Indestructible remarkable cardinals

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The origin of remarkable cardinals

**Definition:** A cardinal $\kappa$ is **remarkable** if in every $\text{Coll}(\omega, < \kappa)$-extension $V[G]$, for every regular $\lambda > \kappa$, there is an embedding $j : H_{\lambda} \rightarrow H_{\lambda}$, for some $V$-regular $\bar{\lambda} < \kappa$, with $\text{cp}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.

**Theorem:** (Schindler, ’00) The following are equiconsistent.
- The theory of $L(\mathbb{R})$ cannot be changed by proper forcing.
- There exists a **remarkable cardinal**.
A generic version of supercompactness

**Theorem:** (Magidor, '71) A cardinal $\kappa$ is supercompact if and only if for every regular $\lambda > \kappa$, there is an embedding $j : H_{\bar{\lambda}} \rightarrow H_{\lambda}$, for some regular $\bar{\lambda} < \kappa$, with $\text{cp}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.

**Absoluteness lemma for countable embeddings:** (Folklore) If $M$ is countable and there is an embedding $j : M \rightarrow N$, then every transitive model $W \models \text{ZFC}^-$ such that $M, N \in W$ and $M$ is countable in $W$ has an embedding $j^* : M \rightarrow N$. If $\text{cp}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$, we can arrange the same to be true for $j^*$. We can also arrange for $j$ and $j^*$ to agree on finitely many values.

**Proof:**

- Construct the tree $T$ of finite partial embeddings from $M$ to $N$.
- $T$ has a branch if and only if there is an embedding from $M$ to $N$.
- The tree $T$ is in $W$.
- The tree $T$ is ill-founded in $V$ and hence also in $W$. □
A generic version of supercompactness (continued)

**Observation:** A cardinal $\kappa$ is remarkable if and only if for every regular $\lambda > \kappa$, there is a set-forcing extension $V[H]$ in which there is an embedding $j : H_\lambda \to H_\lambda$, for some $V$-regular $\bar{\lambda} \prec \kappa$, with $\text{cp}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.

**Proof:** Suppose $j : H_\lambda \to H_\lambda$ exists in $V[H]$.

- Let $G \subseteq \text{Coll}(\omega, <\kappa)$ be $V[H]$-generic.
- $H_\lambda$ is countable in $V[G] \subseteq V[H][G]$.
- $j^* : H_\lambda \to H_\lambda$ exists in $V[G]$ (by the absoluteness lemma).
- Every $\text{Coll}(\omega, <\kappa)$-extension has some $j : H_\lambda \to H_\lambda$ (since $\text{Coll}(\omega, <\kappa)$ is weakly homogeneous). $\square$
Other characterizations of remarkable cardinals

**Definition:** In a \( \text{Coll}(\omega, < \kappa) \)-extension \( V[G] \), an embedding \( j : H_\lambda \rightarrow H_\lambda \) is \((\bar{\mu}, \bar{\lambda}, \mu, \lambda)\)-remarkable if

- \( \bar{\lambda}, \lambda \) are \( V \)-regular,
- \( \text{cp}(j) = \bar{\mu} \) and \( j(\bar{\mu}) = \mu \).

**Note:** A cardinal \( \kappa \) is remarkable if in every \( \text{Coll}(\omega, < \kappa) \)-extension \( V[G] \), for every regular \( \lambda > \kappa \), there is a \((\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)\)-remarkable embedding.

**Lemma:** If \( \kappa \) is remarkable, then in every \( \text{Coll}(\omega, < \kappa) \)-extension \( V[G] \), for every regular \( \lambda > \kappa \) and \( a \in H_\lambda \), there is a \((\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)\)-remarkable \( j : H_\lambda \rightarrow H_\lambda \) with \( a \in \text{ran}(j) \). In particular, there are such \( j \) with \( \text{cp}(j) \) arbitrarily high in \( \kappa \).

**Lemma:** In a \( \text{Coll}(\omega, < \kappa) \)-extension \( V[G] \), if \( j : H_\lambda \rightarrow H_\lambda \) is \((\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)\)-remarkable, then it lifts to

\[
j : H_\lambda[G_{\bar{\kappa}}] \rightarrow H_\lambda[G]
\]

(where \( G_{\bar{\kappa}} \) is the restriction of \( G \) to \( \text{Coll}(\omega, < \bar{\kappa}) \)).
Other characterizations of remarkable cardinals (continued)

**Theorem:** (Schindler, ’00) A cardinal $\kappa$ is remarkable if and only if for every regular $\lambda > \kappa$, there are countable transitive models $M$ and $N$ with embeddings

- $\pi : M \rightarrow H_\lambda$ with $\pi(\kappa) = \kappa$,
- $\sigma : M \rightarrow N$ such that
  - $\text{cp}(\sigma) = \kappa$,
  - $\text{ORD}^M$ is regular in $N$ and $M = H_{\text{ORD}^M}^N$,
  - $\sigma(\kappa) > \text{ORD}^M$. 

\[ \begin{array}{c}
\kappa \\
\sigma(\kappa) \\
\bar{\kappa} \\
\kappa \\
\end{array} \]
Remarkable cardinals in the hierarchy

**Theorem:** (Schindler, '00) Strong cardinals are remarkable.

**Proof:** Suppose $\kappa$ is strong.

- Fix a regular $\lambda > \kappa$ and $j : V \to M^*$ with $\text{cp}(j) = \kappa$, $j(\kappa) > \lambda$, and $H_\lambda \subseteq M^*$.
- $j : H_\lambda \to H_{j(\lambda)}^{M^*}$ and $H_\lambda \subseteq H_{j(\lambda)}^{M^*}$.
- Take countable $\langle X, Y, h, \in \rangle < \langle H_{j(\lambda)}^{M^*}, H_\lambda, j, \in \rangle$.
- Let $\rho : X \to N$ be the collapse map. Then $\rho \upharpoonright Y : Y \to M$ is the collapse of $Y$.
- Define $\pi : \rho^{-1} : M \to H_\lambda$ and $\sigma = \rho \circ h \circ \rho^{-1} : M \to N$. □

**Observation:** Measurable cardinals are not necessarily remarkable.

**Proof:** Remarkable cardinals are totally indescribable and a measurable cardinal is $\Sigma^2_1$-describable. □

**Theorem:** Remarkable cardinals are downward absolute to $L$.

**Proof:** In a $\text{Coll}(\omega, <\kappa)$-extension $V[G]$, suppose $j : H_\lambda \to H_\lambda$ is $(\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)$-remarkable.

- $j : L_\lambda \to L_\lambda$ in $V[G]$.
- $j^* : L_\lambda \to L_\lambda$ exists in $L[G]$ since $L_\lambda$ is countable in $L[G]$ (by the absoluteness lemma). □
Remarkable cardinals in the hierarchy (continued)

**Definition:** A weak $\kappa$-model (for a cardinal $\kappa$) is a transitive $M \models \text{ZFC}^-$ of size $\kappa$ and height above $\kappa$.

Suppose $M$ is a weak $\kappa$-model.

**Observation:** TFAE.

- There exists $j : M \rightarrow N$ with $\text{cp}(j) = \kappa$.
- There exists an $M$-ultrafilter $U$ with a well-founded ultrapower.
  - $U$ is an $M$-ultrafilter if $\langle M, \in, U \rangle \models U$ is a normal ultrafilter.
  - $U = \{ A \in M \mid \kappa \in j(A) \}$.

**Example:** A cardinal $\kappa$ is weakly compact if and only if $2^{<\kappa} = \kappa$ and every $A \subseteq \kappa$ is contained in a weak $\kappa$-model $M$ for which there exists an $M$-ultrafilter on $\kappa$ with a well-founded ultrapower.

**Definition:** An $M$-ultrafilter $U$ is weakly amenable if for every $X \in M$ with $|X|^M \leq \kappa$, $X \cap U \in M$.

- $U$ is partially internal to $M$.
- It is needed to iterate the ultrapower construction.
Remarkable cardinals in the hierarchy (continued)

**Definition:** (G., ’07) A cardinal \( \kappa \) is \( \alpha \)-iterable \((1 \leq \alpha \leq \omega_1)\) if every \( A \subseteq \kappa \) is contained in a weak \( \kappa \)-model \( M \) for which there exists a weakly amenable \( M \)-ultrafilter on \( \kappa \) with \( \alpha \)-many well-founded iterated ultrapowers.

**Theorem:** (G., ’07) A 1-iterable cardinal is a stationary limit of completely ineffable cardinals.

**Theorem:** (G., Welch, ’08) An \( \omega \)-Erdős cardinal implies for every \( n \in \omega \), the consistency of a proper class of \( n \)-iterable cardinals.

**Theorem:** (G., Welch, ’08)
- A remarkable cardinal is 1-iterable and a stationary limit of 1-iterable cardinals.
- If \( \kappa \) is 2-iterable, then there is a proper class of remarkable cardinals in \( V_\kappa \).
Laver-like functions

Suppose $\kappa$ is a large cardinal characterized by the existence of some type of elementary embeddings $j$.

A Laver-like function $\ell : \kappa \to V_\kappa$ is a guessing function with the property that for every set $a$, there is some $j$ of the type characterizing the cardinal such that $j(\ell)(\kappa) = a$.

- (supercompact) For every $a \in H_\theta$, there is a $\theta$-supercompactness embedding $j : V \to M$ with $\text{cp}(j) = \kappa$ and $j(\ell)(\kappa) = a$.
- (strong) For every $a \in V_\theta$, there is a $\theta$-strongness embedding $j : V \to M$ with $\text{cp}(j) = \kappa$ and $j(\ell)(\kappa) = a$.
- (extendible) For every $\alpha > \kappa$ and $a \in V_\alpha$, there is $j : V_\alpha \to V_\beta$ with $\text{cp}(j) = \kappa$ and $j(\ell)(\kappa) = a$.
- (strongly unfoldable) For every $a \in V_\theta$, for every $A \subseteq \kappa$, there is a $\theta$-strong unfoldability embedding $j : M \to N$ ($V_\theta \subseteq N$) with $A \in M$ and $j(\ell)(\kappa) = a$.

Additional assumptions:

- $\text{dom}(\ell)$ is contained in the inaccessible cardinals,
- $\ell " \xi \subseteq V_\xi$.
Laver-like functions (continued)

The existence of Laver-like functions can be forced for most large cardinals, but few have them outright.

**Theorem:**
- (Laver, '78) Every supercompact cardinal has a Laver function.
- (Gitik, Shelah, '89) Every strong cardinal has a strong Laver function.
- (Corazza, '00) Every extendible cardinal has an extendible Laver function.
- (Džamonja, Hamkins, '06) Not every strongly unfoldable cardinal has a strongly unfoldable Laver function.

Laver-like functions play an important role in indestructibility arguments.
Remarkable Laver functions

**Definition**: (Cheng, G., '14) A function $\ell: \kappa \to V_\kappa$ is a remarkable Laver function if for every regular $\lambda > \kappa$ and $a \in H_\lambda$, every $\text{Coll}(\omega, < \kappa)$-forcing extension $V[G]$ has a $(\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)$-remarkable $j: H_{\bar{\lambda}} \to H_\lambda$ such that

- $\ell \upharpoonright \bar{\kappa} + 1 \in H_{\bar{\lambda}}$,
- $\bar{\kappa} \in \text{dom}(\ell)$,
- $j(\ell(\bar{\kappa})) = j(\ell \upharpoonright \bar{\kappa} + 1)(\kappa) = a$.

**Additional assumptions**:

- $j(\ell \upharpoonright \bar{\kappa}) = \ell$,
- $\text{dom}(\ell)$ is contained in the inaccessibles,
- $\ell'' \xi \subseteq V_\xi$. 
Remarkable Laver functions (continued)

Suppose $\kappa$ is remarkable and $V[G]$ is a $\text{Coll}(\omega, <\kappa)$-extension.

**Definition:** In $V[G]$, suppose $\ell : \xi \to V_\kappa$ ($\xi \leq \kappa$). Say that $x$ is $\lambda$-anticipated by $\ell$ (for $\lambda$ regular) if
- $x \in H_\lambda$,
- there is a $(\bar{\xi}, \bar{\lambda}, \xi, \lambda)$-remarkable lift $h : H_\bar{\lambda}[G_{\bar{\xi}}] \to H_\lambda[G_\xi]$ with $h(\ell \upharpoonright \bar{\xi} + 1)(\xi) = x$.

**Lemma:** In $V[G]$, if $\ell : \xi \to V_\kappa$ ($\xi < \kappa$) and there is $\lambda$ for which some $x$ is not $\lambda$-anticipated by $\ell$, then the least such $\lambda < \kappa$.

**Definition:** Fix a well-ordering $W$ of $V_\kappa$ of order-type $\kappa$. In $V[G]$, define $\ell : \kappa \to V_\kappa$ inductively as follows. Suppose $\ell \upharpoonright \xi$ has been defined.
- Suppose there is $\lambda$ such that some $x$ is not $\lambda$-anticipated by $\ell \upharpoonright \xi$.
  - Let $\lambda'$ be least such.
  - Let $a$ be $W$-least set that is not $\lambda'$-anticipated by $\ell \upharpoonright \xi$.

Set $\ell(\xi) = a$.
- Otherwise, $\ell(\xi)$ is undefined.

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Indestructible remarkable cardinals
Remarkable Laver functions exist

**Lemma**: The definition of \( \ell \) is independent of the \( \text{Coll}(\omega, < \kappa) \)-extension \( V[G] \) and \( \ell \in V \).

**Proof**: \( \text{Coll}(\omega, < \kappa) \) is weakly homogeneous. \( \square \)

**Theorem** (Cheng, G., ’14) If \( \kappa \) is remarkable, then there is a remarkable Laver function (with all additional properties).

**Proof**: In a \( \text{Coll}(\omega, < \kappa) \)-extension \( V[G] \), suppose \( \lambda \) is least such that some \( a \) is not \( \lambda \)-anticipated by \( \ell \).

- Fix \( H_\tau[G] \) that is large enough to see this.
- Fix a \((\bar{\kappa}, \bar{\tau}, \kappa, \tau)\)-remarkable lift \( j : H_\bar{\tau}[G_{\bar{\kappa}}] \to H_\tau[G] \) and \( j(\bar{\lambda}) = \lambda \).
- \( j(\ell \upharpoonright \bar{\kappa}) = \ell \) (follows from \( W \in \text{ran}(j) \)).
- By elementarity, \( H_\bar{\tau}[G_{\bar{\kappa}}] \) satisfies that \( \bar{\lambda} \) is least such that some \( x \) is not \( \bar{\lambda} \)-anticipated by \( \ell \upharpoonright \bar{\kappa} \) and it is correct by absoluteness lemma.
- It follows that \( \ell \upharpoonright \bar{\kappa} + 1 \in H_\bar{\tau}[G_{\bar{\kappa}}] \) and \( \ell(\bar{\kappa}) \) is not \( \bar{\lambda} \)-anticipated by \( \ell \upharpoonright \bar{\kappa} \).
- By elementarity, \( j(\ell \upharpoonright \bar{\kappa} + 1)(\kappa) = y \) is not \( \lambda \)-anticipated by \( \ell \).
- But \( y \) is anticipated by \( j' = j \upharpoonright H_\bar{\lambda}[G_{\bar{\kappa}}] \) and \( j' \in H_\tau[G] \). \( \rightarrow \leftarrow \square \)
Remarkable Laver functions in indestructibility arguments

**Lemma:** Suppose $\kappa$ is remarkable and $\ell$ is a remarkable Laver function. In a $\text{Coll}(\omega, <\kappa)$-extension $V[G]$, for every regular $\lambda > \kappa$ and $a \in H_\lambda$, there is a $(\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)$-remarkable $j : H_{\bar{\lambda}} \rightarrow H_\lambda$ such that

- $(\bar{\kappa}, \bar{\lambda}] \cap \text{dom}(\ell) = \emptyset$,
- $\ell(\bar{\kappa}) = \langle \bar{a}, \bar{x} \rangle$, where $j(\bar{a}) = a$.

**Proof:** Let $j(\ell)(\kappa) = \langle a, \lambda + 1 \rangle$ and $j(\bar{a}) = a$.

- $\ell(\bar{\kappa}) \notin V_{\bar{\lambda}}$.
- $\ell" \xi \subseteq V_\xi$. $\Box$
Demonstrating indestructibility of remarkable cardinals

Suppose $\kappa$ is remarkable and $V[G]$ is an extension by a forcing notion $\mathbb{P}$.

**Indestructibility strategy:**

Fix $\pi : M \rightarrow H_\lambda$ ($\pi(\bar{\kappa}) = \kappa$) and $\sigma : M \rightarrow N$ such that

- $\text{cp}(\sigma) = \bar{\kappa}$,
- $\text{ORD}^M$ is regular in $N$ and $M = H^N_{\text{ORD}^M}$,
- $\sigma(\bar{\kappa}) > \text{ORD}^M$.

Lift

- $\pi : M[\bar{G}] \rightarrow H_\lambda[G]$,
- $\sigma : M[\bar{G}] \rightarrow N[H]$,
- preserve that $\text{ORD}^M$ is regular in $N[H]$ and $M[\bar{G}] = H^N_{\text{ORD}^M}[H]$.

**Theorem:** (Lifting Criterion) Suppose $j : M \rightarrow N$ is an embedding of $\text{ZFC}^-$ models having generic extensions $M[G]$ and $N[H]$ by forcing notions $\mathbb{P}$ and $j(\mathbb{P})$ respectively. The embedding $j$ lifts to $j : M[G] \rightarrow N[H]$ if and only if $j'' G \subseteq H$. 

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Choosing a good pair $\pi$ and $\sigma$

Suppose $\kappa$ is remarkable and $\lambda > \kappa$ is regular.

Fix $\delta > \lambda$ and $\rho : M' \to H_\delta$ with $M'$ countable, $\rho(\kappa') = \kappa$, $\rho(\lambda') = \lambda$, $\rho(\ell') = \ell$.

- Fix an $M'$-generic $g \subseteq \text{Coll}(\omega, < \kappa')^{M'}$.
- In $M'[g]$, choose a $(\bar{\kappa}, \bar{\lambda}, \kappa', \lambda')$-remarkable $j : H^{M'}_{\bar{\lambda}} \to H^{M'}_{\lambda'}$ such that $(\bar{\kappa}, \bar{\lambda}) \cap \text{dom}(\ell') = \emptyset$.
- Let $\sigma : H^{M'}_{\bar{\lambda}} \to N$ with $N = \{\sigma(f)(a) \mid a \in (V_{\kappa'} \cup \{\kappa'\})^{< \omega}, f \in H^{M'}_{\bar{\lambda}}\}$.
- Let $M = H^{M'}_{\bar{\lambda}}$, $\sigma : M \to N$ and $\pi = \rho \circ j : M \to H_{\lambda}$.
Indestructibility by $\text{Add}(\kappa, \theta)$

**Theorem:** (Cheng, G., '14) A remarkable cardinal $\kappa$ can be made indestructible by $\text{Add}(\kappa, \theta)$ for every $\theta$.

**Proof:**

- $\mathbb{P}_\kappa$ is the $\kappa$-length Easton support iteration which forces with $\text{Add}(\xi, \mu)^{\mathbb{P}_\xi}$ at stage $\xi$ whenever $\ell(\xi) = \langle \mu, x \rangle$ for some $x$.
- Suppose $\kappa$ is not remarkable in a $\mathbb{P}_\kappa * \text{Add}(\kappa, \theta)$-extension.
- There is a regular $\lambda > \kappa$ and $q \in \mathbb{P}_\kappa * \text{Add}(\kappa, \theta)$ such that $q \models \text{“there are no desired embeddings } \pi \text{ and } \sigma \text{ for } H_\lambda \text{”}$.

- Fix a good pair $\pi : M \rightarrow H_\lambda$ and $\sigma : M \rightarrow N$ with
  - $\pi(q) = q$, $\pi(\overline{\mathbb{P}_\kappa}) = \mathbb{P}_\kappa$, $\pi(\overline{\theta}) = \theta$, $\pi(\overline{\ell}) = \ell$
  - $\sigma(\overline{\mathbb{P}_\kappa}) = \mathbb{P}_{\kappa'}$, $\sigma(\overline{\theta}) = \theta'$, $\sigma(\overline{\ell}) = \ell'$,
  - $(\overline{\kappa}, \text{ORD}^M \cap \text{dom}(\ell')) = \emptyset$ and $\overline{\ell}(\overline{\kappa}) = \langle \overline{\theta}, x \rangle$.
- Fix a $V$-generic $G * g \subseteq \mathbb{P}_\kappa * \text{Add}(\kappa, \theta)$ such that
  - $q \in G * g$,
  - $G * g$ is $\pi$ "$M$-generic" ($\mathbb{P}_\kappa * \text{Add}(\kappa, \theta)$ is countably closed),
- Let $\overline{G} * \overline{g} = \pi^{-1}(G * g)$. 

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Indestructibility by $\text{Add}(\kappa, \theta)$ (continued)

**Lift $\pi$ to $M[\bar{G}][\bar{g}]$:**
- $\pi " \bar{G} * \bar{g} \subseteq G * g$.
- Lift $\pi$ to $\pi : M[\bar{G}][\bar{g}] \to H_\lambda[G][g]$ by the lifting criterion.

**Lift $\sigma$ to $M[\bar{G}]$:**
- Need an $N$-generic $G' \subseteq \bar{P}_\kappa'$ with $\sigma " \bar{G} = \bar{G} \subseteq G'$.
- Factor $\bar{P}_\kappa' = \bar{P}_\kappa * \text{Add}(\bar{\kappa}, \bar{\theta}) * \bar{P}_{\text{tail}}$.
- Choose a $N[\bar{G}][\bar{g}]$-generic $G_{\text{tail}} \subseteq \bar{P}_{\text{tail}}$ ($N[\bar{G}][\bar{g}]$ is countable) and let $G' = \bar{G} * \bar{g} * G_{\text{tail}}$.
- Lift $\sigma$ to $\sigma : M[\bar{G}] \to N[G']$ by the lifting criterion.

**Lift $\sigma$ to $M[\bar{G}][\bar{g}]$:**
- Need an $N[G']$-generic $g' \subseteq \text{Add}(\kappa', \theta')^{N[G']}$ with $\sigma " \bar{g} \subseteq g'$.
Choose any $N[G']$-generic $g' \subseteq \text{Add}(\kappa', \theta')^{N[G']}$. Indestructibility by $\text{Add}(\kappa, \theta)$ (continued)

For $p \in \text{Add}(\kappa', \theta')^{N[G']}$, let $p^*$ be the result of altering $p$ to agree with $\sigma'' \bar{g}$.

Theorem (Woodin):

- Each $p^* \in \text{Add}(\kappa', \theta')^{N[G']}$.\[ \]
- $g^* = \{p^* \mid p \in g'\}$ is $N[G']$-generic for $\text{Add}(\kappa', \theta')^{N[G']}$.\[ \]
- Proof needs (1) $\bar{g} \in N[G']$ and (2) $N = \{\sigma(f)(a) \mid a \in [V_{\kappa'} \cup \{\kappa'\}]^{<\omega}, f \in M\}$.

Lift $\sigma$ to $\sigma : M[\bar{G}][\bar{g}] \rightarrow N[G'][g^*]$ by the lifting criterion.

There is no forcing in $\bar{P}_{\kappa'}$ in $(\bar{\kappa}, \text{ORD}^M)$ and $\bar{P}_{\kappa'}$ is progressively more closed.

- $\text{ORD}^M$ is regular in $N[G'][g^*]$,\[ \]
- $M[\bar{G}][\bar{g}] = H_{\text{ORD}^M}^{N[G'][g^*]}$.\[ \]

Thus, $V[G][g]$ has $\sigma$ and $\pi$ as desired, but $q \in G \ast g$ forces otherwise. $\rightarrow \leftarrow \square$
Indestructibility by \(<\kappa\)-closed \(\leq\kappa\)-distributive forcing

**Theorem:** (Cheng, G., '14) A remarkable cardinal \(\kappa\) can be made indestructible all \(<\kappa\)-closed \(\leq\kappa\)-distributive forcing.

**Proof:**

- \(P_\kappa\) is the \(\kappa\)-length Easton support iteration which forces with \(\dot{Q}_\xi\) at stage \(\xi\) whenever \(\ell(\xi) = \langle \dot{Q}_\xi, x \rangle\), where \(\dot{Q}_\xi\) is a \(P_\xi\)-name for a \(<\xi\)-closed \(\leq\xi\)-distributive poset in \(V^{P_\xi}\) and \(x\) is some set.
- Suppose \(\kappa\) is not remarkable in a \(P_\kappa * \dot{Q}\)-extension, where \(\dot{Q}\) is a \(P_\kappa\)-name for a \(<\kappa\)-closed \(\leq\kappa\)-distributive poset.
- There is a regular \(\lambda > \kappa\) and \(q \in P_\kappa * \dot{Q}\) such that
  \[
  q \Vdash "\text{there are no desired embeddings } \pi \text{ and } \sigma \text{ for } H_\lambda".
  \]
- Fix a good pair \(\pi : M \rightarrow H_\lambda\) and \(\sigma : M \rightarrow N\) with
  
  - \(\pi(\bar{q}) = q, \pi(\bar{P}_\kappa) = P_\kappa, \pi(\dot{Q}_\kappa) = \dot{Q}, \pi(\bar{\ell}) = \ell\)
  
  - \(\sigma(\bar{P}_\kappa) = P_\kappa', \sigma(\dot{Q}_\kappa) = \dot{Q}_\kappa', \sigma(\bar{\ell}) = \ell'\),
  
  - \((\bar{\kappa}, \text{ORD}^M) \cap \text{dom}(\ell') = \emptyset \text{ and } \bar{\ell}(\bar{\kappa}) = \langle \dot{Q}_\bar{\kappa}, x \rangle\).
- Fix a \(V\)-generic \(G * g \subseteq P_\kappa * \dot{Q}\) such that
  
  - \(q \in G * g\),
  
  - \(G * g\) is \(\pi \upharpoonright M\)-generic (\(P_\kappa * \dot{Q}\) is countably closed),
- Let \(\bar{G} * \bar{g} = \pi^{-1}(G * g)\).
Indestructibility by $<\kappa$-closed $\leq\kappa$-distributive forcing (continued)

Lift $\pi$ to $M[\bar{G}][\bar{g}]$:

- $\pi$ " $\bar{G} \ast \bar{g} \subseteq G \ast g$.
- Lift $\pi$ to $\pi : M[\bar{G}][\bar{g}] \to H[\bar{G}][g]$ by the lifting criterion.

Lift $\sigma$ to $M[\bar{G}]$:

- Need an $N$-generic $G' \subseteq \bar{P}$ with $\sigma$ " $\bar{G} = \bar{G} \subseteq G'$.
- Factor $\bar{P} = \bar{P} \ast \dot{Q} \ast \bar{P}$.
- Choose a $N[\bar{G}][\bar{g}]$-generic $G_{\text{tail}} \subseteq \bar{P}_{\text{tail}}$ ($N[\bar{G}][\bar{g}]$ is countable) and let $G' = \bar{G} \ast \bar{g} \ast G_{\text{tail}}$.
- Lift $\sigma$ to $\sigma : M[\bar{G}] \to N[G']$ by the lifting criterion.
Indestructibility by $\langle \kappa \rangle$-closed $\leq \langle \kappa \rangle$-distributive forcing (continued)

**Lift $\sigma$ to $M[\bar{G}][\bar{g}]$:**
- Need an $N[G']$-generic $g' \subseteq (\check{\mathbb{Q}}_{\kappa'})_{G'} = \mathbb{Q}_{\kappa'}$ with $\sigma " \bar{g} \subseteq g'$.
- $g' = \langle \sigma " \bar{g} \rangle$ is the filter generated by $\sigma " \bar{g}$.
- Clearly $g'$ is $\sigma " M[\bar{G}]$-generic.
- Indeed, $g'$ is $N[G']$-generic.
  - $N = \{\sigma(f)(a) \mid a \in [V_{\kappa'} \cup \{\kappa'\}]^{<\omega}, f \in M\}$.
  - $\mathbb{Q}_{\kappa'}$ is $\leq \kappa$-distributive in $N[G']$.
- Lift $\sigma$ to $\sigma : M[\bar{G}][\bar{g}] \rightarrow N[G'][g']$ by the lifting criterion.
- There is no forcing in $\bar{\mathbb{P}}_{\kappa'}$ in $(\check{\kappa}, \text{ORD}^M)$ and $\bar{\mathbb{P}}_{\kappa'}$ is progressively more closed.
  - $\text{ORD}^M$ is regular in $N[G'][g']$,
  - $M[\bar{G}][\bar{g}] = H^{N[G'][g']}_{\text{ORD}^M}$.

Thus, $V[G][g]$ has $\sigma$ and $\pi$ as desired, but $q \in G \ast g$ forces otherwise. $\rightarrow \leftarrow \square$
Indestructible remarkable cardinals

**Theorem:** (Cheng, G., '14) A remarkable $\kappa$ can be made indestructible by all $<\kappa$-closed $\leq \kappa$-distributive forcing and all two-step iterations $\text{Add}(\kappa, \theta) \ast \dot{R}$, where $\dot{R}$ is forced to be $<\kappa$-closed and $\leq \kappa$-distributive.

**Proof:** $P_\kappa$ is the $\kappa$-length Easton-support iteration which forces with $\dot{Q}_\xi$ at stage $\xi$ whenever $\ell(\xi) = \langle \dot{Q}_\xi, x \rangle$ for some set $x$, where $\dot{Q}_\xi$ is a $P_\xi$-name for either

- a $<\xi$-closed $\leq \xi$-distributive forcing, or
- $\text{Add}(\xi, \mu)^{V_{P_\xi}} \ast \dot{R}$, where $\dot{R}$ is forced to be $<\xi$-closed and $\leq \xi$-distributive. □

**Applications**

**Theorem:** Any consistent continuum pattern on the regular cardinals can be realized above a remarkable cardinal.

**Theorem:** It is consistent that $\kappa$ is remarkable, but not weakly compact in $\text{HOD}$. 
Thank you!