INDESTRUCTIBILITY PROPERTIES OF RAMSEY AND RAMSEY-LIKE CARDINALS

VICTORIA GITMAN AND THOMAS A. JOHNSTONE

ABSTRACT. We develop new techniques, for use in indestructibility arguments, of lifting embeddings on transitive set models of ZFC^- which lack closure and of lifting iterations of such embeddings to a forcing extension. We use these techniques to establish basic indestructibility results for Ramsey and the Ramsey-like cardinals – α-iterable, strongly Ramsey, and super Ramsey cardinals – introduced in [Git11]. We show that Ramsey, α-iterable, strongly Ramsey, and super Ramsey cardinals κ are indestructible by small forcing, the canonical forcing of the GCH, and the forcing to add a fast function on κ. We also show that if κ is one of these large cardinals, then there is a forcing extension in which the large cardinal property of κ becomes indestructible by $\mathrm{Add}(\kappa, \theta)$ for any cardinal θ.

The following are consequences of the indestructibility results. If κ is Ramsey, α-iterable, strongly Ramsey, or super Ramsey, then there is a forcing extension preserving this in which the GCH fails at κ. If κ is one of these large cardinals, then there is a forcing extension preserving the large cardinal property of κ in which κ is not even weakly compact in HOD. If κ is Ramsey, then there is a forcing extension in which κ remains virtually Ramsey, but is no longer Ramsey (this answers positively a question posed in [Git11]).

1. Introduction

The study of indestructibility properties of large cardinals was initiated by a seminal result of Lévy and Solovay showing that measurable cardinals cannot be destroyed by small forcing [LS67]. The Lévy-Solovay phenomenon is now known to extend to most large cardinal notions, which means, in particular, that large cardinals cannot decide CH or other independent set theoretic statements that can be manipulated by small forcing. This, taken more generally, is the significance of studying indestructibility properties of large cardinals: it provides a means of verifying which set theoretic properties, among those that can be manipulated by forcing, are compatible with a given large cardinal. There are other applications of indestructibility, such as in separating closely related large cardinal notions by forcing to destroy a part of a large cardinal property, while preserving the rest.

Ramsey cardinals were introduced by Erdős and Hajnal in 1962 [EH62], who defined that a cardinal κ is *Ramsey* if every coloring $f : [\kappa]^{<\omega} \to 2$ of finite tuples of elements of κ into two colors has a homogeneous set of size κ. As we will discuss later, Ramsey cardinals can also be characterized by the existence of indiscernibles for certain structures as well as by the existence of iterable ultrafilters for certain families of subsets of κ of size κ (see Theorems 2.9, 5.2). Historically very little was known about the indestructibility properties of Ramsey cardinals. A folklore

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proof, using their original characterization, shows that Ramsey cardinals are indestructible by small forcing [Kan09] (Section 10). Jensen in [Jen74] hinted at a proof that Ramsey cardinals are indestructible by a product forcing which yields the GCH in the forcing extension. Finally, Welch showed in [Wel88], using a characterization of Ramsey cardinals in terms of the existence of indiscernibles, that they are indestructible by the forcing to code the universe into a real.

Most general techniques for establishing indestructibility properties of a large cardinal require it to have a characterization in terms of the existence of elementary embeddings. The indestructibility arguments then proceed by showing how to lift (extend) the elementary embedding(s) characterizing the large cardinal from the ground model V to the forcing extension $V[G]$, thus verifying that the large cardinal maintains its property there. It is more common to think of the large cardinals including and above measurable cardinals as being characterized by the existence of elementary embeddings. But in fact, even smaller large cardinals that we typically associate with combinatorial definitions, such as weakly compact and indescribable cardinals, have elementary embedding characterizations. These smaller large cardinals κ are usually characterized by the existence of elementary embeddings of weak κ-models (transitive models of ZFC$^-$ of size κ with height above κ) or of κ-models (additionally closed under $<\kappa$-sequences). Mitchell discovered an elementary embeddings characterization of Ramsey cardinals involving the existence of countably complete ultrafilters (Theorem 2.9) for weak κ-models [Mit79], but it was not extensively studied until the first author started to explore it in her dissertation with the purpose of obtaining indestructibility results for Ramsey cardinals [Git07]. In the process, the first author generalized aspects of the Ramsey embeddings to introduce new Ramsey-like large cardinal notions: α-iterable, strongly Ramsey, and super Ramsey cardinals [Git11].[1]

The elementary embeddings characterization of Ramsey cardinals does not easily lend itself to standard indestructibility techniques. The first difficulty is that the embeddings are on weak κ-models, as opposed to κ-models, and these may not even be closed under countable sequences. The second difficulty is that the embeddings are ultrapowers by countably complete ultrafilters and while the lift of an ultrapower embedding remains an ultrapower embedding by a potentially larger ultrafilter, it is not trivial to verify that the larger ultrafilter is still countably complete. Strongly Ramsey and super Ramsey cardinals (Definition 2.11), which, as the name suggests, are a strengthening of Ramsey cardinals, were defined to have embedding characterizations remedying the deficiencies of Ramsey embeddings with respect to indestructibility arguments. The α-iterable cardinals generalized a different aspect of the Ramsey embeddings (Definition 2.14). They are defined by the existence of partially iterable ultrafilters for weak κ-models, a requirement that weakens the Ramsey embeddings characterization because countably complete ultrafilters are fully iterable.

In this article, we prove basic indestructibility results for Ramsey, α-iterable, strongly Ramsey, and super Ramsey cardinals using a mix of old and newly introduced techniques. We use standard techniques to establish indestructibility properties of strongly Ramsey and super Ramsey cardinals, as their definition was directly motivated to make them easily amenable to these techniques. We develop

[1]Since then other Ramsey-like large cardinal notions have been introduced in, for example, [HS18] and [HL21].

techniques for lifting embeddings on models without closure. We show that if the forcing notion is countably closed, then a lift of the ultrapower by a countably complete ultrafilter retains this property in the forcing extension. The combination of these new techniques allows us to prove the same basic indestructibility results for Ramsey cardinals as for strongly and super Ramsey cardinals. For the α-iterable cardinals, we develop techniques for simultaneously lifting entire iterations of embeddings, so that we can verify that the potentially larger ultrafilter associated with the lift of the first ultrapower in the iteration continues to have at least the iterability of the original ultrafilter. The new indestructibility techniques we introduce can potentially be used to establish a variety of indestructibility results for these and similar large cardinal notions. Here, we obtain the following indestructibility results.

Theorem 1.1.

(1) *Ramsey, α-iterable, strongly Ramsey, and super Ramsey cardinals κ are indestructible by:*
  (a) *small forcing,*
  (b) *the canonical forcing of the* GCH*,*
  (c) *the forcing to add a fast function on κ,*

(2) *If κ is one of these large cardinals, then there is a forcing extension in which the large cardinal property of κ becomes indestructible by the forcing* $\mathrm{Add}(\kappa, \theta)$ *for every cardinal θ.*

These indestructibility properties have the following consequences.

Corollary 1.2.

(1) *If κ is Ramsey, α-iterable, strongly Ramsey, or super Ramsey, then there is a forcing extension preserving this in which the* GCH *fails at κ.*
(2) *If κ is Ramsey, α-iterable, strongly Ramsey, or super Ramsey, then there is a forcing extension preserving this in which κ is not even weakly compact in* HOD*.*
(3) *If κ is Ramsey, then there is a forcing extension destroying this, while preserving that κ is virtually Ramsey.*

To establish (2), we use techniques from [CFH15]. The *virtually Ramsey* cardinals from (3) (see Definition 5.3) were introduced in [SW11] as an upper bound on the consistency strength of a variant of Chang's Conjecture studied there. The new indestructibility techniques are introduced in Section 3. Theorem 1.1 and its corollaries are proved in Section 4.

2. Ramsey and Ramsey-like cardinals

Ramsey cardinals and the Ramsey-like cardinals κ studied in [Git11], as well as many other smaller large cardinals, are characterized by the existence of certain elementary embeddings of weak κ-models or κ-models of set theory.

Definition 2.1. Suppose that κ is a cardinal.

(1) A *weak κ-model* is a transitive model of ZFC^- of size κ and height above κ.[2]

[2]The theory ZFC^- consists of the axioms of ZFC without the powerset axiom, with the collection scheme instead of the replacement scheme, and with the statement that every set can be

(2) A *κ-model* is a weak κ-model that is additionally closed under $<\kappa$-sequences.

Natural examples of weak κ-models and κ-models arise as elementary substructures of H_{κ^+}, the collection of all sets of hereditary size $\leq\kappa$. Any elementary substructure of H_{κ^+} of size κ that contains κ as a subset is a weak κ-model and if $\kappa^{<\kappa}=\kappa$, then we can build elementary substructures of H_{κ^+} that are κ-models.

The analogue of a normal ultrafilter[3] in the setting of weak κ-models is the notion of an M-ultrafilter for a weak κ-model M.

Definition 2.2. Suppose that $M\models\mathrm{ZFC}^-$ is transitive and κ is a cardinal in M. A set $U\subseteq\mathcal{P}(\kappa)^M$ is[4] an *M-ultrafilter* if the structure $\langle M,\in,U\rangle$, consisting of M together with a predicate for U, satisfies that U is a normal ultrafilter on κ.

The set U must be viewed as interpreting a predicate over M since in most interesting cases it will not be an element of M. Note that U measures only those subsets of κ that are elements of M and is normal/κ-complete only for sequences that are themselves elements of M. Consequently, even a countable sequence of elements of U might have an empty intersection if the sequence is not from M.

If an ultrafilter is an element of a model of set theory, then the ultrapower construction with it can be iterated along the ordinals. At successor ordinal stages, the iteration proceeds by taking the ultrapower by the image of the ultrafilter under the embedding from the previous stage and direct limits are taken at limit stages. This produces a directed system of *iterated ultrapowers* of the original model. In this situation, it is easy to see that an ultrafilter has a well-founded ultrapower if and only if it is countably complete, and indeed Kunen showed that all iterated ultrapowers of a countably complete ultrafilter are well-founded [Kun70]. If M is a transitive model of ZFC^-, then an M-ultrafilter suffices to carry out the ultrapower construction with the Łoś Theorem holding, but the ultrapower may not be well-founded. To iterate the ultrapower construction by an M-ultrafilter, we must first modify the successor step construction to work with ultrafilters that are external to the model. The modified construction still requires that the ultrafilters be at least partially internal to the model, a concept captured by the notion of weak amenability.

Definition 2.3. Suppose that M is a weak κ-model. An M-ultrafilter U is *weakly amenable* if for every $A\in M$ of size κ in M, the intersection $U\cap A$ is an element of M.

While weak amenability allows us to iterate the ultrapower construction, it does not guarantee the well-foundedness of any of the iterates. There are weakly amenable M-ultrafilters that do not even have a well-founded ultrapower, as well as those all of whose iterated ultrapowers are well-founded. It is shown in [GW11] that it is consistent to have M-ultrafilters realizing all the possibilities in between as well: we can have, for every countable ordinal α, a model M and an M-ultrafilter with exactly α-many well-founded iterated ultrapowers. This covers all possibilities

well-ordered instead of the axiom of choice. See [GHJ16] for the significance of this particular choice of axioms.

[3]Here, we adopt the convention that an ultrafilter on a cardinal κ includes the tail sets. It follows that ultrafilters are necessarily non-principal and normal ultrafilters on κ are κ-complete.

[4]We intend $\mathcal{P}(\kappa)^M$ to refer to the set in V of all the subsets of κ that are elements of M and this set need not exist in M itself.

since Gaifman showed that an M-ultrafilter with ω_1-many well-founded iterated ultrapowers is already *iterable*, having all well-founded iterated ultrapowers [Gai74].

Definition 2.4. Suppose that M is a weak κ-model and $\lambda \leq \kappa$ is a cardinal. An M-ultrafilter is λ*-complete* if every sequence of elements of U of length less than λ has a nonempty intersection. We will call ω_1-complete M-ultrafilters *countably complete*.

Note that, unlike in the definition of completeness for ultrafilters, we do not require that the intersection of the elements of the sequence be an element of the M-ultrafilter, only that the intersection is nonempty. This definition is more appropriate to the context of M-ultrafilters because for sequences external to M there is no reason to assume that the intersection is an element of M, which would be required for the M-ultrafilter to be able to measure it.

Clearly a countably complete M-ultrafilter has a well-founded ultrapower and Kunen showed that a weakly amenable countably complete M-ultrafilter is iterable [Kun70].

In the remark below, we summarize some basic facts concerning elementary embeddings of weak κ-models M and ultrapowers by M-ultrafilters.

Remark 2.5. Suppose that M is a weak κ-model.

(1) An elementary embedding $j : M \to N$[5] with $\operatorname{crit}(j) = \kappa$ is the ultrapower map by an M-ultrafilter on κ if and only if
$$N = \{j(f)(\kappa) \mid f : \kappa \to M, f \in M\}.$$
(2) If $j : M \to N$ is the ultrapower map by an M-ultrafilter on κ and $M^\alpha \subseteq M$ for some $\alpha < \kappa$, then $N^\alpha \subseteq N$.
(3) An elementary embedding $j : M \to N$ with $\operatorname{crit}(j) = \kappa$ gives rise to the M-ultrafilter $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ on κ, which we say is *generated by* κ *via* j. The ultrapower by U is isomorphic to
$$X = \{j(f)(\kappa) \mid f : \kappa \to M \text{ in } M\}$$
via the map $\varphi : [f]_U \mapsto j(f)(\kappa)$ and hence is well-founded.

Definition 2.6. We call an elementary embedding $j : M \to N$ with $\operatorname{crit}(j) = \kappa$ κ*-powerset preserving* if $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$.[6]

The existence of a weakly amenable M-ultrafilter on κ with a well-founded ultrapower is precisely equivalent to the existence of a κ-powerset preserving embedding of M.

Lemma 2.7 (Folklore)**.** *Suppose that M is a weak κ-model.*

(1) *If $j : M \to N$ is a κ-powerset preserving elementary embedding and U is the M-ultrafilter generated by κ via j, then U is weakly amenable.*
(2) *If U is a weakly amenable M-ultrafilter on κ with a well-founded ultrapower and $j : M \to N$ is the ultrapower map by U, then j is κ-powerset preserving.*

[5] Unless explicitly stated otherwise, the embeddings we consider are assumed to be of transitive models.

[6] Note that neither powerset is required to exist in its respective model.

For an extended discussion of weakly amenable M-ultrafilters and relevant proofs see [Kan09] (Section 19).

We now have in place all the preliminaries required for stating the elementary embeddings characterization of Ramsey cardinals and the definitions of the Ramsey-like cardinals that arose from it. But before we do so, it is instructive to recall for comparison the elementary embeddings characterization of weakly compact cardinals.

Theorem 2.8 (Folklore). *A cardinal κ is weakly compact if and only if $\kappa^{<\kappa} = \kappa$ and every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an elementary embedding $j : M \to N$ with $crit(j) = \kappa$ (equivalently, for which there exists an M-ultrafilter on κ with a well-founded ultrapower).*

It is not difficult to show that if κ is weakly compact, then indeed *every* weak κ-model M has an M-ultrafilter with a well-founded ultrapower, and so in particular, embeddings exist for κ-models that are elementary in H_{κ^+} and therefore reflect V to some extent. It is interesting to note that it was shown by Hamkins that the assumption $\kappa^{<\kappa}$ is necessary to Theorem 2.8 as it is possible for a cardinal below the continuum to satisfy just the embedding property [Ham].

Theorem 2.9 (Mitchell [Mit79]). *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable countably complete M-ultrafilter on κ.*

Note that the assumption $\kappa^{<\kappa}$ is absent from the theorem. The reason is that the inaccessibility of κ follows from the weak amenability assumption. For an exposition of the proof adapted from [Dod82], see [Git11]. From the proof presented there it is clear that we can strengthen Theorem 2.9 as follows.

Theorem 2.10. *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable κ-complete M-ultrafilter on κ.*

It is natural to ask, by analogy with weakly compact cardinals, that if κ is Ramsey, whether *every* weak κ-model M has a weakly amenable countably complete M-ultrafilter on κ. This is not the case. By assuming that every $A \subseteq \kappa$ is contained in a κ-model M with a weakly amenable M-ultrafilter on κ (which must then be countably complete), we get a stronger large cardinal notion, and we further strengthen this notion by assuming that $M \prec H_{\kappa^+}$. The assumption that *every* κ-model M has a weakly amenable M-ultrafilter on κ is inconsistent!

Definition 2.11. [Git11] A cardinal κ is *strongly Ramsey* if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a weakly amenable M-ultrafilter on κ. A cardinal κ is *super Ramsey* if we additionally assume that $M \prec H_{\kappa^+}$.

Using Remark 2.5 (2), we obtain a characterization of strongly and super Ramsey cardinals in terms of the existence of κ-powerset preserving embeddings.

Remark 2.12. A cardinal κ is strongly Ramsey if and only if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a κ-powerset preserving embedding $j : M \to N$ where N is a κ-model. The same characterization holds for super Ramsey cardinals with the additional assumption that $M \prec H_{\kappa^+}$.

The first author showed in [Git11] that every strongly Ramsey cardinal is a stationary limit of Ramsey cardinals, every super Ramsey cardinal is a stationary limit of strongly Ramsey cardinals, and every measurable cardinal is a stationary limit of super Ramsey cardinals.

Another aspect of the Ramsey embeddings that can be studied is their iterability properties. For instance, we can ask if κ is weakly compact, whether it follows that every $A \subseteq \kappa$ is contained in a weak κ-model for which there is a weakly amenable M-ultrafilter on κ with a well-founded ultrapower. We can also ask, given a countable ordinal α, whether there is an M-ultrafilter with exactly α-many well-founded iterated ultrapowers. Recall that this behavior is not possible for an ultrafilter that lives inside the model.

Definition 2.13. Suppose that M is a weak κ-model and U is an M-ultrafilter on κ. We say that:

(1) U is 0-*good* if it produces a well-founded ultrapower,
(2) U is 1-*good* if it is 0-good and weakly amenable,
(3) for an ordinal $\alpha > 1$, U is α-*good*, if it produces at least α-many well-founded iterated ultrapowers.

Using the notion of α-good M-ultrafilters, we define the corresponding notion of α-iterable cardinals.

Definition 2.14. For $1 \leq \alpha \leq \omega_1$, a cardinal κ is α-*iterable* if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an α-good M-ultrafilter on κ.

The first author showed in [Git11] that 1-iterable cardinals are limits of completely ineffable cardinals, and hence much stronger than weakly compact cardinals. It was shown in [GW11] and [SW11] that α-iterable cardinals form a hierarchy of strength below Ramsey cardinals. More precisely, for $\alpha < \beta \leq \omega_1$, every β-iterable cardinal is a limit of α-iterable cardinals, and a Ramsey cardinal is a limit of ω_1-iterable cardinals. Thus, in particular, the M-ultrafilter property of being countably complete is stronger than iterability.

We end by noting that in each of the elementary embeddings characterizations of Ramsey, α-iterable, strongly Ramsey, or super Ramsey cardinals, we can equivalently replace "every $A \subseteq \kappa$" by "every $A \in H_{\kappa^+}$" since an element of H_{κ^+} is coded by a binary relation on κ that Mostowski collapses to it and ZFC^- suffices to perform the Mostowski collapse. It also follows that if $j : M \to N$ is κ-powerset preserving, then M and N have the same sets of hereditary size $\leq\kappa$.

3. Old and new lifting techniques

In this section, we review standard techniques for lifting embeddings to a forcing extension and develop new techniques for lifting embeddings of weak κ-models, lifting entire iterations of embeddings, and lifting ultrapowers by countably complete ultrafilters in such a way that the lift continues to be an ultrapower by a countably complete ultrafilter.

Lifting arguments generally rely on two elementary facts, the *lifting criterion* and the *diagonalization criterion*. The lifting criterion states that lifting an embedding $j : M \to N$ to a forcing extension $M[G]$ amounts to finding an N-generic filter H with $j\,"\,G \subseteq H$.

Lemma 3.1 (Lifting Criterion)**.** *Suppose that $j : M \to N$ is an elementary embedding of* ZFC^- *models, $\mathbb{P} \in M$ is a forcing notion, $G \subseteq \mathbb{P}$ is M-generic, and $H \subseteq N$ is $j(\mathbb{P})$-generic. Then the embedding j lifts to an embedding $j : M[G] \to N[H]$ with $j(G) = H$ if and only if $j\,\text{"}\,G \subseteq H$.*

We can weaken the assumption that $\mathbb{P} \in M$ to the assumption that $\mathbb{P}$ is a definable over M, by a formula $\varphi(x, a)$, forcing notion with definable forcing relations, and denote by $j(\mathbb{P})$ the forcing notion given by the interpretation of the formula $\varphi(x, j(a))$ over N.[7]

If the original embedding happened to be an ultrapower map, then its lift to a forcing extension will be an ultrapower map as well.

Remark 3.2. Suppose that $M \models \mathrm{ZFC}^-$, $\mathbb{P} \in M$ is a forcing notion, and $G \subseteq \mathbb{P}$ is M-generic. If $j : M \to N$ is the ultrapower map by an M-ultrafilter, then any lift $j : M[G] \to N[H]$ is the ultrapower map by an $M[G]$-ultrafilter.

The proof follows from Remark 2.5 (1). The diagonalization criterion generalizes a standard argument showing that there is a filter meeting any countable collection of dense subsets of a partial order.

Lemma 3.3 (Diagonalization Criterion (1))**.** *If $\mathbb{P}$ is a forcing notion in a model $M \models \mathrm{ZFC}^-$ and for some cardinal κ the following criteria are satisfied:*

1. *$M^{<\kappa} \subseteq M$,*
2. *$\mathbb{P}$ is $<\kappa$-closed in M,*
3. *M has at most κ many maximal antichains of $\mathbb{P}$,*

then there is an M-generic filter for $\mathbb{P}$.

We shall refer to Lemma 3.3 as diagonalization criterion (1), since below we introduce a second diagonalization criterion that works for models with limited or no closure. First, we need to recall what it means for a filter to be generic for a non-transitive model of set theory.

Definition 3.4. Suppose that $M \models \mathrm{ZFC}^-$ is transitive, $X \prec M$ (not necessarily transitive) and $\mathbb{P} \in X$ is a forcing notion. Then an M-generic filter $G \subseteq \mathbb{P}$ is *X-generic* if $G \cap D \cap X \neq \emptyset$ for every dense subset D of $\mathbb{P}$ in X.

Note that the usual definition of genericity and Definition 3.4 coincide for transitive models X. If $X \prec M \models \mathrm{ZFC}^-$ and G is M-generic for a poset $\mathbb{P} \in X$, then $X[G]$ is defined to be the collection of all τ_G with $\tau \in X$.

Remark 3.5. Suppose that $M \models \mathrm{ZFC}^-$ is transitive and $X \prec M$ (not necessarily transitive).

1. If $\mathbb{P} \in X$ is a forcing notion and $G \subseteq \mathbb{P}$ is M-generic, then G is X-generic if and only if $X[G] \cap M = X$.
2. If $\mathbb{P} * \dot{\mathbb{Q}} \in X$ is a two-step iteration of forcing notions and $G * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is M-generic, then $G * H$ is X-generic if and only if G is X-generic and H is $X[G]$-generic.
3. If $\mathbb{P} \in X$ is a forcing notion and an M-generic filter $G \subseteq \mathbb{P}$ is X-generic, then $X[G] \prec M[G]$.

Although these facts are standard, we will provide a proof for completeness.

[7] It is known that even the forcing relation for atomic formulas may not be definable for a definable forcing notion [HKL+16].

Proof. First, let's prove (3). By the Tarski-Vaught criterion, it suffices to show that if $M[G] \models \exists x\, \varphi(x, \tau_G)$ for some $\mathbb{P}$-name $\tau \in X$, then there is a $\mathbb{P}$-name $\sigma \in X$ such that $M[G] \models \varphi(\sigma_G, \tau_G)$. So fix a formula $\varphi(x, y)$ and a $\mathbb{P}$-name $\tau \in X$. Let

$$D = \{p \in \mathbb{P} \mid p \Vdash \varphi(\sigma, \tau) \text{ for some } \mathbb{P}\text{-name } \sigma \text{ or } p \Vdash \forall x\, \neg\varphi(x, \tau)\}.$$

Clearly, D is dense in $\mathbb{P}$ and $D \in X$ by elementarity. Thus, there is some $p \in G \cap D \cap X$. If $M[G] \models \exists x\, \varphi(x, \tau_G)$, then it must be the case that $p \Vdash \varphi(\sigma, \tau)$ for some $\mathbb{P}$-name σ. But then by elementarity, some such σ must be in X.

Now we can prove (1). Suppose that G is X-generic. First, we will argue that $X[G]$ and X have the same ordinals. Let τ be a $\mathbb{P}$-name in X such that $\tau_G = \beta$ is an ordinal. Let

$$D = \{p \in \mathbb{P} \mid p \Vdash \tau = \check{\alpha} \text{ for some ordinal } \alpha \text{ or } p \Vdash \tau \text{ is not an ordinal}\}.$$

Clearly, D is dense in $\mathbb{P}$ and $D \in X$ by elementarity. Thus, there is some $p \in G \cap D \cap X$. Since $\tau_G = \beta$ is an ordinal, it must be the case that $p \Vdash \tau = \check{\beta}$. But since $p, \tau \in X$ and β is definable from them, $\beta \in X$. Suppose now that $a \in X[G]$. Since $a \in M[G]$ and $M[G] \models \mathrm{ZFC}^-$, it satisfies that a is coded via the Mostowski collapse by a subset of an ordinal α. Thus, by (3), $X[G]$ satisfies this as well, and hence, we can assume that $\alpha \in X$. Thus, it suffices to show that $X[G]$ cannot have a new subset from M of an ordinal $\alpha \in X$. So suppose that τ is a $\mathbb{P}$-name for a subset of an ordinal α and $\tau_G = a \in M$. Let us say that a condition $p \in \mathbb{P}$ *decides* τ if for every $\xi < \alpha$, p decides the statement $\check{\xi} \in \tau$. Let

$$D = \{p \in \mathbb{P} \mid p \text{ decides } \tau \text{ or for all } q \leq p\,, q \text{ does not decide } \tau\}.$$

Clearly, D is dense in $\mathbb{P}$ and $D \in X$. So let $p \in G \cap D \cap X$. Thus, $\xi \in a$ if and only if $p \Vdash \check{\xi} \in \tau$. But then a is definable from p and τ, and hence $a \in X$.

Finally, (2) is an easy consequence of (1). □

We are now ready to state and prove diagonalization criterion (2).

Lemma 3.6 (Diagonalization Criterion (2))**.** *If $\mathbb{P}$ is a forcing notion in a model $M \models \mathrm{ZFC}^-$ and for cardinals $\gamma < \kappa$ the following criteria are satisfied:*

1. *$\mathbb{P}$ is $\leq\kappa$-closed in M,*
2. *there is a sequence $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_\xi \subseteq \cdots$ for $\xi < \gamma$, where each $X_\alpha \in M$ and $|X_\alpha|^M = \kappa$, whose union is M.*
3. *$M^{<\gamma} \subseteq M$,*

then there is an M-generic filter G for $\mathbb{P}$.
If additionally $\mathbb{P} \in X_0$, $X_\alpha \prec M$ and $X_\alpha^{<\kappa} \subseteq X_\alpha$ in M for all non-limit $\alpha < \gamma$, then we can assume that G is X_α-generic for these X_α.
In the case $\gamma = \omega$, to obtain an M-generic filter G, we can weaken (1) *to say that $\mathbb{P}$ is $\leq\kappa$-distributive.*

Proof. Since $\mathbb{P}$ is $\leq\kappa$-closed in M and X_0 has size κ in M, working in M, we construct a κ-length descending sequence of conditions meeting all dense sets of $\mathbb{P}$ that are elements of X_0, and choose a condition $p_0 \in \mathbb{P}$ below the sequence. Now suppose inductively that we are given a condition p_ξ, with $\xi < \gamma$, having the property that it has above it conditions meeting all dense sets of $\mathbb{P}$ that are elements of X_η for $\eta < \xi$. Since $M = \bigcup_{\alpha<\gamma} X_\alpha$, we may choose $\alpha_\xi > \xi$ such that $p_\xi \in X_{\alpha_\xi}$. Working in M, we construct below p_ξ a descending κ-length sequence of conditions meeting all dense sets of $\mathbb{P}$ that are elements of X_{α_ξ}, and choose a condition $p_{\xi+1}$ below the sequence. At limit stages $\lambda < \gamma$, we use $<\gamma$-closure of M together with $<\kappa$-closure of $\mathbb{P}$ in M to find p_λ below the sequence $\langle p_\xi \mid \xi < \lambda\rangle$. Since $M = \bigcup_{\alpha<\gamma} X_\alpha$, any filter G generated by the sequence $\langle p_\xi \mid \xi < \gamma\rangle$ is M-generic.

For the "additionally" part, note that if $\mathbb{P} \in X_{\alpha_\xi}$, $X_{\alpha_\xi} \prec M$ and $X_{\alpha_\xi}^{<\kappa} \subseteq X_{\alpha_\xi}$, then we can modify the construction to meet all dense sets of $\mathbb{P}$ that are elements of X_{α_ξ} inside X_{α_ξ}.

Finally, suppose that $\gamma = \omega$ and $\mathbb{P}$ is $\leq\kappa$-distributive. Observe that, for $n < \omega$, the intersection $\mathcal{D}_n$ of all dense open subsets of $\mathbb{P}$ in X_n is again dense open. Externally, we may then construct an ω-descending sequence $\langle p_n \mid n < \omega\rangle$ with $p_n \in \mathcal{D}_n$, and it is clear that this sequence generates an M-generic filter for $\mathbb{P}$. $\square$

Even though we stated diagonalization criterion (2) very generally, in future arguments we will use it only for the case $\gamma = \omega$, which does not require any closure on the model M and therefore applies to weak κ-models having the structural properties specified in item (2). The genericity requirement for the sets X_α will be used for lifting iterations of embeddings.

We will also make use of the following standard criterions providing conditions under which the closure of a model of set theory extends to its forcing extension. Although, these facts are generally known folklore, we were not able to find proofs of them in standard literature and therefore we provide them here for completeness.

Lemma 3.7 (Ground Closure Criterion)**.** *Suppose that $M \models \mathrm{ZFC}^-$ is transitive, $X \subseteq M$, either (1) X is transitive and $X \models \mathrm{ZFC}^-$ or (2) $X \prec M$, and for some ordinal γ, $X^\gamma \subseteq X$ in M. Suppose further that $\mathbb{P} \in X$ is a forcing notion and M has an X-generic filter $H \subseteq \mathbb{P}$. Then $X[H]^\gamma \subseteq X[H]$ in M.*

Proof. We work in M. Suppose that $\langle a_\xi \mid \xi < \gamma\rangle$ is a sequence with each $a_\xi \in X[H]$. For $\xi < \gamma$, fix a $\mathbb{P}$-name $\dot{a}_\xi$ in X such that $(\dot{a}_\xi)_H = a_\xi$, and observe that $\vec{a} = \langle \dot{a}_\xi \mid \xi < \gamma\rangle \in X$ using that $X^\gamma \subseteq X$. Since H is X-generic, it follows by Remark 3.5 (3) that $X[H]$ is a model of ZFC^-, and so it can recover $\langle a_\xi \mid \xi < \gamma\rangle$ from $\vec{a}$ and H. $\square$

Lemma 3.8 (Generic Closure Criterion)**.** *Suppose that $M \models \mathrm{ZFC}^-$ is transitive, $X \in M$, either (1) X is transitive and $X \models \mathrm{ZFC}^-$ or (2) $X \prec M$, and for some ordinal γ, $X^{<\gamma} \subseteq X$ in M. If $G \subseteq \mathbb{P}$ is M-generic for a forcing notion $\mathbb{P} \in X$ such that $\mathbb{P} \subseteq X$ and has the γ-cc in M, then $X[G]^{<\gamma} \subseteq X[G]$ in $M[G]$.*

If instead we assume for some ordinal γ that $X^\gamma \subseteq X$ in M and $\mathbb{P}$ has the γ^+-cc in M (keeping all other assumptions the same), then $X[G]^\gamma \subseteq X[G]$ in $M[G]$.

Proof. We work in M. Fix $\delta < \gamma$ and suppose that $\langle a_\xi \mid \xi < \delta\rangle$ is a sequence in $M[G]$ such that each $a_\xi \in X[G]$. Choose a $\mathbb{P}$-name $\dot{a}$ in M with $\dot{a}_G = \langle a_\xi \mid \xi < \delta\rangle$, and choose a condition $p \in G$ forcing that $\dot{a}$ is a sequence of length δ of elements of $\check{X}[\dot{G}]$. For $\xi < \delta$, define the dense sets $D_\xi = \{q \leq p \mid \exists \dot{b}_q \in X\ q \Vdash \dot{b}_q = \dot{a}(\check{\xi})\}$ and let A_ξ be some maximal antichain contained in D_ξ. Next, for $\xi < \delta$, define that $S_\xi = \{\langle q, \dot{b}_q\rangle \mid q \in A_\xi\}$. Since $\mathbb{P}$ has the γ-cc, each A_ξ has size less than γ and therefore so does each S_ξ. Also, each $S_\xi \subseteq X$, and therefore, by $X^{<\gamma} \subseteq X$, it is in X. The sequence $\langle S_\xi \mid \xi < \delta\rangle$ is then in X as well.

If $X \models \mathrm{ZFC}^-$ is transitive, then it can construct from $\langle S_\xi \mid \xi < \delta\rangle$ a sequence $\langle \dot{b}_\xi \mid \xi < \delta\rangle$ of mixed names $\dot{b}_\xi$ out of the $\dot{b}_q$'s for $q \in A_\xi$ so that $(\dot{b}_\xi)_G = a_\xi$. If $X \prec M$, then it has such a sequence by elementarity. Thus, X has a $\mathbb{P}$-name τ such that $\tau_G = \langle a_\xi \mid \xi < \delta\rangle$.

If instead we assume that $X^\gamma \subseteq X$ in M and $\mathbb{P}$ has the γ^+-cc, then the same argument verifies that $X[G]^\gamma \subseteq X[G]$ in $M[G]$. □

Next, we define a class of weak κ-models whose embeddings will have the properties required for lifting using diagonalization criterion (2).

Definition 3.9. A weak κ-model M is α-*almost special* if it is the union of a continuous elementary chain of (not necessarily transitive) substructures

$$\kappa \in m_0 \prec m_1 \prec \cdots \prec m_\xi \prec \cdots$$

for $\xi < \alpha$ such that each $m_\xi \in M$, $|m_\xi|^M = \kappa$, and for non-limit ξ, $m_\xi^{<\kappa} \subseteq m_\xi$ in M. A weak κ-model M is α-*special* if the m_ξ are required to be transitive.

Lemma 3.10. *If M is an α-special weak κ-model and $j : M \to N$ is the ultrapower map by a weakly amenable M-ultrafilter on κ, then N is α-almost special. Indeed, if a sequence $\langle m_\xi \mid \xi < \alpha\rangle$ witnesses that M is α-special, then the sequence*

$$\langle x_\xi \mid \xi < \alpha\rangle,$$

where

$$x_\xi = \{j(f)(\kappa) \mid f : \kappa \to m_\xi, f \in m_\xi\},$$

witnesses that N is α-almost special.

Proof. By Remark 2.5 (1), the ultrapower $N = \{j(f)(\kappa) \mid f : \kappa \to M, f \in M\}$ and so it is clear that $N = \bigcup_{\xi<\alpha} x_\xi$. Note that each $x_\xi \cong \mathrm{Ult}(m_\xi, U \cap m_\xi)$ via the map $\varphi : [f]_U \mapsto j(f)(\kappa)$. Thus,

$$\begin{aligned} x_\xi \models \varphi(j(f)(\kappa)) \quad &\leftrightarrow \quad \{\nu < \kappa \mid m_\xi \models \varphi(f(\nu))\} \in U \\ &\leftrightarrow \quad \{\nu < \kappa \mid M \models \varphi(f(\nu))\} \in U \text{ (since } m_\xi \prec M) \\ &\leftrightarrow \quad N \models \varphi(j(f)(\kappa)). \end{aligned}$$

It follows that each $x_\xi \prec N$ and hence for $\xi < \mu$, we have $x_\xi \prec x_\mu$. Suppose that ξ is not a limit ordinal. To verify that x_ξ is closed under $<\kappa$-sequences in N, we fix some $\langle a_\nu \mid \nu < \delta\rangle$ where $\delta < \kappa$ and each $a_\nu \in x_\xi$. Since m_ξ is transitive, it is coded by a binary relation on κ that Mostowski collapses to it, and hence $m_\xi \in N$. Moreover, $j \restriction m_\xi$ is in N as well, since given a bijection $g : \kappa \to m_\xi$ in M, we have that for $a \in m_\xi$, $j(a) = j(g)(\nu)$ where $g(\nu) = a$. Thus, there is a sequence $\vec{f} = \langle f_\nu \mid \nu < \delta\rangle \in N$ such that $f_\nu \in m_\xi$ and $a_\nu = j(f_\nu)(\kappa)$. The sequence $\vec{f}$ is an element of M by κ-powerset preservation, and hence $\vec{f} \in m_\xi$ by closure. Thus, there is $F \in m_\xi$ such that $F(\eta)(\nu) = f_\nu(\eta)$ for all $\nu < \delta$ and $\eta < \kappa$. It is now easy to see that $j(F)(\kappa) = \langle a_\nu \mid \nu < \delta\rangle$. Finally, $\kappa \in x_0$ since $\kappa = j(\mathrm{id})(\kappa)$, where $\mathrm{id} : \kappa \to \kappa$ such that $\mathrm{id}(\alpha) = \alpha$ is in m_0. □

Remark 3.11. If M is an ω-special weak κ-model, $j : M \to N$ is the ultrapower map by a weakly amenable M-ultrafilter on κ, and the x_i are defined as in Lemma 3.10, then $j \restriction m_i \subseteq x_i$ and so in particular, $m_i \cap V_\kappa \subseteq x_i$.

Proof. Observe that if $a \in m_i$, then $j(a) = j(f)(c_a)$, where $c_a : \kappa \to \{a\}$, is in x_i and if a is as well in V_κ, then $j(a) = a$. □

Now we argue that Ramsey and α-iterable cardinals are characterized, as well, by the existence of elementary embeddings for ω-special weak κ-models. For simplicity, we start with 1-iterable cardinals.

Lemma 3.12. *If κ is 1-iterable, then every $A \subseteq \kappa$ is contained in an ω-special weak κ-model M for which there exists a 1-good M-ultrafilter on κ.*

Proof. Fix a weak κ-model $\overline{M}$ containing $V_\kappa \cup \{A\}$ for which there exists a 1-good $\overline{M}$-ultrafilter $\overline{U}$ on κ, and let $\overline{j} : \overline{M} \to \overline{N}$ be the ultrapower map. First, note that $\overline{U}$ remains a 1-good ultrafilter for the substructure $H_{\kappa^+}^{\overline{M}} = \{B \in \overline{M} \mid |\mathrm{Trcl}(B)|^{\overline{M}} \leq \kappa\}$, which is itself a weak κ-model. So we will assume without loss that $\overline{M} = H_{\kappa^+}^{\overline{M}}$. Since $\overline{M}$ satisfies that H_{α^+} exists for all $\alpha < \kappa$ by virtue of containing V_κ, it follows by elementarity that H_{κ^+} exists in $\overline{N}$. Since j is κ-powerset preserving by Lemma 2.7, it must be that $\overline{M} = H_{\kappa^+}^{\overline{N}} \in \overline{N}$. Working inside $\overline{N}$, we build a transitive elementary substructure $m_0 \prec \overline{M}$ of size κ such that $A \in m_0$ and $m_0^{<\kappa} \subseteq m_0$. Note that $m_0 \in M$ since it is of hereditary size κ. Now suppose inductively that we are given $m_i \prec \overline{M}$, for some $i < \omega$, such that $m_i \in \overline{M}$. Note that $\overline{U} \cap m_i \in \overline{M}$ by weak amenability. Working inside $\overline{N}$, we build a transitive elementary substructure $m_{i+1} \prec \overline{M}$ of size κ, such that $m_i, \overline{U} \cap m_i \in m_{i+1}$ and $m_{i+1}^{<\kappa} \subseteq m_{i+1}$. By construction, the union model $M = \bigcup_{i\in\omega} m_i$ is an ω-special weak κ-model and $U = \overline{U} \cap M$ is a weakly amenable M-ultrafilter. The ultrapower of M by U is well-founded as it embeds into $\overline{N}$. Thus, we found an ω-special weak κ-model containing A for which there exists a 1-good M-ultrafilter on κ. $\square$

Lemma 3.13. *If κ is Ramsey, then every $A \subseteq \kappa$ is contained in an ω-special weak κ-model M for which there exists a weakly amenable κ-complete M-ultrafilter on κ.*

Proof. Fix a weak κ-model $\overline{M}$ containing $V_\kappa \cup \{A\}$ for which there exists a weakly amenable κ-complete $\overline{M}$-ultrafilter $\overline{U}$ on κ. We construct an ω-special $M \subseteq \overline{M}$ as in the proof of Lemma 3.12 and note that $U = \overline{U} \cap M$ clearly remains κ-complete. $\square$

Next, we argue that if $\alpha > 1$ and every A is contained in a weak κ-model M for which there exists an α-good M-ultrafilter on κ, then every A is in fact contained in an ω-special such weak κ-model. More specifically, we will show that if $\overline{M}$ is a weak κ-model with an α-good $\overline{M}$-ultrafilter $\overline{U}$ on κ and $M \subseteq \overline{M}$ is constructed as in the proof of Lemma 3.12, then $U = \overline{U} \cap M$ remains α-good. The argument relies chiefly on Lemma 3.8 from [GW11] and we include the proof here for completeness of presentation.

Lemma 3.14. *Suppose that M_0 is a weak κ_0-model with an α-good M_0-ultrafilter U_0 on κ_0 and $N_0 \prec M_0$ is another weak κ_0-model for which $W_0 = U_0 \cap N_0$ remains weakly amenable, then W_0 is α-good.*

Proof. The strategy will be to verify that the iterated ultrapowers of N_0 by W_0 are well-founded by embedding the iteration by W_0 into the iteration by U_0. Let

$$\{j_{\xi\gamma} : M_\xi \to M_\gamma \mid \xi < \gamma < \alpha\}$$

be the directed system of iterated ultrapowers of M_0 with the associated sequence of ultrafilters $\{U_\xi \mid \xi < \alpha\}$ on $\{\kappa_\xi \mid \xi < \alpha\}$ respectively. Also, let

$$\{h_{\xi\gamma} : N_\xi \to N_\gamma \mid \xi < \gamma < \alpha\}$$

be the (not necessarily well-founded) directed system of iterated ultrapowers of N_0 with the associated sequence of ultrafilters $\{W_\xi \mid \xi < \alpha\}$. Let

$$S_0 = \{w \in N_0 \mid w \subseteq W_0\},$$

be the collection of all subsets of W_0 that are elements of N_0, and define

$$S_\xi = \{h_{0\xi}(w) \mid w \in S_0\}.$$

We shall show that the following is a commutative diagram of elementary embeddings between transitive structures:

$$\begin{tikzcd}
M_0 \arrow[r, "j_{01}"] & M_1 \arrow[r, "j_{12}"] & M_2 \arrow[r, "j_{23}"] & \cdots \arrow[r, "j_{\xi\xi+1}"] & M_{\xi+1} \arrow[r, "j_{\xi+1\xi+2}"] & \cdots \\
N_0 \arrow[u, "\rho_0"] \arrow[r, "h_{01}"] & N_1 \arrow[u, "\rho_1"] \arrow[r, "h_{12}"] & N_2 \arrow[u, "\rho_2"] \arrow[r, "h_{23}"] & \cdots \arrow[r, "h_{\xi\xi+1}"] & N_{\xi+1} \arrow[u, "\rho_{\xi+1}"] \arrow[r, "h_{\xi+1\xi+2}"] & \cdots
\end{tikzcd}$$

where

(1) ρ_0 is the identity map,
(2) $\rho_{\xi+1}([f]_{W_\xi}) = [\rho_\xi(f)]_{U_\xi}$,
(3) if λ is a limit ordinal and t is a thread in the direct limit N_λ with domain $[\beta, \lambda)$, then ρ_λ is the map defined by $\rho_\lambda(t) = j_{\beta\lambda}(\rho_\beta(t(\beta)))$,
(4) for all $w \subseteq W_\xi$ with $w \in N_\xi$, there is $\bar{w} \in S_\xi$ such that $w \subseteq \bar{w}$.
(5) $\rho_\xi(w) \subseteq U_\xi$ for all $w \in S_\xi$.

The argument is by induction on ξ. For the base case, note that ρ_0 satisfies condition (5) and W_0 satisfies condition (4) trivially, and N_0 is transitive by assumption. Suppose inductively that the maps ρ_η are elementary for $\eta \leq \xi$, $\rho_\xi : N_\xi \to M_\xi$ satisfies condition (5), and (4) holds for S_ξ: for all $w \subseteq W_\xi$ with $w \in N_\xi$, there is $\bar{w} \in S_\xi$ such that $w \subseteq \bar{w}$. Fix $w^* \subseteq W_{\xi+1}$ in $N_{\xi+1}$. It is easy to see that $w^* = [f]_{W_\xi}$, where $f(\eta) \subseteq W_\xi$ for all $\eta < \bar{\kappa}_\xi$, the critical point of $h_{\xi\xi+1}$. It follows that there is $w \subseteq W_\xi$ in N_ξ such that $w^* \subseteq h_{\xi\xi+1}(w)$. By our inductive assumption, $w \subseteq h_{0\xi}(a)$ for some $a \in S_0$. Thus, $w^* \subseteq h_{\xi\xi+1}(w) \subseteq h_{0\xi+1}(a) \in S_{\xi+1}$.

Define $\rho_{\xi+1}$ as in (2) above. To see that $\rho_{\xi+1}$ is well-defined and elementary, note that if $A \in W_\xi$, then $A \in w$ for some $w \in S_\xi$ and so, in particular, $\rho_\xi(A) \in U_\xi$. The commutativity of the diagram is also clear. It remains to verify that $\rho_{\xi+1}(w) \subseteq U_{\xi+1}$ for all $w \in S_{\xi+1}$. Fix $w \in S_{\xi+1}$ and $\overline{w} \in S_\xi$ such that $w = h_{\xi\xi+1}(\overline{w}) = [c_{\overline{w}}]_{W_\xi}$. Let $\rho_\xi(\overline{w}) = v \subseteq U_\xi$ by the inductive assumption. Thus,

$$\rho_{\xi+1}(w) = [c_v]_{U_\xi} = j_{\xi\xi+1}(v) \subseteq U_{\xi+1}.$$

The last relation follows since $v \subseteq U_\xi$. This completes the inductive step. The limit case also follows easily. □

Lemma 3.15. *If κ is α-iterable, then every $A \subseteq \kappa$ is contained in an ω-special weak κ-model M for which there exists an α-good M-ultrafilter on κ.*

Proof. Fix $A \subseteq \kappa$ and a weak κ-model $\overline{\overline{M}}$ containing $V_\kappa \cup \{A\}$ for which there exists an α-good $\overline{\overline{M}}$-ultrafilter $\overline{\overline{U}}$ on κ. Let $\overline{M} = H^{\overline{\overline{M}}}_{\kappa^+}$ and $\overline{U} = \overline{M} \cap \overline{\overline{U}}$. It is clear that $\overline{U}$ is a weakly amenable $\overline{M}$-ultrafilter. Observe also that $\overline{U}$ is α-good since $\overline{M}$ has all the same functions $f : \kappa \to \overline{M}$ as $\overline{\overline{M}}$, and therefore the iterated ultrapowers of $\overline{M}$ by $\overline{U}$ are restrictions of the corresponding iterated ultrapowers of $\overline{\overline{M}}$ by $\overline{\overline{U}}$. We construct a special weak κ-model $M \subseteq \overline{M}$ as in the proof of Lemma 3.12 with the weakly amenable M-ultrafilter $U = M \cap \overline{U}$, and observe that U is α-good by Lemma 3.14. Thus, we found an ω-special weak κ-model containing A for which there exists an α-good M-ultrafilter on κ. □

For one of the lifting arguments, we will need that super Ramsey cardinals can be characterized by the existence of embeddings for κ-special κ-models.

Lemma 3.16. *If κ is super Ramsey, then every $A \subseteq \kappa$ is contained in a κ-special κ-model $M \prec H_{\kappa^+}$ for which there exists a weakly amenable M-ultrafilter on κ.*

Proof. Fix a κ-model $\overline{M} \prec H_{\kappa^+}$ containing A for which there exists a weakly amenable $\overline{M}$-ultrafilter U on κ and let $j : \overline{M} \to \overline{N}$ be the ultrapower map. Since $\overline{M}$ satisfies that every set has size at most κ, by weak amenability of U, $\overline{M} = H^{\overline{N}}_{\kappa^+}$. Working in $\overline{N}$, we can construct a κ-model $A \in M_0 \prec \overline{M}$. Note that $M_0 \in \overline{M}$ since $\overline{M} = H^{\overline{N}}_{\kappa^+}$. Assume inductively that we have constructed an elementary chain of κ-models $M_\xi \in \overline{M}$ for $\xi < \gamma$ such that whenever $\eta < \xi < \gamma$, then $M_\eta, U \cap M_\eta \in M_\xi$. Suppose first that γ is a limit ordinal. Since $\overline{M}^{<\kappa} \subseteq \overline{M}$, the sequence $\langle M_\xi \mid \xi < \gamma\rangle \in \overline{M}$. Let $M_\gamma = \bigcup_{\xi<\gamma} M_\xi$, and note that $M_\gamma \in \overline{M}$. Next, suppose that $\gamma = \delta+1$. In this case, working in $\overline{N}$, let $M_\gamma \prec \overline{M}$ be any κ-model containing $M_\delta, U \cap M_\delta$. Now, let $M = \bigcup_{\xi<\kappa} M_\xi$. Clearly, $M \prec \overline{M} \prec H_{\kappa^+}$ is a κ-special κ-model containing A. □

In [AGH12], techniques were introduced for lifting entire iterations of embeddings, and we adopt these techniques here to iterations characterizing α-iterable cardinals. Suppose that $j : M \to N$ is the ultrapower by an α-good M-ultrafilter U on κ, $\mathbb{P} \in M$ is a forcing notion and $G \subseteq \mathbb{P}$ is M-generic. Suppose also that we are able to lift j to $j : M[G] \to N[H]$ and the resulting lift is the ultrapower by an $M[G]$-ultrafilter W that is again weakly amenable. We shall show that if the N-generic filter H satisfies an additional technical condition (see Theorem 3.18), then the embeddings in the rest of the iteration lift automatically. The iteration composed of the lifts will turn out to be precisely the α-iteration by W, confirming that W is α-good. The argument relies on the following standard fact about lifts of ultrapower embeddings adapted here to models of ZFC$^-$.

Lemma 3.17. *Suppose that M is a weak κ-model, U is a 0-good M-ultrafilter on κ, $\mathbb{P} \in M$ is a forcing notion and $G \subseteq \mathbb{P}$ is M-generic. If $W \supseteq U$ is a 0-good $M[G]$-ultrafilter on κ, then the ultrapower by W lifts the ultrapower by U if and only if every function $f : \kappa \to M$ in $M[G]$ is W-equivalent to some $g : \kappa \to M$ in M.*

Proof. Suppose that $j : M \to N$ is the ultrapower map by U and $h : M[G] \to K$ is the ultrapower map by W. For the forward direction, suppose that $h \restriction M = j$ and $\tau_G = f : \kappa \to M$ is a function in $M[G]$. Note that the range of f is contained in

$$A = \{a \in M \mid \exists p{\in}\mathbb{P}\, \exists \xi{\in}\kappa\, p \Vdash \tau(\check{\xi}) = \check{a}\}$$

which is a set in M by replacement. Thus, $h(f)(\kappa) \in h(A) = j(A) \subseteq N$ and so $h(f)(\kappa) = j(g)(\kappa) = h(g)(\kappa)$ for some $g \in M$, from which it follows that f is W-equivalent to g. For the backward direction, suppose that every $f : \kappa \to M$ in $M[G]$ is W-equivalent to some $g : \kappa \to M$ in M. Observe that the map φ sending $[f]_U \in N$ to $[f]_W \in K$ is well-defined and an isomorphism of N with a substructure of K. It follows from our assumption that this substructure must be transitive and hence is equal to N, making φ the identity map. Thus, $N \subseteq K$ and $j(a) = [c_a]_U = [c_a]_W = h(a)$ for $a \in M$. We will now argue that $h(G)$ is an N-generic filter for $j(\mathbb{P})$ and $K = N[h(G)]$. Fix a dense set $D \subseteq j(\mathbb{P})$ in N. Then $D = j(f)(\kappa)$ for some $f : \kappa \to M$ in M and we can assume that all $f(\xi) = D_\xi$ are dense in $\mathbb{P}$. Since $M[G]$ satisfies that G meets all dense sets in the range of f, by elementarity, K satisfies that $h(G)$ meets all dense sets in the range of $h(f) = j(f)$, and so in particular, it meets $h(f)(\kappa) = j(f)(\kappa) = D$. Thus, $h(G)$ is N-generic. Clearly $N[h(G)] \subseteq K$ and so it remains to check that $K \subseteq N[h(G)]$. Fix $a \in K$ and let $a = h(f)(\kappa)$ for some $f : \kappa \to M[G]$ in $M[G]$. Let $t \in M[G]$ be some transitive set containing f and κ and let $t = \tau_G$ for some $\mathbb{P}$-name $\tau \in M$. Let $T = \{\sigma \mid \exists p \in \mathbb{P}\, \langle \sigma, p\rangle \in \tau\}$, so that $T[G] \supseteq t$. By elementarity, in K, $h(t)$ is a transitive set containing $h(f)$ and κ, and therefore $a = h(f)(\kappa)$. Also, by elementarity, in K, $h(t) \subseteq h(T)[h(G)] = j(T)[h(G)] \subseteq N[h(G)]$. □

Theorem 3.18. *Suppose that M_0 is an ω-special weak κ_0-model as witnessed by the sequence $\langle m_i \mid i < \omega\rangle$, $\mathbb{P} \in m_0$ is a forcing notion and $G_0 \subseteq \mathbb{P}$ is M_0-generic. Suppose further that the ultrapower $j_{01} : M_0 \to M_1$ by an α-good M_0-ultrafilter U_0 on κ_0 lifts to the ultrapower $j_{01} : M_0[G_0] \to M_1[G_1]$ by a weakly amenable $M_0[G_0]$-ultrafilter W_0 and that the M_1-generic filter G_1 is additionally x_i-generic for all $i \in \omega$ where $x_i = \{j_{01}(f)(\kappa_0) \mid f : \kappa_0 \to m_i, f \in m_i\}$. Then W_0 is α-good.*

Proof. By Lemma 3.17, since $j_{01} : M_0[G_0] \to M_1[G_1]$ is a lift of $j_{01} : M_0 \to M_1$, it follows that every $f : \kappa_0 \to M_0$ in $M_0[G_0]$ is W_0-equivalent to some $g : \kappa_0 \to M_0$ in M_0. The strategy will be to use elementarity to propagate the property that every new function is equivalent to an old function along the entire W_0-iteration, thereby showing that every embedding in the W_0-iteration is a lift of the corresponding embedding in the U_0-iteration, and thus well-founded. It will follow that by lifting just the first step, we have already lifted the entire iteration. The difficulty we must surmount is that elementarity is guaranteed only for the language $\mathcal{L} = \{\in\}$, and it is not immediately clear how to express the "no new functions" property as a first-order statement without the predicate for the ultrafilter. This is where we will use the extra genericity of G_1, which will allow us to capture the property that every new function is equivalent to an old function as a schema of first-order statements in the language $\mathcal{L} = \{\in\}$.

Let $w_i = W_0 \cap m_i[G_0]$, and note that it is an element of $M_0[G_0]$ by the weak amenability of W_0. The critical consequence of the hypothesis that the M_1-generic filter G_1 is x_i-generic for $x_i = \{j_{01}(f)(\kappa_0) \mid f : \kappa_0 \to m_i, f \in m_i\}$ is that $M_0[G_0]$ satisfies the following schema of first-order statements $(*i)$ for $i \in \omega$.

> $(*i)$ Every $f : \kappa_0 \to m_i$ in $m_i[G_0]$ is w_i-equivalent to some $g : \kappa_0 \to m_i$ in m_i.

Let us prove this. First, recall that $\kappa_0 \in x_i$ and $j_{01} \restriction m_i \subseteq x_i$ (Remark 3.11). Since G_1 is x_i-generic, it follows by Remark 3.5 (1) that $x_i[G_1] \cap M_1 = x_i$. Now suppose that $f : \kappa_0 \to m_i$ is any function in $m_i[G_0]$ and $\dot{f} \in m_i$ is a $\mathbb{P}$-name such that $(\dot{f})_{G_0} = f$. Since $\dot{f} \in m_i$, it follows that $j_{01}(\dot{f}) \in x_i$ and hence $j_{01}(f)(\kappa_0) \in x_i[G_1]$. Since $M_0[G_0]$ satisfies that $\operatorname{ran}(f) \subseteq m_i$, we have that $\operatorname{ran}(j_{01}(f)) \subseteq j_{01}(m_i) \subseteq M_1$. Thus $j_{01}(f)(\kappa_0) \in x_i[G_1] \cap M_1 = x_i$ and hence $j_{01}(f)(\kappa_0) = j_{01}(g)(\kappa_0)$ for some $g : \kappa_0 \to m_i$ in m_i completing the proof that $(*i)$ holds in $M_0[G_0]$. It is this schema of statements $(*i)$ that we will propagate along the iteration using elementarity.

Let $\{j_{\xi\gamma} : M_\xi \to M_\gamma \mid \xi < \gamma < \alpha\}$ be the directed system of iterated ultrapowers of M_0 with the associated sequence of ultrafilters $\{U_\xi \mid \xi < \alpha\}$ and the critical sequence $\{\kappa_\xi \mid \xi < \alpha\}$. Clearly, each $M_\xi = \bigcup_{i\in\omega} j_{0\xi}(m_i)$.

By assumption, the first step of the W_0-iteration $j_{01} : M_0[G_0] \to M_1[G_1]$ is the lift of the first step of the U_0-iteration $j_{01} : M_0 \to M_1$. Now assume inductively that every step of the W_0-iteration up to ξ is a lift of the corresponding step of the U_0-iteration:

$$M_0[G_0] \xrightarrow{j_{01}} M_1[G_1] \xrightarrow{j_{12}} \quad \cdots \quad \longrightarrow M_\xi[G_\xi]$$

Let W_ξ be the $M_\xi[G_\xi]$-ultrafilter associated to stage ξ of the iteration above. A standard argument shows that $W_\xi = \bigcup_{i\in\omega} j_{0\xi}(w_i)$. By applying elementarity to the schema of statements $(*i)$, we shall argue that $M_\xi[G_\xi]$ and W_ξ satisfies the characterization of Lemma 3.17 and therefore the next embedding in the W_0-iteration has the form $j_{\xi\xi+1} : M_\xi[G_\xi] \to M_{\xi+1}[G_{\xi+1}]$.

Applying $j_{0\xi}$ to each statement $(*i)$, we obtain the corresponding statement:

$(*i)_\xi$ Every $f : \kappa_\xi \to j_{0\xi}(m_i)$ in $j_{0\xi}(m_i)[G_\xi]$ is $j_{0\xi}(w_i)$-equivalent to some $g : \kappa_\xi \to j_{0\xi}(m_i)$ in $j_{0\xi}(m_i)$.

Since $M_\xi = \bigcup_{i\in\omega} j_{0\xi}(m_i)$, we have that every $f : \kappa_\xi \to M_\xi$ is in one such $j_{0\xi}(m_i)$ and since $W_\xi = \bigcup_{i\in\omega} j_{0\xi}(w_i)$, we have that $j_{0\xi}(w_i) \subseteq W_\xi$. It follows that $M_\xi[G_\xi]$ and W_ξ satisfy the characterization of Lemma 3.17, and so the next embedding in the W_0-iteration has the form $j_{\xi\xi+1} : M_\xi[G_\xi] \to M_{\xi+1}[G_{\xi+1}]$. The limit stages follow easily. □

Using the original definition of Ramsey cardinals, it is not difficult to see that Ramsey cardinals are preserved by small forcing. The next lemma will provide a proof of this using embeddings. First, we need to make the following easy observation. We shall say that a sequence $\vec{y} = \langle y_\eta \mid \eta < \alpha\rangle$ *covers* another sequence $\vec{x} = \langle x_\xi \mid \xi < \beta\rangle$ if for every $\xi < \beta$, there is $\eta < \alpha$ such that $x_\xi = y_\eta$.

Remark 3.19. Suppose that $\mathbb{P}$ is a forcing notion with the κ-cc for a regular cardinal κ and $G \subseteq \mathbb{P}$ is V-generic. If $\vec{x}$ is a sequence of elements of V of length less than κ in $V[G]$, then it is covered by a sequence $\vec{y}$ of length less than κ in V.

Lemma 3.20. *Suppose that κ is inaccessible, M is a weak κ-model, and $j : M \to N$ is the ultrapower map by a weakly amenable κ-complete M-ultrafilter on κ. If $\mathbb{P} \in V_\kappa \cap M$ is a poset and $G \subseteq \mathbb{P}$ is V-generic, then the lift $j : M[G] \to N[G]$ is the ultrapower map by a weakly amenable countably complete (indeed, κ-complete) $M[G]$-ultrafilter in $V[G]$.*

Proof. Suppose that j is the ultrapower map by an M-ultrafilter U. First, we argue that the lift $j : M[G] \to N[G]$ is κ-powerset preserving, and so by Lemma 2.7, it will follow that it is the ultrapower map by a weakly amenable $M[G]$-ultrafilter, call it W. If $B \subseteq \kappa$ in $N[G]$, then N has a nice $\mathbb{P}$-name $\dot{B} \in H^N_{\kappa^+}$ such that $\dot{B}_G = B$. It follows that $\dot{B} \in M$, and hence $B \in M[G]$. Now we argue that W is countably complete. For this, it suffices to show that whenever $\langle A_n \mid n \in \omega\rangle$ is a sequence of subsets of κ in $V[G]$ such that each $A_n \in M[G]$ and $\kappa \in j(A_n)$, then $\bigcap_{n\in\omega} A_n \neq \emptyset$. We start by fixing such a sequence $\langle A_n \mid n \in \omega\rangle$. For each n, we fix a $\mathbb{P}$-name $\dot{A}_n \in M$ with $(\dot{A}_n)_G = A_n$ and fix a condition $p_n \in G$ such that $p_n \Vdash \check{\kappa} \in j(\dot{A}_n)$ over N. For each $n \in \omega$, we let

$$S_n = \{\alpha < \kappa \mid p_n \Vdash \check{\alpha} \in \dot{A}_n \text{ over } M\}.$$

Individually, each $S_n \in M$, and, indeed, $S_n \in U$ since $\kappa \in j(S_n)$. The sequence

$$\langle S_n \mid n < \omega\rangle$$

is an element of $V[G]$, but not necessarily an element of V. Nevertheless, by Remark 3.19, there is a sequence $\langle T_\xi \mid \xi < \beta\rangle$ in V for some $\beta < \kappa$ covering the sequence $\langle S_n \mid n < \omega\rangle$ of $V[G]$. By thinning out if necessary, we may assume that all T_ξ are elements of U. Since U is κ-complete, we have some $\gamma \in \bigcap_{\xi<\beta} T_\xi$. Hence $\gamma \in \bigcap_{n\in\omega} S_n$. It follows that each $p_n \Vdash \check{\gamma} \in \dot{A}_n$ over M and so $\gamma \in (\dot{A}_n)_G = A_n$. Thus, $\bigcap_{n\in\omega} A_n \neq \emptyset$, concluding the proof that W is countably complete. Indeed, an analogous argument shows that W is κ-complete. $\square$

The next lemma states that for countably closed posets, the lift of the ultrapower map by a countably complete ultrafilter to a forcing extension is again the ultrapower map by a countably complete ultrafilter.

Lemma 3.21. *Suppose that κ is inaccessible, M is a weak κ-model and $j : M \to N$ is the ultrapower map by a countably complete M-ultrafilter on κ. Suppose further that $\mathbb{P} \in M$ is a countably closed forcing notion and $G \subseteq \mathbb{P}$ is M-generic. If the ultrapower map j lifts to an elementary embedding $j : M[G] \to N[j(G)]$, then the lift j is the ultrapower map by a countably complete $M[G]$-ultrafilter in $V[G]$.*

Proof. Let $j : M \to N$ be the ultrapower map by the M-ultrafilter U. We need to verify that whenever $\langle A_n \mid n \in \omega\rangle$ is a sequence of subsets of κ in $V[G]$ such that each $A_n \in M[G]$ and $\kappa \in j(A_n)$, then $\bigcap_{n\in\omega} A_n \neq \emptyset$. We fix such a sequence $\langle A_n \mid n \in \omega\rangle$. We also fix $\mathbb{P}$-names $\dot{A}_n \in M$ with $(\dot{A}_n)_G = A_n$, and note that the sequence of names $\langle \dot{A}_n \mid n \in \omega\rangle \in V$ by countable closure of $\mathbb{P}$. Next, we choose a $\mathbb{P}$-name $\dot{S}$ such that $\mathbb{1} \Vdash$ "$\dot{S}$ is an ω-sequence" and for all n, $\mathbb{1} \Vdash \dot{S}(\check{n}) = \dot{A}_n$ over V (the name is constructed from the sequence $\langle \dot{A}_n \mid n \in \omega\rangle$). Now we suppose towards a contradiction that $\bigcap_{n\in\omega} A_n = \emptyset$ and choose a condition $p \in G$ such that $p \Vdash \bigcap \dot{S} = \emptyset$ over V.

Since $p \in G$, we have that $j(p) \in j(G)$. In the filter $j(G)$, we choose an ω-descending sequence of conditions below $j(p)$,

$$j(p) \geq p_0 \geq p_1 \geq \cdots \geq p_n \geq \cdots,$$

such that $p_n \Vdash \check{\kappa} \in j(\dot{A}_n)$ over N. Since $j : M \to N$ is the ultrapower by U, we may fix for each $n \in \omega$, a function $f_n : \kappa \to \mathbb{P}$ such that $p_n = [f_n]_U$. By countable closure of $\mathbb{P}$, the sequence of functions $\langle f_n \mid n \in \omega\rangle \in V$. Now observe that the following sets are in U:

(1) $S_n = \{\xi < \kappa \mid f_n(\xi) \Vdash \check{\xi} \in \dot{A}_n \text{ over } M\}$ for $n \in \omega$,
(2) $T_n = \{\xi < \kappa \mid f_{n+1}(\xi) \leq f_n(\xi)\}$ for $n \in \omega$,
(3) $S = \{\xi < \kappa \mid f_0(\xi) \leq p\}$.

Note that the sequences $\langle S_n \mid n < \omega\rangle$ and $\langle T_n \mid n < \omega\rangle$ are themselves elements of V by countable closure of $\mathbb{P}$. Thus, since U is countably complete in V, we may intersect all these sets to obtain an ordinal $\alpha < \kappa$ such that:

(1) for all $n < \omega$, $f_n(\alpha) \Vdash \check{\alpha} \in \dot{A}_n$ over M,
(2) for all $n < \omega$, $f_{n+1}(\alpha) \leq f_n(\alpha)$,
(3) $f_0(\alpha) \leq p$.

Once again by closure of $\mathbb{P}$, we may fix a condition q below the descending ω-sequence

$$p \geq f_0(\alpha) \geq f_1(\alpha) \geq \cdots \geq f_n(\alpha) \geq \cdots.$$

Clearly, for all $n \in \omega$, $q \Vdash \check{\alpha} \in \dot{A}_n$ over M.

Suppose that $\overline{G} \subseteq \mathbb{P}$ is any V-generic filter containing q. Since for all $n \in \omega$, $q \Vdash \check{\alpha} \in \dot{A}_n$ over M, we have that $\alpha \in (\dot{A}_n)_{\overline{G}}$ for all $n \in \omega$ in $V[\overline{G}]$. Thus, since $\mathbb{1} \Vdash \dot{S}(\check{n}) = \dot{A}_n$ over V, we have that $\bigcap(\dot{S})_{\overline{G}} = \bigcap_{n\in\omega}(\dot{A}_n)_{\overline{G}} \neq \emptyset$ in $V[\overline{G}]$. But now, since $q \leq p$, $q \Vdash \bigcap \dot{S} = \emptyset$ over V. Thus, we have reached a contradiction showing that $j : M[G] \to N[j(G)]$ is the ultrapower by a countably complete $M[G]$-ultrafilter.

□

We end this section with a standard lemma to be used in later arguments.

Lemma 3.22. *Suppose that $\mathbb{P}$ is an iteration of inaccessible length κ such that for all $\alpha < \kappa$, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \in \check{V}_\kappa$ and a direct limit is taken on a stationary set of stages below κ, then $\mathbb{P}$ has size κ and the κ-cc.*

A more detailed exposition of standard lifting techniques, including Lemma 3.22, can be found in [Cum10].

4. Indestructible Ramsey and Ramsey-like cardinals

4.1. **Small forcing.** We start by showing that Ramsey, α-iterable, strongly Ramsey, and super Ramsey cardinals cannot be destroyed by small forcing. Suppose that κ is one of these cardinals and $\mathbb{P}$ is small relative to κ. By replacing $\mathbb{P}$ with an isomorphic copy, we can assume that $\mathbb{P} \in V_\kappa$. Suppose that $G \subseteq \mathbb{P}$ is V-generic. Note that if $A \subseteq \kappa$ in $V[G]$, then it has a $\mathbb{P}$-name in H_{κ^+}. Thus, to verify that κ retains its large cardinal property in $V[G]$, we will show that every embedding $j : M \to N$ of the type characterizing κ can be lifted to $j : M[G] \to N[j(G)]$ and the lift retains the relevant properties. The same strategy will be employed in other indestructibility argument provided that every subset of κ in the forcing extension has a name in H_{κ^+}.

Theorem 4.1. *Ramsey, α-iterable, strongly Ramsey, and super Ramsey cardinals are preserved by small forcing.*

Proof. First, suppose that κ is strongly Ramsey. Suppose that $\mathbb{P} \in V_\kappa$ and $G \subseteq \mathbb{P}$ is V-generic. Let $j : M \to N$ be the ultrapower map by a weakly amenable M-ultrafilter. Then j is κ-powerset preserving embedding and N is a κ-model. Since $\mathbb{P} \in V_\kappa$, we have that $j(\mathbb{P}) = \mathbb{P}$ and $j\,"\,G = G$ and so j lifts to $j : M[G] \to N[G]$ in $V[G]$ by the lifting criterion (Lemma 3.1). The model $M[G]$ is a κ-model in $V[G]$ by the generic closure criterion (Lemma 3.8) since $\mathbb{P}$ has the κ-cc. By Lemma 3.20, the lift is κ-powerset preserving. Note that if $M \prec H_{\kappa^+}$, then $M[G] \prec H_{\kappa^+}[G] = H_{\kappa^+}^{V[G]}$, which gives the argument for super Ramsey cardinals. In the argument for Ramsey cardinals, we start with $j : M \to N$ that is the ultrapower by a weakly amenable κ-complete M-ultrafilter and use Lemma 3.20 to conclude that the lift is the ultrapower by a countably complete $M[G]$-ultrafilter in $V[G]$.

In the argument for α-iterable cardinals, we start with $j : M \to N$ that is the ultrapower of an ω-special weak-κ model M by a weakly amenable α-good M-ultrafilter U. Let $\langle m_i \mid i < \omega\rangle$ witness that M is ω-special and assume without loss of generality that $\mathbb{P} \in m_0$. Let $\langle x_i \mid i < \omega\rangle$, defined as in Lemma 3.10, witness that N is ω-almost special. To argue that the lift is the ultrapower by an α-good $M[G]$-ultrafilter it suffices, by Theorem 3.18, to show that the V-generic filter G is x_i-generic for all $i \in \omega$. Recall that $V_\kappa \cap m_i \subseteq x_i$ and so we have $\mathbb{P} \in x_i$ and $\mathbb{P} \subseteq x_i$, from which it follows that any V-generic filter is x_i-generic. □

4.2. **Canonical forcing of the** GCH. Next, we show that Ramsey, α-iterable, strongly Ramsey, and super Ramsey cardinals are indestructible by the canonical forcing of the GCH. Recall that if κ and θ are cardinals, we call $\mathrm{Add}(\kappa,\theta)$ the poset, generalizing the Cohen poset, which adds θ many subsets to κ with conditions of size less than κ. If κ is regular, then $\mathrm{Add}(\kappa,\theta)$ is $<\kappa$-closed and if $2^{<\kappa}=\kappa$, then it has the κ^+-cc. A forcing iteration $\mathbb{P}$ is said to have *Easton support* if direct limits are taken at inaccessible cardinals and inverse limits are taken everywhere else. The canonical forcing of the GCH is an ORD-length Easton support iteration $\mathbb{P}$ where at stage α if α is an infinite cardinal in $V^{\mathbb{P}_\alpha}$, we force with $\mathrm{Add}(\alpha^+,1)$, and with the trivial poset otherwise. To show that a Ramsey or one of our other cardinals κ is indestructible by the canonical forcing of the GCH, it suffices to argue that it is indestructible by the set forcing $\mathbb{P}_\kappa$, the iteration $\mathbb{P}$ up to κ, since the tail of the forcing is $\leq\kappa$-closed and therefore cannot destroy these large cardinals. Since κ is a limit of inaccessible cardinals, by Lemma 3.22, the iteration $\mathbb{P}_\kappa$ has size κ and the κ-cc. Note also that $\mathbb{P}_\kappa\subseteq V_\kappa$ and therefore every subset of κ in an extension by $\mathbb{P}_\kappa$ has a name in H_{κ^+}.

Theorem 4.2. *Ramsey, α-iterable, strongly Ramsey, and super Ramsey cardinals are indestructible by the canonical forcing of the* GCH.

Proof. First, suppose that κ is strongly Ramsey. Let $G\subseteq\mathbb{P}_\kappa$ be V-generic. Let $j:M\to N$ be a κ-powerset preserving embedding of κ-models. The poset $\mathbb{P}_\kappa$ is an element of M automatically since it is a definable subset of V_κ which is an element of every κ-model.

In order to lift j to $M[G]$, by the lifting criterion, we require an N-generic filter for the poset $j(\mathbb{P}_\kappa)=\mathbb{P}^N_{j(\kappa)}\cong\mathbb{P}_\kappa*\dot{\mathbb{P}}_{\text{tail}}$, the canonical GCH iteration up to $j(\kappa)$ of N, containing $j\,''\,G=G$. The lifting criterion is satisfied by using the V-generic filter G for the $\mathbb{P}_\kappa$ portion of $\mathbb{P}^N_{j(\kappa)}$. Let $(\dot{\mathbb{P}}_{\text{tail}})_G=\mathbb{P}_{\text{tail}}$, and note that it is $\leq\kappa$-closed in $N[G]$ since the first poset in the iteration is $\mathrm{Add}(\kappa^+,1)^{N[G]}$. Since $\mathbb{P}$ has the κ-cc, by the generic closure criterion, $N[G]^{<\kappa}\subseteq N[G]$ in $V[G]$. Thus, by diagonalization criterion (1), $V[G]$ has an $N[G]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$, and so we can lift j to $j:M[G]\to N[G][G_{\text{tail}}]$. By the generic closure criterion, $M[G]$ is a κ-model in $V[G]$. The model $N[G]$ satisfies that $\mathbb{P}_{\text{tail}}$ is $\leq\kappa$-closed and therefore $N[G]$ has the same subsets of κ as $N[G][G_{\text{tail}}]$. The models $M[G]$ and $N[G]$ have the same subsets of κ by the argument from the proof of Lemma 3.20. Note that if $M\prec H_{\kappa^+}$, then $M[G]\prec H_{\kappa^+}[G]=H^{V[G]}_{\kappa^+}$, which gives the argument for super Ramsey cardinals.

In the argument for Ramsey cardinals, we start with an ω-special weak κ-model M such that $V_\kappa\in M$, and $j:M\to N$ that is the ultrapower by a weakly amenable countably complete M-ultrafilter on κ. Let $\langle m_i\mid i<\omega\rangle$ witness that M is ω-special and let $\langle x_i\mid i<\omega\rangle$, defined as in Lemma 3.10, witness that N is ω-almost special. Then $N[G]$ together with the sequence $\langle x_i[G]\mid i<\omega\rangle$ and the poset $\mathbb{P}_{\text{tail}}$ satisfies the requirements of diagonalization criterion 2 (Lemma 3.6) in $V[G]$. Thus, $V[G]$ has an $N[G]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$, and so we can lift j to $j:M[G]\to N[G][G_{\text{tail}}]$. The iteration $\mathbb{P}_\kappa$ is countably closed since the first poset in it is $\mathrm{Add}(\omega_1,1)$, and so, by Lemma 3.21, the lift of j is the ultrapower by a countably complete $M[G]$-ultrafilter in $V[G]$.

In the argument for α-iterable cardinals, we start with an ω-special weak κ-model M such that $V_\kappa\in M$, and $j:M\to N$ that is the ultrapower by a weakly amenable α-good M-ultrafilter on κ. Let $\langle m_i \mid i<\omega\rangle$ witness that M is ω-special and let $\langle x_i \mid i<\omega\rangle$, defined as in Lemma 3.10, witness that N is almost ω-special. Since $V_\kappa \subseteq m_i$, it follows that $V_\kappa \subseteq x_i$, and so in particular, $\mathbb{P}_\kappa \subseteq x_i$. Also, since $\kappa \in x_i$ and $\mathbb{P}_\kappa$ is definable from κ, it is in x_i. Thus, any V-generic filter, and G in particular, is x_i-generic for all i. Using that $\mathbb{P}_\kappa$ has the κ-cc, by the generic closure criterion, we have that all $x_i[G]^{<\kappa} \subseteq x_i[G]$ in $N[G]$. Finally, each $x_i[G] \prec N[G]$ by Remark 3.5(3). Thus, we can use diagonalization criterion (2) to obtain an $N[G]$-generic filter $G_{\rm tail}$ for $\mathbb{P}_{\rm tail}$ that is $x_i[G]$-generic for all i. By Remark 3.5(2), the N-generic filter $G * G_{\rm tail}$ is then x_i-generic for all i. Thus, by Theorem 3.18, $j: M[G] \to N[G][G_{\rm tail}]$ is the ultrapower by an α-good $M[G]$-ultrafilter. □

4.3. **The forcing** $\mathrm{Add}(\kappa,\theta)$**.** To produce a forcing extension in which an α-iterable, Ramsey, or strongly Ramsey κ is indestructible by $\mathrm{Add}(\kappa,\theta)$ for every cardinal θ, we use the standard preparatory forcing to produce a forcing extension in which the large cardinal property of κ becomes indestructible by $\mathrm{Add}(\kappa,1)$, and argue that it is, in fact, already indestructible by $\mathrm{Add}(\kappa,\theta)$ for every θ. A different argument for super Ramsey cardinals will be given later in the section. Let $\mathbb{P}_\kappa$ be the Easton support forcing iteration of length κ where at stage α if α is an uncountable cardinal in $V^{\mathbb{P}_\alpha}$, we force with $\mathrm{Add}(\alpha,1)$, and with the trivial poset otherwise. We start $\mathbb{P}_\kappa$ with $\mathrm{Add}(\omega_1,1)$ rather than $\mathrm{Add}(\omega,1)$ to ensure that it is countably closed. Note that, by Lemma 3.22, $\mathbb{P}_\kappa$ has size κ and the κ-cc, and also that $\mathbb{P}_\kappa \subseteq V_\kappa$. The preparatory forcing will be $\mathbb{P}_\kappa * \mathrm{Add}(\kappa,1)$. Let us argue that every $\mathbb{P}_\kappa$-name for an element of $\mathrm{Add}(\kappa,1)$ is equivalent to one of at most κ-many $\mathbb{P}_\kappa$-names and therefore we may assume that $\mathbb{P}_\kappa * \mathrm{Add}(\kappa,1)$ is a subset of V_κ as well. Every element of $\mathrm{Add}(\kappa,1)$ is a function $f{:}\kappa \to 2$ with bounded support, and since $\mathbb{P}_\kappa$ has the κ-cc, there is $\gamma<\kappa$ such that $\mathbb{P}_\kappa$ forces the support to be a subset of γ. It follows that we can associate every $\mathbb{P}_\kappa$-name for an element of $\mathrm{Add}(\kappa,1)$ with a nice $\mathbb{P}_\kappa$-name for a subset of γ for some $\gamma<\kappa$, and there are only κ many of these. Thus, every subset of κ in an extension by $\mathbb{P}_\kappa * \mathrm{Add}(\kappa,1)$ has a name in H_{κ^+}.

Theorem 4.3. *If κ is Ramsey, α-iterable, strongly Ramsey, or super Ramsey, then there is a forcing extension in which this becomes indestructible by the forcing* $\mathrm{Add}(\kappa,1)$.

Proof. We will first argue for each of our cardinals that they retain their large cardinal property in the forcing extension by $\mathbb{P}_\kappa * \mathrm{Add}(\kappa,1)$.

Suppose that κ is strongly Ramsey. Let $G * g \subseteq \mathbb{P}_\kappa * \mathrm{Add}(\kappa,1)$ be V-generic. Let $j: M \to N$ be a κ-powerset preserving embedding of κ-models. The poset $\mathbb{P}_\kappa * \mathrm{Add}(\kappa,1)$ is an element of M as it is a definable subset of $V_\kappa \in M$. We will lift j in two steps in $V[G][g]$, first to $M[G]$, and then to $M[G][g]$.

To lift j to $M[G]$, we require an N-generic filter for the poset

$$j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \mathrm{Add}(\kappa,1) * \dot{\mathbb{P}}_{\rm tail}$$

containing $j\,``\,G = G$. We satisfy the requirement of diagonalization criterion (1) by using the V-generic filter $G * g$ for the $\mathbb{P}_\kappa * \mathrm{Add}(\kappa,1)$ portion of $j(\mathbb{P}_\kappa)$. Let $(\dot{\mathbb{P}}_{\rm tail})_{G*g} = \mathbb{P}_{\rm tail}$, and note that it is $\leq\kappa$-closed in $N[G][g]$. Since $\mathbb{P}_\kappa$ has the κ-cc, by the generic closure criterion, $N[G]^{<\kappa} \subseteq N[G]$ in $V[G]$. The poset $\mathrm{Add}(\kappa,1)$ is $<\kappa$-closed in $V[G]$ and so $N[G]^{<\kappa} \subseteq N[G]$ in $V[G][g]$. Finally, by the ground closure criterion (Lemma 3.7), we have that $N[G][g]^{<\kappa} \subseteq N[G][g]$ in $V[G][g]$. Thus, by diagonalization criterion (1), $V[G][g]$ has an $N[G][g]$-generic filter $G_{\rm tail}$ and so we can lift j to $j : M[G] \to N[j(G)]$, with $j(G) = G * g * G_{\rm tail}$.

Next, we lift j to $M[G][g]$ in $V[G][g]$. For this portion of the lift, we require an $N[j(G)]$-generic filter for the poset $j(\mathrm{Add}(\kappa, 1)) = \mathrm{Add}(j(\kappa), 1)^{N[j(G)]}$ containing $j\,"\,g = g$. By construction, we have ensured that $g \in N[j(G)]$, from which it follows that $B = \bigcup g$ is a condition in $\mathrm{Add}(j(\kappa), 1)^{N[j(G)]}$. Note that any $N[j(G)]$-generic filter containing B satisfies the lifting criterion, making B a *master condition* for the lift. The poset $\mathrm{Add}(j(\kappa), 1)^{N[j(G)]}$ is $\leq\kappa$-closed in $N[j(G)]$. We argued previously that $N[G][g]^{<\kappa} \subseteq N[G][g]$ in $V[G][g]$ and so, by the ground closure criterion, $N[G][g][G_{\rm tail}]^{<\kappa} \subseteq N[G][g][G_{\rm tail}]$. Thus, by diagonalization criterion (1) applied below B, $V[G][g]$ has an $N[j(G)]$-generic filter $\overline{g}$ containing B, and so we are able to lift j to $j : M[G][g] \to N[j(G * g)]$, with $j(G * g) = G * g * G_{\rm tail} * \overline{g}$, in $V[G][g]$.

By an identical argument as for $N[G][g]$, we conclude that $M[G][g]$ is a κ-model in $V[G][g]$. A set $C \subseteq \kappa$ in $N[j(G * g)]$ could not have been added by the $\leq\kappa$-closed $\mathbb{P}_{\rm tail} * \mathrm{Add}(j(\kappa), 1)$ and hence it is already in $N[G][g]$ and the rest of the argument to verify κ-powerset preservation is as before. Note that if $M \prec H_{\kappa^+}$, then $M[G][g] \prec H_{\kappa^+}[G][g] = H_{\kappa^+}^{V[G][g]}$, which gives the argument for super Ramsey cardinals.

In the argument for Ramsey cardinals, we start with an ω-special weak κ-model M such that $V_\kappa \in M$ and $j : M \to N$ that is the ultrapower by a weakly amenable countably complete M-ultrafilter on κ. Let $\langle m_i \mid i < \omega\rangle$ witness that M is ω-special and let $\langle x_i \mid i < \omega\rangle$, defined as in Lemma 3.10, witness that N is ω-almost special. Then $N[G][g]$ together with the sequence $\langle x_i[G][g] \mid i < \omega\rangle$ and the poset $\mathbb{P}_{\rm tail}$ satisfies the requirements of diagonalization criterion (2) in $V[G][g]$. Thus, $V[G][g]$ has an $N[G][g]$-generic filter $G_{\rm tail}$ for $\mathbb{P}_{\rm tail}$, and so we can lift j to $j : M[G] \to N[j(G * g)]$ with $j(G) = G * g * G_{\rm tail}$. Next, we use diagonalization criterion (2) with the model $N[j(G)]$ and the sequence $\langle x_i[j(G)] \mid i < \omega\rangle$ to obtain an $N[j(G)]$-generic filter $\overline{g}$ for $\mathrm{Add}(j(\kappa), 1)^{N[j(G)]}$ containing B. Finally, it remains to observe that the two-step iteration $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$ is countably closed, and so by Lemma 3.21, the lift $j : M[G][g] \to N[j(G)][\overline{g}]$ is the ultrapower by a countably complete $M[G][g]$-ultrafilter in $V[G][g]$.

In the argument for α-iterable cardinals, we start with an ω-special weak κ-model M such that $V_\kappa \in M$, and $j : M \to N$ that is the ultrapower by a weakly amenable α-good M-ultrafilter on κ. Let $\langle m_i \mid i < \omega\rangle$ witness that M is ω-special and let $\langle x_i \mid i < \omega\rangle$, defined as in Lemma 3.10, witness that N is almost ω-special. Since $V_\kappa \subseteq m_i$, it follows that $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1) \subseteq x_i$. The poset $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$ is definable from κ and therefore is in x_i as well. Thus, $G * g$ is x_i-generic for all i. Exactly following the argument above, which showed $N[G][g]^{<\kappa} \subseteq N[G][g]$ in $V[G][g]$, we conclude that each $x_i[G][g]^{<\kappa} \subseteq x_i[G][g]$ in $N[G][g]$. Each $x_i[G][g] \prec N[G][g]$ by Remark 3.5(3). Thus, we can use diagonalization criterion (2) to obtain an $N[G][g]$-generic filter $G_{\rm tail}$ for $\mathbb{P}_{\rm tail}$ that is $x_i[G][g]$-generic for all i. By Remark 3.5(2), the N-generic filter $j(G) = G * g * G_{\rm tail}$ is x_i-generic for all i. It now follows, by the ground closure criterion, that each $x_i[j(G)]^{<\kappa} \subseteq x_i[j(G)]$ in $N[j(G)]$. Since $j(\kappa) \in x_i$ (Remark 3.11), $\mathrm{Add}(j(\kappa), 1)^{N[j(G)]} \in x_i[j(G)]$. So we can use diagonalization criterion (2) with the model $N[j(G)]$ and the sequence $\langle x_i[j(G)] \mid i < \omega\rangle$ to obtain an $N[j(G)]$-generic filter $\overline{g}$ for $\mathrm{Add}(j(\kappa), 1)^{N[j(G)]}$ containing B that is $x_i[j(G)]$-generic for all i. The combined N-generic filter $j(G * g) = G * g * G_{\rm tail} * \overline{g}$ is therefore x_i-generic for all i, and hence, by Theorem 3.18, the final lift $j : M[G][g] \to N[j(G * g)]$ is the ultrapower by an α-good $M[G][g]$-ultrafilter.

Finally, let's argue for each of these cardinals κ that in a forcing extension $V[G][g]$ by $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$ their respective large cardinal property is indestructible by the forcing $\mathrm{Add}(\kappa, 1)$. Here it remains to observe that the poset $\mathrm{Add}(\kappa, 1)$ is unchanged in any forcing extension by the $<\kappa$-closed $\mathrm{Add}(\kappa, 1)$ and so forcing with $\mathrm{Add}(\kappa, 1)$ over $V[G]$ followed by forcing with $\mathrm{Add}(\kappa, 1)$ again is equivalent to forcing with $\mathrm{Add}(\kappa, 1) \times \mathrm{Add}(\kappa, 1) \cong \mathrm{Add}(\kappa, 1)$. □

If $p \in \mathrm{Add}(\kappa, \theta)$, define that the support, $\mathrm{supp}(p)$, of p is the collection of slices $\xi < \theta$ mentioned in p. If $S \subseteq \theta$, we can factor $\mathrm{Add}(\kappa, \theta)$ as $\mathrm{Add}(\kappa, \theta) \cong \mathbb{Q}_S \times \mathbb{Q}_{\theta \setminus S}$, where

$$\mathbb{Q}_S = \{p \in \mathrm{Add}(\kappa, \theta) \mid \mathrm{supp}(p) \subseteq S\}$$

and

$$\mathbb{Q}_{\theta \setminus S} = \{p \in \mathrm{Add}(\kappa, \theta) \mid \mathrm{supp}(p) \subseteq \theta \setminus S\},$$

and if $G \subseteq \mathrm{Add}(\kappa, \theta)$ is V-generic, we can correspondingly factor $G \cong G_S \times G_{\theta \setminus S}$.

Theorem 4.4. *If κ is α-iterable, Ramsey, or strongly Ramsey, then there is a forcing extension in which its large cardinal property becomes indestructible by the forcing $\mathrm{Add}(\kappa, \theta)$ for every cardinal θ.*

Proof. First, suppose that κ is strongly Ramsey. Using Theorem 4.3, we may assume that κ is indestructible by $\mathrm{Add}(\kappa, 1)$. We shall argue that, in this case, it is also indestructible by $\mathrm{Add}(\kappa, \theta)$ for every cardinal θ. It is immediate that κ is indestructible as well by $\mathrm{Add}(\kappa, \kappa) \cong \mathrm{Add}(\kappa, 1)$. Let $G \subseteq \mathrm{Add}(\kappa, \theta)$ be V-generic and fix $A \subseteq \kappa$ in $V[G]$ and a nice $\mathrm{Add}(\kappa, \theta)$-name $\dot{A} = \bigcup_{\alpha \in \kappa} \{\check{\alpha}\} \times \mathcal{A}_\alpha$ for A, where $\mathcal{A}_\alpha$ are antichains of $\mathrm{Add}(\kappa, \theta)$. Since the poset $\mathrm{Add}(\kappa, \theta)$ has the κ^+-cc, $\bigcup_{\alpha < \kappa} \mathcal{A}_\alpha$ has size at most κ. Let S be the union of all $\mathrm{supp}(p)$ for $p \in \bigcup_{\alpha < \kappa} \mathcal{A}_\alpha$. Observe that S has size at most κ. It follows that $\dot{A}_G = \dot{A}_{G_S}$ is an element of $V[G_S]$ (as defined above) where κ remains strongly Ramsey since $\mathbb{Q}_S \cong \mathrm{Add}(\kappa, \kappa)$. Thus, in $V[G_S] \subseteq V[G]$, the set A is contained a κ-model M for which there exists a κ-powerset preserving embedding $j : M \to N$. Finally, we observe that M continues to be a κ-model in $V[G] = V[G_S][G_{\theta \setminus S}]$ since forcing with $\mathbb{Q}_S$ does not add new $<\kappa$-sequences.

In the argument for Ramsey cardinals, note that if U is a weakly amenable countably complete M-ultrafilter for a weak κ-model M in $V[G_S]$, then it continues to be so in $V[G_S][G_{\theta \setminus S}]$ because the further extension does not add new ω-sequences. In the argument for α-iterable cardinals, note that being a weakly amenable α-good M-ultrafilter for a weak κ-model M is absolute for transitive models. □

The above argument does not readily generalize to the case of super Ramsey cardinals. If κ is a super Ramsey cardinal indestructible by $\mathrm{Add}(\kappa, 1)$ in V, then κ will be super Ramsey in $V[G_S]$, but it may not be super Ramsey in the further extension $V[G] = V[G_S][G_{\theta \setminus S}]$ because $H_{\kappa^+}^{V[G]}$ is different from $H_{\kappa^+}^{V[G_S]}$. The argument we give to show that super Ramsey cardinals can be made indestructible by all forcings $\mathrm{Add}(\kappa, \theta)$ is more complicated. We start with a lemma.

Lemma 4.5. *Suppose that κ is super Ramsey, $2^\kappa = \kappa^+$, and $\theta > \kappa^+$ is a cardinal. If $G \subseteq \mathrm{Add}(\kappa, \theta)$ is V-generic and $s \in H_{\kappa^+}^{V[G]}$, then there is $S \subseteq \theta$ of size κ^+ such that $s \in H_{\kappa^+}^{V[G_S]} \prec H_{\kappa^+}^{V[G]}$.*

Proof. Suppose that $M \in V[G]$ is an elementary substructure of $H_{\kappa^+}^{V[G]}$ of size κ^+. Let $F \in V[G]$ enumerate all subsets of κ from M in a κ^+-sequence, and note that since every element of M is coded by a subset of κ, F codes M. Let $\dot{F}$ be an $\mathrm{Add}(\kappa,\theta)$-name for F such that

$$\mathbb{1} \Vdash \text{“}\dot{F} \text{ is a function on } \check{\kappa}^+ \text{ which codes an elementary substructure of } H_{\check{\kappa}^+}\text{”}.$$

Since $\mathrm{Add}(\kappa,\theta)$ has the κ^+-cc, there is a function f on κ^+, whose range consists of $\mathrm{Add}(\kappa,\theta)$-names, and for every condition $p \in \mathrm{Add}(\kappa,\theta)$ and $\xi < \kappa^+$, there is $\sigma \in \mathrm{ran}(f)$ such that $p \Vdash \dot{F}(\check{\xi}) = \sigma$. For every $\sigma \in \mathrm{ran}(f)$, there is a nice $\mathrm{Add}(\kappa,\theta)$-name $\bar{\sigma}$ for a subset of κ so that we have

$$\mathbb{1} \Vdash \text{“}\sigma \subseteq \check{\kappa} \to \sigma = \bar{\sigma}\text{”}.$$

Thus, we can assume without loss of generality that the range of f consists of nice $\mathrm{Add}(\kappa,\theta)$-names for subsets of κ. Let us call such a function f a *ground model cover of* M.

Let f_0 be a ground model cover of a substructure $M_0 \prec H_{\kappa^+}^{V[G]}$ from $V[G]$ such that $H_{\kappa^+}^V \subseteq M_0$ and $s \in M_0$. Let S_0 be the union of $\mathrm{supp}(p)$ over all conditions p appearing in the $\mathrm{Add}(\kappa,\theta)$-names in the range of f_0, and note that, since $\mathrm{Add}(\kappa,\theta)$ has the κ^+-cc, S_0 has size at most κ^+. Suppose that we are given a ground model cover f_ξ of some M_ξ. Let S_ξ be the union of $\mathrm{supp}(p)$ over all conditions p appearing in the $\mathrm{Add}(\kappa,\theta)$-names in the range of f_ξ, and note that S_ξ has size κ^+. Let $f_{\xi+1}$ be a ground model code for some substructure $M_{\xi+1} \prec H_{\kappa^+}^{V[G]}$ from $V[G]$ such that $H_{\kappa^+}^{V[G_{S_\xi}]} \subseteq M_{\xi+1}$. Note that an $\mathrm{Add}(\kappa,\theta)$-name for some such structure $M_{\xi+1}$ can be constructed from f_ξ, and hence so can the function $f_{\xi+1}$. For limit $\lambda \leq \kappa^+$, we let f_λ be some function whose range is the union of the ranges of f_ξ for $\xi < \lambda$, namely the ground model cover of the structures so far, and we let $S_\lambda = \bigcup_{\xi<\lambda} S_\xi$.

Let $\bar{X}_\xi = \{\sigma_G \mid \sigma \in \mathrm{ran}(f_\xi)\}$ for $\xi \leq \kappa^+$. Let's argue that $\bar{X}_\nu \subseteq \bar{X}_\eta$ for $\nu < \eta$. Fix $\eta \leq \kappa^+$ and suppose that for $\nu < \bar{\eta} < \eta$, we have $\bar{X}_\nu \subseteq \bar{X}_{\bar{\eta}}$. If η is a limit ordinal, then the statement holds by construction. So suppose that $\eta = \xi + 1$. We need to check that $\bar{X}_\xi \subseteq \bar{X}_{\xi+1}$, and it suffices to note that $\bar{X}_\xi \subseteq H_{\kappa^+}^{V[G_{S_\xi}]}$ by construction.

Let X_ξ be the set consisting of all sets coded by subsets of κ in $\bar{X}_\xi$, and note that $X_\xi \subseteq H_{\kappa^+}^{V[G_{S_\xi}]}$. By construction, X_0 contains some structure $M_0 \prec H_{\kappa^+}^{V[G]}$ with $H_{\kappa^+} \subseteq M_0$ and $s \in M_0$. Also, each $X_{\xi+1}$ contains a structure $M_{\xi+1} \prec H_{\kappa^+}^{V[G]}$ such that $H_{\kappa^+}^{V[G_{S_\xi}]} \subseteq M_{\xi+1}$. Let $M = X_{\kappa^+}$. Since $M = \bigcup_{\xi<\kappa^+} M_{\xi+1}$, we have $M \prec H_{\kappa^+}^{V[G]}$. Also, we have $H_{\kappa^+}^{V[G_{S_\xi}]} \subseteq M$ for every $\xi < \kappa^+$. Let's argue that $M = H_{\kappa^+}^{V[G_{S_{\kappa^+}}]}$. Clearly $M \subseteq H_{\kappa^+}^{V[G_{S_{\kappa^+}}]}$. So suppose that $x \in H_{\kappa^+}^{V[G_{S_{\kappa^+}}]}$ and assume without loss of generality that $x \subseteq \kappa$. Let σ be a nice $\mathrm{Add}(\kappa,\theta)$-name for x. Then σ has size κ and therefore the union of all $\mathrm{supp}(p)$ for p appearing in σ must be contained in S_ξ for some $\xi < \kappa^+$. It follows that $x \in H_{\kappa^+}^{V[G_{S_\xi}]} \subseteq M_{\xi+1} \subseteq M$.

Thus, $S = S_{\kappa^+}$ works, and note crucially that $S \in V$.

□

It follows from Lemma 4.5, using an argument analogous to the proof of Theorem 4.4, that to show that a super Ramsey κ can be made indestructible by all $\mathrm{Add}(\kappa,\theta)$, it suffices to show that it can be made indestructible by $\mathrm{Add}(\kappa,\kappa^+)$.

Note that if κ is super Ramsey and $2^\kappa = \delta > \kappa^+$, we can always force to collapse δ to κ^+ while preserving that κ is super Ramsey because $\mathrm{Coll}(\kappa^+, \delta)$ is $\leq\kappa$-closed.

Let $\mathbb{P}_\kappa$ be the Easton support forcing iteration of length κ where at stage α if α is a cardinal in $V^{\mathbb{P}_\alpha}$, we force with $\mathrm{Add}(\alpha, \alpha^+)$, and with the trivial poset otherwise. Note that, by Lemma 3.22, $\mathbb{P}_\kappa$ has size κ and the κ-cc, and also that $\mathbb{P}_\kappa \subseteq V_\kappa$. The preparatory forcing for super Ramsey cardinals κ will be $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^+)$. The argument that follows uses ideas from [Lev95] for building a generic filter from a sequence of increasingly powerful master conditions.

Theorem 4.6. *If κ is super Ramsey, then there is a forcing extension in which this becomes indestructible by the forcing* $\mathrm{Add}(\kappa, \kappa^+)$.

Proof. It suffices to argue that κ remains super Ramsey after the preparatory forcing $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^+)$. Let $G * H \subseteq \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^+)$ be V-generic. Fix $A \subseteq \kappa$ in $V[G]$ and a nice $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^+)$-name $\dot{A}$ for A. The union of all $\mathrm{supp}(p)$ over p appearing in $\dot{A}$ must be bounded below κ^+, and therefore without loss of generality (using an automorphism argument) we can assume that all conditions in $\dot{A}$ come from the first coordinate of $\mathrm{Add}(\kappa, \kappa^+)$. In other words, $\dot{A}$ is an $\mathrm{Add}(\kappa, 1)$-name. So we can fix a κ-model $M \prec H_{\kappa^+}$ containing $\dot{A}$ for which there exists a κ-powerset preserving embedding $j : M \to N$ of κ-models. The poset $\mathbb{P}_\kappa$ is an element of M as it is a definable subset of $V_\kappa \in M$. For the argument below, we need to additionally assume that M is κ-special, using Lemma 3.16.

First, we lift j to $M[G]$. As usual, it suffices to find an N-generic filter for the poset

$$j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^+) * \dot{\mathbb{P}}_{\mathrm{tail}}$$

containing $j\,"G = G$. We will use here the filter $G*h*G_{\mathrm{tail}}$, where h is the restriction of H to $(\kappa^+)^N$ and G_{tail} comes from diagonalization criterion (1). Note that h is $N[G]$-generic for $\mathrm{Add}(\kappa, \kappa^+)^{N[G]}$ because $N[G]$ is a κ-model in $V[G]$ and therefore $\mathrm{Add}(\kappa, \kappa^+)^{N[G]}$ is precisely the restriction of $\mathrm{Add}(\kappa, \kappa^+)^{V[G]}$ to coordinates below $(\kappa^+)^N$. Thus, we have lifted j to $j : M[G] \to N[j(G)]$ with $j(G) = G * h * G_{\mathrm{tail}}$.

The poset $\mathbb{Q} = \mathrm{Add}(\kappa, \kappa^+)^{N[G]}$ is not an element of $M[G]$, but it is definable there as the collection of all partial functions $p\colon \kappa \times \alpha \to 2$ for some $\alpha \in \mathrm{ORD}^M = (\kappa^+)^N$ with domain of size less than κ. Let's argue that $\mathbb{Q}$ is pretame in $M[G]$. Fix a definable collection $\langle D_i \mid i \in I\rangle$ of dense classes of $\mathbb{Q}$ over $M[G]$ for a set $I \in M[G]$, and let it be defined by a formula $\varphi(i, x, a)$. Since $M[G] \prec H_{\kappa^+}^{V[G]} = H_{\kappa^+}[G]$, the formula $\varphi(i, x, a)$ also defines a dense class of $\mathrm{Add}(\kappa, \kappa^+)^{V[G]}$ for every $i \in I$ over $H_{\kappa^+}^{V[G]}$. Thus, $H_{\kappa^+}^{V[G]}$ satisfies that there is a set sequence $\vec{A} = \langle A_i \mid i \in I\rangle$ such that A_i is a maximal antichain contained in the dense class defined by $\varphi(i, x, a)$ for every $i \in I$. By elementarity, there is some such sequence $\vec{A} \in M[G]$.

Although, the forcing relation need not be definable for a class partial order in general, by a theorem of M. Stanley, it is definable for all pretame class partial orders over models of ZFC^-.[8] Similarly the poset $\mathrm{Add}(\kappa, \kappa^+)^{V[G]}$ is a class partial order of $H_{\kappa^+}^{V[G]}$ for which the forcing relation is definable. Now using the fact that h is the restriction of H to $\mathbb{Q}$ and the definablity of the forcing relation, it follows that

$$M[G][h] \prec H_{\kappa^+}[G][H] = H_{\kappa^+}^{V[G][H]}.$$

[8]For the definition of pretameness and a proof of Stanley's theorem see [HKS18].

Thus, it remains to show that we can lift j to $M[G][h]$ and argue that the lift is κ-powerset preserving.

Let $\overline{\mathbb{Q}}$ be the class partial order in $N[j(G)]$ consisting of all partial functions $p : j(\kappa) \times \alpha \to 2$ for some $\alpha \in \mathrm{ORD}^N$ with domain of size less than $j(\kappa)$. By elementarity, the poset $\overline{\mathbb{Q}}$ is pretame in $N[j(G)]$. The lifting criterion holds for the pair $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ because of the definability of the forcing relation. Thus, to lift j to $M[G][h]$, we need an $N[j(G)]$-generic filter $\overline{h}$ for $\overline{\mathbb{Q}}$ such that $j\,"\,h \subseteq \overline{h}$. Let $h(\xi)$ be the subset of κ on coordinate ξ of h. Observe that $j\,"\,h$ consists of the same subsets of κ as h, but now $h(\xi)$ sits on coordinate $j(\xi)$. Let $\langle m_\xi \mid \xi < \kappa\rangle$ witness that M is κ-special. Let $\delta_\xi = m_\xi \cap (\kappa^+)^N$. Observe that each δ_ξ is an initial segment of $(\kappa^+)^N$ since $\kappa \in m_\xi$ and $m_\xi \prec M$, and $(\kappa^+)^N$ is the union of the increasing sequence of the δ_ξ. Let $h_\xi = h \cap m_\xi$ be h restricted to coordinates in δ_ξ, and let $p_\xi = j\,"\,h_\xi$. Next, observe that p_ξ can be constructed from h and $j\,"\,\delta_\xi$, both of which are elements of $N[j(G)]$, and so, in particular, each p_ξ is an element of $\overline{\mathbb{Q}}$. The conditions p_ξ will be the increasingly more powerful master conditions for the lift.

We are now going to construct an $N[j(G)]$-generic filter $\overline{h}$ compatible with $j\,"\,h$ in κ-many steps. Because j is an ultrapower map, the ordinals $j(\delta_\xi)$ are unbounded in ORD^N. Thus, every maximal antichain A of $\overline{\mathbb{Q}}$ is already contained in some $\mathrm{Add}(j(\kappa), j(\delta_\xi))^{N[G]}$. In $V[G][H]$, we enumerate all maximal antichains of $\overline{\mathbb{Q}}$ in $N[j(G)]$ in a κ-sequence $\langle A_\xi \mid \xi < \kappa\rangle$. We start with A_0, which must be contained in some $\mathrm{Add}(j(\kappa), j(\delta_{\xi_0}))^{N[G]}$ for some $\xi_0 < \kappa$. The condition p_{ξ_0} is an element of $\mathrm{Add}(j(\kappa), j(\delta_{\xi_0}))$, and hence we can choose $q_0 \in \mathrm{Add}(j(\kappa), j(\delta_{\xi_0}))$ below p_{ξ_0} and some element of A_0. Note that q_0 is compatible with $j\,"\,h$. Now we suppose inductively that we have defined an increasing sequence of conditions $\langle q_\xi \mid \xi < \alpha\rangle$, for some $\alpha < \kappa$, such that each q_ξ is compatible to $j\,"\,h$ and q_ξ has above it an element of A_ξ. Note that the sequence $\langle q_\xi \mid \xi < \alpha\rangle$ is an element of the κ-model $N[j(G)]$. Let q^*_α be the union of the q_ξ. Choose $\mathrm{Add}(j(\kappa), j(\delta_{\xi_\alpha}))$, with $\xi_\alpha < \kappa$, containing A_α and q^*_α. Let q_α be any condition below q^*_α, p_{ξ_α}, and some element of A_α. It should be clear that that the sequence $\langle q_\xi \mid \xi < \kappa\rangle$ generates an $N[j(G)]$-generic filter $\overline{h}$ that is compatible with $j\,"\,h$ and therefore contains all elements of $j\,"\,h$. Finally, we note that the argument to show that the lift is κ-powerset preserving is standard. □

Corollary 4.7. *If κ is Ramsey, α-iterable, strongly Ramsey, or super Ramsey, then there is a forcing extension preserving this in which $2^\kappa > \kappa^+$. Indeed, there is such a forcing extension in which κ is the first cardinal at which the* GCH *fails.*

Proof. For the "moreover" part, first force the GCH to hold. Then perform the appropriate preparatory forcing to make the cardinal indestructible by all $\mathrm{Add}(\kappa, \theta)$, which preserves the GCH (this is a name counting argument). Finally, force with $\mathrm{Add}(\kappa, \kappa^{++})$. □

Thus, unlike a measurable cardinal, a Ramsey cardinal, or one of our other cardinals, at which the GCH fails for the first time does not have higher consistency strength, and the failure of the GCH need not reflect below. In fact, in [CG15], Cody and Gitman showed that if κ is a strongly Ramsey or a Ramsey cardinal and

F is any Easton function[9] such that $F\,``\,\kappa \subseteq \kappa$, then there is a forcing extension in which κ retains the large cardinal property, and for all regular cardinals δ, we have $2^\delta = F(\delta)$.

4.4. **Fast function forcing.** Recall that a *fast function* is a generically added ordinal-guessing function on a large cardinal that mimics a Laver function on a supercompact cardinal. If a large cardinal κ is characterized by the existence of a certain type of elementary embeddings, then a fast function $f:\kappa \to \kappa$ has the property that for every reasonably chosen ordinal θ,[10] there is an embedding j of the type characterizing the cardinal such that $j(f)(\kappa) = \theta$. The fast function grows "fast enough" to create an arbitrarily large gap between κ and $j(f)(\kappa)$, a useful property for lifting the embedding to a forcing extension. For an inaccessible cardinal κ, the fast function forcing $\mathbb{F}_\kappa$, invented by Woodin, consists of conditions that are partial functions $p:\kappa \to \kappa$, ordered by inclusion, such that $\gamma \in \mathrm{dom}(p)$ implies $p\,``\,\gamma \subseteq \gamma$, and for every inaccessible cardinal $\gamma \leq \kappa$, we have $|\mathrm{dom}(p \upharpoonright \gamma)| < \gamma$. The fast function $f:\kappa \to \kappa$ is the union of the generic filter for $\mathbb{F}_\kappa$. In what follows, we will often identify the generic filter for the fast function forcing with the fast function itself. Note that, since κ is inaccessible, the poset $\mathbb{F}_\kappa \subseteq V_\kappa$.

For $\gamma < \kappa$, let $\mathbb{F}_\gamma$ be the subposet of $\mathbb{F}_\kappa$ consisting of all conditions p whose domain is contained in γ, let $\mathbb{F}_{[\gamma,\kappa)}$ be the subposet of $\mathbb{F}_\kappa$ consisting of all conditions p whose domain is contained in $[\gamma,\kappa)$, and define $\mathbb{F}_{(\gamma,\kappa)}$ analogously. For any $p \in \mathbb{F}_\kappa$ and $\gamma \in \mathrm{dom}(p)$, the poset $\mathbb{F}_\kappa \upharpoonright p$ (consisting of all $q \leq p$ in $\mathbb{F}_\kappa$) factors as the product

$$\mathbb{F}_\gamma \upharpoonright (p \upharpoonright \gamma) \times \mathbb{F}_{[\gamma,\kappa)} \upharpoonright (p \upharpoonright [\gamma,\kappa)),$$

where the second factor is $\leq\delta$-closed for $\delta = \max\{\gamma, p(\gamma)\}$.

In particular, if $p = \{\langle \gamma, \delta\rangle\}$ for some $\gamma \leq \delta$, then $\mathbb{F}_\kappa \upharpoonright p$ factors as $\mathbb{F}_\gamma \times \mathbb{F}_{(\delta,\kappa)}$, and the second factor $\mathbb{F}_{(\delta,\kappa)}$ is $\leq\delta$-closed. For a more detailed exposition on fast functions, see [Ham00].

Note, finally, that $\mathbb{F}_\kappa$ is obviously countably closed (indeed, it is closed up to the first inaccessible cardinal).

Theorem 4.8. *If κ is Ramsey, α-iterable, strongly Ramsey, or super Ramsey, then this is preserved to a forcing extension $V[f]$ by the fast function forcing $\mathbb{F}_\kappa$. Moreover, given a generic fast function f,*

1. *if $j: M \to N$ is a κ-powerset preserving embedding of κ-models, or*
2. *$j: M \to N$ is the ultrapower of an ω-special weak κ-model by an α-good weakly amenable M-ultrafilter, or*
3. *$j: M \to N$ is the ultrapower of an ω-special weak κ-model by a countably complete weakly amenable M-ultrafilter,*

then, for any $\theta < j(\kappa)$, there is a lift $j: M[f] \to N[j(f)]$ satisfying the same properties with $j(f)(\kappa) = \theta$.

[9] An *Easton function* F is a class function on the regular cardinals such that $F(\delta) \leq F(\gamma)$ whenever $\delta < \gamma$, and $\mathrm{cof}(\delta) < F(\delta)$. Easton showed that if the GCH holds and F is an Easton function, then there is a generic extension in which $2^\delta = F(\delta)$ for all regular cardinals δ [Eas70].

[10] The ordinals θ that can reasonably be made the value of $j(f)(\kappa)$ are restricted by the properties of the embedding characterization. For example, if a cardinal is characterized by the existence of well-founded ultrapowers of weak κ-models, as are α-iterable cardinals, then the target of the embedding has size κ and hence θ must be below κ^+.

Proof. First, suppose that κ is strongly Ramsey. The poset $\mathbb{F}_\kappa$ is an element of M since it is a definable subset of $V_\kappa \in M$. First, we verify that $M[f]^{<\kappa} \subseteq M[f]$ in $V[f]$. We cannot apply the generic closure criterion since $\mathbb{F}_\kappa$ does not have the κ-cc. Instead, our strategy will be to show that for arbitrarily large inaccessible cardinals $\alpha < \kappa$, we have $M[f]^\alpha \subseteq M[f]$ in $V[f]$. Fix any $\beta < \kappa$. Observe that there is an inaccessible $\alpha > \beta$ and $\gamma \in \mathrm{dom}(f)$ such that $\gamma < \alpha < f(\gamma) = \delta$ because conditions with this property are dense in $\mathbb{F}_\kappa$. Thus, the condition $p = \{\langle\gamma, \delta\rangle\}$ is in the generic filter. Below p, the poset $\mathbb{F}_\kappa$ factors as $\mathbb{F}_\gamma \times \mathbb{F}_{(\delta,\kappa)}$ and f factors correspondingly as $f_\gamma \times f_{(\delta,\kappa)}$. We argue that $M[f_\gamma][f_{(\delta,\kappa)}]^\alpha \subseteq M[f_\gamma][f_{(\delta,\kappa)}]$ in $V[f]$. Since $\mathbb{F}_\gamma$ clearly has the α^+-cc, by the generic closure criterion, $M[f_\gamma]^\alpha \subseteq M[f_\gamma]$ in $V[f_\gamma]$. Also, since $\mathbb{F}_{(\delta,\kappa)}$ is $\leq\alpha$-closed, $M[f_\gamma]^\alpha \subseteq M[f_\gamma]$ in $V[f]$. Finally, by the ground closure criterion, $M[f_\gamma][f_{(\delta,\kappa)}]^\alpha \subseteq M[f_\gamma][f_{(\delta,\kappa)}]$ in $V[f]$. Thus, $M[f]^\alpha \subseteq M[f]$ in $V[f]$ for arbitrarily large α and so it follows that $M[f]^{<\kappa} \subseteq M[f]$ in $V[f]$.

Next, we fix an ordinal $\theta < j(\kappa)$. We shall lift the embedding j to $M[f]$ so that $j(f)(\kappa) = \theta$. Consider $j(\mathbb{F}_\kappa) = \mathbb{F}^N_{j(\kappa)}$, the poset to add a fast function on $j(\kappa)$ from the perspective of N. Let $p = \{\langle\kappa, \theta\rangle\}$ and factor $\mathbb{F}^N_{j(\kappa)} \upharpoonright p \cong \mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}}$, where $\mathbb{F}_{\text{tail}} = \mathbb{F}^N_{[\kappa,j(\kappa))} \upharpoonright p$ is $\leq\kappa$-closed in N. We will build an N-generic filter for $\mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}}$ containing $j\,``\,f = f$, and then take its upward closure, which contains p, to obtain an N-generic filter for $\mathbb{F}^N_{j(\kappa)}$. This will ensure that $j(f)(\kappa) = \theta$. Unlike in the previous arguments, we cannot apply diagonalization criterion (1) to $N[f]$ and $\mathbb{F}_{\text{tail}}$ in $V[f]$ since $\mathbb{F}_{\text{tail}}$ is not $<\kappa$-closed in $N[f]$ (since the poset is a product and not an iteration). Instead, we exploit the fact that $\mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}} \cong \mathbb{F}_{\text{tail}} \times \mathbb{F}_\kappa$. We use diagonalization criterion (1) to obtain an N-generic filter f_{tail} for $\mathbb{F}_{\text{tail}}$ in V, using that the poset is $<\kappa$-closed in N. Since f is V-generic for $\mathbb{F}_\kappa$, it is $N[f_{\text{tail}}]$-generic and so $f_{\text{tail}} \times f$ is N-generic for $\mathbb{F}_{\text{tail}} \times \mathbb{F}_\kappa$. Thus, we are able to lift $j : M \to N$ to $j : M[f] \to N[f][f_{\text{tail}}]$ in $V[f]$ with $j(f)(\kappa) = \theta$.

We already showed that $M[f]$ is a κ-model, and so it remains to verify that the lift $j : M[f] \to N[f][f_{\text{tail}}]$ is κ-powerset preserving. Suppose that $A \subseteq \kappa$ is in $N[f][f_{\text{tail}}] = N[f_{\text{tail}}][f]$. Then A has a nice $\mathbb{F}_\kappa$-name $\dot{A}$ in $N[f_{\text{tail}}]$, which can itself be coded by a subset of κ. It follows that $\dot{A} \in N$, and hence $\dot{A} \in M$. But then $\dot{A}_f = A \in M[f]$. Note that if $M \prec H_{\kappa^+}$, then $M[f] \prec H_{\kappa^+}[f] = H^{V[f]}_{\kappa^+}$, which gives the argument for super Ramsey cardinals.

For Ramsey cardinals, we use diagonalization criterion (2) in place of diagonalization criterion (1) to build the N-generic filter for $\mathbb{F}_{\text{tail}}$, and use the countable closure of $\mathbb{F}_\kappa$ to verify that the lift is the ultrapower by a countably complete $M[f]$-ultrafilter. For α-iterable cardinals, we use diagonalization criterion (2) to build the N-generic filter for $\mathbb{F}_{\text{tail}}$. □

We can reformulate the elementary embedding characterization of Ramsey cardinals using embeddings instead of M-ultrafilters. Having a weakly amenable countably complete M-ultrafilter is equivalent to having an embedding $j : M \to N$ with critical point κ, $P^M(\kappa) = P^N(\kappa)$, and the property that whenever $\langle A_n \mid n < \omega\rangle$ is a sequence of subsets of κ with $A_n \in M$ and $\kappa \in j(A_n)$ for every $n < \omega$, then $\bigcap_{n<\omega} A_n \neq \emptyset$. Let's say that such embeddings have the *Ramsey property.* The advantage of this reformulation is that we are no longer restricted to ultrapower embeddings whose target has size κ and we can get the following generalization of the fast function property.

Theorem 4.9. *Suppose that κ is Ramsey and f is a V-generic fast function. Then for every $A \subseteq \kappa$ and every ordinal θ, there is an ω-special weak κ-model M containing A and an embedding $j : M \to N$ with the Ramsey property that lifts to $j : M[f] \to N[j(f)]$ having the Ramsey property and $j(f)(\kappa) = \theta$.*

Proof. By Lemma 3.13, we can fix an ω-special weak κ-model M containing A and V_κ for which there exists a weakly amenable countably complete M-ultrafilter U on κ. Let $\langle m_i \mid i < \omega\rangle$ witness that M is ω-special. Let

$$\{j_{\xi\gamma} : M_\xi \to M_\gamma \mid \xi < \gamma \in \mathrm{ORD}\},$$

where $M_0 = M$, be the associated sequence of iterated ultrapowers and

$$\{\kappa_\xi \mid \xi \in \mathrm{ORD}\},$$

where $\kappa_0 = \kappa$, be the critical sequence. First, we argue that every iterate embedding $j_{0\gamma} : M \to M_\gamma$ has the Ramsey property. This follows by assumption for $\gamma = 1$, so suppose $\gamma > 1$ and observe that if $\kappa \in j_{0\gamma}(X)$, then $\kappa = j_{1\gamma}(\kappa) \in j_{0\gamma}(X)$, and so by elementarity $\kappa \in j_{01}(X)$.

If $\theta < \kappa^+$, then we are already done by Theorem 4.8. So we fix an ordinal $\theta \geq \kappa^+$ and choose an ordinal γ such that $\kappa_\gamma \leq \theta < \kappa_{\gamma+1}$. We shall lift the iterate embedding $j_{0\gamma}$ to $j_{0\gamma} : M[f] \to M_\gamma[j(f)]$ having the Ramsey property and $j_{0\gamma}(f)(\kappa) = \theta$. This will require appropriate analogues of Lemma 3.10 and Lemma 3.21 for iterates of an ultrapower map and some standard facts about iterations (see [Kan09] Section 19 for details).

(1) Every element x of M_γ has the form $x = j_{0\gamma}(g)(\kappa_{\gamma_1}, \ldots, \kappa_{\gamma_n})$ for some $n \in \omega$, function $g : [\kappa]^n \to M$, and $\gamma_i < \gamma$ for $i \leq n$.

(2) Given $x \in M$, we have that $M_\gamma \models \varphi(j_{0\gamma}(x), \kappa_{\gamma_1}, \ldots, \kappa_{\gamma_n})$ if and only if

$$\{(\xi_1, \ldots, \xi_n) \in [\kappa]^n \mid M \models \varphi(x, \xi_1, \ldots, \xi_n)\} \in U^n,$$

where U^n is the n-fold product ultrafilter of U.

Sets in U^n and U are connected by the following useful fact:

(3) For every $A \in U^n$, there is $\overline{A} \in U$ such that if $\alpha_1 < \cdots < \alpha_n$ are in $\overline{A}$, then $(\alpha_1, \ldots, \alpha_n) \in A$.

For ease of notation, let $j_{0\gamma} = h$. For the analogue of Lemma 3.10, we define

$$y_i = \{h(f)(\kappa_{\gamma_1}, \ldots, \kappa_{\gamma_n}) \mid n \in \omega, f : [\kappa]^n \to m_i, f \in m_i, \text{ and } \gamma_i < \gamma \text{ for } i \leq n\}.$$

Clearly, $M_\gamma = \bigcup_{i\in\omega} y_i$. Moreover, each y_i is an element of M_γ of size κ_γ, since it is definable from h " m_i, which is in M_γ. Now following the proof of Theorem 4.8, we factor $\mathbb{F}^{M_\gamma}_{h(\kappa)}$ below the condition $p = \{\langle\kappa, \theta\rangle\}$ as $\mathbb{F}_\kappa \times \mathbb{F}_{\mathrm{tail}}$, where $\mathbb{F}_{\mathrm{tail}} = \mathbb{F}^{M_\gamma}_{(\theta, h(\kappa))}$ is $\leq\theta$-closed and hence $\leq\kappa_\gamma$-closed. The model M_γ together with the sequence $\langle y_i \mid i < \omega\rangle$ and the poset $\mathbb{F}_{\mathrm{tail}}$ satisfies the requirements of diagonalization criterion (2) for the cardinal κ_γ. Thus, we are able to lift h to the κ-powerset preserving $h : M[f] \to M_\gamma[h(f)]$ with $h(f)(\kappa) = \theta$ as the proof of Theorem 4.8. It remains to verify that for every sequence $\langle A_n \mid n < \omega\rangle$ of subsets of κ from $V[f]$ with $A_n \in M[f]$ and $\kappa \in h(A_n)$, we have $\bigcap_{n<\omega} A_n \neq \emptyset$.

Following the proof of Lemma 3.21, we fix a sequence $\langle A_n \mid n < \omega\rangle$ of subsets of κ in $V[f]$ such that each A_n is an element of $M[f]$ with $\kappa \in h(A_n)$, and we also fix $\mathbb{F}_\kappa$-names $\dot{A}_n \in M$ such that $A_n = (\dot{A}_n)_f$. Let $\dot{S}$ be an $\mathbb{F}_\kappa$-name such that

$$\dot{S} \Vdash \text{“}\dot{S} \text{ is an } \omega\text{-sequence”}$$

and for all $n < \omega$, $\mathbb{1} \Vdash \dot{S}(\check{n}) = \dot{A}_n$ over V. Suppose towards a contradiction that $\bigcap_{n<\omega} A_n = \emptyset$ and choose a condition $p \in f$ such that $p \Vdash \bigcap \dot{S} = \emptyset$ over V. As in that proof, we choose a descending ω-sequence of conditions $p_n \in j(f)$ below $j(p)$ such that $p_n \Vdash \check{\kappa} \in h(\dot{A}_n)$ over M_γ. Using fact (1), each

$$p_n = h(g_n)(\kappa_{\gamma_0^{(n)}}, \ldots, \kappa_{\gamma_{i_n}^{(n)}}),$$

and we shall assume for convenience that $\gamma_0^{(n)} = 0$, meaning that the first element in the sequence is κ. Define for $n < \omega$, the sets

$$S_n = \{\vec{\xi} \in [\kappa]^{i_n} \mid g_n(\vec{\xi}) \Vdash \xi_0 \in \dot{A}_n \text{ over } M\}.$$

and the sets

$$T_n = \{\vec{\xi} \in [\kappa]^{j_n} \mid g_{n+1}(\vec{\alpha}) \leq g_n(\vec{\beta})\},$$

where $j_n = |\{\gamma_0^{(n)}, \ldots, \gamma_{i_n}^{(n)}, \gamma_0^{(n+1)}, \ldots, \gamma_{i_{n+1}}^{(n+1)}\}|$ and $\vec{\alpha}$ and $\vec{\beta}$ are subsequences of $\vec{\xi}$ corresponding to how elements of $\{\gamma_0^{(n)}, \ldots, \gamma_{i_n}^{(n)}, \gamma_0^{(n+1)}, \ldots, \gamma_{i_{n+1}}^{(n+1)}\}$ intertwine. Finally, define

$$T = \{\vec{\xi} \in [\kappa]^{i_0} \mid g_0(\vec{\xi}) \leq p\}.$$

Using fact (2), each S_n, T_n and T is an element of some product ultrafilter U^m. Let $\overline{S}_n$, $\overline{T}_n$, and $\overline{T}$ be the sets in U given by fact (3) for the sets S_n, T_n, and T respectively. Let I be the intersection of all $\overline{S}_n$, $\overline{T}_n$, and $\overline{T}$. Since the M-ultrafilter U is countably complete, the set I is non-empty, but, in fact, it has size κ (if I was bounded by $\alpha < \kappa$, then we could add $\kappa \setminus \alpha$ to the sets we are intersecting, thus contradicting that U is countably complete). Let α be the countable order-type of the union $\mathcal{K}$ of all $\{\kappa_{\gamma_0^{(n)}}, \ldots \kappa_{\gamma_{i_n}^{(n)}}\}$, and let I_α be an initial segment of I of order-type α. Let $\vec{\gamma}_n$ be a subsequence of I_α of length i_n that corresponds to where $\{\kappa_{\gamma_0^{(n)}}, \ldots \kappa_{\gamma_{i_n}^{(n)}}\}$ sits inside $\mathcal{K}$, and note that by the earlier assumption that κ is the first element of every $\{\kappa_{\gamma_0^{(n)}}, \ldots \kappa_{\gamma_{i_n}^{(n)}}\}$, we have that δ, the first element of I_α, is the first element of every $\vec{\gamma}_n$. It follows that:

(1) for all $n < \omega$, $g_n(\vec{\gamma}_n) \Vdash \delta \in \dot{A}_n$ over M,
(2) for all $n < \omega$, $g_{n+1}(\vec{\gamma}_{n+1}) \leq g_n(\vec{\gamma}_n)$,
(3) $g_0(\vec{\gamma}_0) \leq p$.

The contradiction, showing that $\bigcap_{n<\omega} A_n \neq \emptyset$, is now obtained exactly as in the proof of Lemma 3.21. □

It is not clear how to obtain similar results for our other Ramsey-like cardinals. If κ is strongly Ramsey or super Ramsey, we also have iterate embeddings with large target models. But the target models have no closure, and so it is not clear at all how one would go about building a generic filter for $\mathbb{F}_{\text{tail}}$ for them. The α-iterable cardinals are characterized by the existence of M-ultrafilters with certain iterability properties, and here it is simply not clear how to restate this characterization in terms of embeddings with large target models.

5. Virtually Ramsey cardinals

Virtually Ramsey cardinals were introduced in [SW11] in order to provide an upper bound for the consistency strength of a variant of Chang's conjecture. They are defined by weakening the characterization of Ramsey cardinals in terms of the existence of good sets of indiscernibles (see Definition 5.1). In [GW11], it was shown that if κ is strongly Ramsey, then there is a forcing extension in which κ remains virtually Ramsey but is no longer Ramsey, thus separating the two notions. Here, we improve this result to show that we can obtain such a forcing extension by starting with just a Ramsey cardinal. This answers positively a question of Welch posed in [Git11]. We start with the definition of *good sets of indiscernibles.*

Definition 5.1. Suppose that κ is a cardinal and $A \subseteq \kappa$. A set $I \subseteq \kappa$ is a *good set of indiscernibles* for the structure $\langle L_\kappa[A], A\rangle$ if for all $\gamma \in I$, we have:

(1) $\langle L_\gamma[A\cap\gamma], A\cap\gamma\rangle \prec \langle L_\kappa[A], A\rangle$,
(2) $I \setminus \gamma$ is a set of indiscernibles for the structure $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$.

Theorem 5.2. *A cardinal κ is Ramsey if and only if for every $A \subseteq \kappa$ there is a good set of indiscernibles I for $\langle L_\kappa[A], A\rangle$ such that $|I| = \kappa$.*

See [Dod82] (Section 17) for proof.

For $A \subseteq \kappa$, we define that $\mathscr{I} = \{\alpha < \kappa \mid$ there is an unbounded good set of indiscernibles $I \subseteq \alpha$ for $\langle L_\kappa[A], A\rangle\}$.

Definition 5.3. A cardinal κ is *virtually Ramsey* if for every $A \subseteq \kappa$, the set $\mathscr{I}$ contains a club of κ.

It is easily seen that Ramsey cardinals are virtually Ramsey by noting that if I is a good set of indiscernibles for $\langle L_\kappa[A], A\rangle$ of size κ, then the club of all its limit points is a subset of $\mathscr{I}$. A virtually Ramsey cardinal is Mahlo and one that is weakly compact is already Ramsey (see [GW11] Proposition 6.6 and 6.7). Thus, a natural strategy to separate virtually Ramsey and Ramsey cardinals is to destroy the weak compactness of a Ramsey cardinal while preserving that it is virtually Ramsey. This strategy was implemented in [GW11], using a two-step iteration of Kunen from [Kun78] to destroy and then resurrect a weakly compact cardinal. The argument started with a strongly Ramsey κ and produced a forcing extension in which κ remained virtually Ramsey but was no longer weakly compact. Here we improve on this result by starting with a Ramsey cardinal.

Theorem 5.4. *If κ is Ramsey, then there is a forcing extension in which κ is virtually Ramsey but not Ramsey.*

Sketch of proof. By using Theorem 4.3, we may assume that κ is Ramsey cardinal whose Ramseyness is indestructible by $\mathrm{Add}(\kappa,1)$. Let $\mathbb{Q}$ be the poset to add a κ-Souslin tree having a group of automorphisms with a transitive action, described in [GW11]. If $\mathbb{T}$ is the κ-Souslin tree added by $\mathbb{Q}$ in the forcing extension, then viewing it as a forcing notion and forcing with it adds a branch through $\mathbb{T}$. Letting $\dot{\mathbb{T}}$ be the $\mathbb{Q}$-name for the κ-Souslin tree it adds, we shall force with the iteration $\mathbb{Q}*\dot{\mathbb{T}}$. The critical observation about the iteration $\mathbb{Q}*\dot{\mathbb{T}}$ is that it has size κ and a dense subset that is $<\kappa$-closed, from which it follows that it is forcing equivalent to $\mathrm{Add}(\kappa,1)$. Let $T*B \subseteq \mathbb{Q}*\dot{\mathbb{T}}$ be V-generic and consider the forcing extensions $V[T]$ and $V[T][B]$. It is clear that κ is no longer weakly compact in $V[T]$ since it contains a κ-tree, namely T, without a branch. However, κ is again Ramsey in $V[T][B]$ since, as we observed, $\mathbb{Q}*\dot{\mathbb{T}}$ is forcing equivalent to $\mathrm{Add}(\kappa,1)$, which preserves that κ is Ramsey by assumption. But this implies that κ must have already been virtually Ramsey in $V[T]$ as the forcing $\mathbb{T} = \dot{\mathbb{T}}_T$ is $<\kappa$-distributive and preserves stationary subsets of κ (because it is κ-cc), and therefore cannot create new virtually Ramsey cardinals. To summarize, the cardinal κ is virtually Ramsey in the forcing extension $V[T]$, but it cannot be Ramsey since, in particular, it is not weakly compact. $\square$

For complete details of the argument, see [GW11].

6. Losing the large cardinal property in HOD

In [CFH15], techniques were developed to show that certain large cardinals can lose their large cardinal property in HOD. Using the indestructibility we obtained for Ramsey, α-iterable, strongly Ramsey, and super Ramsey cardinals, we can apply the techniques of [CFH15] to show that these cardinals can lose their large cardinal property in HOD, and indeed need not even be weakly compact there.

Theorem 6.1. *If κ is Ramsey, α-iterable, strongly Ramsey, or super Ramsey, then there is a forcing extension in which κ retains its large cardinal property, but is not even weakly compact in* HOD*.*

Sketch of proof. By using Theorem 4.3, we may assume that the large cardinal property of κ is indestructible by $\mathrm{Add}(\kappa, 1)$. Let $\mathbb{Q}*\dot{\mathbb{T}}$ be Kunen's two-step iteration from the proof of Theorem 5.4 for killing and then resurrecting a weakly compact cardinal. Let $T * B \subseteq \mathbb{Q} * \dot{\mathbb{T}}$ be V-generic and consider the forcing extensions $V[T]$ and $V[T][B]$. In $V[T]$, we force with the standard GCH coding forcing $\mathbb{R}$ to code T into the continuum pattern above κ and let $H \subseteq \mathbb{R}$ be $V[T]$-generic. Note that the $V[T]$-generic B is also $V[T][H]$-generic because $\mathbb{T} = \dot{\mathbb{T}}_T$ is still a κ-Souslin tree in $V[T][H]$. The tree $\mathbb{T}$ remains κ-Soulin because $\mathbb{R}$ is $\leq\kappa$-closed and every branch of a κ-Souslin tree is automatically generic. Thus, H and B are mutually generic, giving us that $V[T][H][B] = V[T][B][H]$. Also, since $\mathbb{T}$ has size κ, and so obviously has the κ^+-cc, $\mathbb{R}$ is $\leq\kappa$-distributive in $V[T][B]$ by Easton's Lemma (if $\mathbb{P}$ has the κ^+-cc and $\mathbb{Q}$ is $\leq\kappa$-closed, then $\mathbb{Q}$ remains $\leq\kappa$-distributive after forcing with $\mathbb{P}$). By our indestructibility assumption and the fact that $\leq\kappa$-distributive forcing cannot destroy Ramsey or Ramsey-like cardinals, we get that κ retains its large cardinal property in $V[T][B][H]$. But we will now argue that κ is not weakly compact in $\mathrm{HOD}^{V[T][B][H]}$. Since $V[T][B][H] = V[T][H][B]$ and the forcing $\mathbb{T}$ is weakly homogeneous[11] in $V[T][H]$, $\mathrm{HOD}^{V[T][B][H]} \subseteq V[T][H]$. The tree $\mathbb{T}$ is an element of $\mathrm{HOD}^{V[T][B][H]}$ because it was coded into the continuum pattern, and it is κ-Souslin there because it is κ-Souslin in $V[T][H]$. Thus, κ is not weakly compact in $\mathrm{HOD}^{V[T][B][H]}$. $\square$

In particular, it follows that none of our Ramsey-like cardinals are downward absolute to arbitrary transitive inner models. In contrast, it was shown in [GW11] that the α-iterable cardinals are downward absolute to L and that strongly Ramsey and super Ramsey cardinals are downward absolute to the core model K.

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(V. Gitman) The City University of New York, CUNY Graduate Center, Mathematics Program, 365 Fifth Avenue, New York, NY 10016
E-mail address: vgitman@gmail.com
URL: https://victoriagitman.github.io/

(T. A. Johnstone) Mathematics, New York City College of Technology, 300 Jay Street, Brooklyn, NY 11201
E-mail address: tjohnstone@citytech.cuny.edu