then there are boundedly many η such that $(T_\eta)_t$ satisfies (1). Next, suppose that there are cofinally many η such that $(T_\eta)_t$ satisfies (2). For each such η, let $(T_\eta)_t = (\mathcal{T}^{(\mu_\eta)}_{\rho_\eta})_t$. Note that the sequence of the μ_η must be weakly increasing. If the sequence stabilizes at some η, then $T_t = (\mathcal{T}^{(\mu_\eta)}_{\rho_\eta})_t$, and hence satisfies condition (1). Otherwise, T_t satisfies condition (2). Finally, if both cases (1) and (2) are bounded below η, then $T_t = (2^{<\kappa})_t$, which places it in (3). Thus, α is a nice level for T. □

Lemma 4.8. *Every maximal antichain $\mathcal{A} \in M_\alpha$ from $\mathbb{P}_\alpha$ remains maximal in $\mathbb{P}_\xi$ for every $\xi \leq \kappa^+$.*

Proposition 4.8 is proved simultaneously with Proposition 4.7 above by induction on $\xi < \kappa^+$.

Proof. By Proposition 3.9 (3), $\mathcal{A}$ remains maximal in $\mathbb{P}_{\alpha+1}$. So suppose inductively that $\mathcal{A}$ remains maximal in $\mathbb{P}_\xi$ for every $\alpha+1 \leq \xi < \lambda$. First, suppose that $\lambda = \eta+1$ is a non-trivial successor stage. Since $M_\alpha \subseteq M_\eta$, $\mathcal{A} \in M_\eta$ and we assumed inductively that it remains maximal in $\mathbb{P}_\eta$. Thus, $\mathcal{A}$ is maximal in $\mathbb{P}_\lambda$ by Proposition 3.9 (3). Next, suppose that λ is a limit of cofinality κ. In this case $\mathbb{P}_\lambda = \bigcup_{\xi<\lambda}\mathbb{P}_\xi$, and so clearly $\mathcal{A}$ remains maximal. Finally, suppose that λ is a limit of cofinality less than κ. Fix a tree $T\in\mathbb{P}_\lambda$. By Proposition 4.7, which we can assume holds at λ since its proof relies on the lemma holding below λ, we can let α be a nice level of T. Fix a node t on level α of T. Clearly, if $T_t = (\mathcal{T}^{(\mu)}_{\rho_\xi})_t$ for some $\mu<\lambda$ or $T_t\in\mathbb{P}_{\min}$, then T_t is compatible with some tree in $\mathcal{A}$. If $T_t = \bigcap_{\xi<\nu}(\mathcal{T}^{(\mu_\xi)}_{\rho_\xi})_t$, where the μ_ξ are bounded below λ, then $T_t\in\mathbb{P}_{\bar\lambda}$ for some $\bar\lambda<\lambda$ by the $<\kappa$-closure property. So by our inductive assumption, T_t is compatible with some tree in $\mathcal{A}$. So suppose that $T_t = \bigcap_{\xi<\nu}(\mathcal{T}^{(\mu_\xi)}_{\rho_\xi})_t$, where the μ_ξ are unbounded below λ. Fix $\alpha\leq\mu_\xi<\lambda$. Then $\mathcal{A}$ remains maximal in $\mathbb{P}_{\mu_\xi}$ by our inductive assumption. By Lemma 4.6, there is a $\beta<\kappa$ such that for every node s on level β of $\mathcal{T}^{(\mu_\xi)}_{\rho_\xi}$, there is $A_s\in\mathcal{A}$ such that $(\mathcal{T}^{(\mu_\xi)}_{\rho_\xi})_s\subseteq A_s$, and we can assume without loss of generality that $\beta>\alpha$. Fix a node s on level β of T_t. Then $T_s\subseteq(\mathcal{T}^{(\mu_\xi)}_{\rho_\xi})_s\subseteq A_s$, and hence T_t is compatible with $A_s\in\mathcal{A}$. □

In particular, we get that each antichain $\{\mathcal{T}^{(\alpha)}_\xi \mid \xi<\kappa\}$ is maximal in $\mathbb{J}(\kappa)$.

Proposition 4.9. *Every maximal antichain of $\mathbb{P}^{<\kappa}_\alpha$ from M_α remains maximal in $\mathbb{P}^{<\kappa}_\xi$ for every $\xi\leq\kappa^+$.*

Proof. Fix a maximal antichain $\mathcal{A}\in M_\alpha$ of $\mathbb{P}^{<\kappa}_\alpha$. It clearly suffices to argue that $\mathcal{A}$ remains maximal in $\mathbb{P}^{<\kappa}_\lambda$ for λ of cofinality less than κ. Fix a condition $p\in\mathbb{P}^{<\kappa}_\lambda$, with $p(\xi)=T_\xi$, and let its domain be contained in $\beta<\kappa$. Let γ be large enough that γ is a nice level for every tree T_ξ with $\xi<\beta$. For each $\xi<\beta$, choose a node t_ξ on level γ of T_ξ. We would like to argue that we can thin out each $(T_\xi)_{t_\xi}$ to a tree $S_\xi\subseteq\mathcal{T}^{(\alpha)}_{\rho_\xi}$ for some $\rho_\xi<\kappa$. If $(T_\xi)_{t_\xi}=(2^{<\kappa})_{t_\xi}$ or $(T_\xi)_{t_\xi}=(\mathcal{T}^{(\eta)}_\rho)_{t_\xi}$ for some $\eta<\alpha$ and $\rho<\kappa$, then $(T_\xi)_{t_\xi}\in\mathbb{P}_\alpha$, and so by density, there is some $\mathcal{T}^{(\alpha)}_{\rho_\xi}$ such that $\mathcal{T}^{(\alpha)}_{\rho_\xi}\subseteq(T_\xi)_{t_\xi}$, and hence we can let $S_\xi=\mathcal{T}^{(\alpha)}_{\rho_\xi}$. If $(T_\xi)_{t_\xi}=(\mathcal{T}^{(\alpha)}_\rho)_{t_\xi}$ for some $\rho<\kappa$, then we can let $S_\xi=(T_\xi)_{t_\xi}$. Finally, assume that $(T_\xi)_{t_\xi}=(\mathcal{T}^{(\eta)}_\rho)_{t_\xi}$ for some $\eta>\alpha$ and $\rho<\kappa$ or $(T_\xi)_{t_\xi}=\bigcap_{\eta<\nu}(\mathcal{T}^{(\eta)}_{\rho_\eta})_{t_\xi}$ for some $\subseteq$-decreasing sequence of the trees $\mathcal{T}^{(\eta)}_{\rho_\xi}$ with $\rho_\xi<\kappa$. In either case, there is some $\eta>\alpha$ such that $(T_\xi)_{t_\xi}\subseteq(\mathcal{T}^{(\eta)}_\rho)_{t_\xi}$. By Proposition 4.6, there is some $t>t_\xi$ and $\nu<\kappa$ such that $(\mathcal{T}^{(\eta)}_\rho)_t\subseteq\mathcal{T}^{(\alpha)}_\nu$. Thus, $(T_\xi)_t\subseteq\mathcal{T}^{(\alpha)}_\nu$, and we can let $S_\xi=(T_\xi)_t$. Thus, by strengthening the condition p, if necessary, we can assume that for every $\xi<\beta$, $p(\xi)\subseteq\mathcal{T}^{(\alpha)}_{\rho_\xi}$ for some $\rho_\xi<\kappa$.

Let
$$X=\{\rho_\xi \mid \xi<\beta\}.$$
Fix a condition $q\in\mathbb{Q}(\mathbb{P}_\alpha)^{<\kappa}$ with $q(\eta)=(S_\eta,\gamma_\eta)$. Enumerate all sequences
$$\{\vec{t}^{\,\nu} \mid \nu<\delta\},$$
with $\delta<\kappa$, such that $\vec{t}^{\,\nu}: X\to\bigcup_{\eta<\beta}S_\eta$ and $\vec{t}^{\,\nu}(\eta)$ is a node on level γ_η of S_η. Let $r_0\in\mathbb{P}^{<\kappa}_\alpha$ be the condition with $r_0(\xi)=(S_{\rho_\xi})_{\vec{t}^{\,0}(\rho_\xi)}$ for every $\xi<\beta$ (and trivial otherwise). Since $\mathcal{A}$ is maximal in $\mathbb{P}^{<\kappa}_\alpha$, there is a condition $a_0\in\mathcal{A}$ compatible with r_0. Let $r_0'\leq r_0,a_0$. For every $\rho_\xi\in X$, let $S^0_{\rho_\xi}$ be the tree we get by replacing $(S_{\rho_\xi})_{\vec{t}^{\,0}(\rho_\xi)}$ with $r'(\xi)$ in S_{ρ_ξ}. Let q_0 be the condition such that for every $\rho_\xi\in X$, $q_0(\rho_\xi)=(S^0_{\rho_\xi},\gamma_{\rho_\xi})$, and otherwise $q_0(\eta)=q(\eta)$. Suppose we

have defined a descending sequence of conditions q_ν, with $q_\nu(\eta) = (S^\nu_\eta, \gamma_\eta)$ for some $\nu < \mu < \delta$. First, for each $\eta < \kappa$, we let $(S^\mu_\eta)' = \bigcap_{\nu<\mu} S^\nu_\eta$. Next, we define q'_μ by $q'_\mu(\eta) = ((S^\mu_\eta)', \gamma_\eta)$. Clearly, $q'_\mu \leq q_\nu$ for all $\nu < \mu$. At this stage we take care of the sequence of nodes $\vec{t}^\mu$. Let $r_\mu \in \mathbb{P}^{<\kappa}_\alpha$ be the condition with $r_\mu(\xi) = (S^\mu_{\rho_\xi})'_{\vec{t}^\mu(\rho_\xi)}$ for every $\beta < \xi$. Since $\mathcal{A}$ is maximal in $\mathbb{P}^{<\kappa}_\alpha$, there is a condition $a_\mu \in \mathcal{A}$ compatible with r_μ. Let $r'_\mu \leq r_\mu, a_\mu$. For every $\rho_\xi \in X$, let $S^\mu_{\rho_\xi}$ be the tree we get by replacing $(S^\mu_{\rho_\xi})'_{\vec{t}^\mu(\rho_\xi)}$ with $r'_\mu(\xi)$ in $(S^\mu_{\rho_\xi})'$. Let q_μ be the condition such that for every $\rho_\xi \in X$, $q_\mu(\rho_\xi) = (S^\mu_{\rho_\xi}, \gamma_{\rho_\xi})$ and otherwise $q_\mu(\eta) = q'_\mu(\eta)$. After δ-many steps, we end up with the descending sequence of conditions q_ν, with $q_\nu(\eta) = (S^\nu_\eta, \gamma_\eta)$, for $\nu < \delta$. We let $S^\delta_\eta = \bigcap_{\nu<\delta} S^\nu_\eta$. Next, we define q_δ by $q_\delta(\eta) = (S^\delta_\eta, \gamma_\eta)$. Clearly, $q_\delta \leq q_\nu$ for all $\nu < \delta$. Thus, conditions q_δ are dense in $\mathbb{P}^{<\kappa}_\alpha$. So some such $q_\delta \in G_\alpha$.

For every $\xi < \beta$, since $p(\xi) \subseteq \mathcal{T}^{(\alpha)}_{\rho_\xi}$, there must be some node t_ξ on level γ_{ρ_ξ} of $\mathcal{T}^{(\alpha)}_{\rho_\xi}$ such that $p(\xi)_{t_\xi} \subseteq (\mathcal{T}^{(\alpha)}_{\rho_\xi})_{t_\xi}$. Let $\vec{t}$ be defined by $\vec{t}(\xi) = t_\xi$. Since $M^{<\kappa}_\alpha \subseteq M_\alpha$, it follows that $\vec{t} \in M_\alpha$. Thus, $\vec{t} = \vec{t}^\nu$ for some $\nu < \delta$. It follows by our construction of q_δ that $(\mathcal{T}^{(\alpha)}_{\rho_\xi})_{t_\xi} \subseteq (S^\delta_{\rho_\xi})_{t_\xi} \subseteq a_\nu(\xi)$. Thus, in particular, $p(\xi) \subseteq a_\nu(\xi)$ for all $\xi < \beta$, and so p is compatible with $a_\nu \in \mathcal{A}$.

□

Proposition 4.10. *Each poset $\mathbb{P}_\xi$, for $\xi < \kappa^+$, has the $<\kappa$-compatibility property. In particular, the poset $\mathbb{J}(\kappa)$ has the $<\kappa$-compatibility property.*

Proof. By Proposition 2.14, $\mathbb{P}_0$ has the $<\kappa$-compatibility property. Suppose inductively that for all $\xi < \lambda$, $\mathbb{P}_\xi$ has the $<\kappa$-compatibility property. If $\lambda = \xi + 1$ for some ξ, then $\mathbb{P}_\lambda$ has the $<\kappa$-compatibility property by Proposition 3.8. If λ is a limit of cofinality κ, then it is obvious that $\mathbb{P}_\lambda$ has the $<\kappa$-compatibility property. So suppose that λ is a limit of cofinality less than κ.

Fix a collection $\{T_\xi \mid \xi < \beta\} \subseteq \mathbb{P}_\lambda$, with $\beta < \kappa$, and suppose that there is a perfect κ-tree $S \subseteq \bigcap_{\xi<\beta} T_\xi$. We need to argue that there is a tree $R \in \mathbb{P}_\lambda$ such that $R \subseteq \bigcap_{\xi<\beta} T_\xi$. Let α' be large enough so that it is a nice level for every T_ξ with $\xi < \beta$. Now consider the collection of all trees $\mathcal{T}^{(\mu)}_\rho$ such that for some $\xi < \beta$ and node t on level α' of T_ξ, (1) $(T_\xi)_t = \mathcal{T}^{(\mu)}_\rho$ or (2) $(T_\xi)_t = \bigcap_{\eta<\nu} (\mathcal{T}^{(\mu_\eta)}_{\rho_\eta})_t$ and $\mu = \mu_\eta$ and $\rho = \rho_\eta$ for some $\eta < \nu < \kappa$. Note that this collection must have size less than κ. By Proposition 4.6, we can choose a large enough level $\alpha > \alpha'$ such that if $\mu < \bar{\mu}$, then for every node t on level α it is decided whether whether $(\mathcal{T}^{(\bar{\mu})}_{\bar{\rho}})_t \subseteq (\mathcal{T}^{(\mu)}_\rho)_t$. Fix a node t on level α of S. Then $S_t \subseteq \bigcap_{\xi<\beta} (T_\xi)_t$, meaning that the intersection contains a perfect κ-tree. By our choice of level α and the fact that the intersection contains a perfect κ-tree, we can intertwine all the trees mentioned in forms (1) or (2) of the trees $(T_\xi)_t$ into a $\subseteq$-decreasing sequence of the form $\{\mathcal{T}^{(\mu_\eta)}_{\rho_\eta} \mid \eta < \gamma\}$ with $\gamma < \kappa$. Thus, this intersection is an element of $\mathbb{P}_\lambda$ by construction. □

Theorem 4.11. *The bounded support κ-length product $\mathbb{J}(\kappa)^{<\kappa}$ of $\mathbb{J}(\kappa)$ has the κ^+-cc. In particular, $\mathbb{J}(\kappa)$ must have the κ^+-cc.*

Proof. Fix a maximal antichain $\mathcal{A}$ of $\mathbb{J}(\kappa)^{<\kappa}$. Choose a transitive model $M \prec L_{\kappa^{++}}$ of size κ^+ with $\mathcal{A} \in M$. We can decompose M as the union of a continuous elementary chain of length κ^+ of substructures of size κ,

$$X_0 \prec X_1 \prec \cdots \prec X_\xi \prec \cdots \prec M,$$

with $\mathcal{A} \in X_0$, such that each successor stage $X_{\xi+1}$ is closed under sequences of length less than κ, $X_{\xi+1}^{<\kappa} \subseteq X_{\xi+1}$. It will follow that each X_α for $\alpha \in \mathrm{Cof}(\kappa)$ is closed under sequences of length less than κ, $X_\alpha^{<\kappa} \subseteq X_\alpha$. By properties of $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$, there must be some $\alpha \in \mathrm{Cof}(\kappa)$ such that $\alpha = \kappa^+ \cap X_\alpha$, $\mathbb{P}_\alpha = \mathbb{J}(\kappa) \cap X_\alpha$ and S_α codes X_α. Let M_α be the transitive collapse of X_α. Then $\mathbb{P}_\alpha$ is the image of $\mathbb{J}(\kappa)$ under the Mostowski collapse and α is the image of κ^+. Let $\bar{\mathcal{A}} = \mathcal{A} \cap X_\alpha$ be the image of $\mathcal{A}$ under the collapse. So at stage α in the construction of $\mathbb{J}(\kappa)$, we chose a forcing extension $M_\alpha[G_\alpha]$ of M_α by $\mathbb{Q}(\mathbb{P}_\alpha)^{<\kappa}$ and let $\mathbb{P}_{\alpha+1} = \mathbb{P}^*_\alpha$ as constructed in $M_\alpha[G_\alpha]$. Thus, $\bar{\mathcal{A}}$ remains maximal in $\mathbb{J}(\kappa)^{<\kappa}$ by Proposition 4.9, and so it must have been the case that $\bar{\mathcal{A}} = \mathcal{A}$, verifying that $\mathcal{A}$ has size κ. □

Proposition 4.12. *Suppose $H \subseteq \mathbb{J}(\kappa)$ is L-generic. Then for any $\alpha < \kappa$ such that M_α is defined, the restriction H_α of H to $\mathbb{P}_\alpha$ is M_α-generic.*

Proof. Fix an α such that M_α is defined. Since every maximal antichain of $\mathbb{P}_\alpha$ from M_α remains maximal in $\mathbb{J}(\kappa)$, it suffices to check that for every $T, S \in H_\alpha$, there is $R \in H_\alpha$ such that $R \subseteq T, S$. First, observe that the collection of all trees R in $\mathbb{P}_\alpha$ such that either $R \subseteq S \cap T$ or R is incompatible with either S or T is dense in $\mathbb{P}_\alpha$. Thus, there is a maximal antichain $\mathcal{A}$ of $\mathbb{P}_\alpha$ such that if $R \in \mathcal{A}$, then either $R \subseteq T \cap S$ or R is incompatible with either S or T. Since $\mathcal{A}$ remains maximal in $\mathbb{J}(\kappa)$, H meets $\mathcal{A}$. Thus, there is $R \in H_\alpha$ such that either $R \subseteq S \cap T$ or R is incompatible with one of them. Now observe that if R is incompatible, say, with S, then, by the $<\kappa$-compatibility property of $\mathbb{P}_\alpha$ (Proposition 4.10), there cannot be perfect κ-tree in the intersection of R and S. Thus, since $R, S, T \in H$, R must be compatible with both S and T, and hence $R \subseteq S, T$.

□

Proposition 4.13. *Suppose $H \subseteq \mathbb{J}(\kappa)^{<\kappa}$ is L-generic. Then for any $\alpha < \kappa$ such that M_α is defined, the restriction H_α of H to $\mathbb{P}_\alpha^{<\kappa}$ is M_α-generic.*

Proof. Again, it suffices to argue that for any $p, q \in H_\alpha$, there is $r \in H_\alpha$ such that $r \leq p, q$. For every $\beta < \kappa$, let $H^{(\beta)}$ be the restriction of H to the β-th coordinate of the product. Then each $H^{(\beta)}$ is L-generic for $\mathbb{J}(\kappa)$. So by Proposition 4.12, each restriction $H^{(\beta)}_\alpha$ of $H^{(\beta)}$ to $\mathbb{P}_\alpha$ is M_α-generic. Also, for each $\beta < \kappa$, $p(\beta), q(\beta) \in H^{(\beta)}_\alpha$. Thus, for each $\beta < \kappa$, there is $r_\beta \in H^{(\beta)}_\alpha$ such that $r_\beta \leq p(\beta), q(\beta)$. Let $r \in \mathbb{P}_\alpha^{<\kappa}$ be defined by $r(\beta) = r_\beta$. Clearly, $r \leq p, q$. We claim that $r \in H$. If not, then there is some $\bar{q} \in H$ such that r is incompatible with $\bar{q}$. This means that there is $\beta < \kappa$ such that $r(\beta)$ is incompatible with $\bar{q}(\beta)$. But this is impossible because $r(\beta) = r_\beta$ and $\bar{q}(\beta)$ are both in $H^{(\beta)}_\alpha$. Thus, $r \in H$ is as desired.

□

5. The Kanovei-Lyubetsky Theorem for $\mathbb{J}(\kappa)^{<\kappa}$

Jensen showed that the poset $\mathbb{J}$ adds a unique generic real over L [Jen70]. As, in the introduction, let $\mathbb{J}^{<\omega}$ be the bounded support ω-length product of the poset $\mathbb{J}$. Lyubetsky and Kanovei extended the unique generics property of Jensen's forcing to $\mathbb{J}^{<\omega}$ by showing that if $G \subseteq \mathbb{J}^{<\omega}$ is L-generic, then the only $\mathbb{J}$-generic reals in $L[G]$ are the (reals obtained from the) slices G_n, for $n < \omega$, of the generic filter G [KL17]. In this section we will show that an appropriate generalization of the Kanovei-Lyubetsky theorem holds for $\mathbb{J}(\kappa)^{<\kappa}$: if $G \subseteq \mathbb{J}(\kappa)^{<\kappa}$ is L-generic, then the only $\mathbb{J}(\kappa)$-generic subsets of κ in $L[G]$ are the (subsets obtained from the) slices G_ξ, for $\xi < \kappa$, of the generic filter G. In particular, this will imply that $\mathbb{J}(\kappa)$ adds a unique generic subset over L.

Suppose that $\mathbb{P}$ is a κ-perfect poset and H is a V-generic filter for the product $\mathbb{P}^{<\kappa}$. We will call h_ξ the subset of κ added by the ξ-th coordinate of H and let $\dot{h}_\xi$ be the canonical name for h_ξ.

For the next theorem we suppose that $\mathbb{P}$ is a κ-perfect poset of size κ that is an element of a κ-suitable model M. We should think of $\mathbb{P}$ as one of those κ-perfect posets $\mathbb{P}_\alpha$ arising at stage α, for a non-trivial successor stage $\alpha+1$, in the construction of the poset $\mathbb{J}(\kappa)$ and of M as the model M_α from that stage.

Theorem 5.1. *In M, suppose that $\dot{h}$ is a $\mathbb{P}^{<\kappa}$-name for a subset of κ such that for all $\xi < \kappa$, $\mathbb{1}_{\mathbb{P}^{<\kappa}} \Vdash \dot{h} \neq \dot{h}_\xi$. Then in every forcing extension $M[G]$ by $\mathbb{Q}(\mathbb{P})^{<\kappa}$, for every generic perfect κ-tree $\mathcal{T}_\xi$, with $\xi < \kappa$, conditions forcing that $\dot{h} \notin [\mathcal{T}_\xi]$ are dense in $\mathbb{P}^{*<\kappa}$.*

Proof. Fix a condition $p \in \mathbb{P}^{*<\kappa}$ and $\gamma < \kappa$. We need to argue that there is a condition $p' \leq p$ such that $p' \Vdash \dot{h} \notin [\mathcal{T}_\gamma]$. Since $\mathbb{U}^{<\kappa}$ is dense in $\mathbb{P}^{*<\kappa}$, we can assume, by strengthening p, if necessary, that $p \in \mathbb{U}^{<\kappa}$. Let d be the domain of p. For every $\eta \in d$, let $p(\eta) = (\mathcal{T}_{\xi_\eta})_{t_\eta}$. By strengthening p further, if necessary, we can assume that for every $\alpha < \kappa$ and η in the domain of p, the nodes t_η for $\xi_\eta = \alpha$ form an antichain. Observe that even though the condition p is not in M, the sequences $\{\xi_\eta \mid \eta \in d\}$ and $\{t_\eta \mid \eta \in d\}$ are in M by closure. We will use both sequences in the construction that takes place in M below.

Fix $q \in \mathbb{Q}(\mathbb{P})^{<\kappa}$, with $q(\alpha) = (T_\alpha, \rho_\alpha)$. By strengthening q, if necessary, we can assume that for every $\alpha < \kappa$, if $\alpha = \xi_\eta$, then ρ_α is above the level of t_η. If there is $\alpha = \xi_\eta$ such that $t_\eta \notin T_\alpha$, then let $\bar{q} = q$. So assume that for every $\alpha = \xi_\eta$, $t_\eta \in T_\alpha$. For every $\eta \in d$, choose some node t'_η on level ρ_{ξ_η} of T_{ξ_η}.

For every $\eta \in d$ such that $\xi_\eta = \gamma$, we let $\eta_{t'_\eta} = \eta$. Let $a_0 \in \mathbb{P}^{<\kappa}$ be a condition such that:

1. For every $\eta \in d$, $a_0(\eta) = (T_{\xi_\eta})_{t'_\eta}$.
2. For every node s on level ρ_γ of T_γ, if there is no η in the domain of p such that $\xi_\eta = \gamma$ and $t'_\eta = s$, then there is η_s in the domain of a_0 such that $a_0(\eta_s) = (T_\gamma)_s$.

Let δ be above the domain of a_0. By assumption $\mathbb{1}_{\mathbb{P}^{<\kappa}} \Vdash \dot{h} \neq \dot{h}_\xi$ for every $\xi < \kappa$. So there is a condition $a_1 \in \mathbb{P}^{<\kappa}$ such that $a_1 \leq a_0$ and

$$a_1 \Vdash_{\mathbb{P}^{<\kappa}} \dot{h} \notin [a_1(0)].$$

More precisely there is a condition $a'_0 \leq a_0$ which decides for some node $s \in 2^{<\kappa}$ that $s \in \dot{h}_0$ and $s \notin \dot{h}$, and then we can choose a condition $a_1 \leq a'_0$ such that $s \notin a_1(0)$.

Next, we let $a_2 \leq a_1$ be some condition such that

$$a_2 \Vdash_{\mathbb{P}^{<\kappa}} \dot{h} \notin [a_2(1)].$$

Continuing in this manner, we construct a descending sequence of conditions

$$\{a_\xi \mid \xi < \delta\}$$

such that lower bounds are taken at limit stages, using $<\kappa$-closure, and at successor stages $a_{\xi+1} \Vdash_{\mathbb{P}^{<\kappa}} \dot{h} \notin [a_{\xi+1}(\xi)]$. Let a be a lower bound of the a_ξ for $\xi < \delta$. Clearly, for every $\xi < \delta$,

$$a \Vdash_{\mathbb{P}^{<\kappa}} \dot{h} \notin [a(\xi)].$$

For every $\eta \in d$, we let $R^{(\eta)} = a(\eta)$. For every node s on level ρ_γ of T_γ, we let $R^{(\eta_s)} = a(\eta_s)$. Fix $\alpha < \kappa$, with $\alpha \neq \gamma$. We let $\bar{T}_\alpha$ be the tree T_α, where for every $\xi_\eta = \alpha$, $(T_\alpha)_{t'_\eta}$ is replaced by $R^{(\eta)}$. Next, we let $\bar{T}_\gamma$ be the tree T_γ, where $(T_\gamma)_{t'_\eta}$ is replaced by $R^{(\eta)}$ for every $\xi_\eta = \gamma$, and we additionally replace $(T_\gamma)_{\eta_s}$ with $R^{(\eta_s)}$ for every node s on level ρ_γ. Let $\bar{q}$ be the condition such that for every $\eta \in d$, $\bar{q}(\xi_\eta) = (\bar{T}_{\xi_\eta}, \rho_\eta)$, $\bar{q}(\gamma) = (\bar{T}_\gamma, \rho_\gamma)$, and for the rest of the $\xi < \kappa$, $\bar{q}(\xi) = q(\xi)$. Clearly, $\bar{q} \leq q$. Thus, conditions of the form $\bar{q}$ are dense in $\mathbb{Q}(\mathbb{P})^{<\kappa}$, and hence some such $\bar{q} \in G$. It follows that for every $\eta \in d$, $(\mathcal{T}_{\xi_\eta})_{t'_\eta} \leq a(\eta)$. Thus, p and a are compatible. Let $p' \leq p, a$. Now we argue that $a \Vdash_{\mathbb{P}^{*<\kappa}} \dot{h} \notin [\mathcal{T}_\gamma]$.

Let H^* be any $M[G]$-generic filter for $\mathbb{P}^{*<\kappa}$ containing a. By Proposition 3.12, it follows that H^* restricts to an M-generic filter H for $\mathbb{P}^{<\kappa}$. Obviously, $a \in H$. Thus, in $M[H]$, $\dot{h}_H$ is not a branch through any $a(\eta)$ for $\eta < \delta$. In particular, for every node s on level ρ_γ of $\mathcal{T}_\gamma$, $\dot{h}_H$ is not a branch through $(\mathcal{T}_\gamma)_s$, and hence $\dot{h}_H$ is not a branch through $\mathcal{T}_\gamma$ in $M[H]$. But this is absolute, and therefore $\dot{h}_H$ is not a branch through $\mathcal{T}_\gamma$ in $M[G][H^*]$.

Thus, it follows that $p' \Vdash \dot{h} \notin [\mathcal{T}_\gamma]$, completing the proof.

□

Theorem 5.2. *Suppose that $H \subseteq \mathbb{J}(\kappa)^{<\kappa}$ is L-generic. If $h \in L[H]$ is L-generic for $\mathbb{J}(\kappa)$, then $h = h_\xi$ for some $\xi < \kappa$.*

Proof. Suppose towards a contradiction that h is not one of the h_ξ. Let $\dot{h}$ be a nice $\mathbb{J}(\kappa)^{<\kappa}$-name for h such that for all $\xi < \kappa$, $\mathbb{1}_{\mathbb{J}(\kappa)^{<\kappa}} \dot{h} \neq \dot{h}_\xi$.

Choose some transitive model $M \prec L_{\kappa^{++}}$ of size κ^+ with $\dot{h} \in M$. We can decompose M as the union of a continuous elementary chain of structures of size κ

$$X_0 \prec X_1 \prec \cdots \prec X_\xi \prec \cdots \prec M,$$

with $\dot{h} \in X_0$, such that each successor stage $X_{\xi+1}$ is closed under sequences of length less than κ, $X_{\xi+1}^{<\kappa} \subseteq X_{\xi+1}$. By properties of $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$, there is some $\alpha \in \mathrm{Cof}(\kappa) \cap \kappa^+$ such that $\alpha = \kappa^+ \cap X_\alpha$, $\mathbb{P}_\alpha = \mathbb{J}(\kappa) \cap X_\alpha$, and D_α codes X_α. Let M_α be the Mostowski collapse of X_α. Then $\mathbb{P}_\alpha$ is the image of $\mathbb{J}(\kappa)$ under the collapse and α is the image of κ^+. Clearly $\dot{h}$ is fixed by the collapse because it is a nice name and all antichains of $\mathbb{J}(\kappa)^{<\kappa}$ have size $\leq\kappa$ (by Theorem 11.8). So at stage α in the construction of $\mathbb{J}(\kappa)$, we chose a forcing extension $M_\alpha[G_\alpha]$ by $\mathbb{Q}(\mathbb{P}_\alpha)^{<\kappa}$ and let $\mathbb{P}_{\alpha+1} = \mathbb{P}^*_\alpha$ as constructed in M_α. By elementarity, M_α satisfies that $\mathbb{1}_{\mathbb{P}_\alpha^{<\kappa}} \Vdash \dot{h} \neq \dot{h}_\xi$ for all $\xi < \kappa$. Thus, by Theorem 5.1, for every $\xi < \kappa$, $\mathbb{P}_{\alpha+1}^{<\kappa}$ has a maximal antichain $\mathcal{A}_\xi \in M_\alpha[G_\alpha]$ consisting of conditions q such that $q \Vdash_{\mathbb{P}_{\alpha+1}^{<\kappa}} \dot{h} \notin [\mathcal{T}_\xi^{(\alpha)}]$. By Proposition 4.9, all maximal antichains $\mathcal{A}_\xi$, as well as the antichain $\{\mathcal{T}_\xi^{(\alpha)} \mid \xi < \kappa\}$, remain maximal in $\mathbb{J}(\kappa)^{<\kappa}$.

So let's argue that if $q \in \mathcal{A}_\xi$, then $q \Vdash_{\mathbb{J}(\kappa)^{<\kappa}} \dot{h} \notin [\mathcal{T}_\xi^{(\alpha)}]$. Let $\bar{H} \subseteq \mathbb{J}(\kappa)^{<\kappa}$ be an L-generic filter containing q. Let $\beta + 1 > \alpha + 1$ be the first non-trivial successor stage so that M_β is defined. By Proposition 4.13, $\bar{H}$ restricts to an M_β-generic filter $\bar{H}_\beta$ for $\mathbb{P}_\beta^{<\kappa}$, but in this case, we have $\mathbb{P}_\beta = \mathbb{P}_{\alpha+1}$. Thus, since $M_\alpha[G_\alpha] \subseteq M_\beta$ by Proposition 4.5, $\bar{H}_\beta$ is $M_\alpha[G_\alpha]$-generic. Since $q \in \bar{H}_\beta$, it follows that $\dot{h}_{\bar{H}_\beta} \notin [\mathcal{T}_\xi^{(\alpha)}]$ holds in $M_\alpha[G_\alpha][\bar{H}_\beta]$, but this statement is absolute, and so it also holds in $L[\bar{H}]$.

Since H must meet every maximal antichain $\mathcal{A}_\xi$, it holds in $L[H]$ that $\dot{h}_H = h$ is not a branch through any $\mathcal{T}^{(\alpha)}_\xi$. So, since $\{\mathcal{T}^{(\alpha)}_\xi \mid \xi < \kappa\}$ is a maximal antichain, h cannot be L-generic for $\mathbb{J}(\kappa)$. □

Corollary 5.3. *The poset $\mathbb{J}(\kappa)$ adds a unique generic subset of κ.*

6. Jensen forcing outside the constructible universe

While $\diamondsuit$ is crucial to the definition of Jensen's poset $\mathbb{J}$, working in the constructible universe L is not. We will describe below a general construction of a poset, which we will still refer to as $\mathbb{J}$, with the properties of Jensen's poset in any universe V having a $\diamondsuit$-sequence. The more general poset $\mathbb{J}$ will still add a unique real and have the ccc. The only property we lose by not working in L is that the unique generic real may no longer be a Π^1_2-singleton.

Fix a $\diamondsuit$-sequence $\vec{D} = \langle D_\alpha \mid \alpha < \omega_1\rangle$. Let $\mathcal{M}$ be the collection of all countable transitive models $M \models \text{ZFC}^-$. Given some $M \in \mathcal{M}$, let $\mathcal{G}_M$ be a countable set of subsets of M such that for every poset $\mathbb{P} \in M$, $\mathcal{G}_M$ has an M-generic filter for $\mathbb{P}$. Let f be a function mapping each M in $\mathcal{M}$ to $\mathcal{G}_M$. Note that $f \in H_{\omega_2}$. Suppose that we are at stage α in the construction of the poset $\mathbb{J}$ and we have already defined $\mathbb{P}_\alpha$. If D_α codes a set X_α such that the Mostowski collapse of X_α is a model $M_\alpha \models \text{ZFC}^- +$ "$\mathcal{P}(\omega)$ exists" such that

1. $\alpha = (\omega_1)^{M_\alpha}$
2. $\mathbb{P}_\alpha \in M_\alpha$,
3. $\langle D_\xi \mid \xi < \alpha\rangle \in M_\alpha$,
4. $f \restriction (M_\alpha \cap \mathcal{M}) \in M_\alpha$,

then we choose some $G_\alpha \in \mathcal{G}_{M_\alpha}$, and let $\mathbb{P}_{\alpha+1}$ be $\mathbb{P}^*_\alpha$ as constructed in $M_\alpha[G_\alpha]$. Assumptions (3) and (4) ensure that $M_\beta, M_\beta[G_\beta] \in M_\alpha$ for $\beta < \alpha$.

Now it suffices to observe that given some $f, \vec{D}, \mathbb{P}^J \in N \prec H_{\omega_2}$ with $|N| = \omega_1$, if we decompose N as the union of a chain of countable elementary substructures

$$X_0 \prec X_1 \cdots \prec \cdots X_\alpha \prec \cdots \prec N$$

with $f, \vec{D}, \mathbb{P}^J \in X_0$, then there is some X_α whose collapse M_α satisfies the properties above. But this is clearly true, by applying $\diamondsuit$ to a subset of ω_1 coding $\mathbb{J}$, $\vec{D}$, and f. This will ensure that we can still argue that $\mathbb{J}$ has the ccc as well as a unique generic.

The analysis carries over to the case of the forcing $\mathbb{J}(\kappa)$, which can similarly be defined outside of L provided that the universe has a $\diamondsuit_{\kappa^+}(\text{Cof}(\kappa))$-sequence.

7. Finite iterations of κ-perfect posets

In this section we introduce the notion of a *finite iteration of κ-perfect posets*, which is going to be a finite forcing iteration in which every initial segment of the iteration forces that the next stage is a κ-perfect poset. We will follow the presentation in Section 4 of [FGK19], but we will need to make significant changes because several key constructions there relied on closure under unions, which we lose with κ-perfect posets.

Definition 7.1. A *finite iteration of κ-perfect posets* is a finite iteration

$$\mathbb{P}_n = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \cdots * \dot{\mathbb{Q}}_{n-1}$$

such that $\mathbb{Q}_0$ is a κ-perfect poset and for $1 \leq i < n$,

$$\mathbb{1}_{\mathbb{P}_i} \Vdash \dot{\mathbb{Q}}_i \text{ is a } \kappa\text{-perfect poset.}$$

Given conditions p and q in $\mathbb{P}_n$ such that $p \leq q$ and $q \leq p$, we will always just write $p = q$ because the two conditions have the same forcing properties, e.g. they force the same statements and one is in the generic filter if and only if the other one is.

Recall that we only defined the notion of a κ-perfect poset for an inaccessible κ. Thus, to formally define finite iterations of κ-perfect posets, we need to argue by induction on $n < \omega$ that an iteration of κ-perfect posets of length n preserves the inaccessibility of κ. Since κ-perfect posets are $<\kappa$-closed, we can assume inductively that an iteration of length n of κ-perfect posets is $<\kappa$-closed, and hence, in particular, preserves the inaccessibility of κ. Let $\dot{\mathbb{Q}}_n$ be a $\mathbb{P}_n$-name for a κ-perfect poset. Since $\mathbb{P}_n \Vdash \dot{\mathbb{Q}}_n$ is $<\kappa$-closed, the iteration $\mathbb{P}_{n+1}$ is $<\kappa$-closed as well.

Indeed, given a descending sequence $\vec{p} = \{p_\xi \mid \xi < \beta\}$, with $\beta < \kappa$, of conditions in $\mathbb{P}_n$, let's construct explicitly a condition $p \leq p_\xi$ for all $\xi < \beta$ such that if any condition $p' \leq p_\xi$ for all $\xi < \beta$, then $p' \leq p$. Although, such a condition p won't be unique because of the arbitrary choices of names in the construction, it will be the case that any two such conditions p and q will have the property that $p \leq q$ and $q \leq p$. Hence, since we identify such conditions, we can call p a *greatest lower bound* of $\vec{p}$. We will construct p by induction on $i < n$. Let

$$p(0) = \bigcap_{\xi<\beta} p_\xi(0),$$

which is in $\mathbb{Q}_0$ by the $<\kappa$-intersection property. Suppose inductively that we have defined $p \restriction i$ for some $i < n$ such that $p \restriction i \leq p_\xi \restriction i$ for all $\xi < \beta$. Thus, $p \restriction i \Vdash p_\xi(i) \leq p_\eta(i)$ for any pair $\xi \leq \eta < \beta$. By the $<\kappa$-intersection property, there is a $\mathbb{P}_i$-name $\dot{p}$ such that

$$p \restriction i \Vdash \dot{p} = \bigcap_{\xi<\eta} p_\xi(i) \in \dot{\mathbb{Q}}_i.$$

We define $p(i) = \dot{p}$. Clearly, the condition p has the desired properties.

Suppose that $G \subseteq \mathbb{P}_n$ is V-generic. For $1 \leq i < n$, let G_i be the restriction of G to $\mathbb{P}_i$. Let $G(0) = G_1$ and for $1 \leq i < n$, let $G(i) = \{p(i)_{G_i} \mid p \in G\}$. Let h_i be the unique subset of κ determined by $G(i)$. It is not difficult to see that the sequence $\vec{h} = \langle h_0, \ldots, h_{n-1}\rangle$, of subsets of κ, determines G. Elements of G_1 are trees with h_0 as a branch, and inductively, elements of G_{i+1} are conditions $p \in \mathbb{P}_{i+1}$ such that $p \restriction i \in G_i$ and h_i is a branch through $p(i)_{G_i}$.

Let $\mathrm{succ}(\kappa) = \{\xi < \kappa \mid \xi \text{ is a successor ordinal}\}$. Next, we define the analogue of the fusion poset $\mathbb{Q}(\mathbb{P})$ for the finite iterations $\mathbb{P}_n$ of κ-perfect posets.

Definition 7.2. We associate to each finite iteration $\mathbb{P}_n$ of κ-perfect posets the poset $\mathbb{Q}(\mathbb{P}_n)$ whose conditions are pairs (p, F) with $p \in \mathbb{P}_n$ and $F : n \to \mathrm{succ}(\kappa)$ ordered so that $(p_2, F_2) \leq (p_1, F_1)$ whenever $p_2 \leq p_1$, for every $i < n$, we have $F_2(i) \geq F_1(i)$, and

$$p_2 \restriction i \Vdash p_1(i) \cap 2^{F_1(i)} = p_2(i) \cap 2^{F_1(i)}.$$

We call $\mathbb{Q}(\mathbb{P}_n)$ the *fusion poset* for $\mathbb{P}_n$.

Proposition 7.3. *The fusion poset $\mathbb{Q}(\mathbb{P}_n)$ is $<\kappa$-closed.*

Proof. Suppose that $\{(p_\xi, F_\xi) \mid \xi < \beta\}$, with $\beta < \kappa$, is a descending sequence of conditions in $\mathbb{Q}(\mathbb{P}_n)$. Let p be a greatest lower bound of $\{p_\xi \mid \xi < \beta\}$. Let

$$F(i) = (\sup_{\xi<\beta} F_\xi(i)) + 1.$$

Clearly, $p(0) \cap 2^{F_\xi(0)} = p_\xi(0) \cap 2^{F_\xi(0)}$ and

$$p \restriction i \Vdash p(i) \cap 2^{F_\xi(i)} = p_\xi(i) \cap 2^{F_\xi(i)}$$

for all $\xi < \beta$ and $i < n$. Hence $(p, F) \leq (p_\xi, F_\xi)$ for all $\xi < \beta$. □

Fusion arguments with names for perfect κ-trees require that we have some information about a fixed level α of the tree. We will now argue that there are densely many conditions in $\mathbb{Q}(\mathbb{P}_n)$ where this is the case.

Suppose $p \in \mathbb{P}_n$ and $\sigma : n \to 2^{<\kappa}$. Following [FGK19], let's define, by induction on n, what it means for σ to *lie* on p. For $n = 1$, we shall say that σ *lies* on p whenever $\sigma(0) \in p(0)$. If σ lies on p, we shall denote by $p \mid \sigma$ the condition $p(0)_{\sigma(0)}$. Note that $p \mid \sigma \leq p$. So suppose that we have defined when σ lies on p for $p \in \mathbb{P}_n$, and for a σ which lies on p, we have defined $p \mid \sigma$ so that $p \mid \sigma \leq p$. Let $p \in \mathbb{P}_{n+1}$. We define that σ lies on p if $\sigma \restriction n$ lies on $p \restriction n$ and $(p \restriction n) \mid (\sigma \restriction n) \Vdash \sigma(n) \in p(n)$. If σ lies on p, we shall denote by $p \mid \sigma$ the condition $\bar{p}$ such that $\bar{p} \restriction n = (p \restriction n) \mid (\sigma \restriction n)$ and $\bar{p}(n) = \dot{T}$, where $\dot{T}$ is a $\mathbb{P}_n$-name that is interpreted as $p(n)_{\sigma(n)}$ by any $\mathbb{P}_n$-generic filter containing $(p \restriction n) \mid (\sigma \restriction n)$. Note that even though we are being vague here about the choice of the names $\dot{T}$, we can actually make these choices canonically provided we fix well-orderings of the posets $\mathbb{P}_n$ ahead of time. Clearly this gives that $p \mid \sigma \leq p$. Note that if σ and σ' lie on p are such that $\sigma \restriction i = \sigma' \restriction i$ for some $i < n$, then $(p \mid \sigma) \restriction i = (p \mid \sigma') \restriction i$.

Definition 7.4. Let $F : n \to \text{succ}(\kappa)$, $\sigma : n \to 2^{<\kappa}$, and $p \in \mathbb{P}_n$. We shall say that σ *lies on levels* F if $\sigma(i) \in 2^{F(i)}$ for all $i < n$. If σ lies on levels F and lies on p, we shall say that σ *lies* on (p, F). Given a condition $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$, we will let X^p_F denote the set of all σ which lie on (p, F). We shall say that a condition $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$ is *determined* if for every σ which lies on levels F, either

(1) σ lies on p ($\sigma \in X^p_F$), or
(2) $\sigma(0) \notin p(0)$, or
(3) there is $1 \leq i < n$ such that $\sigma \restriction i$ lies on $p \restriction i$ and

$$(p \restriction i) \mid (\sigma \restriction i) \Vdash \sigma(i) \notin p(i).$$

Note that a set X^p_F for a condition $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$ has size less than κ, since for $\alpha < \kappa$, $2^\alpha < \kappa$ and F has a finite domain.

Proposition 7.5. *If $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$ is determined, then the set*

$$\{p \mid \sigma \mid \sigma \in X^p_F\}$$

is a maximal antichain below p in $\mathbb{P}_n$.

Proof. We will argue by induction on n. Suppose $n = 1$. Fix $(p, F) \in \mathbb{Q}(\mathbb{P}_1)$ and let $q \leq p$. Then there is $t_0 \in 2^{F(0)}$ such that $q(0)_{t_0} \subseteq p(0)_{t_0}$. Thus, σ with $\sigma(0) = t_0$ lies on (p, F) and q is compatible with $p \mid \sigma$. So suppose inductively that the statement holds for n. Fix a determined condition $(p, F) \in \mathbb{Q}(\mathbb{P}_{n+1})$ and let $q \leq p$. By our inductive assumption, $q \restriction n$ is compatible with $(p \restriction n) \mid \sigma$ for some σ which lies on $(p \restriction n, F \restriction n)$. So let

$$r \leq q \restriction n, (p \restriction n) \mid \sigma.$$

Since $q \restriction n \Vdash q(n) \leq p(n)$, it follows that $r \Vdash q(n) \leq p(n)$. Thus, there is $r' \leq r$ and some $t \in 2^{F(n)}$ such that $r' \Vdash q(n)_t \leq p(n)_t$. Let $\tau : n+1 \to 2^{<\kappa}$ be such that $\tau \restriction n = \sigma$ and $\tau(n) = t$. Since (p, F) is determined, it must be the case that

$$(p \restriction n) \mid (\sigma \restriction n) = (p \restriction n) \mid (\tau \restriction n) \Vdash t \in p(n).$$

It follows that τ lies on (p, F). Let $r'' \in \mathbb{P}_{n+1}$ be such that $r'' \restriction n = r'$ and $r''(n) = q(n)_t$. Then r'' witnesses that q is compatible with $p \mid \tau$. □

Proposition 7.6. *Suppose $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$ is determined and σ lies on $(p \restriction i, F \restriction i)$ for some $1 \leq i < n$. Then there is τ lying on (p, F) such that $\tau \restriction i = \sigma$.*

Proof. Let q be $(p \restriction i) \mid \sigma$ concatenated with the tail of p for $i \leq j < n$. Clearly, $q \leq p$. By Proposition 7.5, q is compatible with $p \mid \tau$ for some τ which lies on (p, F). But then clearly $\tau \restriction i = \sigma$. □

Proposition 7.7. *Suppose that $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$ is determined and σ lies on $(p \restriction i, F \restriction i)$ for some $i < n$. Then*

1. *$p(0)$ is the union of $p(0)_{\tau(0)}$ for τ that lie on (p, F).*
2. *$(p \restriction i) \mid \sigma$ forces that $p(i)$ is the union of the $p \mid \tau(i)$ for τ that lie on (p, F) with $\tau \restriction i = \sigma$.*

In particular, $(p \restriction i) \mid \sigma$ decides the $F(i)$-th level of $p(i)$.

Proof. First, let's prove (1). If τ lies on (p, F), then by definition, $\tau(0) \in p(0)$. So suppose that t is a node on level $F(0)$ of $p(0)$. By Proposition 7.6, there must be a τ which lies on (p, F) with $\tau(0) = t$. Next, let's prove (2). If $\tau \restriction i = \sigma$, then by definition,

$$(p \restriction i) \mid \sigma = (p \mid \tau) \restriction i \Vdash \tau(i) \in p(i).$$

Let t be a node on level $F(i)$ such that there is no τ which lies on (p, F) with $\tau \restriction i = \sigma$ and $\tau(i) = t$. Let τ' lie on F be such that $\tau' \restriction i = \sigma$ and $\tau'(i) = t$. Since τ' does not lie on the determined condition (p, F), there must be some $j < n-1$ such that $\tau' \restriction j$ lies on $(p \restriction j, F \restriction j)$ and $(p \restriction j) \mid (\tau' \restriction j) \Vdash \tau'(j) \notin p(j)$. Clearly, $j \geq i$. If $j = i$, then $(p \restriction i) \mid \sigma \Vdash t \notin p(i)$, and so we are done. But if $j > i$, then σ' lies on $(p \restriction i+1, F \restriction i+1)$, where $\sigma' \restriction i = \sigma$ and $\sigma'(i) = t$, which cannot be the case because then, by Proposition 7.6, we would be able to extend σ' to a τ which lies on (p, F). □

Proposition 7.8. *Suppose that $\{(p_\xi, F) \mid \xi < \beta\}$, with $\beta < \kappa$, is a descending sequence of determined conditions in $\mathbb{Q}(\mathbb{P}_n)$. If p is a greatest lower bound of the sequence $\{p_\xi \mid \xi < \beta\}$, then (p, F) is determined.*

Proof. Since the statement clearly holds for $n = 1$, we can assume inductively that it holds for some n and suppose that we have a descending sequence of conditions $\{(p_\xi, F) \mid \xi < \beta\}$, with $\beta < \kappa$, in $\mathbb{Q}(\mathbb{P}_{n+1})$. Suppose that σ lies on F. If $\sigma \restriction n$ does not lie on $(p \restriction n, F \restriction n)$, then by our inductive assumption, there is some $i < n-1$ such that $\sigma \restriction i$ lies on $(p \restriction i, F \restriction i)$ and $(p \restriction i) \mid (\sigma \restriction i) \Vdash \sigma(i) \notin p(i)$. So we can assume that $\sigma \restriction n$ lies on $(p \restriction n, F \restriction n)$. Thus, $\sigma \restriction n$ lies on every $(p_\xi \restriction n, F \restriction n)$ for $\xi < \beta$. Since every (p_ξ, F) is determined, $(p_\xi \mid \sigma) \restriction n$ decides whether $\sigma(n) \in p_\xi(n)$. If every $(p_\xi \mid \sigma) \restriction n \Vdash \sigma(n) \in p_\xi(n)$, then $(p \restriction n) \mid (\sigma \restriction n) \Vdash \sigma(n) \in p(n)$. But if some $(p_\xi \mid \sigma) \restriction n \Vdash \sigma(n) \notin p_\xi(n)$, then $(p \restriction n) \mid (\sigma \restriction n) \Vdash \sigma(n) \notin p(n)$. □

We will soon show that determined conditions are dense in $\mathbb{Q}(\mathbb{P}_n)$. But first we need to develop some machinery for working with them.

Proposition 7.9. *If $p \leq q$ are conditions in $\mathbb{P}_n$ such that σ lies on both p and q, then $p \mid \sigma \leq q \mid \sigma$.*

Proof. Since σ lies on p, it follows that $\sigma(0) \in p(0)$ and since σ also lies on q, it follows that $\sigma(0) \in q(0)$. Also, since $p \leq q$, it follows that $p(0) \subseteq q(0)$. Thus, in particular, $p(0)_{\sigma(0)} \subseteq q(0)_{\sigma_0}$. This shows that $(p \mid \sigma) \upharpoonright 1 \leq (q \mid \sigma) \upharpoonright 1$. So suppose inductively that $(p \mid \sigma) \upharpoonright i \leq (q \mid \sigma) \upharpoonright i$ for some $1 \leq i < n$. Since σ lies on p, it follows that $(p \mid \sigma) \upharpoonright i \Vdash \sigma(i) \in p(i)$ and since σ lies on q, it follows that $(q \mid \sigma) \upharpoonright i \Vdash \sigma(i) \in q(i)$. So $(p \mid \sigma) \upharpoonright i \Vdash \sigma(i) \in q(i)$. Also, we have $(p \mid \sigma) \upharpoonright i \Vdash p(i) \leq q(i)$. Thus, in particular, $(p \mid \sigma) \upharpoonright i \Vdash p(i)_{\sigma(i)} \leq q(i)_{\sigma(i)}$. Hence, $(p \mid \sigma) \upharpoonright i+1 \leq (q \mid \sigma) \upharpoonright i+1$. □

The statement reverses for determined conditions with the same set of σ lying on them.

Proposition 7.10. *Suppose that (p, F) and (q, F) are determined conditions such that $X^p_F = X^q_F$. If $p \mid \sigma \leq q \mid \sigma$ for every $\sigma \in X^p_F$, then $(p, F) \leq (q, F)$.*

Proof. Clearly, $p(0) \subseteq q(0)$ and $p(0) \cap 2^{F(0)} = q(0) \cap 2^{F(0)}$. Thus,

$$(p \upharpoonright 1, F \upharpoonright 1) \leq (q \upharpoonright 1, F \upharpoonright 1).$$

So suppose inductively that for some $1 \leq i < n$,

$$(p \upharpoonright i, F \upharpoonright i) \leq (q \upharpoonright i, F \upharpoonright i).$$

Let $G \subseteq \mathbb{P}_i$ be V-generic with $p \upharpoonright i \in G$. By Proposition 7.5, there is $\sigma \in X^p_F$ such that $(p \mid \sigma) \upharpoonright i \in G$. So also, $(q \mid \sigma) \upharpoonright i \in G$. Then $p(i)_G$ is the union of the $p \mid \sigma'(i)_G$ with $\sigma \upharpoonright i = \sigma' \upharpoonright i$ and $\sigma' \in X^p_F$ and $q(i)_G$ is the union of the $q \mid \sigma'(i)_G$ with $\sigma \upharpoonright i = \sigma' \upharpoonright i$ and $\sigma' \in X^p_F$. Fix σ' with $\sigma \upharpoonright i = \sigma' \upharpoonright i$ and $\sigma' \in X^p_F$. Then $p \mid \sigma' \leq q \mid \sigma'$ by assumption. Thus, $(p \mid \sigma) \upharpoonright i \Vdash p \mid \sigma'(i) \subseteq q \mid \sigma'(i)$. It follows that $p(i)_G \subseteq q(i)_G$ and $p(i)_G \cap 2^{F(i)} = q(i)_G \cap 2^{F(i)}$. Thus,

$$(p \upharpoonright i+1, F \upharpoonright i+1) \leq (q \upharpoonright i+1, F \upharpoonright i+1).$$

□

Next, we will introduce a kind of normal form for determined conditions.

Definition 7.11. Suppose that $\sigma : n \to 2^{<\kappa}$. Let's call a condition $p \in \mathbb{P}_n$ a *σ-condition* if

1. $p(0) \subseteq (2^{<\kappa})_{\sigma(0)}$,
2. for all $1 \leq i < n$, $p \upharpoonright i \Vdash p(i) \subseteq (2^{<\kappa})_{\sigma(i)}$.

Suppose that X_F is a collection of $\sigma : n \to 2^{<\kappa}$ lying on levels $F : n \to \text{succ}(\kappa)$. An *$X_F$-assignment* is a function $\varphi : X_F \to \mathbb{P}_n$ such that each $\varphi(\sigma)$ is a σ-condition and, for $1 \leq i < n$, $\varphi(\sigma) \upharpoonright i = \varphi(\sigma') \upharpoonright i$ whenever $\sigma \upharpoonright i = \sigma' \upharpoonright i$.

To motivate these definitions, consider a determined condition $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$. In this case, the map φ_p defined by $\varphi_p(\sigma) = p \mid \sigma$ for all $\sigma \in X^p_F$ is clearly an X^p_F-assignment. Thus, a determined condition (p, F) gives us a natural X^p_F-assignment. Because κ-perfect posets are not closed under unions, we will not get that every X_F-assignment can be used to build a determined condition. This can already be seen to fail for $\mathbb{P}_1$, where $\mathbb{Q}_0 = \mathbb{P}_{\min}$ because of the counterexample to forming unions from Proposition 2.6. However, given a determined condition (p, F) and an X^p_F-assignment φ such that $\varphi(\sigma) \leq p \mid \sigma$ for every $\sigma \in X^p_F$, we will be able to build a determined condition $(q, F) \leq (p, F)$ from φ. This will give us a way of strengthening determined conditions.

Suppose that X_F and φ are as in Definition 7.11. Observe that given any $\sigma, \sigma' \in X_F$, either $\sigma(0) = \sigma'(0)$ and so $\varphi(\sigma)(0) = \varphi(\sigma')(0)$, or $\sigma(0) = s \neq t = \sigma'(0)$ are two nodes on level $F(0)$ such that $\varphi(\sigma)(0) \subseteq (2^{<\kappa})_s$ and $\varphi(\sigma')(0) \subseteq (2^{<\kappa})_t$. In particular, if $\sigma(0) \neq \sigma'(0)$, then the conditions $\varphi(\sigma)(0)$ and $\varphi(\sigma')(0)$ are incompatible. More generally, if $\sigma \neq \sigma'$, then either $\sigma(0) \neq \sigma'(0)$ or there is some least $i \geq 1$ such that $\sigma \restriction i = \sigma' \restriction i$ and there are nodes $\sigma(i) = s \neq t = \sigma'(i)$ on level $F(i)$ such that $\varphi(\sigma) \restriction i \Vdash \varphi(\sigma)(i) \subseteq (2^{<\kappa})_s$ and $\varphi(\sigma) \restriction i = \varphi(\sigma') \restriction i \Vdash \varphi(\sigma')(i) \subseteq (2^{<\kappa})_t$. It follows, in particular, that for any $i < n$, unless $\sigma \restriction i = \sigma' \restriction i$, the conditions $\varphi(\sigma) \restriction i$ and $\varphi(\sigma') \restriction i$ are incompatible.

Proposition 7.12. *Suppose that $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$ is determined and that φ is an X^p_F-assignment such that $\varphi(\sigma) \leq p \mid \sigma$ for every $\sigma \in X^p_F$. Then there is a condition $(q, F) \in \mathbb{Q}(\mathbb{P}_n)$ such that*

1. *(q, F) is determined,*
2. *$X^q_F = X^p_F$,*
3. *for every $\sigma \in X^q_F$, $q \mid \sigma = \varphi(\sigma)$,*
4. *$(q, F) \leq (p, F)$.*

Proof. Let $q(0)$ be the union of the $\varphi(\sigma)(0)$ for $\sigma \in X^p_F$. Since $q(0)$ is obviously the condition $p(0)$ slimmed down by $\vec{T} = \{\varphi(\sigma)(0) \mid \sigma \in X^p_F\}$, we have that $q(0) \in \mathbb{Q}_0$ by the weak union property.

Let $q(1)$ be the (canonical) $\mathbb{P}_1$-name for the tree which is the union of the collection of trees given by the interpretation of the name

$$\{\langle \varphi(\sigma)(1), \varphi(\sigma)(0)\rangle \mid \sigma \in X^p_F\}.$$

Fix a V-generic filter $G \subseteq \mathbb{P}_1$ containing $q(0)$. We need to argue that $q(1)_G \in (\mathbb{Q}_1)_G$ and $q(1)_G \leq p(1)_G$. If $\varphi(\sigma)(0)$ and $\varphi(\sigma')(0)$ are in G, then $\sigma(0) = \sigma'(0) = s$ for some s, and so $\varphi(\sigma)(0) = \varphi(\sigma')(0)$. Fix σ such that $\varphi(\sigma)(0) \in G$. Thus, the interpretation $q(1)_G$ is the tree which is the union of the $\varphi(\sigma')(1)_G$ for $\sigma'(0) = \sigma(0)$. Since $\varphi(\sigma)(0) \leq (p \mid \sigma) \restriction 1$, it follows that $(p \mid \sigma) \restriction 1 \in G$. Therefore $p(1)_G$ is the union of the $p \mid \sigma'(1)$ for $\sigma'(0) = \sigma(0)$ by Proposition 7.7, and $\varphi(\sigma')(1) \leq p \mid \sigma'(1)$ for every σ' with $\sigma'(0) = \sigma(0)$. Thus, $q(1)_G$ is the tree $p(1)_G$ slimmed down by $\vec{T} = \{\varphi(\sigma')(1) \mid \sigma'(0) = \sigma(0) \text{ and } \sigma' \in X^p_F\}$, and hence $q(1)_G \in (\mathbb{Q}_1)_G$ by the weak union property. Also, $q(1)_G \leq p(1)_G$. Thus, $q \restriction 2 \leq p \restriction 2$.

Now generally, for some $1 \leq i < n$, suppose that we have defined $q \restriction i$ such that $q \restriction i \leq p \restriction i$ and for every $\sigma \in X^p_F$, $(q \restriction i) \mid (\sigma \restriction i) = \varphi(\sigma) \restriction i$. Let $q(i)$ be the $\mathbb{P}_i$-name for the tree which is the union of the collection of trees given by the interpretation of the name

$$\{(\varphi(\sigma)(i), \varphi(\sigma) \restriction i) \mid \sigma \in X^p_F\}.$$

Fix a V-generic filter $G \subseteq \mathbb{P}_i$ with $q \restriction i \in G$. We need to argue that $q(i)_G \in (\mathbb{Q}_i)_G$ and $q(i)_G \leq p(i)_G$. Since for $\sigma, \sigma' \in X^p_F$, $\varphi(\sigma) \restriction i$ and $\varphi(\sigma') \restriction i$ are incompatible whenever $\sigma \restriction i \neq \sigma' \restriction i$, if $\varphi(\sigma) \restriction i$ and $\varphi(\sigma') \restriction i$ are both in G, then $\sigma' \restriction i = \sigma \restriction i = \tau$ for some τ. By Proposition 7.5 and our inductive assumption, we can fix σ such that $\varphi(\sigma) \restriction i \in G$. Thus, the interpretation $q(i)_G$ is the tree which is the union of the $\varphi(\sigma')(i)_G$ for $\sigma' \restriction i = \sigma \restriction i$. Since $\varphi(\sigma) \restriction i \leq (p \mid \sigma) \restriction i$, it follows that $(p \mid \sigma) \restriction i \in G$. Therefore $p(i)_G$ is the union of the $p \mid \sigma'(i)$ for $\sigma' \restriction i = \sigma \restriction i$ by Proposition 7.7, and $\varphi(\sigma')(i) \leq p \mid \sigma'(i)$ for every σ' with $\sigma' \restriction i = \sigma \restriction i$. Thus, $q(i)_G$

is the tree $p(i)_G$ slimmed down by $\vec{T} = \{\varphi(\sigma')(i) \mid \sigma' \restriction i = \sigma \restriction i \text{ and } \sigma' \in X^p_F\}$, and hence $q(i)_G \in (\mathbb{Q}_i)_G$ by the weak union property. Also, $q(i)_G \leq p(i)_G$. Thus, we get $q \restriction i+1 \leq p \restriction i+1$. Let q be the condition resulting from this construction.

First, we argue that every $\sigma \in X^p_F$ lies on (q, F) and simultaneously show (3). So fix some $\sigma \in X^p_F$. By construction $\sigma(0) \in q(0)$ and $q(0)_{\sigma(0)} = \varphi(\sigma)(0)$. So assume inductively that for some $1 \leq i < n$,

(1) $\sigma \restriction i$ lies on $q \restriction i$,
(2) $(q \mid \sigma) \restriction i = \varphi(\sigma) \restriction i$.

Suppose that $G \subseteq \mathbb{P}_i$ is a V-generic filter containing $(q \mid \sigma) \restriction i = \varphi(\sigma) \restriction i$. By definition of q, we have

$$(q(i)_G)_{\sigma(i)} = \varphi(\sigma)(i)_G.$$

So $\sigma \restriction i+1$ lies on $q \restriction i+1$ and we have

$$(q \mid \sigma) \restriction i+1 = \varphi(\sigma) \restriction i+1.$$

Now suppose that $\tau : n \to 2^{<\kappa}$ lies on (q, F). By definition of q, it follows that $\tau(0) = \sigma(0)$ for some $\sigma \in X^p_F$. So suppose inductively that for some $1 \leq i < n$, there is $\sigma \in X^p_F$ such that $\tau \restriction i = \sigma \restriction i$. Since τ lies on q, it follows that $(q \mid \sigma) \restriction i \Vdash \tau(i) \in q(i)$. But then there is some σ' such that $\sigma \restriction i = \sigma' \restriction i$ and $\tau(i) = \sigma'(i)$. So in the last step, we will obtain $\sigma \in X^p_F$ such that $\sigma = \tau$. This completes the proof of (2).

Next, let's argue that (q, F) is determined. Fix $\sigma : n \to 2^{<\kappa}$ lying on levels F such that $\sigma \notin X^q_F = X^p_F$. Let $i < n-1$ be largest such that $\sigma \restriction i$ lies on $p \restriction i$ and

$$(p \restriction i) \mid (\sigma \restriction i) \Vdash \sigma(i) \notin p(i).$$

Then $\sigma \restriction i$ lies on $q \restriction i$ and $(q \restriction i) \mid (\sigma \restriction i)$ forces that $q(i)$ is the union of the $\varphi(\sigma')(i)$ with $\sigma \restriction i = \sigma' \restriction i$. It follows that

$$(q \restriction i) \mid (\sigma \restriction i) \Vdash \sigma(i) \notin q(i),$$

because otherwise i would not be the largest with the above property.

Finally, since $X^p_F = X^q_F$ and for every $\sigma \in X^p_F$, $q \mid \sigma = \varphi(\sigma) \leq p \mid \sigma$, it follows, by Proposition 7.10, that $(q, F) \leq (p, F)$. □

We shall call the condition q constructed in Proposition 7.12 the *amalgamation* of φ. Thus, we get the promised normal form for determined conditions.

Corollary 7.13. *Suppose that (p, F) is determined and q is the amalgamation of the X^p_F-assignment φ_p. Then $q = p$.*

Proof. By Proposition 7.12, $q \leq p$, $X^p_F = X^q_F$, and $q \mid \sigma = p \mid \sigma$ for every $\sigma \in X^p_F$. Thus, by Proposition 7.10, $p \leq q$. □

Proposition 7.14. *Suppose that (p, F) is a determined condition and $q \leq p \mid \sigma$ for some $\sigma \in X^p_F$. Then there is a determined condition $(p', F) \leq (p, F)$ such that $p' \mid \sigma = q$.*

Proof. We will define an X^p_F-assignment φ, such that $\varphi(\sigma) = q$ and $\varphi(\tau) \leq p \mid \tau$ for every $\tau \in X^p_F$, whose amalgamation will be p'. Fix $\tau \in X^p_F$. Let $i < n$ be largest such that $\tau \restriction i = \sigma \restriction i$. Let $\varphi(\tau)$ be $q \restriction i$ concatenated with the tail of $p \mid \tau$ for $i \leq j < n$. Clearly φ is an X^p_F-assignment and $\varphi(\tau) \leq p \mid \tau$ for every $\tau \in X^p_F$. Let p' be the amalgamation of φ. By Proposition 7.12, $(p', F) \leq (p, F)$. By definition, $\varphi(\sigma) = q$ and, by Proposition 7.12, $q = p' \mid \sigma$. □

Proposition 7.15. *Suppose that $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$ is determined. Suppose further that $\psi(\sigma, q)$ is a property of a condition $q \in \mathbb{P}_n$ and $\sigma : n \to 2^{<\kappa}$ such that for every $\sigma \in X^p_F$, the set of all q such that $\psi(\sigma, q)$ holds is dense open below $p \mid \sigma$. Then there is an X^p_F-assignment φ such that for every $\sigma \in X^p_F$, $\varphi(\sigma) \leq p \mid \sigma$ and $\psi(\sigma, \varphi(\sigma))$.*

Proof. Using the inaccessibility of κ, the set X^p_F has size less than κ. So we can enumerate $X^p_F = \{\sigma_\xi \mid \xi < \beta\}$ for some $\beta < \kappa$. Let $q_0 \leq p \mid \sigma_0$ be such that $\psi(\sigma_0, q_0)$ holds. Suppose inductively that for some $\gamma < \beta$, we have constructed $q_\xi \leq p \mid \sigma_\xi$ for $\xi < \gamma$ such that $\psi(\sigma_\xi, q_\xi)$ holds and for all $\xi_1 < \xi_2 < \gamma$ if $\sigma_{\xi_1} \upharpoonright i = \sigma_{\xi_2} \upharpoonright i$ for some $1 \leq i < n$, then $q_{\xi_2} \upharpoonright i \leq q_{\xi_1} \upharpoonright i$. Let's argue that we can construct q_γ to maintain this inductive hypothesis. Let $i = 1$ and let

$$Y_1 = \{\xi < \gamma \mid \sigma_\xi(0) = \sigma_\gamma(0)\}.$$

If $Y_1 = \emptyset$, let $q'_\gamma(0) = p(0)_{\sigma_\gamma(0)}$. Otherwise, by our inductive assumption, the conditions $q_\xi(0)$ for $\xi \in Y_1$ form a $\subseteq$-descending sequence below $p(0)_{\sigma_\gamma(0)}$. Let $q'_\gamma(0) = \bigcap_{\xi \in Y_1} q_\xi(0)$, which is an element of $\mathbb{Q}_0$ by the $<\kappa$-intersection property. Suppose inductively that for some $i < n$, we have constructed $q'_\gamma \upharpoonright i$ such that

$$q'_\gamma \upharpoonright i \leq (p \mid \sigma_\gamma) \upharpoonright i,$$

and for all $\xi < \gamma$ if $\sigma_\xi \upharpoonright j = \sigma_\gamma \upharpoonright j$ for some $1 \leq j \leq i$, then $q'_\gamma \upharpoonright j \leq q_\xi \upharpoonright j$. Let

$$Y_i = \{\xi < \gamma \mid \sigma_\xi \upharpoonright i+1 = \sigma_\gamma \upharpoonright i+1\}.$$

If $Y_i = \emptyset$, let $q'_\gamma(i) = p \mid \sigma_\gamma(i)$. Since $q'_\gamma \upharpoonright i \leq (p \mid \sigma_\gamma) \upharpoonright i$, we have

$$q'_\gamma \upharpoonright i+1 \leq (p \mid \sigma_\gamma) \upharpoonright i+1.$$

Otherwise, $Y_i \neq \emptyset$. Observe that in this case, the condition $q'_\gamma \upharpoonright i$ forces that the $q_\xi(i)$ for $\xi \in Y_i$ form a $\subseteq$-descending sequence below $p(i)_{\sigma_\gamma(i)}$. Thus, we can let $q'_\gamma(i)$ be a $\mathbb{P}_i$-name that is forced by $q'_\gamma \upharpoonright i$ to be the intersection of the $q_\xi(i)$ for $\xi \in Y_i$. Finally, let q_γ be any condition below q'_γ such that $\psi(\sigma_\gamma, q_\gamma)$ holds. Thus, we have constructed a sequence of conditions $\{q_\xi \mid \xi < \beta\}$ such that

1. $q_\xi \leq p \mid \sigma_\xi$,
2. $\psi(\sigma_\xi, q_\xi)$ holds, and
3. for every $\xi_1 < \xi_2 < \beta$, if there is $i < n$ such that $\sigma_{\xi_1} \upharpoonright i = \sigma_{\xi_2} \upharpoonright i$, then $q_{\xi_2} \upharpoonright i \leq q_{\xi_1} \upharpoonright i$.

Now fix any $\sigma \in X^p_F$. We will define the corresponding condition $\varphi(\sigma)$. Let $\varphi(\sigma)(0)$ be the intersection of the $\subseteq$-descending (below $p(0)_{\sigma(0)}$) sequence of conditions $q_\xi(0)$ for $\xi < \beta$ such that $\sigma(0) = \sigma_\xi(0)$. Suppose we have defined $\varphi(\sigma) \upharpoonright i$ for some $i < n$ such that $\varphi(\sigma) \upharpoonright i \leq q_\xi \upharpoonright i$ for every $\xi < \beta$ whenever $\sigma_\xi \upharpoonright i = \sigma \upharpoonright i$. Let $\varphi(\sigma)(i)$ be a $\mathbb{P}_i$-name for the intersection of the $\subseteq$-descending sequence $q_\xi(i)$ for $\xi < \beta$ such that $\sigma_\xi \upharpoonright i+1 = \sigma \upharpoonright i+1$. Note that, for $\sigma, \tau \in X^p_F$, if $\sigma \upharpoonright i = \tau \upharpoonright i$, then $\varphi(\sigma) \upharpoonright i = \varphi(\tau) \upharpoonright i$. Clearly, the collection of the $\varphi(\sigma)$ is an X^p_F-assignment, for every $\gamma < \beta$, $\varphi(\sigma_\gamma) \leq q_\gamma \leq p \mid \sigma_\gamma$, and so by open denseness, $\psi(\sigma_\gamma, \varphi(\sigma_\gamma))$ holds. □

The corollary below follows by Proposition 7.12.

Corollary 7.16. *Suppose that $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$ is determined and $\psi(\sigma, q)$ is a property of a condition $q \in \mathbb{P}_n$ and $\sigma : n \to 2^{<\kappa}$ such that for every $\sigma \in X^p_F$, the set* *of all q such that $\psi(\sigma, q)$ holds is dense open below $p \mid \sigma$. Then there is a determined condition $(q, F) \leq (p, F)$ such that $X^q_F = X^p_F$ and $\psi(\sigma, q \mid \sigma)$ holds for every $\sigma \in X^p_F$.*

Proposition 7.17. *If $(p, F) \in \mathbb{Q}(\mathbb{P}_n)$, then there is $(q, F) \leq (p, F)$ such that (q, F) is determined. In particular, determined conditions (q, F) are dense in $\mathbb{Q}(\mathbb{P}_n)$.*

Proof. We will argue by induction on n. The case $n = 1$ is obvious. So suppose the statement is true for $\mathbb{Q}(\mathbb{P}_n)$ for some n. Let $(p, F) \in \mathbb{Q}(\mathbb{P}_{n+1})$. By our inductive assumption, there is a determined condition $(q', F \restriction n) \leq (p \restriction n, F \restriction n)$. Fix a $\sigma \in X^{q'}_{F\restriction n}$. Enumerate the level $F(n)$ of $2^{<\kappa}$ as $\{t_\xi \mid \xi < \beta\}$ with $\beta < \kappa$. Build a $\leq$-descending sequence of conditions $r_\xi \leq q' \mid \sigma$ for $\xi < \beta$ such that r_ξ decides whether $t_\xi \in p(n)$. Fix r below this sequence. Then $r \leq q' \mid \sigma$ and r decides whether $t \in p(n)$ for every node t on level $F(n)$. Let $\psi(\sigma, r)$ be the property that $r \leq q' \mid \sigma$ and r decides $t \in p(n)$ for every node t on level $F(n)$. We just showed that conditions r satisfying $\psi(\sigma, r)$ are open dense below $q' \mid \sigma$ for every σ. Thus, by Corollary 7.16, there is a determined condition $(q'', F \restriction n) \leq (q', F \restriction n)$ such that $X^{q''}_{F\restriction n} = X^{q'}_{F\restriction n}$ and $\psi(\sigma, q'' \mid \sigma)$ holds for every $\sigma \in X^{q'}_{F\restriction n}$. Let q be q'' concatenated with $p(n)$. We claim that (q, F) is determined. Suppose that σ does not lie on (q, F). If $\sigma \restriction n$ does not lie on $F \restriction n$, then we are done because $(q \restriction n, F \restriction n)$ is determined. So suppose that $\sigma \restriction n$ lies on $F \restriction n$. Then $q \mid (\sigma \restriction n)$ decides $t \in p(n)$ for every t on level $F(n)$ by construction. Thus, in particular, $q \mid (\sigma \restriction n)$ decides $\sigma(n) \in p(n)$, and hence $q \mid (\sigma \restriction n) \Vdash \sigma(n) \notin q(n)$. □

8. Growing finite iterations of κ-perfect posets

In the construction of the poset $\mathbb{J}(\kappa)$, at nontrivial successor stages $\alpha + 1$, we used the κ-many perfect κ-trees obtained from a partially generic filter for $\mathbb{Q}(\mathbb{P}_\alpha)^{<\kappa}$ to grow the κ-perfect poset $\mathbb{P}_\alpha$ to $\mathbb{P}_{\alpha+1}$. What we would like to do now is to find an appropriate generalization of this construction for growing a finite iteration $\mathbb{P}_n$ of κ-perfect posets using partially generic filters for (a version of) the associated fusion poset $\mathbb{Q}(\mathbb{P}_n)$. We will start by showing how to grow a finite iteration of κ-perfect posets in a fully generic forcing extension of V.

Given a finite iteration $\mathbb{P}_n = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \cdots * \dot{\mathbb{Q}}_{n-1}$ of κ-perfect posets, we would like to be able, in a well-chosen forcing extension $V[G]$, to extend it to a finite iteration $\mathbb{P}^*_n = \mathbb{Q}^*_0 * \dot{\mathbb{Q}}^*_1 * \cdots * \dot{\mathbb{Q}}^*_{n-1}$ of κ-perfect posets with the following properties:

(1) $\mathbb{Q}_0 \subseteq \mathbb{Q}^*_0$,
(2) For all $1 \leq i < n$, $\mathbb{1}_{\mathbb{P}^*_i}$ forces over $V[G]$ that $\dot{\mathbb{Q}}_i$ is a κ-perfect poset and $\dot{\mathbb{Q}}^*_i$ extends it.
(3) $\mathbb{P}_n \subseteq \mathbb{P}^*_n$.
(4) Every maximal antichain $\mathcal{A} \in V$ of $\mathbb{P}_n$ remains maximal in $\mathbb{P}^*_n$.
(5) Every $V[G]$-generic filter $H^* \subseteq \mathbb{P}^*_n$ restricts to an V-generic filter H for $\mathbb{P}_n$.

The next theorem (generalizing a construction from [FGK19], which itself generalized [Abr84]) holds the main idea for constructing $\mathbb{P}^*_n$. The set-up for the theorem is left intentionally vague with the details forthcoming in the next section.

Suppose that $\mathbb{P}_n = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \cdots * \dot{\mathbb{Q}}_{n-1}$ is a finite iteration of κ-perfect posets. We assume that we are working in a generic extension $V[G]$ of V by a yet unexplained forcing notion. We carry out the construction of $\mathbb{P}^*_n$ in n-steps. In the 0-th step, we extend $\mathbb{Q}_0$, and at each step $1 \leq i < n$, we construct a $\mathbb{P}^*_i$-name $\dot{\mathbb{Q}}^*_i$ for a κ-perfect poset extending $\dot{\mathbb{Q}}_i$. We extend $\mathbb{Q}_0$ to $\mathbb{Q}_0^*$, as before, using a carefully chosen V-generic filter $\bar{G}_1 \subseteq \mathbb{Q}(\mathbb{Q}_0)^{<\kappa} = \mathbb{Q}(\mathbb{P}_1)^{<\kappa}$ from $V[G]$. By Proposition 3.11, every $V[G]$-generic filter for $\mathbb{Q}_0^*$ (which is, in particular, $V[\bar{G}_0]$-generic) restricts to a V-generic filter for $\mathbb{Q}_0$, verifying condition (5). So suppose inductively that we already extended $\mathbb{P}_i$ to $\mathbb{P}_i^*$ satisfying requirements (1)-(5). In $V[G]$, we carefully choose a V-generic filter $\bar{G}_{i+1} \subseteq \mathbb{Q}(\mathbb{P}_{i+1})$. Let $H^* \subseteq \mathbb{P}_i^*$ be $V[G]$-generic. By condition (5), H^* restrict to a V-generic filter H for $\mathbb{P}_i$. Thus, $\mathbb{Q}_i = (\dot{\mathbb{Q}}_i)_H = (\dot{\mathbb{Q}}_i)_{H^*}$ is a κ-perfect poset in $V[H] \subseteq V[H^*]$. Let

$$K = \{(p(i)_H, F(i)) \in \mathbb{Q}(\mathbb{Q}_i) \mid (p, F) \in \bar{G}_{i+1}\}.$$

Provided that the poset $\mathbb{P}_i^*$ contains a kind of master condition for $\bar{G}_{i+1}$, we will be able to conclude that K is $V[H]$-generic for $\mathbb{Q}(\mathbb{Q}_i)$. Let $\tau(\bar{G}_{i+1})$ be a $\mathbb{P}_i^*$-name for a subset of $\mathbb{Q}(\dot{\mathbb{Q}}_i)$ such that in any forcing extension $V[G][H^*]$ by $\mathbb{P}_i^*$, $\tau(\bar{G}_{i+1})$ gets interpreted as K.

Theorem 8.1. *Suppose that $\bar{p} \in \mathbb{P}_i^*$ is such that for every $(p, F) \in \bar{G}_{i+1}$, $\bar{p} \leq p \restriction i$. Then*

$$\bar{p} \Vdash \tau(\bar{G}_{i+1}) \text{ is a } V[\dot{H}]\text{-generic filter for } \mathbb{Q}(\dot{\mathbb{Q}}_i),$$

where $\dot{H}$ is the canonical name for the restriction of the generic filter to $\mathbb{P}_i$.

Proof. Suppose $H^* \subseteq \mathbb{P}_i^*$ is $V[G]$-generic with $\bar{p} \in H^*$. Let H be the restriction of H^* to $\mathbb{P}_i$. Let $\mathbb{Q}_i = (\dot{\mathbb{Q}}_i)_H$ and

$$K = \tau(\bar{G}_{i+1})_H = \{(p(i)_H, F(i)) \in \mathbb{Q}(\mathbb{Q}_i) \mid (p, F) \in \bar{G}_{i+1}\}.$$

Note that by our assumption on the condition $\bar{p}$, it follows that for every $(p, F) \in \bar{G}_{i+1}$, $p \restriction i \in H$.

First, we argue that K is a filter on $\mathbb{Q}(\mathbb{Q}_i)$. Fix $(p, F) \in \bar{G}_{i+1}$, and suppose that $(p(i)_H, F(i)) \leq (T, \alpha) \in \mathbb{Q}(\mathbb{Q}_i)$. It follows that $\alpha \leq F(i)$ and $p(i)_H \cap 2^\alpha = T \cap 2^\alpha$. Fix a $\mathbb{P}_i$-name $\dot{T}$ for T such that

$$\mathbb{1}_{\mathbb{P}_i} \Vdash p(i) \subseteq \dot{T} \text{ and } p(i) \cap 2^\alpha = \dot{T} \cap 2^\alpha$$

over V. Let

$$p' = p \restriction i \cup \{(i, \dot{T})\} \text{ and } F' = F \cup \{(i, \alpha)\}.$$

Clearly, $(p, F) \leq (p', F')$, which means that $(p', F') \in \bar{G}_{i+1}$ because it is a filter. It follows that $p'(i)_H = T$, and so $(T, \alpha) \in K$.

Next, we fix (p, F) and (p', F') both in $\bar{G}_{i+1}$ and argue that $(p(i)_H, F(i))$ and $(p'(i)_H, F'(i))$ are compatible in K. Since $\bar{G}_{i+1}$ is a filter, there is $(q, J) \in \bar{G}_{i+1}$ below both (p, F) and (p', F'). It follows that

$$J(i) \geq F(i), F'(i),\ q \restriction i \Vdash_{\mathbb{P}_i} q(i) \subseteq p(i), p'(i),$$

$$q \restriction i \Vdash_{\mathbb{P}_i} q(i) \cap 2^{F(i)} = p(i) \cap 2^{F(i)} \text{ and } q(i) \cap 2^{F'(i)} = p'(i) \cap 2^{F'(i)},$$

and $(q(i)_H, J(i)) \in K$. Since, by our observation above, $q \restriction i \in H$, we have

$$q(i)_H \subseteq p(i)_H, p'(i)_H,$$

and

$$q(i)_H \cap {}^{F(i)}2 = p(i)_H \cap 2^{F(i)} \text{ and } q(i)_H \cap {}^{F'(i)}2 = p'(i)_H \cap 2^{F'(i)}.$$

So $(q(i)_H, J(i)) \leq (p(i)_H, F(i)), (p'(i)_H, F'(i))$.

Finally, we have to see that K is $V[H]$-generic. So suppose $D \in V[H]$ is dense open in $\mathbb{Q}(\mathbb{Q}_i)$. Let $\dot{D} \in V$ be a $\mathbb{P}_i$-name for D such that

$$\mathbb{1}_{\mathbb{P}_i} \Vdash_{\mathbb{P}_i} \dot{D} \text{ is dense open in } \mathbb{Q}(\dot{\mathbb{Q}}_i).$$

In V, define
$$E = \{(p, F) \in \mathbb{Q}(\mathbb{P}_{i+1}) \mid p \upharpoonright i \Vdash (p(i), F(i)) \in \dot{D}\}.$$
We claim that E is dense open in $\mathbb{Q}(\mathbb{P}_{i+1})$. It is easy to see that E is open, so let's argue that it is dense. Fix some $(q, J) \in \mathbb{Q}(\mathbb{P}_{i+1})$. By Proposition 7.17, we can assume, by moving to a stronger condition, that (q, J) is determined. Since $\dot{D}$ is forced to be dense in $\mathbb{Q}(\dot{\mathbb{Q}}_i)$, there must be some pair of $\mathbb{P}_i$-names $(\dot{T}, \dot{\alpha})$ such that $\dot{T}$ is a $\mathbb{P}_i$-name for an element of $\dot{\mathbb{Q}}_i$, $\dot{\alpha}$ is a $\mathbb{P}_i$-name for an ordinal, and
$$q \upharpoonright i \Vdash (\dot{T}, \dot{\alpha}) \in \dot{D} \text{ and } (\dot{T}, \dot{\alpha}) \leq (q(i), J(i)).$$
The set of conditions which decide the value of $\dot{\alpha}$ is dense open below $q \upharpoonright i$ in $\mathbb{P}_i$. So, by Corollary 7.16, there a determined condition $(p', F') \leq (q \upharpoonright i, J \upharpoonright i)$ in $\mathbb{Q}(\mathbb{P}_i)$ such that for every $\sigma \in X^{p'}_{F'}$, $p' \mid \sigma$ decides that $\dot{\alpha} = \alpha(\sigma)$. Let $\alpha < \kappa$ be a successor ordinal above all the $\alpha(\sigma)$. Then
$$p' \Vdash (\dot{T}, \alpha) \leq (\dot{T}, \dot{\alpha}) \leq (q(i), J(i)) \text{ and } (\dot{T}, \alpha) \in \dot{D}.$$
Let $p = p' \cup \{(i, \dot{T})\}$ and $F = F' \cup \{(i, \alpha)\}$. Clearly $(p, F) \in E$. Let's argue that $(p, F) \leq (q, J)$. By construction $p \leq q$ and each $\alpha(\sigma) \geq J(i)$, so $\alpha \geq J(i)$. Finally, $p' = p \upharpoonright i \Vdash (\dot{T}, \alpha) \leq (q(i), J(i))$, and so $p \upharpoonright i \Vdash \dot{T} \cap 2^{J(i)} = q(i) \cap 2^{J(i)}$.

Fix some $(p, F) \in E \cap \bar{G}_{i+1}$, and recall that $p \upharpoonright i \in H$. Thus,
$$(p(i)_H, F(i)) \in D \cap K,$$
completing the argument that K is a $V[H]$-generic filter for $\mathbb{Q}(\mathbb{Q}_i)$. □

Next, we are going to obtain a stronger version of Theorem 8.1 that tells us how to get a $V[H]$-generic filter for $\mathbb{Q}(\mathbb{Q}_i)^{<\kappa}$, which is really what we need to extend $\mathbb{Q}_i$ to $\mathbb{Q}_i^*$ in $V[G][H^*]$. For this, we will need to enlarge our fusion poset $\mathbb{Q}(\mathbb{P}_{i+1})$.

Definition 8.2. Let $\bar{\mathbb{Q}}_i(\mathbb{P}_{i+1})$ be the following modification of $\mathbb{Q}(\mathbb{P}_{i+1})$. Conditions in $\bar{\mathbb{Q}}(\mathbb{P}_{i+1})$ are pairs (p, F) such that $(p \upharpoonright i, F \upharpoonright i) \in \mathbb{Q}(\mathbb{P}_i)$, $p(i)$ is some $<\kappa$-length sequence $\{\dot{T}_\xi \mid \xi < \beta\}$ with $p \upharpoonright i \Vdash \dot{T}_\xi \in \dot{\mathbb{Q}}_i$ for all $\xi < \beta$, and $F(i) = f : \beta \to \text{succ}(\kappa)$. The ordering is $(p', F') \leq (p, F)$ whenever

(1) $(p' \upharpoonright i, F' \upharpoonright i) \leq (p \upharpoonright i, F \upharpoonright i)$, and
for $\xi < \beta$,
(2) $F'(i)(\xi) \geq F(i)(\xi)$,
(3) $p' \upharpoonright i \Vdash p'(i)(\xi) \cap 2^{F(i)(\xi)} = p(i)(\xi) \cap 2^{F(i)(\xi)}$.

An argument as in the proof of Proposition 7.3 verifies that $\bar{\mathbb{Q}}(\mathbb{P}_{i+1})$ is $<\kappa$-closed.

Let us now use the same set-up as that preceding Theorem 8.1, but use instead a carefully chosen V-generic filter $\bar{G}_{i+1} \subseteq \bar{\mathbb{Q}}(\mathbb{P}_{i+1})$ from $V[G]$. Let K be the collection of all
$$\{((\dot{T}_\xi)_H, f(\xi)) \mid \xi < \beta\}$$
such that $(p, F) \in \bar{G}_{i+1}$ with $p(i) = \{\dot{T}_\xi \mid \xi < \beta\}$ and $F(i) = f$. Let $\tau(\bar{G}_{i+1})$ be a $\mathbb{P}_i^*$-name for a subset of $\mathbb{Q}(\dot{\mathbb{Q}}_i)^{<\kappa}$ such that in any forcing extension $V[H^*]$ by $\mathbb{P}_i^*$ $\tau(\bar{G}_{i+1})$ gets interpreted as K.

Theorem 8.3. *Suppose $\bar{p} \in \mathbb{P}_i^*$ is such that for every $(p, F) \in \bar{G}_{i+1}$, $\bar{p} \leq p \upharpoonright i$. Then*
$$\bar{p} \Vdash \tau(\bar{G}_{i+1}) \textit{ is an } V[\dot{H}]\textit{-generic filter for } \mathbb{Q}(\dot{\mathbb{Q}}_i)^{<\kappa},$$

where $\dot{H}$ is the canonical name for the restriction of the generic filter to $\mathbb{P}_i$.

The proof is essentially the same as of Theorem 8.1.

Using Theorem 8.3, we can let $\dot{\mathbb{Q}}_i^*$ be the canonical $\mathbb{P}_i^*$-name for the extension of $\dot{\mathbb{Q}}_i$ formed in $V[\dot{H}][\tau(\bar{G}_{i+1})]$, where $\dot{H}$ is the restriction of the generic filter to $\mathbb{P}_i$. In the next section, we will show how to obtain the required V-generic filter G so that the inductive assumptions hold for $\mathbb{P}_i^*$.

9. Tree iterations of κ-perfect posets

In this section, we will introduce the notion of a tree iteration of κ-perfect posets along some tree $\mathscr{T}$.

Definition 9.1. An *ω-iteration of κ-perfect posets* is a sequence

$$\vec{P} = \langle \mathbb{P}_n \mid n < \omega \rangle,$$

where $\mathbb{P}_0 = \{\emptyset\}$ is a trivial poset and each $\mathbb{P}_n$, for $n \geq 1$, is a finite iteration of κ-perfect posets with the coherence requirement that for $0 < m < n$, $\mathbb{P}_n \restriction m = \mathbb{P}_m$.

The initial poset $\mathbb{P}_0$ is included in the sequence to make the subsequent definitions more uniform. For this reason, we will also make the ad hoc definition that $\mathbb{Q}(\mathbb{P}_0) = \{\emptyset\}$. Note that an ω-iteration of κ-perfect posets is not a forcing iteration or even a poset, it is simply a coherent sequence of finite iterations.

A *tree iteration* is a non-linear forcing iteration along some tree. Given a tree of height ω, the tree iteration of κ-perfect posets will use an ω-iteration $\vec{P} = \langle \mathbb{P}_n \mid n < \omega \rangle$ of κ-perfect posets with conditions assigned to nodes of the tree on level n coming from the poset $\mathbb{P}_n$. Conditions will be assigned to the nodes coherently so that if a node s on level $n > m$ extends a node t on level m, then the condition p on node s will be such that $p \restriction m$ is on node t. Given a tree $\mathscr{T}$ and a node $s \in \mathscr{T}$, we will denote by $\operatorname{lev}(s)$ the level of s in $\mathscr{T}$.

Definition 9.2. Let $\vec{P} = \langle \mathbb{P}_n \mid n < \omega \rangle$ be an ω-iteration of κ-perfect posets and let $\mathscr{T}$ be a tree of height ω. A *$\mathscr{T}$-iteration of κ-perfect posets* is the following partial order $\mathbb{P}(\vec{P}, \mathscr{T})$. Conditions in $\mathbb{P}(\vec{P}, \mathscr{T})$ are functions f_X whose domain is a subtree X of $\mathscr{T}$ of size less than κ such that:

1. For every node s on level n of X, $f_X(s) \in \mathbb{P}_n$.
2. Whenever $s \leq t$ are two nodes in X, then $f_X(t) \restriction \operatorname{lev}(s) = f_X(s)$.

The ordering is defined to be $f_Y \leq f_X$ whenever Y extends X and for every node $s \in X$, $f_Y(s) \leq f_X(s)$.

Note that since $\mathscr{T}$ has height ω and the trees X are only required to have size less than κ, they can potentially have height ω. So the trees X can be tall, but not wide.

The analogue of the fusion posets $\mathbb{Q}(\mathbb{P})$ and $\mathbb{Q}(\mathbb{P}_n)$ for $\mathbb{P}(\vec{P}, \mathscr{T})$ will be the fusion poset $\mathbb{Q}(\vec{P}, \mathscr{T})$.

Definition 9.3. Conditions in the fusion poset $\mathbb{Q}(\vec{P}, \mathscr{T})$ are functions f_X whose domain is a subtree X of $\mathscr{T}$ of size less than κ such that:

1. For every node s on level n of X, $f_X(s) \in \mathbb{Q}(\mathbb{P}_n)$.
2. Whenever $s \leq t$ are two nodes in X, with $f_X(s) = (p_s, F_s)$ and $f_X(t) = (p_t, F_t)$, then $p_t \restriction \operatorname{lev}(s) = p_s$ and $F_t \restriction \operatorname{lev}(s) = F_s$.

The ordering is defined to be $f_Y \leq f_X$ whenever Y extends X and for every node $s \in X$, $f_Y(s) \leq f_X(s)$.

Proposition 9.4. *The posets $\mathbb{P}(\vec{\mathbb{P}}, \mathscr{T})$ and $\mathbb{Q}(\vec{\mathbb{P}}, \mathscr{T})$ are $<\kappa$-closed.*

Proof. We will give the argument for $\mathbb{P}(\vec{\mathbb{P}}, \mathscr{T})$ because the argument for $\mathbb{Q}(\vec{\mathbb{P}}, \mathscr{T})$ is analogous. Suppose that $\{f^{(\xi)}_{X_\xi} \mid \xi < \beta\}$, with $\beta < \kappa$, is a descending sequence of conditions in $\mathbb{P}(\vec{P}, \mathscr{T})$. Let $X = \bigcup_{\xi<\beta} X_\xi$, and observe that X is a tree of size less than κ. We will build a condition f_X below all the $f^{(\xi)}_{X_\xi}$ for $\xi < \beta$. We start by building level 1 of f_X. Fix a node s on level 1 of X. Since the conditions $f^{(\xi)}_{X_\xi}(s)$ with $s \in X_\xi$ form a descending sequence in $\mathbb{P}_1$, we can let

$$f_X(s) = \bigcap_{s \in X_\xi} f^{(\xi)}_{X_\xi}(s).$$

Next, let's build level 2 of f_X. Fix a node s on level 2 of X. First, observe that since $f_X(s)(0) \leq f^{(\xi)}_{X_\xi}(s)(0)$ whenever $s \in X_\xi$, $f_X(s)(0)$ forces that the conditions $f^{(\xi)}_{X_\xi}(s)(1)$ with $s \in X_\xi$ form a descending sequence. Thus, we can let $f_X(s)(1)$ be a $\mathbb{P}_1$-name for the intersection of the $f^{(\xi)}_{X_\xi}(s)(1)$ with $s \in X_\xi$. Continuing in this manner, we can build the entire condition f_X in ω-many steps. □

We will call the lower bound constructed as in the proof of Proposition 9.4 is a *greatest lower bound* of the sequence. Note that, as was the case with the iterations $\mathbb{P}_n$, a greatest lower bound is not unique, but any two greatest lower bounds f_X and f'_X have the property that $f_X \leq f'_X$ and $f'_X \leq f_X$, so for the purposes of forcing we can basically assume uniqueness.

We shall say that a condition $f_Y \leq f_X$ *trivially extends* a condition f_X if

(1) for $s \in X$, $f_Y(s) = f_X(s)$,
(2) for $s \in Y \setminus X$, if $i \geq 0$ is largest such that $s \restriction i \in X$, then $f_Y(s)$ is $f_X(s \restriction i)$ concatenated with a trivial tail.

Essentially, a trivial extension of a condition extends the domain without adding any extra information about what is happening on the larger domain. We make an analogous definition for $\mathbb{Q}(\vec{\mathbb{P}}, \mathscr{T})$.

Proposition 9.5. *Suppose $G \subseteq \mathbb{P}(\vec{\mathbb{P}}, \mathscr{T})$ is V-generic and $f_X \in G$. If f_Y trivially extends f_X, then $f_Y \in G$. An analogous statement holds for $\mathbb{Q}(\vec{\mathbb{P}}, \mathscr{T})$.*

Proof. Let f_Y trivially extend f_X. It suffices to show that f_Y is compatible with every condition in G. So fix $g_Z \in G$. Let $h_W \leq f_X, g_Z$. Let $\bar{W} = W \cup Y$ and let $h_{\bar{W}}$ be the trivial extension of h_W to $\bar{W}$. Clearly, $h_{\bar{W}} \leq f_Y, g_Z$. The proof for $\mathbb{Q}(\vec{\mathbb{P}}, \mathscr{T})$ is analogous. □

Proposition 9.6. *Suppose $f_X \in \mathbb{P}(\vec{P}, \mathscr{T})$, $f_X(s) = p$, and $q \leq p$. Then there is a condition $g_X \leq f_X$ in $\mathbb{P}(\vec{P}, \mathscr{T})$ with $g_X(s) = q$. An analogous statement holds for $\mathbb{Q}(\vec{P}, \mathscr{T})$.*

Proof. Define g_X as follows. Fix a node $t \in X$. If $t \leq s$, then let $g_X(t) = q \restriction \mathrm{lev}(t)$. If $s \leq t$, then let $g_X(t)$ be q concatenated with the tail of $f_X(t)$. Otherwise, let t' be the highest node that is compatible with both t and s. Let $g_X(t)$ be $q \restriction \mathrm{lev}(t')$ concatenated with the tail of $f_X(t)$. The proof for $\mathbb{Q}(\vec{\mathbb{P}}, \mathscr{T})$ is analogous. □

Proposition 9.7. *Suppose $G \subseteq \mathbb{P}(\vec{\mathbb{P}}, \mathscr{T})$ is V-generic. Then for every node s on level n of $\mathscr{T}$, $G_s = \{f_X(s) \mid f_X \in G\}$ is V-generic for $\mathbb{P}_n$. An analogous statement holds for $\mathbb{Q}(\vec{\mathbb{P}}, \mathscr{T})$.*

Proof. Fix a node s on level n of $\mathscr{T}$. Suppose that $p \in G_s$ and $p \leq q$. Let $f_X \in G$ be such that $f_X(s) = p$. Define a condition g_X as follows. First, $g_X(s) = q$. If $t \leq s$, then $g_X(t) = q \restriction \mathrm{lev}(s)$. If $t \geq s$, then $g_X(t)$ is q concatenated with a trivial tail. Otherwise, let t' be the highest node that is compatible with both t and s, and set $g_X(t)$ to be $q \restriction \mathrm{lev}(t')$ concatenated with a trivial tail. Clearly, $g_X \geq f_X$. Thus, $g_X \in G$, and hence $q \in G_s$. Next, suppose that $p, q \in G_s$. Let $f_X \in G$ with $f_X(s) = p$ and $g_Y \in G$ with $g_Y(s) = q$. Then there is $h_Z \in G$ such that $h_Z \leq f_X, g_Y$. It follows that $h_Z(s) \in G_s$ and $h_Z(s) \leq p, q$. Finally, fix a maximal antichain $\mathcal{A}$ of $\mathbb{P}_n$. Let X_s be the stem in $\mathscr{T}$ ending in s. Consider the collection of conditions $f^a_{X_s}$, for $a \in \mathcal{A}$, defined by $f^a_{X_s}(s) = a$. It suffices to see that the conditions $f^a_{X_s}$ form a maximal antichain in $\mathbb{P}(\vec{\mathbb{P}}, \mathscr{T})$. So fix any condition $f_X \in \mathbb{P}(\vec{\mathbb{P}}, \mathscr{T})$. If $X \cap X_s = \{\emptyset\}$, then clearly f_X is compatible with any condition $f^a_{X_s}$. So suppose that $\emptyset \neq t \in X \cap X_s$, and t is on the highest level with this property. Extend $f_X(t)$ by adding a trivial tail, if necessary, to a get a condition $p \in \mathbb{P}_n$. Since $\mathcal{A}$ is maximal in $\mathbb{P}_n$, some $a \in \mathcal{A}$ is compatible with p. Let $q \leq p, a$. By Proposition 9.6, there is a condition $g_X \leq f_X$ such that $g_X(t) = q \restriction \mathrm{lev}(t)$. Let X' be X together with nodes from t to s (if any) and let $g_{X'}$ extend the condition g_X such that $g_{X'}(s) = q$. Clearly, $g_{X'} \leq g_X$ and $g_{X'} \leq f^a_{X_s}$ as well. □

Let us call a condition $f_X \in \mathbb{Q}(\vec{P}, \mathscr{T})$ *determined* if for every $s \in X$, $f_X(s)$ is determined.

Proposition 9.8. *The set of all determined conditions is dense in $\mathbb{Q}(\vec{P}, \mathscr{T})$.*

Proof. Fix $f_X \in \mathbb{Q}(\vec{P}, \mathscr{T})$, and for every node $s \in X$, let $f_X(s) = (p_s, F_s)$. We will construct a determined condition $g_X \leq f_X$ by taking a greatest lower bound of a descending sequence of conditions $g^{(n)}_X$ for $2 \leq n < \omega$ such that $g^{(n)}_X \leq f_X$ has determined conditions on all nodes on levels $\leq n$.

Enumerate the nodes on level 2 of X as $\{s_\xi \mid \xi < \beta_2\}$ for some $\beta_2 < \kappa$. Let $(p^0_{s_0}, F_{s_0}) \leq (p_{s_0}, F_{s_0})$ be any determined condition, which exists by Proposition 7.17. If $s_0 \restriction 1 \neq s_1 \restriction 1$, then we let $(p^1_{s_1}, F_{s_1}) \leq (p_{s_1}, F_{s_1})$ be any determined condition. Otherwise, $s_0 \restriction 1 = s_1 \restriction 1$ (and hence $p_{s_0} \restriction 1 = p_{s_1} \restriction 1$). Let $p'_{s_1} = (p^0_{s_0}, p_{s_1}(1)) \leq p_{s_1}$, and let $(p^1_{s_1}, F_{s_1}) \leq (p'_{s_1}, F_{s_1})$ be any determined condition. Now more generally, suppose inductively that for some $\eta < \beta_2$, we have chosen, for every $\xi < \eta$, determined conditions $(p^\xi_{s_\xi}, F_{s_\xi}) \leq (p_{s_\xi}, F_{s_\xi})$ such that if for $\xi_1 < \xi_2 < \eta$, $s_{\xi_1} \restriction 1 = s_{\xi_2} \restriction 1$, then $p^{\xi_2}_{s_{\xi_2}} \restriction 1 \leq p^{\xi_1}_{s_{\xi_1}} \restriction 1$. First, we define a condition $q_{s_\eta} \leq p_{s_\eta} \restriction 1$ as follows. If there is no $\xi < \eta$ such that $s_\xi \restriction 1 = s_\eta \restriction 1$, then $q_{s_\eta} = p_{s_\eta} \restriction 1$. Otherwise, let q_{s_η} be below the descending sequence of conditions $p^\xi_{s_\xi} \restriction 1$ for $\xi < \eta$ such that $s_\xi \restriction 1 = s_\eta \restriction 1$. Let $p'_{s_\eta} = (q_{s_\eta}, p_{s_\eta}(1))$, and let $(p^\eta_{s_\eta}, F_\eta) \leq (p'_{s_\eta}, F_{s_\eta})$ be any determined condition. Next, let $s = s_\eta$ be any node on level 2 of X. We will define the corresponding condition $p^*_s \leq p_s$ as follows. Let $p^*_s(0)$ be the intersection of all conditions $p^\xi_{s_\xi}(0)$ with $s_\xi \restriction 1 = s \restriction 1$ and let $p^*_s(1) = p^\eta_{s_\eta}(1)$. Observe that by Proposition 7.8, the condition (p^*_s, F_s) is determined. Let $g^{(2)}_X$ be the condition, where we start with f_X and for every node s on level 2 of X, we replace (p_s, F_s) with (p^*_s, F_s), and change the conditions on the other levels in the obvious ways to maintain coherence.

Suppose inductively that we have defined a descending sequence of conditions $g^{(j)}_X$ for $2 \leq j \leq n$ such that all conditions on level j of $g^{(j)}_X$ are determined, and for every node $s \in X$, we still have $g^{(j)}_X(s)(1) = F_s$. For a node $s \in X$ and $j \leq n$, let $g^{(j)}_X(s) = (p_{(j,s)}, F_s)$.

Enumerate the nodes on level $n+1$ of X as $\{s_\xi \mid \xi < \beta_{n+1}\}$ for some $\beta_{n+1} < \kappa$. Let $(p^0_{(n,s_0)}, F_{s_0}) \leq (p_{(n,s_0)}, F_{s_0})$ be any determined condition. Suppose inductively that for some $\eta < \beta_{n+1}$, we have chosen, for every $\xi < \eta$, determined conditions $(p^\xi_{(n,s_\xi)}, F_{s_\xi}) \leq (p_{(n,s_\xi)}, F_{s_\xi})$ such that if for $\xi_1 < \xi_2 < \eta$, there is $1 \leq k \leq n$ such that $s_{\xi_1} \upharpoonright k = s_{\xi_2} \upharpoonright k$, then we have $p^{\xi_2}_{(n,s_{\xi_2})} \upharpoonright k \leq p^{\xi_1}_{(n,s_{\xi_1})} \upharpoonright k$. First we define a condition $q_{s_\eta} \leq p_{(n,s_\eta)} \upharpoonright n$ as follows. If there is no $\xi < \eta$ such that $s_\xi \upharpoonright 1 = s_\eta \upharpoonright 1$, then $q_{s_\eta} = p_{(n,s_\eta)} \upharpoonright n$. Otherwise, let $q_{s_\eta}(0)$ be the intersection of all $p^\xi_{(n,s_\xi)}(0)$, for $\xi < \eta$, with $s_\eta \upharpoonright 1 = s_\xi \upharpoonright 1$. Suppose inductively that we have defined $q_{s_\eta} \upharpoonright m$ for some $m < n$ such that if for some $\xi < \eta$, there is $k \leq m$ such that $s_\xi \upharpoonright k = s_\eta \upharpoonright k$, then $q_{s_\eta} \upharpoonright k \leq p^\xi_{(n,s_\xi)} \upharpoonright k$. If there is no $\xi < \eta$ such that $s_\xi \upharpoonright m+1 = s_\eta \upharpoonright m+1$, then we let q_{s_η} be $q_{s_\eta} \upharpoonright m$ concatenated with the tail of $p_{(n,s_\eta)} \upharpoonright n$. Otherwise, let $q_\eta(m)$ be a name for the intersection of all $p^\xi_{(n,s_\xi)}(m)$ with $s_\xi \upharpoonright m{+}1 = s_\eta \upharpoonright m{+}1$. Let $p'_{(n,s_\eta)}$ be q_{s_η} concatenated with $p_{(n,s_\eta)}(n)$, and let $(p^\eta_{(n,s_\eta)}, F_{s_\eta}) \leq (p'_{(n,s_\eta)}, F_{s_\eta})$ be any determined condition. Next, let $s = s_\eta$ be any node on level $n+1$ of X. We will define the corresponding condition $p^*_{(n,s)} \leq p_{(n,s)}$ as follows. Let $p^*_{(n,s)}(0)$ be the intersection of the $p^\xi_{(n,s_\xi)}(0)$ over all $\xi < \beta_{n+1}$ with $s_\xi \upharpoonright 1 = s \upharpoonright 1$. Suppose inductively that we have defined $p^*_{(n,s_\eta)} \upharpoonright m$ for some $m < n$ such that if for some $\xi < \beta_{n+1}$, there is $k \leq m$ such that $s_\xi \upharpoonright k = s \upharpoonright k$, then $p^*_{(n,s)} \upharpoonright k \leq p^\xi_{(n,s_\xi)} \upharpoonright k$. Then we can let $p^*_{(n,s)}(m)$ be the name for the intersection of all $p^\xi_{(n,s_\xi)}(m)$, for $\xi < \beta_{n+1}$, with $s_\xi \upharpoonright m+1 = s \upharpoonright m+1$. Finally, let $p^*_{(n,s)}(n) = p^\eta_{(n,s_\eta)}(n)$. By Proposition 7.8, the condition (p^*_s, F_s) is determined. Let $g^{(n+1)}_X$ be the condition, where we start with $g^{(n)}_X$ and for every node s on level n of X, we replace $(p_{(n,s)}, F_s)$ with $(p^*_{(n,s)}, F_s)$, and change the conditions on the other levels to maintain coherence.

Now, let g_X be a greatest lower bound of the descending sequence

$$\{g^{(n)}_X \mid n < \omega\}.$$

By Proposition 7.8, for every node $s \in X$, $g_X(s)$ is determined. □

We will apply the construction of the above proof repeatedly in the next section whenever we have a condition f_X in a poset $\mathbb{P}(\vec{P}, \mathscr{T})$ (or $\mathbb{Q}(\vec{P}, \mathscr{T})$) and we would like to strengthen it by strengthening all the conditions $f_X(s)$ to satisfy some dense property of the posets $\mathbb{P}_n$ (or $\mathbb{Q}(\mathbb{P}_n)$). We will carry out the construction by building a descending sequence of conditions $g^{(n)}_X$, where all nodes on level n of X have the required dense property. The condition $g^{(n)}_X$ will be built by strengthening the nodes on level n coherently to maintain the tree structure.

We will initially consider tree iterations along the tree $\kappa^{<\omega}$, and then later extend our results to tree iterations along the tree $(\kappa^+)^{<\omega}$.

It is easy to see that the poset $\mathbb{Q}(\mathbb{Q}_0)^{<\kappa}$ completely embeds into $\mathbb{Q}(\vec{P},\kappa^{<\omega})$ via the map sending a condition to the corresponding tree of height ≤ 2.

Proposition 9.9. *The poset $\mathbb{Q}(\mathbb{Q}_0)^{<\kappa}$ completely embeds into $\mathbb{Q}(\vec{P},\kappa^{<\omega})$ via the following map φ:*

(1) *$\varphi(\mathbb{1}_{\mathbb{Q}(\mathbb{Q}_0)^{<\kappa}}) = f_X$, where X consists of the root node s and $f_X(s)=\emptyset$.*
For $p \neq \mathbb{1}_{\mathbb{Q}(\mathbb{Q}_0)^{<\kappa}}$,
(2) *$\varphi(p) = f_X$, where X consists of the root node together with nodes $\langle\xi\rangle$ for $p(\xi)\neq \mathbb{1}_{\mathbb{Q}_0}$, such that $f_X(\langle\xi\rangle) = p(\xi)$.*

More generally, for each node s on level n of $\kappa^{<\omega}$, $\bar{\mathbb{Q}}(\mathbb{P}_{n+1})$ completely embeds into $\mathbb{Q}(\vec{P},\kappa^{<\omega})$ via the map sending a condition to the corresponding tree of height $\leq n+2$ whose stem stretches up to s.

Proposition 9.10. *Fix a node s on level n of $\kappa^{<\omega}$. The poset $\bar{\mathbb{Q}}(\mathbb{P}_{n+1})$ completely embeds into $\mathbb{Q}(\vec{P},\kappa^{<\omega})$ via the following map φ_s:*

(1) *$\varphi_s(\mathbb{1}_{\bar{\mathbb{Q}}(\mathbb{P}_{n+1})}) = f_X$, where X is the branch ending in s, such that $f_X(s) = \mathbb{1}_{\mathbb{Q}(\mathbb{P}_n)}$.*
For $(p,F)\neq \mathbb{1}_{\bar{\mathbb{Q}}(\mathbb{P}_{n+1})}$,
(2) *$\varphi_s((p,F)) = f_X$, where X consists of the branch ending in s together with nodes $\langle s^\frown\xi\rangle$ for non-trivial $(p(n)(\xi),F(n)(\xi))$, such that $f_X(s) = (p\restriction n, F\restriction n)$ and*
$$f_X(s^\frown\xi) = (p\restriction n^\frown p(n)(\xi), F\restriction n^\frown F(n)(\xi)).$$

Suppose $G\subseteq \mathbb{Q}(\vec{P},\kappa^{<\omega})$ is V-generic and fix some node s on level n of $\kappa^{<\omega}$. We will use the notation G_s for the V-generic filter for $\bar{\mathbb{Q}}(\mathbb{P}_{n+1})$ added by G via the embedding φ_s and we will use the notation $G_\emptyset$ for the V-generic filter for $\mathbb{Q}(\mathbb{Q}_0)^{<\kappa}$ added by G via the embedding φ.

10. Growing finite iterations of κ-perfect posets: part II

Suppose that $\vec{P} = \langle \mathbb{P}_n \mid n<\omega\rangle$ is an ω-iteration of κ-perfect posets. Let $G\subseteq \mathbb{Q}(\vec{P},\kappa^{<\omega})$ be V-generic. For the remainder of the section we will work in $V[G]$.

We will argue that we can grow each iteration $\mathbb{P}_n$ to an iteration $\mathbb{P}^*_n$ of κ-perfect posets satisfying requirements (1)-(5) from Section 8:

(1) $\mathbb{Q}_0\subseteq \mathbb{Q}^*_0$,
(2) For all $1\leq i<n$, $\mathbb{1}_{\mathbb{P}^*_i}$ forces that $\dot{\mathbb{Q}}_i$ is a κ-perfect poset and $\dot{\mathbb{Q}}^*_i$ extends it.
(3) $\mathbb{P}_n\subseteq \mathbb{P}^*_n$.
(4) Every maximal antichain $\mathcal{A}\in V$ of $\mathbb{P}_n$ remains maximal in $\mathbb{P}^*_n$.
(5) Every $V[G]$-generic filter $H^*\subseteq \mathbb{P}^*_n$ restricts to a V-generic filter H for $\mathbb{P}_n$.

It is straightforward to extend $\mathbb{Q}_0$ to $\mathbb{Q}^*_0$. By Proposition 9.9, G adds a V-generic filter $G_\emptyset$ for $\mathbb{Q}(\mathbb{Q}_0)^{<\kappa}$. Let $\mathcal{T}^0_\xi$ for $\xi<\kappa$ be the generic perfect κ-trees added by $G_\emptyset$ and construct $\mathbb{Q}^*_0$ as in Section 3. Thus, $\mathbb{Q}^*_0$ is a κ-perfect poset in $V[G_\emptyset]$. Since $V^{<\kappa}\subseteq V$ in $V[G]$, it follows that $V[G_\emptyset]^{<\kappa}\subseteq V[G_\emptyset]$ in $V[G]$ by Proposition 4.3. Thus, $\mathbb{Q}^*_0$ is a κ-perfect poset in $V[G]$. Recall that $\{\mathcal{T}^0_\xi\mid \xi<\kappa\}$ is a maximal antichain in $\mathbb{Q}^*_0$ and every maximal antichain $\mathcal{A}\in V$ of $\mathbb{Q}_0$ remains maximal in $\mathbb{Q}^*_0$. Since a $V[G]$-generic filter H^* for $\mathbb{Q}^*_0$ is, in particular, $V[G_\emptyset]$-generic for $\mathbb{Q}^*_0$, by Proposition 3.11, H^* restricts to a V-generic filter for $\mathbb{Q}_0$.

Now let's show how to extend $\dot{\mathbb{Q}}_1$ to a $\mathbb{P}_1^*$-name $\dot{\mathbb{Q}}_1^*$ for a κ-perfect poset. By Proposition 9.10, for each node s on level 1 of the tree $\kappa^{<\omega}$, G adds a V-generic filter G_s for $\bar{\mathbb{Q}}(\mathbb{P}_2)$. Observe that each $\mathcal{T}_\xi^0 \leq p \upharpoonright 1$ for all p with $(p,F) \in G_{\langle\xi\rangle}$. Thus, by Theorem 8.3, whenever $H^* \subseteq \mathbb{P}_1^*$ is a $V[G]$-generic filter containing $\mathcal{T}_\xi^0$, then the interpretation $\tau(G_{\langle\xi\rangle})_H$ is a $V[H]$-generic filter for $\mathbb{Q}((\dot{\mathbb{Q}}_1)_H)^{<\kappa}$, where H is the restriction of H^* to $\mathbb{P}_1$. Let τ be a mixed $\mathbb{P}_1^*$-name that is interpreted as $\tau(G_{\langle\xi\rangle})_H$, whenever $\mathcal{T}_\xi^0 \in H^*$. Since the conditions $\mathcal{T}_\xi^0$ form a maximal antichain in $\mathbb{P}_1^*$,

$$\mathbb{1}_{\mathbb{P}_1^*} \Vdash \tau \text{ is an } V[\dot{H}]\text{-generic filter for } \mathbb{Q}(\dot{\mathbb{Q}}_1)^{<\kappa},$$

where $\dot{H}$ is the canonical name for the restriction of the generic filter to $\mathbb{P}_1$. So let $\dot{\mathbb{Q}}_1^*$ be a $\mathbb{P}_1^*$-name for the κ-perfect poset constructed from $\dot{\mathbb{Q}}_1$ and τ as in Section 3. This argument gives that $\dot{\mathbb{Q}}_1^*$ is forced to be a κ-perfect poset in $V[\dot{H}][\tau]$, and we again use a closure argument together with Proposition 3.11 to conclude that it is forced to be a κ-perfect poset in any $\mathbb{P}_1^*$-extension of $V[G]$.

For each $\eta < \kappa$, we can choose a $\mathbb{P}_1^*$-name $\dot{\mathcal{T}}_\eta^1$ for the perfect κ-tree on coordinate η of τ. The pairs $(\mathcal{T}_\xi^0, \dot{\mathcal{T}}_\eta^1)$ for $\xi, \eta < \kappa$ form a maximal antichain in $\mathbb{P}_2^*$ because the collection of $\mathcal{T}_\xi^0$ is a maximal antichain and each $\mathcal{T}_\xi^0$ forces that the trees $\dot{T}_\eta^1$, for $\eta < \kappa$, form a maximal antichain in $\dot{\mathbb{Q}}_1^*$. Also, clearly for every $(p,F) \in G_{\langle\xi,\eta\rangle}$, we have $(\mathcal{T}_\xi^0, \dot{\mathcal{T}}_\eta^1) \leq p \upharpoonright 2$. This will allow us to apply Theorem 8.3 to grow $\dot{\mathbb{Q}}_2$ to $\dot{\mathbb{Q}}_2^*$. Finally, let's argue that $\mathbb{P}_2$ is actually a subset of $\mathbb{P}_2^*$. Suppose p is a condition in $\mathbb{P}_2$. Then $p(0) \in \mathbb{P}_1$, and hence $p(0) \in \mathbb{P}_1^*$ as well. Also, clearly $p(1)$ is a $\mathbb{P}_1^*$-name. So we need to argue that $p(0) \Vdash_{\mathbb{P}_1^*} p(1) \in \dot{\mathbb{Q}}_1^*$. Fix a $V[G]$-generic filter $H^* \subseteq \mathbb{P}_1^*$ with $p(0) \in H^*$, and consider the extension $V[H]$, where H is the restriction of H^* to $\mathbb{P}_1$. Since $p(0) \Vdash_{\mathbb{P}_1} p(1) \in \dot{\mathbb{Q}}_1$, in $V[H]$, we have $p(1)_H \in (\dot{\mathbb{Q}}_1)_H \subseteq (\dot{\mathbb{Q}}_1^*)_{H^*}$. It remains to show that conditions (4) and (5) hold. We will provide an inductive proof of this later in Theorem 10.1.

To get some intuition for the construction, let's fix a $V[G]$-generic filter $H^* \subseteq \mathbb{P}_1^*$ and see what a condition $\mathcal{T}_\eta^1 = (\dot{\mathcal{T}}_\eta^1)_{H^*}$ looks like. Since $\{\mathcal{T}_\xi^0 \mid \xi < \kappa\}$ is a maximal antichain of $\mathbb{P}_1^*$, there is a unique ξ such that $\mathcal{T}_\xi^0 \in H^*$. Let

$$K = \{(p(1)_H, F(1)) \mid f_X \in G \text{ and } f_X(\langle\xi,\eta\rangle) = (p,F)\}.$$

Then $\mathcal{T}_\eta^1$ is the union of $T \cap 2^\alpha$ for $(T,\alpha) \in K$.

For now to finish the construction, we assume that properties (1)-(5) hold for $\mathbb{P}_n^*$. We will additionally assume that:

(1) For each node s on level n of $\kappa^{<\omega}$ there is a condition
$$(\mathcal{T}_{s(0)}^0, \dot{\mathcal{T}}_{s(1)}^1, \ldots, \dot{\mathcal{T}}_{s(n-1)}^{n-1}) \in \mathbb{P}_n^*$$
which is below all $p \upharpoonright n$ with $(p,F) \in G_s$.

(2) The collection of all such conditions $(\mathcal{T}_{s(0)}^0, \dot{\mathcal{T}}_{s(1)}^1, \ldots, \dot{\mathcal{T}}_{s(n-1)}^{n-1})$ is a maximal antichain in $\mathbb{P}_n^*$.

With this set-up, we extend $\dot{\mathbb{Q}}_n$ to $\dot{\mathbb{Q}}_n^*$ identically to the case $n = 1$ above, using Theorem 8.3. It is also easy to see inductively that $\mathbb{P}_n$ is a subset of $\mathbb{P}_n^*$.

To give the promised argument for conditions (4) and (5), we first need to define the analogue of $\mathbb{U} \subseteq \mathbb{P}^*$ from Section 2 for $\mathbb{P}_n^*$. For $n = 1$, let $\mathbb{U}_1 = \mathbb{U}$. For $n > 1$, let $\mathbb{U}_n$ be the collection of all conditions $p = (q_0, \ldots, \dot{q}_{n-1})$ such that

(1) there are $\xi_0 < \kappa$ and $t_0 \in 2^{<\kappa}$ such that $p_0 = (\mathcal{T}_{\xi_0}^0)_{t_0}$,

(2) for all $1 \leq i < n$, there are $\xi_i < \kappa$ and $t_i \in 2^{<\kappa}$ such that $p \restriction i \Vdash \dot{q}_i = (\dot{\mathcal{T}}^i_{\xi_i})_{t_i}$.

We will abuse notation by writing conditions in $\mathbb{U}_n$ as

$$p = ((\mathcal{T}^0_{\xi_0})_{t_0}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi_{n-1}})_{t_{n-1}}).$$

Given $\sigma : n \to 2^{<\kappa}$ and $\tau : n \to 2^{<\kappa}$, we will say that σ *extends* τ if for every $0 \leq i < n$, $\mathrm{lev}(\sigma(i)) \geq \mathrm{lev}(\tau(i))$ and $\sigma(i) \restriction \mathrm{lev(i)} = \tau(i)$. Suppose that $X \subseteq \kappa^{<\omega}$ is a tree. We will say that a collection $S = \{\sigma_s \mid s \in X\}$ of $\sigma_s : n \to 2^{<\kappa}$ for s on level n of X is *coherent* if whenever $s \leq t$ are nodes in X, then $\sigma_s \restriction \mathrm{lev}(s) = \sigma_t$. Given coherent collections $S = \{\sigma_s \mid s \in X\}$, $T = \{\tau_s \mid s \in X\}$, we will say that S *extends* T if for every $s \in X$, σ_s extends τ_s.

Theorem 10.1.

(1) *$\mathbb{U}_n$ is dense in $\mathbb{P}^*_n$.*
(2) *Every maximal antichain of $\mathbb{P}_n$ from V remains maximal in $\mathbb{P}^*_n$.*
(3) *Every $V[G]$-generic filter $H^* \subseteq \mathbb{P}^*_n$ restricts to a V-generic filter H for $\mathbb{P}_n$.*

Proof. We can suppose inductively that (1), (2), and (3) hold for some n because we already showed that these conditions hold for $n = 1$.

First, we prove (1). Fix $p \in \mathbb{P}^*_{n+1}$. By construction, there is $q \leq p \restriction n$ such that for some $\xi_n < \kappa$ and $t_n \in 2^{<\kappa}$,

$$q \Vdash (\dot{\mathcal{T}}^n_{\xi_n})_{t_n} \subseteq p(n).$$

By the inductive assumption, there is some condition $((\mathcal{T}^0_{\xi_0})_{t_0}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi_{n-1}})_{t_{n-1}}) \leq q$. Thus,

$$((\mathcal{T}^0_{\xi_0})_{t_0}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi_{n-1}})_{t_{n-1}}, (\dot{\mathcal{T}}^n_{\xi_n})_{t_n}) \leq p.$$

Next, let's prove (2). Let $\mathcal{A} \in V$ be a maximal antichain of $\mathbb{P}_{n+1}$. By (1), it suffices to show that every element $((\mathcal{T}^0_{\xi_0})_{t_0}, \ldots, (\dot{\mathcal{T}}^n_{\xi_n})_{t_n}) \in \mathbb{U}_{n+1}$ is compatible with $\mathcal{A}$. Let $s = \langle \xi_0, \xi_1, \ldots, \xi_n \rangle$ and let $\sigma = \langle t_0, t_1, \ldots, t_n \rangle$.

Let $f_X \in \mathbb{Q}(\vec{P}, \kappa^{<\omega})$. By strengthening f_X, if necessary, we can assume that $s \in X$. Let $f_X(s) = (p, F)$. By strengthening f_X, if necessary, we can assume that for all $0 \leq i < n+1$, $F(i) \geq \mathrm{lev}(t_i)$. Finally, by strengthening f_X, if necessary, we can assume that f_X is determined. If (p, F) does not have a $\bar{\sigma}$ lying on it that extends σ, then we let $\bar{f}_X = f_X$. So suppose that there is some such $\bar{\sigma}$. Since $\mathcal{A}$ is maximal in $\mathbb{P}_{n+1}$, there is $a \in \mathcal{A}$ compatible with $p \mid \bar{\sigma}$. So we can choose some $r \leq p \mid \bar{\sigma}, a$. By Proposition 7.14, there is a condition $(\bar{p}, \bar{F}) \leq (p, F)$ such that $\bar{p} \mid \bar{\sigma} = r \leq a$. By Proposition 9.6 (2), there is a condition $\bar{f}_X \leq f_X$ such that $\bar{f}_X(s) = (\bar{p}, \bar{F})$. Since conditions $\bar{f}_X$ are dense in $\mathbb{Q}(\vec{P}, \kappa^{<\omega})$, some such condition $\bar{f}_X \in G$. Since σ lies on $(\mathcal{T}^0_{\xi_0}, \ldots, \dot{\mathcal{T}}^n_{\xi_n})$ and (p, F) is determined, it follows that there must be some $\bar{\sigma}$ lying on (p, F) extending σ. Thus, we are in the second case, and so $\bar{f}_X(s) \mid \bar{\sigma} \leq a$ for some $a \in \mathcal{A}$. Thus, $((\mathcal{T}^0_{\xi_0})_{\bar{\sigma}(0)}, \ldots, (\dot{\mathcal{T}}^n_{\xi_n})_{\bar{\sigma}(n)}) \leq a$.

Finally, let's prove (3). Suppose that $H^* \subseteq \mathbb{P}^*_{n+1}$ is $V[G]$-generic. We have that $V[G][H^*] = V[G][K^*][k^*]$, where

(1) $K^* = \{p \restriction n \mid p \in H^*\}$ is $V[G]$-generic for $\mathbb{P}^*_n$,
(2) $k^* = \{p(n)_{K^*} \mid p \in H^*\}$ is $V[G][K^*]$-generic for $\mathbb{Q}^*_n = (\dot{\mathbb{Q}}^*_n)_{K^*}$.

By the inductive assumption, K^* restricts to a V-generic filter K for $\mathbb{P}_n$. Let $\mathbb{Q}_n = (\dot{\mathbb{Q}}_n)_K$. Then $\mathbb{Q}^*_n$ is constructed in $V[K][\bar{G}]$, where $\bar{G}$ is the $V[K]$-generic filter for $\mathbb{Q}(\mathbb{Q}_n)^{<\kappa}$ constructed as above in $V[G][K^*]$. Thus, k^* restricts to a $V[K]$-generic filter k for $\mathbb{Q}_n$.

Let H be the restriction of H^* to $\mathbb{P}_{n+1}$. Since every maximal antichain of $\mathbb{P}_{n+1}$ from V remains maximal in $\mathbb{P}^*_{n+1}$, it remains to check that H is a filter. Clearly, H is upward closed. So suppose that $p, q \in H$. Then $p \upharpoonright n, q \upharpoonright n \in K$ and $p(n)_K, q(n)_K \in k$. Thus, there is a condition $u \in k$ such that $u \leq p(n)_K, q(n)_K$. Let $\dot{u}$ be a $\mathbb{P}_n$-name such that $\dot{u}_K = u$, and let $r \leq p \upharpoonright n, q \upharpoonright n$ be a condition in K forcing that $\dot{u} \leq p(n), q(n)$. Let $r = r^* \upharpoonright n$ with $r^* \in H^*$. Also, since $u \in k$, there must be a condition $a \in H^*$ such that $a(n)_{K^*} = u$. Thus, there is a condition $b \in K^*$ forcing that $\dot{u} = a(n)$. Let $b = b^* \upharpoonright n$ with $b^* \in H^*$. Let $c \leq r^*, a, b^*$ be in H^*. Let's argue that $c \leq r^\frown \dot{u}$. We have $c \upharpoonright n \leq r$ by construction. We have $c \upharpoonright n \leq b$, which means that $c \upharpoonright n \Vdash \dot{u} = a(n)$. Also, we have $c \leq a$, which means $c \upharpoonright n \Vdash c(n) \leq a(n)$. Thus, $c \upharpoonright n \Vdash c(n) \leq \dot{u}$. Thus, $r^\frown \dot{u} \in H^*$, and also, by construction, $r^\frown \dot{u} \leq p, q$. □

Let
$$\vec{P}^* = \langle \mathbb{P}^*_n \mid n < \omega \rangle$$
be the ω-iteration made up of the extended iterations $\mathbb{P}^*_n$. Let $\vec{\mathbb{U}}$ be the subset of $\mathbb{P}(\vec{P}^*, \kappa^{<\omega})$ consisting of conditions f_X such that for all $s \in X$ on level n, $f_X(s) \in \mathbb{U}_n$.

Proposition 10.2. *$\vec{\mathbb{U}}$ is dense in $\mathbb{P}(\vec{P}^*, \kappa^{<\omega})$.*

Proof. Let $f_X \in \mathbb{P}(\vec{P}^*, \kappa^{<\omega})$. For every node s on level 1 of X, let
$$(\mathcal{T}^0_{\xi^s_0})_{t^s_{0,1}} \leq f_X(s).$$
Let $g^{(1)}_X \leq f_X$ be the condition we get by replacing $f_X(s)$ with $(\mathcal{T}^0_{\xi^s_0})_{t^s_{0,1}}$ for every node s level 1 of X, and changing the conditions on the upper levels to maintain coherence.

Next, we enumerate the nodes on level 2 of X as $\{s_\eta \mid \eta < \beta\}$ for some $\beta < \kappa$. We can clearly choose a condition
$$((\mathcal{T}^0_{\xi^{s_0 \upharpoonright 1}_0})_{t^0_0}, (\dot{\mathcal{T}}^1_{\xi^{s_0}_1})_{t^0_1}) \leq g^{(1)}_X(s_0)$$
with $t^0_0 \geq t^{s_0\upharpoonright 1}_{0,1}$. Suppose inductively that for some $\eta < \nu < \beta$, we have chosen conditions
$$((\mathcal{T}^0_{\xi^{s_\eta \upharpoonright 1}_0})_{t^\eta_0}, (\dot{\mathcal{T}}^1_{\xi^{s_\eta}_1})_{t^\eta_1}) \leq g^{(1)}_X(s_\eta)$$
such that for $\eta_1 < \eta_2 < \nu$, if $s_{\eta_1} \upharpoonright 1 = s_{\eta_2} \upharpoonright 1$, then $t^{\eta_2}_0 \geq t^{\eta_1}_0$. Let
$$((\mathcal{T}^0_{\xi^{s_\nu \upharpoonright 1}_0})_{t^\nu_0}, (\dot{\mathcal{T}}^1_{\xi^{s_\nu}_1})_{t^\nu_1}) \leq g^{(1)}_X(s_\nu)$$
be such that $t^\nu_0 \geq t^{s_\nu\upharpoonright 1}_{0,1}$ is above all t^η_0 with $s_\eta \upharpoonright 1 = s_\nu \upharpoonright 1$ (if any exist).

For every node s on level 1 of X, let $t^s_{0,2}$ be above all t^η_0 with $s = s_\eta \upharpoonright 1$, and note that $t^s_{0,2} \geq t^s_{0,1}$. For every node $s = s_\eta$ on level 2 of X, let $t^s_{1,2} = t^\eta_1$. Let $g^{(2)}_X \leq g^{(1)}_X$ be the condition we get by replacing $g^1_X(s)$ with $((\mathcal{T}^0_{\xi^{s\upharpoonright 1}_0})_{t^{s\upharpoonright 1}_{0,2}}, (\dot{\mathcal{T}}^1_{\xi^s_1})_{t^s_{1,2}})$ for every node s on level 2 of X, and changing the conditions on the other levels to maintain coherence.

We proceed as in the proof of Proposition 9.8 to construct a descending sequence of conditions $g^{(n)}_X$ for $1 \leq n < \omega$ such that for every node s on level n of $g^{(n)}_X$, $g^{(n)}_X(s) \in \mathbb{U}_n$. When doing this argument for higher levels, we make use of the fact that $\mathbb{U}_n$ is dense in $\mathbb{P}_n$ to obtain the desired conditions. For every node s on level n of X and $i<n$, let t_i^s be above all $t_{i,m}^s$ for $n\leq m<\omega$, and let

$$g_X(s)=((\mathcal{T}^0_{\xi_0^s})_{t_0^s},\ldots,(\dot{\mathcal{T}}^{n-1}_{\xi^s_{n-1}})_{t^s_{n-1}}).$$

Clearly, $g_X\in\vec{\mathbb{U}}$ is below f_X.

□

Lemma 10.3. *Every maximal antichain of* $\mathbb{P}(\vec{P},\kappa^{<\omega})$ *from* V *remains maximal in* $\mathbb{P}(\vec{P}^*,\kappa^{<\omega})$.

Proof. Let $\mathcal{A}$ be a maximal antichain of $\mathbb{P}(\vec{P},\kappa^{<\omega})$ from V. By Proposition 10.2, it suffices to show that every element of $\vec{\mathbb{U}}$ is compatible with $\mathcal{A}$. Fix $f^*_X\in\vec{\mathbb{U}}$, and for a node s on level n of X, let

$$f^*_X(s)=((\mathcal{T}^0_{\xi_0^s})_{t_0^s},\ldots,(\dot{\mathcal{T}}^{n-1}_{\xi^s_{n-1}})_{t^s_{n-1}}).$$

For every node $s\in X$ of length $n<\omega$, let

$$\sigma_s=\langle t_0^s,\ldots,t^s_{n-1}\rangle$$

and let

$$\bar{s}=\langle\xi_0^s,\ldots,\xi^s_{n-1}\rangle.$$

Observe that if $s\restriction i=t\restriction i$ for some $1\leq i<n$, then $\bar{s}\restriction i=\bar{t}\restriction i$ and $\sigma_s\restriction i=\sigma_t\restriction i$. First, we are going to thin out the condition f^*_X to a condition g_X having the property that if i is such that $s\restriction i=t\restriction i$ and $s(i)\neq t(i)$, then either $\bar{s}(i)\neq\bar{t}(i)$ or $\sigma_s(i)$ and $\sigma_t(i)$ are incompatible nodes. The condition g_X is constructed essentially as in the proof of Proposition 10.2 by working with the levels of X successively.

Fix some $\xi=\xi_0^r$ for a node r on level 1 of X. Let α_ξ be a large enough level of $\mathcal{T}^0_\xi$ such that for every node s on level 1 of X with $\xi_0^s=\xi$, there is a node $t^s_{0,1}$ on level α_ξ above t_0^s so that if $s_1\neq s_2$, then $t^{s_1}_{0,1}\neq t^{s_2}_{0,1}$. Repeat this for every $\xi=\xi_0^s$ for some node s on level 1 of X. Let $g_X^{(1)}\leq f^*_X$ be the condition we get by replacing $f^*_X(s)$ with $(\mathcal{T}_{\xi_0^s})_{t^s_{0,1}}$ for every node s on level 1 of X, and changing the conditions on the other levels to maintain coherence.

Next, we define the condition $g^{(2)}_X\leq g^{(1)}_X$ by modifying conditions on nodes on level 2 of X. Let $\{\bar{s}_\eta\mid\eta<\rho\}$, with $\rho<\kappa$, be an enumeration of all sequences $\bar{s}$ for s a node on level 2 of X. Consider $\bar{s}_0=\langle\xi_0^{s_0},\xi_1^{s_0}\rangle$. Let

$$\alpha_0=\xi_0^{s_0},\ t_0=t^{s_0}_{0,1},\ \text{and}\ \beta_0=\xi_1^{s_0}.$$

Let $G\subseteq\mathbb{P}_1$ be a V-generic filter with $(\mathcal{T}^0_{\alpha_0})_{t_0}\in G$ and let $\mathcal{T}^1_{\beta_0}=(\dot{\mathcal{T}}^1_{\beta_0})_G$. In $V[G]$, choose a large enough level γ_0 of $\mathcal{T}^1_{\beta_0}$ such that for every node s on level 2 of X with $\bar{s}=\langle\alpha_0,\beta_0\rangle$ and $t^s_{0,1}=t_0$, there a node $t^s_{1,2}\geq t^s_1$ on level γ_0 so that if $s_1\neq s_2$, then $t^{s_1}_{1,2}\neq t^{s_2}_{1,2}$. Let $t^*_0\geq t_0$ be a node such that $(\mathcal{T}^0_{\alpha_0})_{t^*_0}$ forces the above statement about level γ_0. More generally, given $\bar{s}_\eta$, we let $\alpha_\eta=\xi_0^{s_\eta}$, $t_\eta=t^{s_\eta}_{0,1}$, and $\beta_\eta=\xi_1^{s_\eta}$. Suppose inductively that for some $\nu<\rho$, we have chosen for every $\eta<\nu$ a level γ_η and a node $t^*_\eta\geq t_\eta$ such that $(\mathcal{T}^0_{\alpha_\eta})_{t^*_\eta}$ forces that on level γ_η, for every node s with $\bar{s}=\langle\alpha_\eta,\beta_\eta\rangle$ there is a node $t^s_{1,2}\geq t^s_1$ so that if $s_1\neq s_2$, then $t^{s_1}_{1,2}\neq t^{s_2}_{1,2}$. We will assume that the choices so far satisfy that if $\eta_1<\eta_2<\nu$ are such that $\alpha_{\eta_1}=\alpha_{\eta_2}$ and $t_{\eta_1}=t_{\eta_2}$, then $t^*_{\eta_2}\geq t^*_{\eta_1}$. We choose a node $t^*_\nu\geq t_\nu$ above all t^*_η with $\eta<\nu$ such that $\alpha_\eta=\alpha_\nu$ and $t_\eta=t_\nu$ (if any exist) and a level γ_ν as above. Now let s be any node on level 1 of X. Let $t^s_{0,2}$ be above all t^*_η with $\xi^s_0=\alpha_\eta$ and $t_\eta=t^s_{0,1}$. Let

$g_X^{(2)} \leq g_X^{(1)}$ be the condition we get by replacing $g_X^{(1)}(s)$ with $((\mathcal{T}^0_{\xi_0^s})_{t^{s\restriction 1}_{0,2}}, (\dot{\mathcal{T}}^1_{\xi_1^s})_{t^s_{1,2}})$ for every node s on level 2 of X, and changing the other nodes to maintain coherence.

We proceed in this fashion to construct a descending sequence of conditions $g_X^{(n)}$ for $1 \leq n < \omega$ such that all nodes on level n of X have the required disjointness property. When doing this argument for higher levels, we make use of the fact that $\mathbb{U}_n$ is dense in $\mathbb{P}_n$ to obtain the desired conditions. For every node s on level n of X and $i < n$, let t_i^{s*} be above all $t^s_{i,m}$ for $n \leq m < \omega$, and let

$$g_X(s) = ((\mathcal{T}^0_{\xi_0^s})_{t_0^{s*}}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi^s_{n-1}})_{t^{s*}_{n-1}}).$$

Clearly, the condition g_X has the required disjointness property. Thus, by thinning out f^*_X, we can assume without loss of generality that f^*_X already has the required disjointness property, which we will from now refer to as the *incompatibility requirement*.

Observe that even though the condition f^*_X is not an element of V, the following are all in V by closure.

(1) X,
(2) $\{\sigma_s \mid s \in X\}$,
(3) $\{\bar{s} \mid s \in X\}$.

Fix $g_Y \in \mathbb{Q}(\vec{P}, \kappa^{<\omega})$ and let $g_Y(s) = (p_s, F_s)$. By strengthening g_Y, if necessary, we can assume that $\bar{s} \in Y$ for every $s \in X$, g_Y is determined, and for every $s \in X$ on level n, for every $0 \leq i < n$, $F_{\bar{s}}(i) \geq \text{lev}(\sigma_s(i))$. If there is no coherent collection $\{\bar{\sigma}_s \mid s \in X\}$ extending $\{\sigma_s \mid s \in X\}$ such that for every $s \in X$, $\bar{\sigma}_s$ lies on $g_Y(\bar{s})$, then we let $\bar{g}_Y = g_Y$. So suppose that there is such a coherent collection $\{\bar{\sigma}_s \mid s \in X\}$. Define a condition f_X by $f_X(s) = p_{\bar{s}} \mid \bar{\sigma}_s$, and observe that it is a valid condition in $\mathbb{P}(\vec{P}, \kappa^{<\omega})$ since (1) for $s, t \in X$, whenever $s \restriction i = t \restriction i$, then $\bar{s} \restriction i = \bar{t} \restriction i$, and (2) $\{\bar{\sigma}_s \mid s \in X\}$ is a coherent collection. Since $\mathcal{A}$ is maximal in $\mathbb{P}(\vec{P}, \kappa^{<\omega})$, there is a condition $a_{X'} \in \mathcal{A}$ compatible with f_X. Let b_B be a condition such that $b_B \leq f_X, a_{X'}$.

Let $\bar{X} = \{\bar{s} \mid s \in X\}$, and observe that it is a tree. Thus, $g_{\bar{X}}$, the restriction of g_Y to $\bar{X}$, is a valid condition in $\mathbb{P}(\vec{P}, \kappa^{<\omega})$. Fix a node $\bar{s} \in \bar{X}$ of length n. We are going to define an $X^{p_{\bar{s}}}_{F_{\bar{s}}}$-assignment $\varphi_{\bar{s}}$ with $\varphi_{\bar{s}}(\sigma) \leq p_{\bar{s}} \mid \sigma$ for every $\sigma \in X^{p_{\bar{s}}}_{F_{\bar{s}}}$. Fix $\sigma \in X^{p_{\bar{s}}}_{F_{\bar{s}}}$. Let i be largest such that there is some node $t \in X$ with $\bar{t} = \bar{s} \restriction i'$, for some $i \leq i' \leq n$, such that $\sigma \restriction i = \bar{\sigma}_t \restriction i$. Fix one such node t, and define $\varphi_{\bar{s}}(\sigma) = b_B(t) \restriction i$ concatenated with the tail of $p_{\bar{s}} \restriction \sigma$ after i. We need to argue that this definition does not depend on our choice of the node t. Suppose that there are nodes $t_1, t_2 \in X$ as above such that $\bar{\sigma}_{t_1} \restriction i = \bar{\sigma}_{t_2} \restriction i$. Then, by the incompatibility requirement, $t_1 \restriction i = t_2 \restriction i$, which means that $b_B(t_1) \restriction i = b_B(t_2) \restriction i$. This verifies that $\varphi_{\bar{s}}$ is well-defined. Next, let's check that $\varphi_{\bar{s}}$ is an $X^{p_{\bar{s}}}_{F_{\bar{s}}}$-assignment. Suppose for some $\sigma, \tau \in X^{p_{\bar{s}}}_{F_{\bar{s}}}$ that $\sigma \restriction j = \tau \restriction j$ for some $j > 0$. If there is no $i > 0$ such that $\sigma \restriction i = \bar{\sigma}_t \restriction i$ for some $\bar{t} = \bar{s} \restriction i'$ with $i' \geq i$, then $\varphi_{\bar{s}}(\sigma) = p_{\bar{s}} \mid \sigma$ and $\varphi_{\bar{s}}(\tau) = p_{\bar{s}} \mid \tau$. Thus, $\varphi_{\bar{s}}(\sigma) \restriction j = \varphi_{\bar{s}}(\tau) \restriction j$. So fix the largest $i > 0$ such that for some node $t \in X$, with $\bar{t} = \bar{s} \restriction i'$ and $i' \geq i$, $\sigma \restriction i = \bar{\sigma}_t \restriction i$, and fix some such node t. Thus, $\varphi_{\bar{s}}(\sigma) = b_B(t) \restriction i$ concatenated with the tail of $p_{\bar{s}} \mid \sigma$. If $i < j$, then $\varphi_{\bar{s}}(\tau) = b_B(t) \restriction i$ concatenated with the tail of $p_{\bar{s}} \mid \tau$, and so $\varphi_{\bar{s}}(\sigma) \restriction j = \varphi_{\bar{s}}(\tau) \restriction j$. So assume that $i \geq j$. Let $k \geq j$ be largest such that there is $r \in X$ with $\bar{r} = \bar{s} \restriction i''$ and $\tau \restriction k = \bar{\sigma}_r \restriction k$, and fix some such r. Then $\bar{\sigma}_r \restriction j = \sigma \restriction j = \bar{\sigma}_t \restriction j$. It follows, by the incompatibility requirement, that $t \restriction j = r \restriction j$. Thus $b_B(t) \restriction j = b_B(r) \restriction j$, from which it follows that $\varphi_{\bar{s}}(\sigma) \restriction j = \varphi_{\bar{s}}(\tau) \restriction j$. Repeat this for every node $\bar{s} \in \bar{X}$.

Let $\bar{g}_{\bar{X}}$ be defined by $\bar{g}_{\bar{X}}(\bar{s}) = (p'_{\bar{s}}, F_{\bar{s}})$, where $p'_{\bar{s}}$ is the amalgamation of the $X^{p_{\bar{s}}}_{F_{\bar{s}}}$-assignment $\varphi_{\bar{s}}$. Let's argue that $\bar{g}_{\bar{X}}$ is a valid condition in $\mathbb{P}(\vec{P}, \kappa^{<\omega})$. It follows directly from the definition of the assignments $\varphi_{\bar{s}}$ that $\varphi_{\bar{s}\restriction i}(\sigma \restriction i) = \varphi_{\bar{s}}(\sigma) \restriction i$ for every node $\bar{s} \in \bar{X}$ of length n and $i < n$. Thus, by Proposition 7.10, it follows that $p'_{\bar{s}} \restriction i = p'_{\bar{s}\restriction i}$. Clearly, $\bar{g}_{\bar{X}} \leq g_{\bar{X}}$. So finally, we extend $\bar{g}_{\bar{X}}$ to the tree Y to obtain the condition $\bar{g}_Y$ so that $\bar{g}_Y \leq g_Y$.

Since conditions $\bar{g}_Y$ are dense in $\mathbb{P}(\vec{P}, \kappa^{<\omega})$, some such condition $\bar{g}_Y \in G$. Since σ_s lies on $((\mathcal{T}^0_\xi)_{t^s_0}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi^s_{n-1}})_{t^s_{n-1}})$ for every $s \in X$, it follows using that g_Y is determined, that there must have been a coherent system $\{\bar{\sigma}_s \mid s \in X\}$ extending $\{\sigma_s \mid s \in X\}$ such that $\bar{\sigma}_s$ lies on $g_Y(\bar{s})$. It follows that $\bar{g}_Y(\bar{s}) \mid \bar{\sigma}_t \leq a_{X'}(t)$ for every $\bar{t} = \bar{s}$. Thus, for every $s \in X$, $f^*_X(s) \leq a_{X'}(s)$. Let $b_{X'}$ be the condition such that $b_{X'} \restriction X = f^*_X$ and for a node $s \in X' \setminus X$, $b_{X'}(s) = f^*_X(s \restriction i)$ concatenated with the tail of $a_{X'}(s)$ after i, where i is the largest such that $s \restriction i \in X$. Clearly, $b_{X'} \leq f^*_X, a_{X'}$.

□

Proposition 10.4. *Every $V[G]$-generic filter H^* for $\mathbb{P}(\vec{P}^*, \kappa^{<\omega})$ restricts to a V-generic filter H for $\mathbb{P}(\vec{P}, \kappa^{<\omega})$.*

Proof. By Lemma 10.3, H meets every maximal antichain of $\mathbb{P}(\vec{P}, \kappa^{<\omega})$ from V. Thus, it suffices to argue that H is a filter. Clearly, H is upward closed. So suppose that $f_X, g_Y \in H$. Let $Z = X \cup Y$. Then by Proposition 9.5, we can trivially extend f_X and g_Y to domain Z, and the resulting conditions will be in H. Thus, for simplicity, we can assume to begin with that we have conditions $f_X, g_X \in H$, namely that the two conditions have the same domain. Let X_n be the restriction of X to the first n levels.

For every node s on level n of $\kappa^{<\omega}$, let H^*_s consist of conditions $k_Y(s)$ for $k_Y \in H^*$. Then H^*_s is a $V[G]$-generic filter for $\mathbb{P}^*_n$ by Proposition 9.7. Let H_s be the restriction of H^*_s to $\mathbb{P}_n$, which is V-generic by Theorem 10.1 (3). For every node $s \in X_1$, let $p_s \in H_s$ be below $f_X(s), g_X(s)$. Define f_{X_1} by $f_{X_1}(s) = p_s$. Let's argue that $f_{X_1} \in H^*$. It suffices to argue that it is compatible with every condition in H^*. So fix $h_Y \in H^*$, and suppose that h_Y is not compatible with f_{X_1}. First, suppose that $h_Y(s)$ is compatible with $f_{X_1}(s)$ for every node $s \in X_1$. For every node $s \in X_1$, let $r_s \leq h_Y(s), f_{X_1}(s)$. Let $Z = X_1 \cup Y$, and let $\bar{h}_Z$ be the following condition. If $s \in Z$, but $s(0) \notin X_1$, then $\bar{h}_Z(s) = h_Y(s)$. So suppose that $s(0) \in X_1$. Then $\bar{h}_Z(s)$ is $r_{s(0)}$ concatenated with the tail of $h_Y(s)$. Clearly, $\bar{h}_Z \leq h_Y, f_{X_1}$. Thus, there must be some node $s \in X_1$ such that $h_Y(s)$ is not compatible with $f_{X_1}(s)$, but this is impossible since there must be a condition $h'_Z \in H^*$ with $h'_Z(s) = f_{X_1}(s)$. Thus, $f_{X_1} \in H^*$. By Proposition 9.5, the condition $f^{(1)}_X$ trivially extending f_{X_1} is in H^*.

Next, we enumerate $X_2 = \{s_\xi \mid \xi < \beta\}$ with $\beta < \kappa$. We start by considering s_0. Let p'_{s_0} be any condition in H_{s_0} below $f_X(s_0), g_X(s_0), f_{X_1}(s_0)$. Suppose inductively that we have chosen conditions $p'_{s_\xi} \in H_{s_\xi}$ for $\xi < \eta < \beta$ so that for all $\xi_1 < \xi_2 < \eta$ if $s_{\xi_1} \restriction 1 = s_{\xi_2} \restriction 1$, then $p'_{s_{\xi_1}} \restriction 1 \leq p'_{s_{\xi_2}} \restriction 1$. Let p''_{s_η} be any condition H_{s_η} below $f_X(s_\eta), g_X(s_\eta), f_{X_1}(s_\eta)$. In particular, we have $p''_{s_\eta}(0) \in H_{s_\eta \restriction 1}$. Recall that since $\mathbb{P}_1$ is $<\kappa$-closed, each V-generic filter H_s, for $s \in X_1$, is also $<\kappa$-closed. Thus, we can choose a condition $q_\eta \in H_{s_\eta \restriction 1}$ below $p''_{s_\eta}(0)$ and below all $p'_{s_\xi} \restriction 1$ with $\xi < \eta$

and $s_\xi \restriction 1 = s_\eta \restriction 1$. Let $p'_{s_\eta} = (q_\eta, p''_{s_\eta}(1))$, and observe that $p'_{s_\eta} \in H_{s_\eta}$ because it is compatible with every condition in it. Fix a node $s \in X_2$. Let q be any condition in $H_{s\restriction 1}$ below all q_ξ with $s_\xi \restriction 1 = s \restriction 1$. Let $p_s = (q, p'_s(1))$, and observe that $p_s \in H_s$ because it is compatible with all conditions in it. Let f_{X_2} be the condition with $f_{X_2}(s) = p_s$ for every $s \in X_2$. By the same argument as above, we have $f_{X_2} \in H^*$. Let $f^{(2)}_X \in H^*$ trivially extend f_{X_2}. Clearly, $f^{(2)}_X \leq f^{(1)}_X$.

Proceeding in this fashion, we construct a descending sequence of conditions $f^{(n)}_X \in H^*$. Let $\bar{f}_X$ be a greatest lower bound of the $f^{(n)}_X$. Since $\mathbb{P}(\vec{P}^*, \kappa^{<\omega})$ is $<\kappa$-closed, the generic H^* is also $<\kappa$-closed. Thus, there some condition $f'_Y \in H^*$ below the descending sequence of the $f^{(n)}_X$. But then $f'_Y \leq \bar{f}_X$, and hence $\bar{f}_X \in H^*$. Thus, $\bar{f}_X \in H$, and clearly $\bar{f}_X \leq f_X, g_X$. □

11. An ω-iteration of $\mathbb{J}(\kappa)$

In this section, we will construct in L an ω-iteration $\vec{P}^{\mathbb{J}(\kappa)} = \langle \mathbb{J}(\kappa)_n \mid n < \omega \rangle$ of κ-perfect forcing notions such that the finite iterations $\mathbb{J}(\kappa)_n$ will have unique generics and the κ^+-cc. The construction will take κ^+-many steps and will use a $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$-sequence. Let $\vec{D} = \{D_\alpha \mid \alpha \in \mathrm{Cof}(\kappa)\}$ be the canonical $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$-sequence from Section 6. As in the construction of the poset $\mathbb{J}(\kappa)$, at successor stages, as dictated by $\vec{D}$, we will force over κ-suitable models to grow the ω-iteration we have constructed so far while sealing maximal antichains.

We begin by arguing that whether a poset $\mathbb{P}_n$ is a finite iteration of κ-perfect posets is absolute for κ-suitable models.

Proposition 11.1. *Suppose that $\mathbb{P}_n = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \cdots * \dot{\mathbb{Q}}_{n-1}$ is an n-length forcing iteration and M is a κ-suitable model with $\mathbb{P}_n \in M$. Then $\mathbb{P}_n$ is a finite iteration of κ-perfect posets if and only if M satisfies that $\mathbb{P}_n$ is a finite iteration of κ-perfect posets.*

Proof. Suppose that $\mathbb{P}_n$ is a finite iteration of κ-perfect posets. Clearly, $\mathbb{Q}_0$ is a κ-perfect poset in M. So let's assume that $\mathbb{P}_i$ for some $1 \leq i < n$ is a finite iteration of κ-perfect posets in M. Let's suppose towards a contradiction that $\mathbb{P}_{i+1}$ is not a finite iteration of κ-perfect posets in M. Then it must be the case that, in M, some condition $p \in \mathbb{P}_i$ forces that $\mathbb{Q}_i$ is not a κ-perfect poset. Let $H \subseteq \mathbb{P}_i$ be V-generic with $p \in H$. Then $\mathbb{Q}_i = (\dot{\mathbb{Q}}_i)_H$ is a κ-perfect poset in V, but then $\mathbb{Q}_i$ is a also a κ-perfect poset in $M[H]$, which contradicts our assumption that p forced this not to be the case. Thus, $\mathbb{P}_n$ is a finite iteration of κ-perfect posets in M.

In the other direction, suppose that, in M, $\mathbb{P}_n$ is a finite iteration of κ-perfect posets. By the closure of M, $\mathbb{Q}_0$ is a κ-perfect poset in V. So let's assume that $\mathbb{P}_i$ for some $1 \leq i < n$ is a finite iteration of κ-perfect posets in V. Let $H \subseteq \mathbb{P}_i$ be V-generic. Then $\mathbb{Q}_i = (\dot{\mathbb{Q}}_i)_H$ is a κ-perfect poset in $M[H]$. Since $\mathbb{P}_i$ is $<\kappa$-closed, we have $M^{<\kappa} \subseteq M$ in $V[H]$, and hence, since $H \in V[H]$, $M[H]^{<\kappa} \subseteq M[H]$ in $V[H]$ by Proposition 4.3. It follows that $\mathbb{Q}_i$ is a κ-perfect poset in $V[H]$. Thus, $\mathbb{P}_n$ is a finite iteration of κ-perfect posets in V. □

Now we are ready to construct the ω-iteration $\vec{P}^{\mathbb{J}(\kappa)}$. Let $\vec{P}_0 = \langle \mathbb{P}^{(0)}_n \mid n < \omega \rangle$ be the ω-iteration of κ-perfect posets where $\mathbb{Q}^{(0)}_0 = \mathbb{P}_{\min}$ and each $\dot{\mathbb{Q}}^{(0)}_n = \check{\mathbb{P}}_{\min}$. Note that the poset $\mathbb{P}^{V[G]}_{\min}$ of a forcing extension by a $<\kappa$-closed forcing is the same as the poset $\mathbb{P}^V_{\min}$ of the ground model. Suppose that the ω-iteration

$$\vec{P}_\alpha = \langle \mathbb{P}^{(\alpha)}_n \mid n < \omega\rangle$$

of κ-perfect posets has been defined with

$$\mathbb{P}^{(\alpha)}_n = \mathbb{Q}^{(\alpha)}_0 * \dot{\mathbb{Q}}^{(\alpha)}_1 * \cdots * \dot{\mathbb{Q}}^{(\alpha)}_{n-1}.$$

We let $\vec{P}_\alpha = \vec{P}_{\alpha+1}$, unless the following happens. Suppose that $\alpha \in \mathrm{Cof}(\kappa)$ and D_α codes a well-founded binary relation $E \subseteq \alpha\times\alpha$ such that the collapse of E is a κ-suitable model M_α with $\vec{P}_\alpha \in M_\alpha$ and $\alpha = \omega_1^{M_\alpha}$. In this case, we take the L-least M_α-generic filter $G_\alpha \subseteq \mathbb{Q}(\vec{P}_\alpha, \kappa^{<\omega})$ and let $\vec{P}_{\alpha+1} = \vec{P}^*_\alpha$ as constructed in $M_\alpha[G_\alpha]$. Suppose that λ is a limit stage and we need to obtain the ω-iteration $\vec{P}_\lambda$. We let $(\mathbb{Q}^{(\lambda)}_0)'$ be the union of the $\mathbb{Q}^{(\xi)}_0$ for $\xi < \lambda$. If λ has cofinality κ, we let $\mathbb{Q}^{(\lambda)}_0 = (\mathbb{Q}^{(\lambda)}_0)'$, and otherwise, we let $\mathbb{Q}^{(\lambda)}_0$ be the the closure of $(\mathbb{Q}^{(\lambda)}_0)'$ under the $<\kappa$-intersection property and the weak union property. Suppose that we have now defined $\mathbb{P}^{(\lambda)}_n$. We let $(\dot{\mathbb{Q}}^{(\lambda)}_n)'$ be a $\mathbb{P}^{(\lambda)}_n$-name for the poset that is the union of the $\dot{\mathbb{Q}}^{(\xi)}_n$ for $\xi < \lambda$. If λ has cofinality κ, we let $\dot{\mathbb{Q}}^{(\lambda)}_n = (\dot{\mathbb{Q}}^{(\lambda)}_n)'$, and otherwise, we let $\dot{\mathbb{Q}}^{(\lambda)}_n$ be the $\mathbb{P}^{(\lambda)}_n$-name for the closure of $(\dot{\mathbb{Q}}^{(\lambda)}_n)'$ under the $<\kappa$-intersection property and the weak union property. In order for this limit definition to make sense, we need to verify that each $\dot{\mathbb{Q}}^{(\xi)}_n$ is a $\mathbb{P}^{(\lambda)}_n$-name for a κ-perfect poset. This will follow from the following more general lemma. First, however, we need the following standard proposition about maximal antichains in a two-step forcing iteration.

Proposition 11.2. *Suppose that $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration. Then the following are equivalent for $\mathcal{A} \subseteq \mathbb{P} * \dot{\mathbb{Q}}$.*

1. *$\mathcal{A}$ is a maximal antichain of $\mathbb{P} * \dot{\mathbb{Q}}$.*
2. *For every V-generic filter $G \subseteq \mathbb{P}$, the set $\bar{\mathcal{A}} = \{\dot{q}_G \mid (p, \dot{q}) \in \mathcal{A}$ and $p \in G\}$ is a maximal antichain of $\dot{\mathbb{Q}}_G$.*

Proof. Suppose first that (1) holds and let $G \subseteq \mathbb{P}$ be V-generic. First, let's argue that $\bar{\mathcal{A}}$ is an antichain. Fix $q, r \in \bar{\mathcal{A}}$. Fix $p_1, p_2 \in G$ and names $\dot{q}, \dot{r}$ such that $(p_1, \dot{q}), (p_2, \dot{r}) \in \mathcal{A}$ and $q = \dot{q}_G$, $r = \dot{r}_G$. Suppose towards a contradiction that q and r are compatible. Then there is $\bar{p} \in G$, with $\bar{p} \le p_1, p_2$, forcing that $\dot{c} \in \dot{\mathbb{Q}}$ and $\dot{c} \le \dot{q}, \dot{r}$. But then $(\bar{p}, \dot{c}) \le (p_1, \dot{q}), (p_2, \dot{r})$, which contradicts that $\mathcal{A}$ is an antichain. Thus, $\bar{\mathcal{A}}$ is an antichain. Next, fix any condition $r \in \dot{\mathbb{Q}}_G$ and let $\dot{r}_G = r$. Let $p \in G$ be a condition such that $p \Vdash \dot{r} \in \dot{\mathbb{Q}}$. Fix any condition $\bar{p} \le p$, and note that $(\bar{p}, \dot{r}) \in \mathbb{P} * \dot{\mathbb{Q}}$. Since $\mathcal{A}$ is maximal, $(\bar{p}, \dot{r})$ is compatible with some $(a, \dot{b}) \in \mathcal{A}$. Thus, there is a condition $(d, \dot{c}) \in \mathbb{P} * \dot{\mathbb{Q}}$ such that $(d, \dot{c}) \le (\bar{p}, \dot{r}), (a, \dot{b})$. It follows that conditions d for which there is a name $\dot{c}$ and a condition $(a, \dot{b}) \in \mathcal{A}$ such that $(d, \dot{c}) \le (a, \dot{b}), (\bar{p}, \dot{r})$ are dense below p. Thus, some such condition $d \in G$. But then the associated a belongs to G as well, which means that $\dot{b}_G \in \bar{\mathcal{A}}$. Finally, $d \Vdash \dot{c} \le \dot{r}, \dot{b}$. Thus, $\dot{c}_G \le \dot{b}_G, r$.

Next, suppose that (2) holds. First, let's argue that $\mathcal{A}$ is an antichain. Let $(a, \dot{b}), (c, \dot{d}) \in \mathcal{A}$, and suppose towards a contradiction that there is a condition $(p, \dot{q}) \le (a, \dot{b}), (c, \dot{d})$. Let $G \subseteq \mathbb{P}$ be a V-generic filter with $p \in G$. Then $a, c \in G$, and hence $b = \dot{b}_G$ and $d = \dot{d}_G$ are in $\bar{\mathcal{A}}$. Also, we have $p \Vdash \dot{q} \le \dot{b}, \dot{d}$, which means that $\dot{q}_G \le b, d$, but this is impossible by our assumption that $\bar{\mathcal{A}}$ is an antichain. Thus, $\mathcal{A}$ must be an antichain. It remains to argue that $\mathcal{A}$ is maximal. Fix a condition $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ and fix a V-generic filter $G \subseteq \mathbb{P}$ with $p \in G$. Let $q = \dot{q}_G$.

Since $\bar{\mathcal{A}}$ is maximal in $\dot{\mathbb{Q}}_G$, there must be some $b \in \bar{\mathcal{A}}$ such that q is compatible with b. Let $(a, \dot{b}) \in \mathcal{A}$ such that $a \in G$ and $\dot{b}_G = b$. There has to be a name $\dot{c}$ and $r \in G$, below p and a, forcing that $\dot{c} \leq \dot{q}, \dot{b}$. But then $(r, \dot{c}) \leq (p, \dot{q}), (a, \dot{b})$, verifying that these conditions are compatible. □

Given a non-trivial successor stage $\xi + 1$ in the construction of the ω-iteration $\vec{P}^{\mathbb{J}(\kappa)}$, we will denote by $\xi^* + 1$ the next immediate non-trivial successor stage. Because we are trying to construct the ω-iterations inside κ-suitable models, we are going to need to argue inductively that the iterations $\mathbb{P}_n^{(\eta)}$ for $1 \leq n < \omega$ and $\eta < \kappa^+$ can fit into such models, meaning that as sets they have transitive closure of size at most κ.

Lemma 11.3. *For every $\xi \leq \eta < \kappa^+$ and $1 \leq n < \omega$,*

1. *$\dot{\mathbb{Q}}_n^{(\xi)}$ is a $\mathbb{P}_n^{(\eta)}$-name for a κ-perfect poset.*
2. *$\mathbb{P}_n^{(\eta)}$ forces that $\dot{\mathbb{Q}}_n^{(\xi)} \subseteq \dot{\mathbb{Q}}_n^{(\eta)}$.*
3. *$\mathbb{P}_n^{(\xi)} \subseteq \mathbb{P}_n^{(\eta)}$.*
4. *The transitive closure of $\mathbb{P}_n^{(\eta)}$ has size κ.*
5. *Every maximal antichain $\mathcal{A} \in M_\xi$ of $\mathbb{P}_n^{(\xi)}$ remains maximal in $\mathbb{P}_n^{(\eta)}$,*
6. *Every L-generic filter H for $\mathbb{P}_n^{(\eta)}$ restricts to an M_ξ-generic filter H_ξ for $\mathbb{P}_n^{(\xi)}$.*

Proof. The sequence of posets $\mathbb{Q}_0^{(\eta)}$ for $\eta < \kappa^+$ is constructed by using, at non-trivial successor stages $\xi + 1$, the models $M_\xi[\bar{G}_\xi]$, where $\bar{G}_\xi$ is the M_ξ-generic filter for $\mathbb{Q}(\mathbb{Q}_0^{(\xi)})^{<\kappa}$ obtained from the first level of G_ξ (as explained in Section 10), precisely as in the definition of the poset $\mathbb{J}(\kappa)$. Thus, all the properties derived in Section 6 hold for this sequence. Clearly, we have properties (3) and (4). Property (5) holds by Proposition 4.8, and the proof of Proposition 4.12 shows that property (6) holds.

Next, let's consider the posets $\mathbb{P}_2^{(\eta)}$ for $\eta < \kappa^+$. By construction, $\dot{\mathbb{Q}}_1^{(0)} = \check{\mathbb{P}}_{\min}$. Given a non-trivial successor stage $\xi = \bar{\xi}+1$, the $\mathbb{P}_1^{(\xi)}$-name $\dot{\mathbb{Q}}_1^{(\xi)}$ is constructed in the model $M_{\bar{\xi}}[G_{\bar{\xi}}]$ so that whenever K is $M_{\bar{\xi}}[G_{\bar{\xi}}]$-generic for $\mathbb{P}_1^{(\xi)}$, then $\mathbb{Q}_1^{(\xi)} = (\dot{\mathbb{Q}}_1^{(\xi)})_K$ is a κ-perfect poset in $M_{\bar{\xi}}[K_{\bar{\xi}}][\bar{G}_{\bar{\xi}}]$, where $K_{\bar{\xi}}$ is the restriction of K to $\mathbb{P}_1^{(\bar{\xi})}$ and $\bar{G}_{\bar{\xi}}$ is the $M_{\bar{\xi}}[K_{\bar{\xi}}]$-generic filter for $\mathbb{Q}(\mathbb{Q}_1^{(\bar{\xi})})^{<\kappa}$ obtained from $G_{\bar{\xi}}$ (as explained in Section 10). Let's suppose inductively that properties (1)-(6) hold for $\mathbb{P}_2^{(\xi)}$ for all pairs $\xi \leq \rho < \eta$ in L. We need to verify that they continue to hold for all pairs $\xi \leq \eta$. First, let's suppose that $\eta = \bar{\eta} + 1$ is a non-trivial successor stage. We immediately get (1)-(6) for $\xi = \bar{\eta}$ by the results in Section 10. Next, fix a non-trivial successor stage $\xi = \bar{\xi} + 1 < \bar{\eta}$. Fix an L-generic filter $K \subseteq \mathbb{P}_1^{(\eta)}$. Let $\mathbb{Q}_1^{(\bar{\eta})} = (\dot{\mathbb{Q}}_1^{(\bar{\eta})})_K$ and let $\mathbb{Q}_1^{(\xi)} = (\dot{\mathbb{Q}}_1^{(\xi)})_K$. By what we already showed, $K_{\bar{\xi}}$, the restriction of K to $\mathbb{P}_1^{(\bar{\xi})}$, is $M_{\bar{\xi}}$-generic, and $K_{\bar{\xi}^*}$, the restriction of K to $\mathbb{P}_1^{(\bar{\xi}^*)} = \mathbb{P}_1^{(\xi)}$, is $M_{\bar{\xi}^*}$-generic. In particular, $K_{\bar{\xi}^*}$ is $M_{\bar{\xi}}[G_{\bar{\xi}}]$-generic for $\mathbb{P}_1^{(\xi)}$. By what we wrote above, it follows that $\mathbb{Q}_1^{(\xi)}$ is a κ-perfect poset in $M_{\bar{\xi}}[K_{\bar{\xi}}][\bar{G}_{\bar{\xi}}]$, and hence by closure considerations also in $L[K]$. This proves (1). By the inductive assumption $\mathbb{P}_1^{(\bar{\eta})}$ forces that $\dot{\mathbb{Q}}_1^{(\xi)} \subseteq \dot{\mathbb{Q}}_1^{(\bar{\eta})}$, and, by what we already showed, $K_{\bar{\eta}}$, the restriction of K to $\mathbb{P}_1^{(\bar{\eta})}$, is $M_{\bar{\eta}}$-generic. Since the inductive assumption can apply equally well to the model $M_{\bar{\eta}}$, we get that $\mathbb{Q}_1^{(\xi)} \subseteq \mathbb{Q}_1^{(\bar{\eta})}$ in $M_{\bar{\eta}}[K_{\bar{\eta}}] \subseteq L[K]$. This proves (2), and (3) follows easily. Next, we fix a maximal antichain $\mathcal{A} \in M_\xi$ of $\mathbb{P}_2^{(\xi)}$. By our inductive assumption applied to $M_{\bar\eta}$, $\mathcal{A}$ remains maximal in $\mathbb{P}_2^{(\bar\eta)}$, but then by what we already showed above $\mathcal{A}$ remains maximal in $\mathbb{P}_2^{(\eta)}$. Finally, suppose that $H \subseteq \mathbb{P}_2^{(\eta)}$ is L-generic. By above, H restricts to a $\mathbb{P}_2^{(\bar\eta)}$-generic filter for $M_{\bar\eta}$, and so to obtain the statement for ξ, we just use the inductive assumption applied to $M_{\bar\eta}$.

Next, suppose that $\eta < \kappa^+$ is a limit. Let $K \subseteq \mathbb{P}_1^{(\eta)}$ be L-generic and, for $\xi < \eta$, let K_ξ be the restriction of K to $\mathbb{P}_1^{(\xi)}$, which we showed above is M_ξ-generic. By considering a large enough M_ρ together with K_ρ and applying our inductive assumptions inside M_ρ, we get that the sequence $\mathbb{Q}_1^{(\xi)} = (\dot{\mathbb{Q}}_1^{(\xi)})_K$ for $\xi < \eta$ is constructed analogously to the poset $\mathbb{J}(\kappa)$ using the models $M_\xi[K_\xi][\bar G_\xi]$. Thus, we can conclude that (1)-(3) hold, and $\mathbb{Q}_1^{(\eta)} = (\dot{\mathbb{Q}}_1^{(\eta)})_K$ is either the union of the sequence of the $\mathbb{Q}_1^{(\xi)}$ or the closure of the sequence under the weak union property and the $<\kappa$-intersection property depending on the cofinality of η. Since each $\dot{\mathbb{Q}}_1^{(\xi)}$ is constructed in a transitive model of size κ, it follows that the union of the M_ξ, for $\xi < \eta$, has all the names needed for the union of the $\dot{\mathbb{Q}}_1^{(\xi)}$. If η has cofinality less than κ, then from these we can explicitly construct the κ-many names of elements yielded by the closure operations at stages of cofinality less than κ. Thus, $\mathbb{P}_2^{(\eta)}$ has transitive closure of size κ, verifying (3). Next, suppose that $\mathcal{A} \in M_\xi$ is a maximal antichain of $\mathbb{P}_2^{(\xi)}$ for some non-trivial stage $\xi + 1 < \eta$. We need to argue that $\mathcal{A}$ remains maximal in $\mathbb{P}_2^{(\eta)} = \mathbb{P}_1^{(\eta)} * \dot{\mathbb{Q}}_1^{(\eta)}$. By Proposition 11.2, it suffices to argue that

$$\bar{\mathcal{A}} = \{\dot q_K \mid (p, \dot q) \in \mathcal{A} \text{ and } p \in K\}$$

is a maximal antichain of $\mathbb{Q}_1^{(\eta)}$. Since $\mathcal{A}$ is maximal in $\mathbb{P}_2^{(\xi)} = \mathbb{P}_1^{(\xi)} * \dot{\mathbb{Q}}_1^{(\xi)}$, it follows, by Proposition 11.2, that $\bar{\mathcal{A}} \in M_\xi[H_\xi][\bar G_\xi]$ is maximal in $\mathbb{Q}_1^{(\xi)} = (\dot{\mathbb{Q}}_1^{(\xi)})_K$. Since we already argued above that $\mathbb{Q}_1^{(\eta)}$ is constructed analogously to the poset $\mathbb{J}(\kappa)$, we can apply the results of Section 6 to conclude that $\bar{\mathcal{A}}$ remains maximal in $\mathbb{Q}_1^{(\eta)}$. Finally, we argue that, for every non-trivial successor stage $\xi + 1 < \eta$, every L-generic filter H for $\mathbb{P}_2^{(\eta)}$ restricts to an M_ξ-generic filter for $\mathbb{P}_2^{(\xi)}$. Fix an L-generic filter $H = H' * h$ for $\mathbb{P}_2^{(\eta)}$. By what we already showed, for every $\nu < \eta$, H' restricts to an M_ν-generic filter H'_ν for $\mathbb{P}_1^{(\nu)}$. Also, the sequence of the posets $\mathbb{Q}_1^{(\nu)} = (\dot{\mathbb{Q}}_1^{(\nu)})_H$ for $\nu < \eta$ is constructed in $L[H']$ analogously to the $\mathbb{J}(\kappa)$ sequence using the models $M_\nu[H_\nu][\bar G_\nu]$. Since h is clearly $M_\xi[H_\xi][\bar G_\xi]$-generic, it follows, by Proposition 4.8, that h restricts to an $M_\xi[H_\xi]$-generic filter for $\mathbb{Q}_1^{(\xi)}$. Now we just repeat the argument from the proof of Theorem 10.1 (3) to argue that the restriction of H to $\mathbb{P}_2^{(\xi)}$ is M_ξ-generic.

The more general argument for $n > 2$ proceeds identically by induction. □

For $n < \omega$, let $\mathbb{J}(\kappa)_n = \mathbb{P}_n^{(\kappa^+)}$ be the n-length iteration defined as follows. Let $\mathbb{Q}_0^{(\kappa^+)} = \bigcup_{\xi<\kappa^+} \mathbb{Q}_0^{(\xi)}$. Suppose that we have now defined $\mathbb{P}_n^{(\kappa^+)}$. Let $\dot{\mathbb{Q}}_n^{(\kappa^+)}$ be a $\mathbb{P}_n^{(\kappa^+)}$-name for the poset that is the union of the $\dot{\mathbb{Q}}_n^{(\xi)}$ for $\xi < \kappa^+$. The proof of Lemma 11.3 gives that the definition makes sense and gives the following corollary.

Corollary 11.4. *For every $\xi < \kappa^+$ and $1 \leq n < \omega$,*

(1) *$\dot{\mathbb{Q}}_n^{(\xi)}$ is a $\mathbb{P}_n^{(\kappa^+)}$-name for a κ-perfect poset.*

(2) $\mathbb{P}_n^{(\xi)} \subseteq \mathbb{P}_n^{(\kappa^+)}$.
(3) *Every maximal antichain $\mathcal{A} \in M_\xi$ of $\mathbb{P}_n^{(\xi)}$ remains maximal in $\mathbb{P}_n^{(\kappa^+)}$,*
(4) *Every L-generic filter H for $\mathbb{P}_n^{(\kappa^+)}$ restricts to an M_ξ-generic filter for $\mathbb{P}_n^{(\xi)}$.*

Let $\xi+1$ be a non-trivial successor stage in the construction and $n<\omega$. We will denote by $(\mathcal{T}^{0,\xi}_{\rho_0},\ldots,\dot{\mathcal{T}}^{n-1,\xi}_{\rho_{n-1}})$ the sequences of generic κ-trees arising at that stage.

Proposition 11.5. *Suppose that $\lambda \leq \kappa^+$ is a limit ordinal and $\xi+1<\lambda$ is a non-trivial successor stage. If $p \in \mathbb{P}_n^{(\lambda)}$, then there is $q \leq p$ and a unique sequence $(\rho_0,\ldots,\rho_{n-1})$ such that $q \leq (\mathcal{T}^{0,\xi}_{\rho_0},\ldots,\dot{\mathcal{T}}^{n-1,\xi}_{\rho_{n-1}})$.*

Proof. The assertion is true for $n=1$ by the proof of Proposition 4.8 and the fact that two different generic κ-trees have a bounded intersection. So suppose that the assertion is true for some $n<\omega$, and fix a condition $(p,\dot{q}) \in \mathbb{P}_{n+1}^{(\lambda)}$ with $p \in \mathbb{P}_n^{(\lambda)}$. By our inductive assumption, there is a condition $p_1 \leq p$ and a unique sequence $(\rho_0,\ldots,\rho_{n-1})$ such that $p_1 \leq (\mathcal{T}^{0,\xi}_{\rho_0},\ldots,\dot{\mathcal{T}}^{n-1,\xi}_{\rho_{n-1}})$. Consider the condition $(p_1,\dot{q})$. Let H be an L-generic filter for $\mathbb{P}_n^{(\lambda)}$ containing p_1. In $L[H]$, the poset $\mathbb{Q}_n^{(\lambda)} = (\dot{\mathbb{Q}}_n^{(\lambda)})_H$ is constructed analogously to the poset $\mathbb{J}(\kappa)$. This means that, by the proof of Proposition 4.8, the condition $q=\dot{q}_H \leq \mathcal{T}^{n,\xi}_\rho = (\dot{\mathcal{T}}^{n,\xi}_\rho)_H$ for some $\rho<\kappa$. Thus, there is a condition $p_2 \leq p_1$ forcing that $\dot{q} \leq \dot{\mathcal{T}}^{n,\eta}_\rho$. It follows that $(p_2,\dot{q}) \leq (\mathcal{T}^{0,\eta}_{\rho_0},\ldots,\dot{\mathcal{T}}^{n-1,\eta}_{\rho_{n-1}},\dot{\mathcal{T}}^{n,\eta}_{\rho_n})$, where $\rho_n=\rho$, and $p_2 \leq p$. By the inductive assumption, the initial sequence $(\rho_0,\ldots,\rho_{n-1})$ is unique and using that any two generic κ-trees have a bounded intersection, we get that ρ_n must be unique as well. □

Lemma 11.6. *Suppose that $\lambda \leq \kappa^+$ is a limit ordinal and $\xi+1<\lambda$ is a non-trivial successor stage. Then every maximal antichain $\mathcal{A} \in M_\xi$ of $\mathbb{P}(\vec{P}_\xi,\kappa^{<\omega})$ remains maximal in $\mathbb{P}(\vec{P}_\lambda,\kappa^{<\omega})$.*

Proof. Fix a maximal antichain $\mathcal{A} \in M_\xi$ of $\mathbb{P}(\vec{P}_\xi,\kappa^{<\omega})$. Let $f_X \in \mathbb{P}(\vec{P}_\lambda,\kappa^{<\omega})$ with $f_X(s)=(p_{0,s},\ldots,\dot{p}_{n-1,s})$. Let X_n be the restriction of X to the first n-levels. Let g_{X_1} be defined by $g_{X_1}(s)=p^1_{0,s} \leq p_{0,s}$ such that $p^1_{0,s} \leq \mathcal{T}^{0,\xi}_{\rho^s_0}$ for some ρ^s_0. Using our standard construction (e.g. as in the proof of Proposition 10.4) to define tree conditions coherently, we let g_{X_2} be defined by

$$g_{X_2}(s)=(p^2_{0,s},\dot{p}^2_{1,s}) \leq (p^1_{0,s\restriction 1},\dot{p}_{1,s})$$

such that

$$(p^2_{0,s},\dot{p}^2_{1,s}) \leq (\mathcal{T}^{0,\xi}_{\rho^s_0},\dot{\mathcal{T}}^{1,\xi}_{\rho^s_1})$$

for some (ρ^s_0,ρ^s_1). By uniqueness given in Proposition 11.5, it must be the case that $\rho_0^{s\restriction 1}=\rho^s_0$. Continuing in this manner, we define a descending sequence of conditions g_{X_n} below f_X such that

$$g_{X_n}(s) \leq (\mathcal{T}^{0,\xi}_{\rho^s_0},\ldots\dot{\mathcal{T}}^{n-1,\xi}_{\rho^s_{n-1}}),$$

and for all $0 \leq i<n$, $m<\text{len}(s)$, $\rho^s_i=\rho_i^{s\restriction m}$. Define h_X by

$$h_X(s)=(\mathcal{T}^{0,\xi}_{\rho^s_0},\ldots\dot{\mathcal{T}}^{n-1,\xi}_{\rho^s_{n-1}}),$$

and note that by the above coherence property, we have that h_X is a valid condition in $\mathbb{P}(\vec{P}_{\xi+1}, \kappa^{<\omega})$. Let g_X be a lower bound of the g_{X_n} for $n < \omega$. Then $g_X \leq f_X$ and $g_X \leq h_X$. For $s \in X$, let

$$g_X(s) = (q_{0,s}, \ldots, \dot{q}_{n-1,s}).$$

For every $s \in X$, let $\bar{s} = \langle \rho_0^s, \ldots, \rho_{n-1}^s \rangle$.

Next, we are going to use our usual level-by-level construction (e.g. from the proof of Proposition 10.2) to obtain a condition $g'_X \leq g_X$ satisfying for every $s \in X$ on level n:

(1) $g'_X(s) = ((q_{0,s})_{t_0^s}, \ldots, (\dot{q}_{n-1,s})_{t_{n-1}^s})$,
(2) if s, r are nodes on level $n+1$ are such that $\bar{s} \restriction n = \bar{r} \restriction n$, then t_n^s and t_n^r are incompatible nodes.

For every $s \in X$ on level n, let

$$\sigma_s = \langle t_0^s, \ldots, t_{n-1}^s \rangle.$$

Let $h'_X \leq h_X$ be the condition defined by

$$h'_X(s) = ((\mathcal{T}^{0,\xi}_{\rho_0^s})_{t_0^s}, \ldots (\dot{\mathcal{T}}^{n-1,\xi}_{\rho_{n-1}^s})_{t_{n-1}^s}),$$

and note that h'_X satisfies the incompatibility requirement from the proof of Lemma 10.3. Thus, by the proof of that Lemma, there is a coherent collection $\{\bar{\sigma}_s \mid s \in X\}$ extending $\{\sigma_s \mid s \in X\}$, with $\bar{\sigma}_s = \langle \bar{t}_0^s, \ldots, \bar{t}_{n-1}^s \rangle$, and a condition condition $a_{X'} \in \mathcal{A}$, with $X \subseteq X'$, such that for every $s \in X$,

$$((\mathcal{T}^{0,\xi}_{\rho_0^s})_{\bar{t}_0^s}, \ldots (\dot{\mathcal{T}}^{n-1,\xi}_{\rho_{n-1}^s})_{\bar{t}_{n-1}^s}) \leq a_{X'}(s).$$

It follows that for every $s \in X$,

$$((q_{0,s})_{\bar{t}_0^s}, \ldots, (\dot{q}_{n-1,s})_{\bar{t}_{n-1}^s}) \leq a_{X'}(s).$$

It follows that $a_{X'}$ is compatible with f_X.

□

Lemma 11.7. *Suppose that $\lambda \leq \kappa^+$ is a limit ordinal and $\xi + 1 < \lambda$ is a non-trivial successor stage. Then every L-generic filter $H \subseteq \mathbb{P}(\vec{P}_\lambda, \kappa^{<\omega})$ restricts to an M_ξ-generic filter $H \subseteq \mathbb{P}(\vec{P}_\xi, \kappa^{<\omega})$.*

Proof. By Lemma 11.6, H meets every maximal antichain of $\mathbb{P}(\vec{P}_\xi, \kappa^{<\omega})$ from M_ξ. So it suffices to show that any two conditions in H are compatible in H. For every node s on level n of $\kappa^{<\omega}$, let H_s consist of conditions $f_X(s)$ for $f_X \in H$. Then H_s is an L-generic filter for $\mathbb{P}_n^{(\lambda)}$ by an argument analogous to the proof of Proposition 9.7. Now we proceed exactly as in the proof of Proposition 10.4. □

It will be useful for future arguments to assume that conditions in $\mathbb{J}(\kappa)_n$ are coded by subsets of κ.

Theorem 11.8. *The tree iteration $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$ has the κ^+-cc. In particular, all the iterations $\mathbb{J}(\kappa)_n$ have the κ^+-cc.*

Proof. Fix a maximal antichain $\mathcal{A}$ of $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$. Choose a transitive model $M \prec L_{\kappa^{++}}$ of size κ^+ with $\mathcal{A} \in M$. We can decompose M as the union of a continuous elementary chain of length κ^+ of substructures of size κ,

$$X_0 \prec X_1 \prec \cdots \prec X_\xi \prec \cdots \prec M,$$

with $\mathcal{A} \in X_0$, such that each successor stage $X_{\xi+1}$ is closed under sequences of length less than κ, $X_{\xi+1}^{<\kappa} \subseteq X_{\xi+1}$. It will follow that each X_α for $\alpha \in \mathrm{Cof}(\kappa)$ is closed under sequences of length less than κ, $X_\alpha^{<\kappa} \subseteq X_\alpha$. By properties of $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$, there must be some $\alpha \in \mathrm{Cof}(\kappa)$ such that $\alpha = \kappa^+ \cap X_\alpha$, $\mathbb{P}_n^{(\alpha)} = \mathbb{P}_n^{\mathbb{J}(\kappa)} \cap X_\alpha$ for all $n < \omega$ and D_α codes X_α. Let M_α be the transitive collapse of X_α. Then for all $n < \omega$, $\mathbb{P}_n^{(\alpha)}$ is the image of $\mathbb{P}_n^{\mathbb{J}(\kappa)}$ under the Mostowski collapse and α is the image of κ^+. Let $\bar{\mathcal{A}} = \mathcal{A} \cap X_\alpha$ be the image of $\mathcal{A}$ under the collapse. So at stage α in the construction of $\vec{P}^{\mathbb{J}(\kappa)}$ we chose a forcing extension $M_\alpha[G_\alpha]$ of M_α by $\mathbb{Q}(\vec{P}_\alpha, \kappa^{<\omega})$ and let $\vec{P}_{\alpha+1} = \vec{P}_\alpha^*$ be constructed in $M_\alpha[G_\alpha]$. Thus, $\bar{\mathcal{A}}$ remains maximal in $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$ by Lemma 11.6, and so it must have been the case that $\bar{\mathcal{A}} = \mathcal{A}$, verifying that $\mathcal{A}$ has size κ. □

12. The Kanovei-Lyubetsky Theorem for tree iterations of $\mathbb{J}(\kappa)$

In this section, we extend the "unique generics" property of the poset $\mathbb{J}(\kappa)$ and its bounded support products of length κ to finite iterations and tree iterations. We will show that if $G \subseteq \mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$ is L-generic, then the only finite sequences $\langle A_0, \ldots, A_n\rangle \in L[G]$ of subsets of κ that are L-generic for $\mathbb{J}(\kappa)_n$ are the sequences added on the nodes of the tree $\kappa^{<\omega}$ by G.

Suppose that $\vec{P} = \langle \mathbb{P}_n \mid n < \omega\rangle$ is an ω-iteration of κ-perfect posets and H is a generic filter for $\mathbb{P}(\vec{P}, \kappa^{<\omega})$. Given a node s on level n of $\kappa^{<\omega}$, let A_s be the n-length sequence of generic subsets of κ added by H on node s and let $\dot{A}_s$ be the canonical $\mathbb{P}(\vec{P}, \kappa^{<\omega})$-name for A_s.

For the next lemma, suppose that $\vec{P} = \langle \mathbb{P}_n \mid n < \omega\rangle$ is an ω-iteration of κ-perfect posets that is an element of a κ-suitable model M. We should think of $\vec{P}$ as one of the ω-iterations $\vec{P}_\alpha$ arising at stage α, for a non-trivial successor stage $\alpha+1$, in the construction of the ω-iteration $\vec{P}^{\mathbb{J}(\kappa)}$ and we should think of M as the model M_α from that stage.

Lemma 12.1. *In M, suppose that $\dot{A}$ is a $\mathbb{P}(\vec{P}, \kappa^{<\omega})$-name for an m-length sequence of subsets of κ such that for all nodes s on level m of $\kappa^{<\omega}$,*

$$\mathbb{1} \Vdash_{\mathbb{P}(\vec{P}, \kappa^{<\omega})} \dot{A}_s \neq \dot{A}.$$

Then in a forcing extension $M[G]$ by $\mathbb{Q}(\vec{P}, \kappa^{<\omega})$, for every node s on level m of $\kappa^{<\omega}$, the set of conditions forcing the statement

$\Phi(s) :=$ *"If $\dot{A}$ is $M[G]$-generic for $\mathbb{P}^*_m$, then $(\mathcal{T}^0_{s(0)}, \dot{\mathcal{T}}^1_{s(1)}, \ldots, \dot{\mathcal{T}}^{m-1}_{s(m-1)})$ is not in the filter determined by $\dot{A}$."*

is dense in $\mathbb{P}(\vec{P}^*, \kappa^{<\omega})$.

Proof. Fix a condition $f^*_X \in \mathbb{P}(\vec{P}^*, \kappa^{<\omega})$ and a node d on level m of $\kappa^{<\omega}$. By strengthening f^*_X, if necessary, we can assume:

(1) $f^*_X \in \vec{\mathbb{U}}$,
(2) f^*_X satisfies the incompatibility requirement from the proof of Lemma 10.3.

We need to find a condition $h_{X'} \leq f^*_X$ such that $h_{X'} \Vdash \Phi(d)$. For a node s on level n of X, let

$$f^*_X(s) = ((\mathcal{T}^0_{\xi^s_0})_{t^s_0}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi^s_{n-1}})_{t^s_{n-1}}),$$

$$\bar{s} = \langle \xi^s_0, \ldots, \xi^s_{n-1}\rangle \text{ and } \sigma_s = \langle t^s_0 \ldots, t^s_{n-1}\rangle.$$

Observe that by closure, X, $\{\bar{s} \mid s \in X\}$, and $\{\sigma_s \mid s \in X\}$ are all in V.

Fix $g_Y \in \mathbb{Q}(\vec{P}, \kappa^{<\omega})$, and let $g_Y(s) = (p_s, F_s)$. By strengthening g_Y, if necessary, we can assume that

(1) $\bar{s} \in Y$ for every $s \in X$,
(2) $d \in Y$,
(3) g_Y is determined,
(4) for every $s \in X$ on level n, for every $0 \leq i < n$, $F_{\bar{s}}(i) \geq \text{lev}(\sigma_s(i))$.

If there is no coherent collection $\{\bar{\sigma}_s \mid s \in X\}$ extending $\{\sigma_s \mid s \in X\}$ such that for every $s \in X$, $\bar{\sigma}_s$ lies on $g_Y(\bar{s})$, then we let $\bar{g}_Y = g_Y$. So suppose that there is such a coherent collection $\{\bar{\sigma}_s \mid s \in X\}$.

Let us assume (this need not be the case, but will turn out to be the case by density) that for every node s on level n of X,

$$((\mathcal{T}^0_{\xi^s_0})_{\bar{\sigma}_s(0)}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi^s_{n-1}})_{\bar{\sigma}_s(n-1)})$$

is a condition in $\mathbb{P}^*_n$. Let f'_X be defined by

$$f'_X(s) = ((\mathcal{T}^0_{\xi^s_0})_{\bar{\sigma}_s(0)}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi^s_{n-1}})_{\bar{\sigma}_s(n-1)}),$$

which is a valid condition because $\{\bar{\sigma}_s \mid s \in X\}$ is coherent. Now we are going to strengthen f'_X further to a condition $f'_{X'}$ so that for every σ which lies on (p_d, F_d), there is a node $s \in X'$ such that

$$f'_{X'}(s) = ((\mathcal{T}^0_{d(0)})_{\sigma(0)}, \ldots, (\dot{\mathcal{T}}^{m-1}_{d(m-1)})_{\sigma(m-1)}),$$

while maintaining the incompatibility requirement. Enumerate all σ which lie on (p_d, F_d) as $\{\sigma_\xi \mid \xi < \beta\}$ for some $\beta < \kappa$. We start with σ_0. First, suppose that there is no $s \in X$ such that $\bar{s}(0) = d(0)$. In this case, we add a new node r_0 of length m to our tree X along with the nodes $r_0 \restriction i$ for $1 \leq i < m$ such that $r_0(0) \notin X$. Suppose next that $d(0) = \bar{s}(0)$ for some $s \in X$. Thus, either $\sigma_0(0) = \sigma_{\bar{s}}(0)$ or $\sigma_0(0)$ is incompatible with $\sigma_{\bar{s}}(0)$. If $\sigma_0(0)$ is incompatible with $\sigma_{\bar{s}}(0)$, we do exactly what we did in the previous case, adding a new node r_0 of length m to our tree X along with the nodes $r_0 \restriction i$ for $1 \leq i < m$ such that $r_0(0) \notin X$. Finally, if $\sigma_0(0) = \sigma_{\bar{s}}(0)$, we don't modify level 1 of X, and move on to consider $d(1)$. Again, first suppose that there is no $\bar{t}$ extending $\bar{s}$ with $\bar{t}(1) = \bar{s}(1)$. In this case, we add a node r_0 of length m to our tree X along with the nodes $r_0 \restriction i$ for $2 \leq i < m$ such that $r_0(0) = \bar{s}(0)$ and $r_0 \restriction 2 \notin X$. The rest of the cases are identical as well. Let f'_{X_0} be the condition where X_0 is X together with the node r_0 and its predecessors that we added such that

$$f'_{X_0}(r_0) = ((\mathcal{T}^0_{d(0)})_{\sigma(0)}, \ldots, (\dot{\mathcal{T}}^{m-1}_{d(m-1)})_{\sigma(m-1)}).$$

Note that the new condition f'_{X_0} still satisfies the incompatibility requirement. Suppose we have constructed conditions f'_{X_ξ} for $\xi < \eta < \beta$. First, we let $X'_\eta = \bigcup_{\xi<\eta} X_\xi$ and let $f''_{X'_\eta}$ be the union of the f'_{X_ξ} for $\xi < \eta$. Then we extend $f''_{X'_\eta}$ to f'_{X_η} as in the first step. Let $X' = \bigcup_{\xi<\beta} X_\xi$ and let $f'_{X'}$ be the union of the f'_{X_ξ} for $\xi < \beta$. It should be clear from the construction that $f'_{X'}$ is as desired.

Define a condition $f_{X'} \in \mathbb{P}(\vec{P}, \kappa^{<\omega})$ by $f_{X'}(s) = p_{\bar{s}} \mid \bar{\sigma}_s$ for $s \in X$ and $f_{X'}(r_\xi) = p_d \mid \sigma_\xi$ for $\xi < \beta$.

Next, we are going to strengthen $f_{X'}$ to a condition a_Z in $\mathbb{P}(\vec{P}, \kappa^{<\omega})$ such that for every node s on level m of X, a_Z forces over $\mathbb{P}(\vec{P}, \kappa^{<\omega})$ the statement:

"There is $i < m-1$ such that $\dot{A}\upharpoonright i$ is M-generic for $\mathbb{P}_i$ and $\dot{A}(i) \notin [(a_Z(s)(i))_{\dot{A}\upharpoonright i}]$."

Fix a node s on level m of X and consider an forcing extension $M[H]$ by $\mathbb{P}(\vec{P}, \kappa^{<\omega})$ with $f_{X'} \in H$. By assumption, we have

$$A = \dot{A}_H \neq (\dot{A}_s)_H = A_s.$$

So there is $i < m-1$ such that $A \upharpoonright i = A_s \upharpoonright i$ and $A(i) \neq A_s(i)$. So we can strengthen $f_{X'}$ to a condition a_{Z_0} forcing that there is $i < m-1$ such that $\dot{A}\upharpoonright i$ is M-generic and $\dot{A}(i) \notin [(a_{Z_0}(s)(i))_{\dot{A}\upharpoonright i}]$. By repeating this for all the $<\kappa$-many nodes on level m and taking a lower bound of the conditions, we obtain the required condition a_Z.

Let $\bar{g}_Y \leq g_Y$, with $\bar{g}_Y(s) = (p'_s, F_s)$, be the determined condition constructed as in the proof of Lemma 10.3 such that:

(1) for every node $s \in X$, $p'_{\bar{s}} \mid \bar{\sigma}_s \leq a_Z(s)$,
(2) for every σ which lies on (p_d, F_d), if $s \in X'$ on level m is such that

$$f'_{X'}(s) = ((\mathcal{T}^0_{d(0)})_{\sigma(0)}, \ldots, (\dot{\mathcal{T}}^{n-1}_{d(n-1)})_{\sigma(n-1)}),$$

then $p'_d \mid \sigma \leq a_Z(s)$.

Since conditions $\bar{g}_Y$ are dense in $\mathbb{Q}(\vec{P}, \kappa^{<\omega})$, some such condition $\bar{g}_Y \in G$. Since σ_s lies on $((\mathcal{T}^0_\xi)_{t^s_0}, \ldots, (\dot{\mathcal{T}}^{n-1}_{\xi^s_{n-1}})_{t^s_{n-1}})$ for every $s \in X$, it follows using that g_Y is determined, that there must have been a coherent system $\{\bar{\sigma}_s \mid s \in X\}$ extending $\{\sigma_s \mid s \in X\}$ such that $\bar{\sigma}_s$ lies on $g_Y(\bar{s})$. If $x \in S$, then

$$f'_{X'}(s) \leq \bar{g}_Y(\bar{s}) \mid \bar{\sigma}_s \leq a_Z(s)$$

and if $s \in X' \setminus X$ on level m, then (for some σ)

$$f'_{X'}(s) = ((\mathcal{T}^0_{d(0)})_{\sigma(0)}, \ldots, (\dot{\mathcal{T}}^{m-1}_{d(m-1)})_{\sigma(m-1)}) \leq \bar{g}_Y(d) \mid \sigma \leq a_Z(s).$$

Thus, $f'_{X'}$ is compatible with a_Z, and we can let $\bar{a}_{\bar{Z}}$ be some condition below both of them. The condition $\bar{a}_{\bar{Z}}$ will be as required provided that we can verify that a_Z forces the statement $\Phi(d)$ over $\mathbb{P}(\vec{P}^*, \kappa^{<\omega})$ because since $\bar{a}_{\bar{Z}} \leq a_Z$, it will force the statement as well.

Suppose $H^* \subseteq \mathbb{P}(\vec{P}^*, \kappa^{<\omega})$ is an $M[G]$-generic filter containing $\bar{a}_{\bar{Z}}$. Now let's suppose towards a contradiction that

$$p = (\mathcal{T}^0_{d(0)}, \dot{\mathcal{T}}^1_{d(1)}, \ldots, \dot{\mathcal{T}}^{n-1}_{d(n-1)})$$

is in the filter determined by $A = \dot{A}_{H^*}$. Thus, there is some σ which lies on p such that for for all $i < m$, $A(i)$ is a branch through $(p \mid \sigma(i))_{A\upharpoonright i}$. By construction, we have that $p \mid \sigma \leq a_Z(s)$ for some s on level m of $\kappa^{<\omega}$. Let H be the restriction of H^* to an M-generic filter for $\mathbb{P}(\vec{P}, \kappa^{<\omega})$, and note that $a_Z \in H$. Thus, there is some $i < m-1$ such that $A(i)$ is not a branch through $(a_Z(s)(i))_{A\upharpoonright i}$, which is the desired contradiction. □

Theorem 12.2. *Suppose $H \subseteq \mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$ is L-generic. If an m-length sequence $A \in L[H]$ of subsets of κ is L-generic for $\mathbb{J}(\kappa)_m$, then $A = A_s$ for some node s on level m of $\kappa^{<\omega}$.*

Proof. Let's suppose that A is not one of the A_s for s on level m of $\kappa^{<\omega}$. Let $\dot{A}$ be a nice $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$-name for A such that for all nodes s on level m of $\kappa^{<\omega}$,

$$\mathbb{1} \Vdash_{\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})} \dot{A} \neq \dot{A}_s.$$

Choose some transitive $M \prec L_{\kappa^{++}}$ of size κ^+ with $\dot{A} \in M$. We can decompose M as the union of a continuous elementary chain of substructures of size κ

$$X_0 \prec X_1 \prec \cdots \prec X_\alpha \prec \cdots \prec M$$

with $\dot{A} \in X_0$. By the properties of $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$, there is some α such that $\alpha = \kappa^+ \cap X_\alpha$, $\mathbb{P}_n^{(\alpha)} = \mathbb{P}_n^{\mathbb{J}(\kappa)} \cap X_\alpha$ for all $n < \omega$, and D_α codes X_α. Let M_α be the transitive collapse of X_α. Then, for every $n < \omega$, $\mathbb{P}_n^{(\alpha)}$ is the image of $\mathbb{P}_n^{\mathbb{J}(\kappa)}$ under the collapse, and α is the image of κ^+. The name $\dot{A}$ is fixed by the collapse by our assumption that we can always code conditions in $\mathbb{P}(\vec{\mathbb{P}}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$ by subsets of κ and because all antichains of $\mathbb{P}(\vec{\mathbb{P}}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$ have size κ. So at stage α in the construction of $\vec{P}^{\mathbb{J}(\kappa)}$, we chose a forcing extension $M_\alpha[G]$ of M_α by $\mathbb{Q}(\vec{P}_\alpha, \kappa^{<\omega})$ and let $\vec{P}_{\alpha+1} = \vec{P}^*_\alpha$ be constructed in $M_\alpha[G]$.

By elementarity, M_α satisfies that

$$\mathbb{1} \Vdash_{\mathbb{P}(\vec{P}_\alpha, \kappa^{<\omega})} \dot{A} \neq \dot{A}_s$$

for all s on level m of $\kappa^{<\omega}$. Thus, by Lemma 12.1, for every s on level m of $\kappa^{<\omega}$, $\mathbb{P}(\vec{\mathbb{P}}_{\alpha+1}, \kappa^{<\omega})$ has a maximal antichain $\mathcal{A}_s$ consisting of conditions f_X forcing the statement:

$\Phi(s) :=$"If $\dot{A}$ is $M_\alpha[G]$-generic for $\mathbb{P}_m^{(\alpha+1)}$, then $(\dot{\mathcal{T}}^{0,\alpha}_{s(0)}, \ldots, \dot{\mathcal{T}}^{m-1,\alpha}_{s(m-1)})$ is not in the filter determined by $\dot{A}$."

It follows by Lemma 11.6 that every antichain $\mathcal{A}_s$ as well as the antichain

$$\mathcal{A} = \{(\dot{\mathcal{T}}^{0,\alpha}_{s(0)}, \ldots, \dot{\mathcal{T}}^{m-1,\alpha}_{s(m-1)}) \mid s \in \kappa^m\}$$

remain maximal in $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$. So, fixing some $s \in \kappa^m$, let's argue that if $f_X \in \mathcal{A}_s$, then f_X forces in $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$ that if $\dot{A}$ is L-generic for $\mathbb{P}_m^{\mathbb{J}(\kappa)}$, then $(\dot{\mathcal{T}}^{0,\alpha}_{s(0)}, \dot{\mathcal{T}}^{1}_{s(1)}, \ldots, \dot{\mathcal{T}}^{m-1,\alpha}_{s(m-1)})$ is not in the filter determined by $\dot{A}$.

Let $\bar{H} \subseteq \mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, \kappa^{<\omega})$ be an L-generic filter containing f_X. Let $\alpha^* + 1$ be the next non-trivial successor stage after $\alpha + 1$. Then $\bar{H}$ restricts to an M_{α^*}-generic filter $\bar{H}_{\alpha^*}$ for $\mathbb{P}(\vec{P}_{\alpha^*}, \kappa^{<\omega})$ by Lemma 11.7. Since $\vec{P}_{\alpha+1} = \vec{P}_{\alpha^*}$, it follows that $\bar{H}_{\alpha^*}$ is M_{α^*}-generic for $\mathbb{P}(\vec{P}_{\alpha+1}, \kappa^{<\omega})$. Finally, since $M_\alpha[G_\alpha] \subseteq M_{\alpha^*}$, it follows that $\bar{H}_{\alpha^*}$ is $M_\alpha[G_\alpha]$-generic for $\mathbb{P}(\vec{P}_{\alpha+1}, \kappa^{<\omega})$. Let $A = \dot{A}_{\bar{H}}$ and suppose that it is L-generic for $\mathbb{P}_m^{\mathbb{J}(\kappa)}$. Since $f_X \in \bar{H}_{\alpha^*}$, it follows that $M_\alpha[G_\alpha][\bar{H}_{\alpha^*}]$ satisfies that $(\dot{\mathcal{T}}^{0,\alpha}_{s(0)}, \dot{\mathcal{T}}^{1,\alpha}_{s(1)}, \ldots, \dot{\mathcal{T}}^{m-1,\alpha}_{s(m-1)})$ is not in the filter determined by A by Lemma 12.1. But then this is true in $L[\bar{H}]$ as well.

Since H must meet every $\mathcal{A}_s$, it holds in $L[H]$ that if A is L-generic for $\mathbb{P}_m^{\mathbb{J}(\kappa)}$, then it does not meet the maximal antichain $\mathcal{A}$, and so A, in fact, cannot be L-generic. □

Corollary 12.3. *For every $n < \omega$, the iteration $\mathbb{J}(\kappa)_n$ adds a unique generic sequence of length n of subsets of κ.*

13. Iterating $P^{\mathbb{J}(\kappa)}$ along the tree $(\kappa^+)^{<\omega}$

We will now argue that the tree iteration $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, (\kappa^+)^{<\omega})$, where we iterate along the κ^+-sized tree $(\kappa^+)^{<\omega}$, shares all the key properties of the tree iteration

$\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},\kappa^{<\omega})$, namely it has the κ^+-cc and Theorem 12.2, concerning the uniqueness of generic filters for $\mathbb{J}(\kappa)_n$, continues to holds.

The following proposition is easy to verify.

Proposition 13.1. *Suppose that $\mathscr{T}$ is a tree of height ω, f_Y, g_Z are conditions in the poset $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},\mathscr{T})$, and $Y\cap Z = X$. If $f_Y\upharpoonright X$ is compatible with $g_Z\upharpoonright X$, then f_Y is compatible with g_Z.*

Theorem 13.2. *The poset $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$ has the κ^+-cc.*

Proof. Let's suppose to the contrary that there is an antichain $\mathcal{A}$ in $\mathbb{P}(\vec{P}^{\kappa J},(\kappa^+)^{<\omega})$ of size κ^+. By a Δ-system argument, there must be some subtree $X\subseteq(\kappa^+)^{<\omega}$ and a subset $\mathcal{A}'\subseteq\mathcal{A}$ of size κ^+ such that for any f_Y and g_Z in $\mathcal{A}'$, $Y\cap Z=X$. Given $f_Y\in\mathcal{A}'$, let f_X be the restriction of f_Y to X. By Proposition 13.1, if $f_Y\neq g_Z$ are in $\mathcal{A}'$, then f_X and g_X are incompatible conditions. Thus, the collection of all such f_X is an antichain of size κ^+ in $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$. But then since X has size less than κ, there must be a corresponding antichain of size κ^+ in $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},\kappa^{<\omega})$, which is impossible by Theorem 11.8. □

The following proposition is not difficult to verify.

Proposition 13.3. *Suppose that $\mathscr{T}$ is a subtree of $(\kappa^+)^{<\omega}$ and $H\subseteq\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$ is L-generic. Then the restriction $H_{\mathscr{T}}$ of H to $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},\mathscr{T})$ is also L-generic.*

Suppose that H is L-generic for $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$. Given a node s on level n of $(\kappa^+)^{<\omega}$, let A_s be the n-length sequence of generic subsets of κ added by H on node s and let $\dot{A}_s$ be the canonical $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$-name for A_s.

Theorem 13.4. *Suppose $H\subseteq\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$ is L-generic. If an n-length sequence $A\in L[H]$ of subsets of κ is L-generic for $\mathbb{J}(\kappa)_n$, then $A=A_s$ for some node s on level n of $(\kappa^+)^{<\omega}$.*

Proof. Suppose that A is an n-length sequence of subsets of κ in $L[H]$ that is L-generic for $\mathbb{P}^{\mathbb{J}(\kappa)}_n$. Let $\dot{A}$ be a nice $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$-name for A. Since $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$ has the κ^+-cc by Theorem 13.2, it follows that conditions in the name $\dot{A}$ use only κ-many nodes of $(\kappa^+)^{<\omega}$. Thus, $\dot{A}$ is a $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},\mathscr{T})$-name, where $\mathscr{T}$ is a subtree of $(\kappa^+)^{<\omega}$ of size κ. We can assume without loss of generality that $\mathscr{T}$ is isomorphic to $\kappa^{<\omega}$, and so $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},\mathscr{T})$ is isomorphic to $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},\kappa^{<\omega})$. Let $H_{\mathscr{T}}$ be the restriction of H to $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},\mathscr{T})$, which is L-generic by Proposition 13.3. Thus, $A\in L[H_{\mathscr{T}}]$, from which it follows, by Theorem 12.2, that $A=A_s$ for some $s\in\mathscr{T}$. □

14. A symmetric model of $\mathrm{ZF}+\mathrm{AC}_\kappa+\neg\mathrm{DC}$

We will construct a symmetric submodel of a forcing extension $L[G]$ by $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)},(\kappa^+)^{<\omega})$ in which $\mathrm{ZF}+\mathrm{AC}_\kappa$ holds, but the axiom of dependent choices DC fails. The subsets of V_κ of this model will yield a model of Kelley-Morse second-order set theory in which $\mathrm{KM}+\mathrm{CC}$ holds, but DC fails. Let's start with a brief discussion of the method of constructing symmetric submodels of a forcing extension, which goes all the way back to Cohen's pioneering work on forcing (see, for instance, [Jec03]).

Suppose that $\mathbb{P}$ is a forcing notion. Recall that if π is an automorphism of $\mathbb{P}$, then we can apply π recursively to conditions in a $\mathbb{P}$-name σ to obtain the $\mathbb{P}$-name $\pi(\sigma)$. It is not difficult to see, by induction on complexity of formulas, that for every formula φ and condition $p \in \mathbb{P}$,

$$p \Vdash \varphi(\sigma) \text{ if and only if } \pi(p) \Vdash \varphi(\pi(\sigma)).$$

Fix some group $\mathcal{G}$ of automorphisms of $\mathbb{P}$. Recall that a filter $\mathscr{F}$ on subgroups of a group $\mathcal{G}$ is *normal* if whenever $g \in \mathcal{G}$ and $\mathcal{K} \in \mathscr{F}$, then $g\mathcal{K}g^{-1} \in \mathscr{F}$. Let's fix a normal filter $\mathscr{F}$ on the subgroups of $\mathcal{G}$. The subgroup of $\mathcal{G}$ fixing a particular $\mathbb{P}$-name σ, consisting of automorphisms π such that $\pi(\sigma) = \sigma$, is called $\mathrm{sym}(\sigma)$. If $\mathrm{sym}(\sigma)$ is in $\mathscr{F}$, then we say that σ is a *symmetric* $\mathbb{P}$-name. We recursively define that a $\mathbb{P}$-name is *hereditarily symmetric* when it is symmetric and all names inside it are hereditarily symmetric. Let HS be the collection of all hereditarily symmetric $\mathbb{P}$-names. Let $G \subseteq \mathbb{P}$ be V-generic. The *symmetric model*

$$N = \{\sigma_G \mid \sigma \in \mathrm{HS}\} \subseteq V[G]$$

associated to the group of autmorphisms $\mathcal{G}$ and the normal filter $\mathscr{F}$ consists of the interpretations of all hereditarily symmetric $\mathbb{P}$-names. It is a standard result that $N \models \mathrm{ZF}$ (see, for instance, [Jec03]).

We work over L, although, following the analysis of Section 6, we can work over any model V with an inaccessible cardinal κ in which $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$ holds. Let $\mathbb{P} = \mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, (\kappa^+)^{<\omega})$. Let $\mathrm{Aut}((\kappa^+)^{<\omega})$ be the automorphism group of the tree $(\kappa^+)^{<\omega}$. Every automorphism $\pi \in \mathrm{Aut}((\kappa^+)^{<\omega})$ induces an automorphism π^* of $\mathbb{P}$ defined by $\pi^*(f_X) = f_{\pi``X}$ with $f_{\pi``X}(t) = f_X(\pi^{-1}(t))$. Let

$$\mathcal{G} = \{\pi^* \mid \pi \in \mathrm{Aut}((\kappa^+)^{<\omega})\}.$$

Next, we need to select an appropriate filter on the subgroups of $\mathcal{G}$.

Definition 14.1. A subtree T of the tree $(\kappa^+)^{<\omega}$ is *useful* if it

(1) has size κ,
(2) has no infinite branches.

Given a useful tree T, let $\mathcal{G}_T$ be the subgroup of $\mathcal{G}$ consisting of all automorphisms π^* such that π point-wise fixes T. Let $\mathscr{F}$ be the filter on the subgroups of $\mathcal{G}$ generated by all such subgroups $\mathcal{G}_T$ with T a useful tree. To see that $\mathscr{F}$ is normal, observe that if T is a useful tree and $\mathcal{G}_T \subseteq K \in \mathscr{F}$, then $\pi `` T$ is useful and $\mathcal{G}_{\pi``T} \subseteq \pi^* K \pi^{*-1}$.

Now, let $G \subseteq \mathbb{P}$ be L-generic and let

$$N = \{\sigma_G \mid \sigma \in \mathrm{HS}\} \subseteq L[G]$$

be the symmetric model associated to $\mathcal{G}$ and $\mathscr{F}$.

In $L[G]$, consider the tree $\mathcal{T}$, isomorphic to $(\kappa^+)^{<\omega}$, whose nodes are the generic sequences of subsets of κ for the posets $\mathbb{J}(\kappa)_n$ added by G. Given a node $t \in (\kappa^+)^{<\omega}$, let σ_t be the canonical $\mathbb{P}$-name for the generic, for $\mathbb{J}(\kappa)_n$, sequence of subsets of κ added on node t by G. Let $\dot{\mathcal{T}} = \{(\mathrm{op}(\sigma_s, \sigma_t), \mathbb{1}_{\mathbb{P}}) \mid s \leq t \text{ in } (\kappa^+)^{<\omega}\}$, where $\mathrm{op}(\sigma_s, \sigma_t)$ is the canonical $\mathbb{P}$-name for the ordered pair of (the interpretations of) σ_s and σ_t. Clearly, $\dot{\mathcal{T}}_G = \mathcal{T}$ (when we view $\mathcal{T}$ as the ordering of the tree). Fix any $\pi^* \in \mathcal{G}$, and observe that $\pi^*(\dot{\mathcal{T}}) = \{(\mathrm{op}(\sigma_{\pi(s)}, \sigma_{\pi(t)}), \mathbb{1}_{\mathbb{P}}) \mid s \leq t \text{ in } (\kappa^+)^{<\omega}\} = \dot{\mathcal{T}}$. Also, any automorphism π^* with $\pi(s) = s$ fixes σ_s. This shows that $\dot{\mathcal{T}} \in \mathrm{HS}$, and hence $\mathcal{T}$ is in the symmetric model N.

Suppose that T is any subtree of $(\kappa^+)^{<\omega}$ in L. If f_X is a condition in $\mathbb{P}$, we will denote by $f_{X\cap T}$, the restriction of f_X to nodes in T, i.e. $f_{X\cap T}$ has domain $X \cap T$ and $f_{X\cap T}(t) = f_X(t)$. We let

$$G_T = \{f_X \in G \mid X \subseteq T\},$$

and recall that, by Proposition 13.3, $G_T \subseteq \mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, T)$ is L-generic. Note that if T is useful, then the canonical name for G_T is fixed by all elements of the subgroup $\mathcal{G}_T$, and therefore $G_T \in N$. Thus, each $L[G_T] \subseteq N$.

If σ is a $\mathbb{P}$-name and $\mathcal{G}_T \subseteq \text{sym}(\sigma)$ for a useful tree T, we will say that T *witnesses* that σ is symmetric.

Proposition 14.2. *Suppose that σ is a symmetric $\mathbb{P}$-name, as witnessed by a useful tree T, and $f_X \Vdash \varphi(\sigma, \check{a})$. Then $f_{X\cap T} \Vdash \varphi(\sigma, \check{a})$.*

Proof. Suppose it is not the case that $f_{X\cap T} \Vdash \varphi(\sigma, \check{a})$. Then there is a condition $g_Y \leq f_{X\cap T}$ which forces $\neg\varphi(\sigma, \check{a})$. Let $\pi \in \text{Aut}((\kappa^+)^{<\omega})$ be an automorphism which switches those nodes in $Y \setminus T$ with some nodes outside both T and X. Clearly, $\pi^* \in \mathcal{G}_T$, and so $\pi^*(\sigma) = \sigma$. It follows that $\pi^*(g_Y) \Vdash \neg\varphi(\sigma, \check{a})$. But, by construction, $\pi^*(g_Y)$ is compatible with f_X, which is the desired contradiction. Thus, we have shown that $f_{X\cap T} \Vdash \varphi(\sigma, \check{a})$. □

Proposition 14.3. *Suppose that σ is a symmetric $\mathbb{P}$-name, as witnessed by a useful tree T, and $A = \sigma_G$ is a set of ordinals. Then $A \in L[G_T]$.*

Proof. Let f_X be some condition in G forcing that σ is a set of ordinals. Define a name

$$\sigma^* = \{(\xi, g_{Y\cap T}) \mid g_Y \leq f_X, g_Y \Vdash \xi \in \sigma\}.$$

We will argue that $f_X \Vdash \sigma = \sigma^*$. Let $H \subseteq \mathbb{P}$ be some L-generic filter containing f_X. Suppose $\xi \in \sigma_H$. Then there is $g_Y \in H$ such that $g_Y \leq f_X$ and $g_Y \Vdash \xi \in \sigma$, from which it follows that $(\xi, g_{Y\cap T}) \in \sigma^*$ and $g_{Y\cap T} \in H$. So we have $\xi \in \sigma^*_H$. Next, suppose that $\xi \in \sigma^*_H$. Then there is a condition $g_Y \Vdash \xi \in \sigma$ such that $(\xi, g_{Y\cap T}) \in \sigma^*$ and $g_{Y\cap T} \in H$. But by Proposition 14.2, it follows that $g_{Y\cap T} \Vdash \xi \in \sigma$, and so $\xi \in \sigma_H$. □

Lemma 14.4. DC *fails in N.*

Proof. We will argue that $\mathcal{T}$ does not have an infinite branch in N, and hence DC fails. Suppose to the contrary that N has an infinite branch b through $\mathcal{T}$. Via coding, we can view b as a subset of κ. Fix a symmetric name $\dot{b}$ for b, as witnessed by a useful tree T. By Proposition 14.3, we can assume that the name $\dot{b}$ mentions only conditions with domains contained in T. Recall that for a node $s \in (\kappa^+)^{<\omega}$, A_s is the L-generic sequence of subsets of κ for $\mathbb{J}(\kappa)_{\text{len}(s)}$ added by G on node s and $\dot{A}_s$ is the canonical $\mathbb{P}$-name for A_s. Let $\bar{b}$ be the branch through the tree $(\kappa^+)^{<\omega}$ which corresponds to b via the obvious isomorphism. Since T doesn't have infinite branches by assumption, $\bar{b}$ cannot be a branch through T. Thus, there is some natural number n such that $\bar{b} \upharpoonright n \subseteq T$ and $\bar{b}(n) = s$ is outside T.

Fix a condition

$$f_X \Vdash \dot{b}(n) = \dot{A}_s,$$

with $f_X \in G$, and assume without loss of generality that $s \in X$. Let

$$s = \langle s_0, \ldots, s_{n-1}, s_n \rangle.$$

Fix any condition $f'_{X'} \leq f_X$. Let

$$t = \langle s_0, \dots, s_{n-1}, t_n \rangle$$

be a node in $(\kappa^+)^{<\omega}$ such that $t \notin (X' \cup T)$, which must exist since $X' \cup T$ has size κ. Let π be an automorphism of $(\kappa^+)^{<\omega}$ which maps $((\kappa^+)^{<\omega})_s$ onto $((\kappa^+)^{<\omega})_t$, while fixing everything outside these subtrees. In particular, π fixes T. Let $Y = X' \cup \pi " X'$ and let g_Y be the condition defined as follows. If $r \in X'$, then $g_Y(r) = f'_{X'}(r)$. Otherwise, $r = \pi(\bar{r})$ for $\bar{r} \in X'$, and in this case, $g_Y(r) = f'_{X'}(\bar{r})$. In other words, conditions on nodes in $X' \cap ((\kappa^+)^{<\omega})_s$ are copied over to $((\kappa^+)^{<\omega})_t$. Thus, such conditions g_Y are dense below f_X, and so some such $g_Y \in G$. Let $H = \pi " G$, which is also L-generic for $\mathbb{P}$. Observe that $\dot{b}_H = b$ since the name $\dot{b}$ only mentioned conditions with domain in T, and π fixes T. Observe also that $f_X \in H$ since it is above $\pi^*(g_Y)$. So it must be the case that $b(n) = (\dot{A}_s)_H$. But this is impossible because $(\dot{A}_s)_H = A_t$ and $(\dot{A}_s)_G = A_s$ and, by genericity, $A_s \neq A_t$. □

The proof of the next lemma relies mainly on the fact that $\mathbb{P}$ has the κ^+-cc and uses very little else about what the conditions in $\mathbb{P}$ look like. So to make the notation nicer, we will switch away from the previous convention and call conditions in $\mathbb{P}$ standard names like p and q.

Lemma 14.5. AC_κ *holds in* N.

Proof. Suppose that $F \in N$ is a family of size κ of non-empty sets. Let $\dot{F}$ be a hereditarily symmetric name for F with the useful tree S witnessing that $\dot{F}$ is symmetric, and let $q \in G$ force that $\dot{F}$ is a family of size κ of non-empty sets. We would like to build a name $\dot{C} \in \mathrm{HS}$ such that q forces that $\dot{C}$ is a choice function for $\dot{F}$. By Proposition 14.2, we can assume that $\mathrm{dom}(q) \subseteq S$. We will adopt the following strategy. First, we will build a mixed name $\dot{C}_0 \in \mathrm{HS}$ (over an antichain below q) and a useful tree T_0 extending S, witnessing that $\dot{C}_0$ is symmetric, such that $q \Vdash \dot{C}_0 \in \dot{F}(0)$. Next, we will build a mixed name $\dot{C}_1 \in \mathrm{HS}$ and a useful tree T_1 extending T_0, witnessing that $\dot{C}_1$ is symmetric, such that $q \Vdash \dot{C}_1 \in \dot{F}(1)$. Proceeding in this fashion, we will build names $\dot{C}_\xi \in \mathrm{HS}$, for $\xi < \kappa$, and an increasing sequence of useful trees T_ξ such that $q \Vdash \dot{C}_\xi \in \dot{F}(\xi)$. Provided we can ensure in the course of the construction that $T = \bigcup_{\xi \in \omega} T_\xi$ does not have an infinite branch, we will be able to build from the names $\dot{C}_\xi$ a hereditarily symmetric name $\dot{C}$, witnessed by T to be symmetric, that is forced by q to be a choice function for $\dot{F}$.

Let D_0 be the dense set below q of conditions p such that for some name $\dot{c}_p \in \mathrm{HS}$, $p \Vdash \dot{c}_p \in \dot{F}(0)$. We will thin out D_0 to a maximal antichain over which we can mix the names $\dot{c}_p$ to get the desired name $\dot{C}_0 \in \mathrm{HS}$. Choose some condition $p_0^{(0)} \in D_0$ and a name $\dot{c}_0^{(0)} \in \mathrm{HS}$, witnessed by a useful tree $S_0^{(0)}$ to be symmetric, such that

$$p_0^{(0)} \Vdash \dot{c}_0^{(0)} \in \dot{F}(0).$$

First, observe that by unioning up S and $S_0^{(0)}$, we can assume without loss of generality that $S \subseteq S_0^{(0)}$. Since $\mathrm{dom}(q) \subseteq S$, $p_0^{(0)} \restriction S_0^{(0)} \leq q$. Thus, by Proposition 14.2, we can assume without loss of generality that $\mathrm{dom}(p_0^{(0)}) \subseteq S_0^{(0)}$. Next, we choose some condition $p_1 \in D_0$, incompatible to $p_0^{(0)}$, and a name $\dot{c}_1 \in \mathrm{HS}$, witnessed by a useful tree S_1 to be symmetric, such that $p_1 \Vdash \dot{c}_1 \in \dot{F}(0)$. By Proposition 14.2, we can assume that $\mathrm{dom}(p_1) \subseteq S_1 \cup S_0^{(0)}$. Note that we cannot restrict the domain of p_1 to just S_1 because then we might lose that p_1 is incompatible with $p_0^{(0)}$. At this point, it is not a good idea to union up $S_0^{(0)}$ and S_1 because if we keep doing this, we might end up with an infinite branch in the final tree. Instead, we will use a different name that is witnessed to be symmetric by an isomorphic copy of S_1 whose intersection with $S_0^{(0)}$ is contained in S. Let π be an automorphism which moves nodes in $(S_1 \cap S_0^{(0)}) \setminus S$ outside of $S_0^{(0)}$ so that π fixes S and $\pi \text{ " } S_1 = S_1^{(0)}$, where $S_0^{(0)} \cap S_1^{(0)}$ is contained in S. Since π fixes S, we have $\pi^* \in \mathcal{G}_S$. So we have

$$\pi^*(p_1) \Vdash \pi^*(\dot{c}_1) \in \dot{F}(0).$$

Also, $\pi^*(p_1) \leq q$ since $\mathrm{dom}(q) \subseteq S$. Let

$$p_1' = p_1 \upharpoonright S_0^{(0)} \text{ and } p_1^{(0)} = \pi^*(p_1) \cup p_1'.$$

Thus, we have:

(1) $p_1^{(0)} \in D_0$ since $\pi^*(p_1), p_1' \leq q$ and $p_1^{(0)} \leq \pi^*(p_1) \Vdash \pi^*(\dot{c}_1) \in \dot{F}(0)$.
(2) $p_1^{(0)}$ is not compatible with $p_0^{(0)}$ since $\mathrm{dom}(p_0^{(0)}) \subseteq S_0^{(0)}$ by assumption and so the incompatibility of p_1 and $p_0^{(0)}$ must have occurred because of nodes in $S_0^{(0)}$.
(3) $\mathrm{dom}(p_1^{(0)}) \subseteq S_0^{(0)} \cup S_1^{(0)}$ since $\mathrm{dom}(p_1') \subseteq S_0^{(0)}$ and $\mathrm{dom}(p_1) \subseteq S_1 \cup S_0^{(0)}$, which implies that $\mathrm{dom}(\pi^*(p_1)) \subseteq S_1^{(0)} \cup S_0^{(0)}$.

Let $\dot{c}_1^{(0)} = \pi^*(\dot{c}_1)$. Thus, we have

$$p_1^{(0)} \Vdash \dot{c}_1^{(0)} \in \dot{F}(0).$$

Continuing in this manner, we keep building a sequence of incompatible conditions $p_\xi^{(0)} \in D_0$ such that

$$p_\xi^{(0)} \Vdash \dot{c}_\xi^{(0)} \in \dot{F}(0),$$

$S_\xi^{(0)} \cap \bigcup_{\eta<\xi} S_\eta^{(0)} \subseteq S$, and $\mathrm{dom}(p_\xi^{(0)}) \subseteq \bigcup_{\eta\leq\xi} S_\eta^{(0)}$. This process must terminate after some β_0-many steps, with $\beta_0 < \kappa^+$ because the poset $\mathbb{P}$ has the κ^+-cc. Let

$$A_0 = \{p_\xi^{(0)} \mid \xi < \beta_0\}$$

be the resulting maximal antichain contained in D_0. Let $T_0 = \bigcup_{\xi<\beta_0} S_\xi^{(0)}$, and observe that by the disjointness of the $S_\xi^{(0)}$ modulo S, we have that T_0 cannot have an infinite branch. Since T_0 clearly has size κ, it is useful.

Let's argue that the tree T_0, because it contains $\bigcup_{\xi<\alpha} \mathrm{dom}(p_\xi^{(0)})$, witnesses that the mixed name $\dot{C}_0$ of the names $\dot{c}_\xi^{(0)}$ for $\xi < \beta_0$ over the antichain A_0 is symmetric. Recall that

$$\dot{C}_0 = \bigcup_{\xi<\beta_0} \{(\tau, r) \mid r \leq p_\xi^{(0)}, r \Vdash \tau \in \dot{c}_\xi^{(0)}, \tau \in \mathrm{dom}(\dot{c}_\xi^{(0)})\}.$$

Fix an automorphism π point-wise fixing T_0. It suffices to argue that whenever $(\tau, r) \in \dot{C}_0$, then $(\pi^*(\tau), \pi^*(r)) \in \dot{C}_0$. So suppose $(\tau, r) \in \dot{C}_0$ and fix $p_\xi^{(0)}$, with $\xi < \beta_0$, witnessing this. Since $r \leq p_\xi^{(0)}$, it follows that $\pi^*(r) \leq p_\xi^{(0)}$; since $r \Vdash \tau \in \dot{c}_\xi^{(0)}$, it follows that $\pi^*(r) \Vdash \pi^*(\tau) \in \dot{c}_\xi^{(0)}$; and finally, since $\dot{c}_\xi^{(0)}$ is symmetric and $\tau \in \mathrm{dom}(\dot{c}^{(0)}_\xi)$, it follows that $\pi^*(\tau) \in \mathrm{dom}(\dot{c}^{(0)}_\xi)$. Now observe that, since each $\dot{c}^{(0)}_\xi \in \mathrm{HS}$, we have $\dot{C}_0 \in \mathrm{HS}$.

Next, we let D_1 be the dense set below q of conditions p such that for some name $\dot{c}_p \in \mathrm{HS}$, $p \Vdash \dot{c}_p \in \dot{F}(1)$. We repeat the process for D_1, building a maximal antichain

$$A_1 = \{p^{(1)}_\xi \mid \xi < \beta_1\},$$

with $\beta_1 < \kappa^+$, contained in D_1 and trees $S^{(1)}_\xi$, for $\xi < \beta_1$, such that

$$p^{(1)}_\xi \Vdash \dot{c}^{(1)}_\xi \in \dot{F}(1).$$

At the same time, we ensure that for any $\xi < \beta_1$, $S^{(1)}_\xi \cap (\bigcup_{\eta<\xi} S^{(1)}_\eta \cup T_0) \subseteq S$. Let $T_1 = \bigcup_{\xi<\beta_1} S^{(1)}_\xi$, and observe that $T_0 \cup T_1$ is useful. Let $\dot{C}_1$ be a mixed name of the names $\dot{c}^{(1)}_\xi$ over the antichain A_1, and observe that $\dot{C}_1 \in \mathrm{HS}$ as witnessed by T_1.

We continue this process for every $\xi < \kappa$ and let $T = \bigcup_{\xi<\kappa} T_\xi$, which is clearly useful and witnesses that the canonical name for the sequence of the $\dot{C}_\xi$, for $\xi < \kappa$, is symmetric. □

15. A model of KM + CC in which DC_ω fails

Let $G \subseteq \mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, (\kappa^+)^{<\omega})$ be L-generic, and let N be the symmetric submodel of $L[G]$ constructed in Section 14. Let $\mathscr{V} = (L_\kappa, \in, V^N_{\kappa+1})$. We will argue that $\mathscr{V} \models \mathrm{KM} + \mathrm{CC} + \neg\mathrm{DC}_\omega$.

Theorem 15.1. *The model $\mathscr{V} = (L_\kappa, \in, V^N_{\kappa+1}) \models \mathrm{KM} + \mathrm{CC}$ and DC_ω fails in $\mathscr{V}$ for a Π^1_1-assertion.*

Proof. Since $L \subseteq N \subseteq L[G]$, it follows that $V^N_\kappa = V^{L[G]}_\kappa$. Since the forcing $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, (\kappa^+)^{<\omega})$ is $<\kappa$-closed, $L_\kappa = V^{L[G]}_\kappa$. Thus, $V^N_\kappa = L_\kappa \models \mathrm{ZFC}$. The class replacement axiom holds in $\mathscr{V}$ since for every $A \in V^{L[G]}_{\kappa+1} \supseteq V^N_{\kappa+1}$ and $\alpha < \kappa$, we have $A \cap L_\alpha \in V^{L[G]}_\kappa = L_\kappa$. The global well-order axiom holds in $\mathscr{V}$ since $L_{\kappa+1} \subseteq V^N_{\kappa+1}$ has a well-ordering of L_κ . Finally, comprehension holds for all second-order assertions since $V^N_{\kappa+1} = P^N(L_\kappa)$ and $N \models \mathrm{ZF}$. Thus, $\mathscr{V} \models \mathrm{KM}$. The choice scheme CC holds in $\mathscr{V}$ because $V^N_{\kappa+1} = P^N(L_\kappa)$ and AC_κ holds in N. The generic tree $\mathcal{T} \in V^N_{\kappa+1}$ witnesses that DC_ω fails in $\mathscr{V}$. So it remains to show that the tree $\mathcal{T}$ is Π^1_1-definable over $\mathscr{V}$.

By Theorem 13.4, the elements of $\mathcal{T}$ are precisely the n-length sequences of subsets of κ that are L-generic for $\mathbb{J}(\kappa)_n$ for some $n < \omega$. Since the relation on $\mathcal{T}$ is sequence end-extension, it suffices to show that the property of being an n-length sequence of subsets of κ that is L-generic for $\mathbb{J}(\kappa)_n$ is Π^1_1 over $\mathscr{V}$.

By Corollary 11.4 (4), any L-generic n-length sequence of subsets of κ for $\mathbb{J}(\kappa)_n$ is also M_ξ-generic for $\mathbb{P}^{(\xi)}_n$. Since $\mathbb{P}(\vec{P}^{\mathbb{J}(\kappa)}, (\kappa^+)^{<\omega})$ has the κ^+-cc by Theorem 13.2, the converse holds as well: if an n-length sequence of subsets of κ is M_ξ-generic for $\mathbb{P}^{(\xi)}_n$ for all non-trivial stages $\xi + 1$, then it is fully L-generic for $\mathbb{J}(\kappa)_n$. Next, observe that whether an n-length sequence A of subsets of κ is M_ξ-generic for $\mathbb{P}^{(\xi)}_n$ can be verified in any $L_\alpha[A] \models \mathrm{ZFC}^-$ with $\alpha > \xi$. Putting it all together we get that an n-length sequence A of subsets of κ is L-generic for $\mathbb{J}(\kappa)_n$ if and only if for every $L_\alpha[A] \models \mathrm{ZFC}^-$, with $\alpha < \kappa^+$, $L_\alpha[A]$ satisfies that A is M_ξ-generic for

$\mathbb{P}_n^{(\xi)}$ for every non-trivial stage $\xi+1$. Thus, an n-length sequence A of subsets of κ is L-generic for $\mathbb{J}(\kappa)_n$ if and only if $\mathscr{V}$ satisfies that for every class X, Y, Z, if X is a well-order and Y codes $L_X[A]$ and Z is a truth predicate for $L_X[A]$ and $L_X[A] \models \mathrm{ZFC}^-$ (according to Z), then A is M_ξ-generic for $\mathbb{P}_n^{(\xi)}$ for every non-trivial successor stage $\xi+1$ in $L_X[A]$. Since the if…then statement is clearly first-order, the assertion is Π^1_1. □

16. Open Questions

Our result showing that KM + CC does not prove DC_ω assumed the existence of an inaccessible cardinal. But consistency-wise KM + CC is weaker than an inaccessible cardinal.

Question 16.1. Can we construct a model of KM+CC in which DC_ω fails starting with the assumption that there is a model of KM + CC?

Another natural family of questions involve whether we can further separate the principles DC_α from each other.

Question 16.2. Is it consistent that there is a model of $\mathrm{KM}+\mathrm{CC}+\mathrm{DC}_\omega$ in which DC_{ω_1} fails?

Question 16.3. Is it consistent that there is a model of KM + CC in which DC_α holds for every regular cardinal α, but $\mathrm{DC}_{\mathrm{Ord}}$ fails?

We conjecture that both of the above questions will be resolved positively if the construction in this article can be generalized to trees $\kappa^{<\lambda}$ for cardinals $\omega < \lambda \leq \kappa$.

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