

# INDESTRUCTIBILITY PROPERTIES OF RAMSEY AND RAMSEY-LIKE CARDINALS

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ABSTRACT. We develop new techniques, for use in indestructibility arguments, of lifting embeddings on transitive set models of  $ZFC^-$  which lack closure and of lifting iterations of such embeddings. We use these techniques to establish basic indestructibility results for Ramsey and Ramsey-like cardinals –  $\alpha$ -iterable, strongly Ramsey, and super Ramsey cardinals – introduced in [Git11]. We show that Ramsey and Ramsey-like cardinals  $\kappa$  are indestructible by small forcing, the canonical forcing of the GCH, and the forcing to add a fast function on  $\kappa$ . We also show that if  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension in which the large cardinal property of  $\kappa$  becomes indestructible by  $\text{Add}(\kappa, \theta)$  for any cardinal  $\theta$ .

The following are consequences of the indestructibility results. If  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension preserving this in which the GCH fails at  $\kappa$ . If  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension preserving this in which  $\kappa$  is not even weakly compact in HOD. If  $\kappa$  is Ramsey, then there is a forcing extension in which  $\kappa$  remains virtually Ramsey, but is no longer Ramsey (this answers positively a question posed in [Git11]).

## 1. INTRODUCTION

The study of indestructibility properties of large cardinals was initiated by a seminal result of Lévy and Solovay showing that measurable cardinals cannot be destroyed by small forcing [LS67]. The Lévy-Solovay phenomena is now known to extend to most large cardinal notions, which means, in particular, that large cardinals cannot decide CH or other independent set theoretic statements that can be manipulated by small forcing. This, taken more generally, is the significance of studying indestructibility properties of large cardinals: it provides a means of verifying which set theoretic properties, among those that can be manipulated by forcing, are compatible with a given large cardinal. There are other applications of indestructibility, such as in separating closely related large cardinal notions by forcing to destroy a part of a large cardinal property, while preserving the rest.

Ramsey cardinals were introduced by Erdős and Hajnal in 1962 [EH62], who defined that a cardinal  $\kappa$  is *Ramsey* if every coloring  $f : [\kappa]^{<\omega} \rightarrow 2$  of finite tuples of elements of  $\kappa$  into two colors has a homogeneous set of size  $\kappa$ . As we will discuss later, Ramsey cardinals can also be characterized by the existence of indiscernibles for certain structures as well as by the existence of iterable ultrafilters for certain families of subsets of  $\kappa$  of size  $\kappa$  (see Theorems 2.4, 4.11). Historically very little was known about the indestructibility of properties of Ramsey cardinals. A folklore proof, using their original characterization, shows that Ramsey cardinals are indestructible by small forcing [Kan09] (Section 10). Jensen in [Jen74] hinted at a proof that Ramsey cardinals are indestructible by a product forcing which yields

the GCH in the forcing extension. Finally, Welch showed in [Wel88], using a characterization of Ramsey cardinals in terms of the existence of indiscernibles, that they are indestructible by the forcing to code the universe into a real.

Most general techniques for establishing indestructibility properties of a large cardinal require it to have a characterization in terms of the existence of elementary embeddings. The indestructibility arguments then proceed by showing how to lift (extend) the elementary embedding(s) characterizing the large cardinal from the ground model  $V$  to the forcing extension  $V[G]$ , thus verifying that the large cardinal maintains its property there. It is more common to think of the large cardinals including and above measurable cardinals as being characterized by the existence of elementary embeddings. But in fact, even smaller large cardinals that we typically associate with combinatorial definitions, such as weakly compact and indescribable cardinals, have elementary embedding characterizations. These smaller large cardinals  $\kappa$  are usually characterized by the existence of elementary embeddings of weak  $\kappa$ -models (transitive models of  $\text{ZFC}^-$  of size  $\kappa$  with height above  $\kappa$ ) or of  $\kappa$ -models (additionally closed under  $<\kappa$ -sequences). Mitchell discovered an elementary embeddings characterization of Ramsey cardinals involving the existence of  $\omega_1$ -intersecting ultrafilters (Theorem 2.4) for weak  $\kappa$ -models [Mit79], but it was not extensively studied until Gitman started to explore it in her dissertation with the purpose of obtaining indestructibility results for Ramsey cardinals [Git07]. In the process, Gitman generalized aspects of the Ramsey embeddings to introduce new large cardinal notions, the *Ramsey-like* cardinals:  $\alpha$ -iterable, strongly Ramsey, and super Ramsey cardinals [Git11].

The elementary embeddings characterization of Ramsey cardinals does not easily lend itself to standard indestructibility techniques. The first difficulty is that the embeddings are on weak  $\kappa$ -models, as opposed to  $\kappa$ -models, and these may not even be closed under countable sequences. The second difficulty is that the embeddings are ultrapowers by  $\omega_1$ -intersecting ultrafilters and while the lift of an ultrapower embedding remains an ultrapower embedding by a potentially larger ultrafilter, it is not trivial to verify that the larger ultrafilter is still  $\omega_1$ -intersecting. Strongly Ramsey and super Ramsey cardinals (Definition 2.6), which, as the name suggests, are a strengthening of Ramsey cardinals, were defined to have embedding characterizations remedying the deficiencies of Ramsey embeddings with respect to indestructibility arguments. The  $\alpha$ -iterable cardinals generalized a different aspect of the Ramsey embeddings (Definition 2.9). They are defined by the existence of partially iterable ultrafilters for weak  $\kappa$ -models, a requirement that weakens the Ramsey embeddings characterization because  $\omega_1$ -intersecting ultrafilters are fully iterable.

In this article, we prove basic indestructibility results for Ramsey and Ramsey-like cardinals using a mix of old and newly introduced techniques. We use standard techniques to establish indestructibility properties of strongly Ramsey and super Ramsey cardinals, as their definition was directly motivated to make them easily amenable to these techniques. We develop techniques for lifting embeddings on models without closure. We show that if the forcing is countably closed, then a lift of the ultrapower by an  $\omega_1$ -intersecting ultrafilter retains this property in the forcing extension. The combination of these new techniques allows us to prove the same basic indestructibility results for Ramsey cardinals as for strongly and super Ramsey cardinals. For the  $\alpha$ -iterable cardinals, we develop techniques for

simultaneously lifting entire iterations of embeddings, so that we can verify that the potentially larger ultrafilter associated with the lift of the first ultrapower in the iteration continues to have at least the iterability of the original ultrafilter. The new indestructibility techniques we introduce can potentially be used to establish a variety of indestructibility results for these and similar large cardinal notions. Here, we obtain the following indestructibility results.

**Theorem 1.1.**

- (1) *Ramsey and Ramsey-like cardinals  $\kappa$  are indestructible by:*
  - (a) *small forcing,*
  - (b) *the canonical forcing of the GCH,*
  - (c) *the forcing to add a fast function on  $\kappa$ ,*
- (2) *If  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension in which the large cardinal property of  $\kappa$  becomes indestructible by the forcing  $\text{Add}(\kappa, \theta)$  for every cardinal  $\theta$ .*

These indestructibility properties have the following consequences.

**Corollary 1.2.**

- (1) *If  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension preserving this in which the GCH fails at  $\kappa$ .*
- (2) *If  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension preserving this in which  $\kappa$  is not even weakly compact in HOD.*
- (3) *If  $\kappa$  is Ramsey, then there is a forcing extension destroying this, while preserving that  $\kappa$  is virtually Ramsey.*

To establish (3), we use techniques from [CFH15]. The *virtually Ramsey* cardinals from (4) (see Definition 4.12) were introduced in [SW11] as an upper bound on the consistency strength of a variant of Chang's Conjecture studied there. The new indestructibility techniques are introduced in Section 3. Theorem 1.1 and its corollaries are proved in Section 4.

## 2. RAMSEY AND RAMSEY-LIKE CARDINALS

Ramsey and Ramsey-like cardinals, as well as other smaller large cardinals,  $\kappa$  are characterized by the existence of certain elementary embeddings of weak  $\kappa$ -models or  $\kappa$ -models of set theory. Suppose that  $\kappa$  is a cardinal. A *weak  $\kappa$ -model* is a transitive model of  $\text{ZFC}^{-1}$  of size  $\kappa$  and height above  $\kappa$ . A weak  $\kappa$ -model that is additionally closed under  $<\kappa$ -sequences is called a  *$\kappa$ -model*. Natural examples of weak  $\kappa$ -models and  $\kappa$ -models arise as elementary substructures of  $H_{\kappa^+}$ , the collection of all sets of hereditary size  $\leq \kappa$ . Any elementary substructure of  $H_{\kappa^+}$  of size  $\kappa$  and containing  $\kappa$  as a subset is a weak  $\kappa$ -model and if  $\kappa^{<\kappa} = \kappa$ , then we can build elementary substructures of  $H_{\kappa^+}$  that are  $\kappa$ -models.

The analogue of a  $\kappa$ -complete ultrafilter<sup>2</sup> in the setting of weak  $\kappa$ -models is the notion of an  $M$ -ultrafilter for a weak  $\kappa$ -model  $M$ . If  $M$  is a transitive model of

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<sup>1</sup>The theory  $\text{ZFC}^-$  consists of the axioms of ZFC without the powerset axiom, with the collection scheme instead of the replacement scheme, and with the statement that every set can be well-ordered instead of the axiom of choice. See [GHJ] for the significance of this particular choice of axioms.

<sup>2</sup>Here, we adopt the convention that an ultrafilter on a cardinal  $\kappa$  includes the tail sets. It follows that ultrafilters are necessarily non-principal and a normal ultrafilter on  $\kappa$  is  $\kappa$ -complete.

$\text{ZFC}^-$  and  $\kappa$  is a cardinal in  $M$ , then a set  $U \subseteq \mathcal{P}(\kappa)^M$  is<sup>3</sup> said to be an  $M$ -ultrafilter if the structure  $\langle M, \in, U \rangle$ , consisting of  $M$  together with a predicate for  $U$ , satisfies that  $U$  is a normal ultrafilter on  $\kappa$ . The set  $U$  must be viewed as interpreting a predicate over  $M$  since in most interesting cases it will not be an element of  $M$ . Note that  $U$  measures only those subsets of  $\kappa$  that are elements of  $M$  and is  $\kappa$ -complete only for sequences that are themselves elements of  $M$ . Consequently, even a countable sequence of elements of  $U$  might have an empty intersection if the sequence is not in  $M$ .

If an ultrafilter is an element of a model of set theory, then the ultrapower construction with it can be iterated along the ordinals. At successor ordinal stages, the iteration proceeds by taking the ultrapower by the image of the ultrafilter under the embedding from the previous stage and direct limits are taken at limit stages. This produces a directed system of *iterated ultrapowers* of the original model. In this situation, it is easy to see that an ultrafilter has a well-founded ultrapower if and only if it is countably complete, and indeed Kunen showed that all iterated ultrapowers of a countably complete ultrafilter are well-founded [Kun70]. If  $M$  is a transitive model of  $\text{ZFC}^-$ , then an  $M$ -ultrafilter suffices to carry out the ultrapower construction, but the ultrapower may not be well-founded. To iterate the ultrapower construction by an  $M$ -ultrafilter, we must first modify the successor step construction to work with ultrafilters that are external to the model. The modified construction still requires that the ultrafilters be at least partially internal to the model, a concept captured by the notion of weak amenability. An  $M$ -ultrafilter  $U$  on  $\kappa$ , for a transitive  $M \models \text{ZFC}^-$ , is *weakly amenable* if for every  $A \in M$  of size  $\kappa$  in  $M$ , the intersection  $U \cap A$  is an element of  $M$ . While weak amenability allows us to iterate the ultrapower construction, it does not guarantee the well-foundedness of any of the iterates. There are weakly amenable  $M$ -ultrafilters that do not even have a well-founded ultrapower, as well as those all of whose iterated ultrapowers are well-founded. It is shown in [GW11] that it is consistent to have  $M$ -ultrafilters realizing all the possibilities in between as well: we can have, for every countable ordinal  $\alpha$ , a model  $M$  and an  $M$ -ultrafilter with exactly  $\alpha$ -many well-founded iterated ultrapowers. This covers all possibilities since Gaifman showed that an  $M$ -ultrafilter with  $\omega_1$ -many well-founded iterated ultrapowers is already *iterable*, having all well-founded iterated ultrapowers [Gai74]. For a cardinal  $\lambda$ , call an  $M$ -ultrafilter  $U$   $\lambda$ -*intersecting* if every sequence of elements of  $U$  of length less than  $\lambda$  has a nonempty intersection. Clearly an  $\omega_1$ -intersecting<sup>4</sup>  $M$ -ultrafilter has a well-founded ultrapower and Kunen showed that a weakly amenable  $\omega_1$ -intersecting  $M$ -ultrafilter is iterable [Kun70].

In the remarks below, we summarize some basic facts concerning elementary embeddings of weak  $\kappa$ -models  $M$  and ultrapowers by  $M$ -ultrafilters.

**Remarks 2.1.** Suppose that  $\kappa$  is a cardinal and  $M$  is a weak  $\kappa$ -model.

<sup>3</sup>We intend  $\mathcal{P}(\kappa)^M$  to refer to the set in  $V$  of all the subsets of  $\kappa$  that are elements of  $M$  and this set need not exist in  $M$  itself.

<sup>4</sup>The common terminology is to call such ultrafilters *countably complete* but the authors find this confusing since in the case of  $M$ -ultrafilters the intersection need not be an element of the ultrafilter but merely nonempty.

- (1) An elementary embedding  $j : M \rightarrow N^5$  with critical point  $\kappa$  is the ultrapower by an  $M$ -ultrafilter on  $\kappa$  if and only if

$$N = \{j(f)(\kappa) \mid f : \kappa \rightarrow M, f \in M\}.$$

- (2) If  $j : M \rightarrow N$  is the ultrapower by an  $M$ -ultrafilter on  $\kappa$  and  $M^\alpha \subseteq M$  for some  $\alpha < \kappa$ , then  $N^\alpha \subseteq N$ .
- (3) An elementary embedding  $j : M \rightarrow N$  with critical point  $\kappa$  gives rise to the  $M$ -ultrafilter  $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$  on  $\kappa$ , which we say is *generated by  $\kappa$  via  $j$* . The ultrapower by  $U$  is isomorphic to

$$X = \{j(f)(\kappa) \mid f : \kappa \rightarrow M \text{ in } M\}$$

via the map  $\varphi : [f]_U \mapsto j(f)(\kappa)$  and hence is well-founded.

We call an elementary embedding  $j : M \rightarrow N$  with critical point  $\kappa$ ,  $\kappa$ -*powerset preserving* if  $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$ .<sup>6</sup> The existence of weakly amenable  $M$ -ultrafilters on  $\kappa$  with well-founded ultrapowers is precisely equivalent to the existence of  $\kappa$ -powerset preserving embeddings of  $M$ .

**Lemma 2.2.** *Suppose that  $\kappa$  is a cardinal and  $M$  is a weak  $\kappa$ -model.*

- (1) *If  $j : M \rightarrow N$  is a  $\kappa$ -powerset preserving elementary embedding and  $U$  is generated by  $\kappa$  via  $j$ , then  $U$  is weakly amenable.*
- (2) *If  $U$  is a weakly amenable  $M$ -ultrafilter on  $\kappa$  with the well-founded ultrapower  $j : M \rightarrow N$ , then  $j$  is  $\kappa$ -powerset preserving.*

For an extended discussion of weakly amenable  $M$ -ultrafilters and relevant proofs see [Kan09] (Section 19).

We now have in place all the preliminaries required for stating the elementary embeddings characterization of Ramsey cardinals and the definitions of the Ramsey-like cardinals that arose from it. But before we do so, it is instructive to recall for comparison the elementary embeddings characterization of weakly compact cardinals.

**Theorem 2.3.** *A cardinal  $\kappa$  is weakly compact if and only if  $\kappa^{<\kappa} = \kappa$  and every  $A \subseteq \kappa$  is contained in a weak  $\kappa$ -model  $M$  for which there exists an elementary embedding  $j : M \rightarrow N$  with critical point  $\kappa$  (equivalently, for which there exists an  $M$ -ultrafilter on  $\kappa$  with a well-founded ultrapower).*

It is not difficult to show that if  $\kappa$  is weakly compact, then indeed every weak  $\kappa$ -model  $M$  has an  $M$ -ultrafilter with a well-founded ultrapower, and so in particular, embeddings exist for  $\kappa$ -models that are elementary in  $H_{\kappa^+}$ , and by virtue of that reflect  $V$  to some extent.

**Theorem 2.4.** *A cardinal  $\kappa$  is Ramsey if and only if every  $A \subseteq \kappa$  is contained in a weak  $\kappa$ -model  $M$  for which there exists a weakly amenable  $\omega_1$ -intersecting  $M$ -ultrafilter on  $\kappa$ .*

For an exposition of the proof adapted from [Dod82], see [Git11]. From the proof presented there it is clear that we can strengthen Theorem 2.4 as follows.

<sup>5</sup>Unless explicitly stated otherwise, the embeddings we consider are assumed to be of transitive models.

<sup>6</sup>Note that neither powerset is required to exist in its respective model.

**Theorem 2.5.** *A cardinal  $\kappa$  is Ramsey if and only if every  $A \subseteq \kappa$  is contained in a weak  $\kappa$ -model  $M$  for which there exists a weakly amenable  $\kappa$ -intersecting  $M$ -ultrafilter on  $\kappa$ .*

It is natural to ask, by analogy with weakly compact cardinals, that if  $\kappa$  is Ramsey, whether every weak  $\kappa$ -model  $M$  has a weakly amenable  $\omega_1$ -intersecting  $M$ -ultrafilter on  $\kappa$ . This is not the case. By assuming that every  $A \subseteq \kappa$  is contained in a  $\kappa$ -model  $M$  with a weakly amenable  $M$ -ultrafilter on  $\kappa$  (which must be  $\omega_1$ -intersecting), we get a stronger large cardinal notion, and we further strengthen this notion by assuming that  $M \prec H_{\kappa^+}$ . The assumption that every  $\kappa$ -model  $M$  has a weakly amenable  $M$ -ultrafilter on  $\kappa$  is inconsistent!

**Definition 2.6.** A cardinal  $\kappa$  is *strongly Ramsey* if every  $A \subseteq \kappa$  is contained in a  $\kappa$ -model  $M$  for which there exists a weakly amenable  $M$ -ultrafilter on  $\kappa$ . A cardinal  $\kappa$  is *super Ramsey* if we additionally assume that  $M \prec H_{\kappa^+}$ .

Using Remarks 2.1 (2), we obtain a characterization of strongly and super Ramsey cardinals in terms of the existence of  $\kappa$ -powerset preserving embeddings.

**Remark 2.7.** A cardinal  $\kappa$  is strongly Ramsey if and only if every  $A \subseteq \kappa$  is contained in a  $\kappa$ -model  $M$  for which there exists a  $\kappa$ -powerset preserving embedding  $j : M \rightarrow N$  with  $N$  a  $\kappa$ -model. The same characterization holds for super Ramsey cardinals with the additional assumption that  $M \prec H_{\kappa^+}$ .

Gitman showed in [Git11] that every strongly Ramsey cardinal is a stationary limit of Ramsey cardinals, every super Ramsey cardinal is a stationary limit of strongly Ramsey cardinals, and every measurable cardinal is a stationary limit of super Ramsey cardinals.

Another aspect of the Ramsey embeddings that can be studied is their interability properties. For instance, we can ask if  $\kappa$  is weakly compact, whether it follows that every  $A \subseteq \kappa$  is contained in a weak  $\kappa$ -model for which there is a weakly amenable  $M$ -ultrafilter on  $\kappa$  with a well-founded ultrapower. We can also ask, given a countable ordinal  $\alpha$ , whether there is an  $M$ -ultrafilter with exactly  $\alpha$ -many well-founded iterated ultrapowers. Recall that this behavior is not possible for an ultrafilter that lives inside the model.

**Definition 2.8.** Suppose that  $\kappa$  is a cardinal,  $M$  is a weak  $\kappa$ -model, and  $U$  is an  $M$ -ultrafilter on  $\kappa$ . We say that:

- (1)  $U$  is *0-good* if it produces a well-founded ultrapower,
- (2)  $U$  is *1-good* if it is 0-good and weakly amenable,
- (3) for an ordinal  $\alpha > 1$ ,  $U$  is  *$\alpha$ -good*, if it produces at least  $\alpha$ -many well-founded iterated ultrapowers.

Using the notion of  $\alpha$ -good  $M$ -ultrafilters, we define the corresponding notion of  $\alpha$ -iterable cardinals.

**Definition 2.9.** For  $1 \leq \alpha \leq \omega_1$ , a cardinal  $\kappa$  is  *$\alpha$ -iterable* if every  $A \subseteq \kappa$  is contained in a weak  $\kappa$ -model  $M$  for which there exists an  $\alpha$ -good  $M$ -ultrafilter on  $\kappa$ .

Gitman showed in [Git11] that 1-iterable cardinals are limits of completely ineffable cardinals and hence much stronger than weakly compact cardinals. It is shown in [GW11] and [SW11] that  $\alpha$ -iterable cardinals form a hierarchy of strength

below Ramsey cardinals. More precisely, for  $\alpha < \beta \leq \omega_1$ , every  $\beta$ -iterable cardinal is a limit of  $\alpha$ -iterable cardinals, and a Ramsey cardinal is a limit of  $\omega_1$ -iterable cardinals. Thus, in particular, the  $M$ -ultrafilter property of being  $\omega_1$ -intersecting is stronger than iterability.

We end by noting that in each of the elementary embedding characterizations of Ramsey or Ramsey-like cardinals, we can equivalently replace “every  $A \subseteq \kappa$ ” by “every  $A \in H_{\kappa^+}$ ” since an element of  $H_{\kappa^+}$  is coded by a binary relation on  $\kappa$  that Mostowski collapses to it and  $\text{ZFC}^-$  suffices to perform the Mostowski collapse. It also follows that if  $j : M \rightarrow N$  is  $\kappa$ -powerset preserving, then  $M$  and  $N$  have the same sets of hereditary size  $\leq \kappa$ .

### 3. OLD AND NEW LIFTING TECHNIQUES

In this section, we review standard techniques for lifting embeddings to a generic extension and develop new techniques for lifting embeddings of weak  $\kappa$ -models, lifting entire iterations of embeddings, and lifting ultrapowers by  $\omega_1$ -intersecting ultrafilters in such a way that the lift continues to be the ultrapower by an  $\omega_1$ -intersecting ultrafilter.

Lifting arguments generally rely on two elementary facts, the *lifting criterion* and the *diagonalization criterion*. The lifting criterion states that lifting an embedding  $j : M \rightarrow N$  to the generic extension  $M[G]$  amounts to finding an  $N$ -generic filter  $H$  with  $j \restriction G \subseteq H$ .

**Lemma 3.1** (Lifting Criterion). *Suppose that  $j : M \rightarrow N$  is an elementary embedding of  $\text{ZFC}^-$  models having generic extensions  $M[G]$  and  $N[H]$  by forcing notions  $\mathbb{P}$  and  $j(\mathbb{P})$  respectively. The embedding  $j$  lifts to an embedding  $j : M[G] \rightarrow N[H]$  with  $j(G) = H$  if and only if  $j \restriction G \subseteq H$ .*

If the original embedding happened to be an ultrapower map, then its lift to the generic extension will be an ultrapower map as well.

**Remark 3.2.** Suppose that  $M \models \text{ZFC}^-$ ,  $\mathbb{P}$  is a forcing notion in  $M$  and  $G \subseteq \mathbb{P}$  is  $M$ -generic. If  $j : M \rightarrow N$  is the ultrapower by an  $M$ -ultrafilter, then any lift  $j : M[G] \rightarrow N[H]$  is the ultrapower by an  $M[G]$ -ultrafilter.

The proof follows from Remarks 2.1 (1). The diagonalization criterion generalizes a standard argument showing that there is a filter meeting any countable collection of dense subsets of a partial order.

**Lemma 3.3** (Diagonalization Criterion (1)). *If  $\mathbb{P}$  is a forcing notion in a model  $M \models \text{ZFC}^-$  and for some cardinal  $\kappa$  the following criteria are satisfied:*

- (1)  $M^{<\kappa} \subseteq M$ ,
- (2)  $\mathbb{P}$  is  $<\kappa$ -closed in  $M$ ,
- (3)  $M$  has at most  $\kappa$  many maximal antichains of  $\mathbb{P}$ ,

*then there is an  $M$ -generic filter for  $\mathbb{P}$ .*

We shall refer to Lemma 3.3 as diagonalization criterion (1), since below we introduce a second diagonalization criterion that works for models with limited or no closure. First, we need to recall what it means for a filter to be generic for a non-transitive model of set theory.

**Definition 3.4.** Suppose that  $X \models \text{ZFC}^-$  is a (not necessarily transitive) set in a transitive (set or class) model  $M \models \text{ZFC}^-$  and  $\mathbb{P} \in X$  is a forcing notion. Then an

$M$ -generic filter  $G \subseteq \mathbb{P}$  is  $X$ -generic if  $G \cap D \cap X \neq \emptyset$  for every dense subset  $D$  of  $\mathbb{P}$  in  $X$ .

Note that the usual definition of genericity and Definition 3.4 coincide for transitive models  $X$ . If  $X \in M \models \text{ZFC}^-$  and  $G$  is  $M$ -generic for a poset  $\mathbb{P} \in X$ , then  $X[G]$  is defined to be the collection of all  $\tau_G$  with  $\tau \in X$ .

**Remarks 3.5.** Suppose that  $X \models \text{ZFC}^-$  is a (not necessarily transitive) set in a transitive (set or class) model  $M \models \text{ZFC}^-$ .

- (1) If  $\mathbb{P} \in X$  is a forcing notion and  $G \subseteq \mathbb{P}$  is  $M$ -generic, then  $G$  is  $X$ -generic if and only if  $X[G] \cap M = X$ .
- (2) If  $\mathbb{P} * \dot{\mathbb{Q}} \in X$  is a two-step iteration of forcing notions and  $G * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$  is  $M$ -generic, then  $G * H$  is  $X$ -generic if and only if  $G$  is  $X$ -generic and  $H$  is  $X[G]$ -generic.
- (3) If  $X \prec M$ ,  $\mathbb{P} \in X$  is a forcing notion, and  $G \subseteq \mathbb{P}$  is  $M$ -generic, then  $X[G] \prec M[G]$ .

For proof of (1) see [She98], (2) follows easily from (1), and (3) is discussed in [Ham].

We are now ready to state and prove diagonalization criterion (2).

**Lemma 3.6** (Diagonalization Criterion (2)). *If  $\mathbb{P}$  is a forcing notion in a model  $M \models \text{ZFC}^-$  and for cardinals  $\gamma < \kappa$  the following criteria are satisfied:*

- (1)  $\mathbb{P}$  is  $\leq \kappa$ -closed in  $M$ ,
- (2) there is a sequence  $X_0 \subseteq X_1 \subseteq \dots \subseteq X_\xi \subseteq \dots$  for  $\xi < \gamma$  such that each  $X_\alpha \in M$  and  $|X_\alpha|^M = \kappa$ , whose union is  $M$ .
- (3)  $M^{< \gamma} \subseteq M$ ,

then there is an  $M$ -generic filter  $G$  for  $\mathbb{P}$ .

If additionally  $\mathbb{P} \in X_0$ ,  $X_\alpha \prec M$  and  $X_\alpha^{< \kappa} \subseteq X_\alpha$  in  $M$  for all non-limit  $\alpha < \gamma$ , then  $G$  is  $X_\alpha$ -generic for these  $X_\alpha$ . In the case  $\gamma = \omega$ , if we weaken (1) to say that  $\mathbb{P}$  is just  $\leq \kappa$ -distributive, then still there is an  $M$ -generic filter for  $\mathbb{P}$ .

*Proof.* Since  $\mathbb{P}$  is  $\leq \kappa$ -closed in  $M$  and  $X_0$  has size  $\kappa$  in  $M$ , working in  $M$ , we construct a  $\kappa$ -length descending sequence of conditions meeting all dense sets of  $\mathbb{P}$  that are elements of  $X_0$ , and choose a condition  $p_0 \in \mathbb{P}$  below the sequence. Now suppose inductively that we are given a condition  $p_\xi$ , with  $\xi < \gamma$ , having the property that it has above it conditions meeting all dense sets of  $\mathbb{P}$  that are elements of  $X_\eta$  for  $\eta < \xi$ . Since  $M = \bigcup_{\alpha < \gamma} X_\alpha$ , we may choose  $\alpha_\xi > \xi$  such that  $p_\xi \in X_{\alpha_\xi}$ . Working in  $M$ , we construct below  $p_\xi$  a descending  $\kappa$ -length sequence of conditions meeting all dense sets of  $\mathbb{P}$  that are elements of  $X_{\alpha_\xi}$ , and choose a condition  $p_{\xi+1}$  below the sequence. At limit stages  $\lambda < \gamma$ , we use  $< \gamma$ -closure of  $M$  together with  $< \kappa$ -closure of  $\mathbb{P}$  in  $M$  to find  $p_\lambda$  below the sequence  $\langle p_\xi \mid \xi < \lambda \rangle$ . Since  $M = \bigcup_{\alpha < \gamma} X_\alpha$ , any filter  $G$  generated by the sequence  $\langle p_\xi \mid \xi < \gamma \rangle$  is  $M$ -generic. For the ‘‘additionally’’ part, note that if  $\mathbb{P} \in X_{\alpha_\xi}$ ,  $X_{\alpha_\xi} \prec M$  and  $X_{\alpha_\xi}^{< \kappa} \subseteq X_{\alpha_\xi}$ , then we can modify the construction to meet all dense sets of  $\mathbb{P}$  that are elements of  $X_{\alpha_\xi}$  inside  $X_{\alpha_\xi}$ . Finally, suppose that  $\gamma = \omega$  and  $\mathbb{P}$  is  $\leq \kappa$ -distributive. Observe that, for  $n < \omega$ , the intersection  $\mathcal{D}_n$  of all dense open subsets of  $\mathbb{P}$  in  $X_n$  is again dense open. Externally, we may then construct an  $\omega$ -descending sequence  $\langle p_n \mid n < \omega \rangle$  with  $p_n \in \mathcal{D}_n$ , and it is clear that this sequence generates an  $M$ -generic filter for  $\mathbb{P}$ .  $\square$



Even though we stated diagonalization criterion (2) very generally, in future arguments we will use it only for the case  $\gamma = \omega$ , which does not require any closure on the model  $M$  and therefore applies to weak  $\kappa$ -models having the structural properties specified in item (2). The genericity requirement for the sets  $X_\alpha$  will be used for lifting iterations of embeddings.

We will also make use of the following standard criterions providing conditions under which the closure of a model of set theory extends to its generic extension. Although, these facts are generally known, we were not able to find proofs of them in standard literature and therefore we provide them here for completeness.

**Lemma 3.7** (Ground Closure Criterion). *Suppose that  $X \models \text{ZFC}^-$  is a (not necessarily transitive) set in a transitive (set or class) model  $M \models \text{ZFC}^-$ . Suppose further that for some ordinal  $\gamma$ ,  $X^\gamma \subseteq X$  in  $M$  and  $M$  has an  $X$ -generic filter  $H \subseteq \mathbb{P}$  for a forcing notion  $\mathbb{P} \in X$ . Then  $X[H]^\gamma \subseteq X[H]$  in  $M$ .*

*Proof.* We work in  $M$ . Suppose that  $\langle a_\xi \mid \xi < \gamma \rangle$  is a sequence with each  $a_\xi \in X[H]$ . For  $\xi < \gamma$ , fix a  $\mathbb{P}$ -name  $\dot{a}_\xi$  in  $X$  such that  $(\dot{a}_\xi)_H = a_\xi$ , and observe that  $\bar{a} = \langle \dot{a}_\xi \mid \xi < \gamma \rangle \in X$  using  $X^\gamma \subseteq X$ . Since  $H$  is  $X$ -generic, it follows that  $X[H]$  is a model of  $\text{ZFC}^-$ , and so it can recover  $\langle a_\xi \mid \xi < \gamma \rangle$  from  $\bar{a}$  and  $H$ .  $\square$

**Lemma 3.8** (Generic Closure Criterion). *Suppose that  $X \models \text{ZFC}^-$  is a (not necessarily transitive) set in a transitive (set or class) model  $M \models \text{ZFC}^-$ . Suppose further that for some ordinal  $\gamma$ ,  $X^{<\gamma} \subseteq X$  in  $M$ . If  $G \subseteq \mathbb{P}$  is  $M$ -generic for a forcing notion  $\mathbb{P} \in X$  such that  $\mathbb{P} \subseteq X$  and has the  $\gamma$ -cc in  $M$ , then  $X[G]^{<\gamma} \subseteq X[G]$  in  $M[G]$ . The same statement holds if  $X^\gamma \subseteq X$  and  $\mathbb{P}$  has the  $\gamma^+$ -cc in  $M$ .*

*Proof.* We work in  $M$ . Fix  $\delta < \gamma$  and suppose that  $\langle a_\xi \mid \xi < \delta \rangle$  is a sequence in  $M[G]$  such that each  $a_\xi \in X[G]$ . Choose a  $\mathbb{P}$ -name  $\dot{a}$  in  $M$  with  $(\dot{a})_G = \langle a_\xi \mid \xi < \delta \rangle$  and choose a condition  $p \in G$  forcing that  $\dot{a}$  is a sequence of length  $\delta$  of elements of  $X[\dot{G}]$ . For  $\xi < \delta$ , define the dense sets  $D_\xi = \{q \leq p \mid \exists \dot{b}_q \in X \ q \Vdash \dot{b}_q = \dot{a}(\check{\xi})\}$  and let  $A_\xi$  be the corresponding maximal antichains. Next, for  $\xi < \delta$ , define that  $S_\xi = \{\langle q, \dot{b}_q \rangle \mid q \in A_\xi\}$ . Since  $\mathbb{P}$  has the  $\gamma$ -cc, each  $A_\xi$  has size less than  $\gamma$  and therefore so does each  $S_\xi$ . Also, each  $S_\xi \subseteq X$ , and therefore, by  $X^{<\gamma} \subseteq X$ , it is in  $X$ . The sequence  $\langle S_\xi \mid \xi < \delta \rangle$  is then in  $X$  as well. Now, working in  $X$ , construct, for each  $\xi$ , a mixed name  $\dot{b}_\xi$  of  $\dot{b}_q$  so that  $(\dot{b}_\xi)_G = a_\xi$ , and note that  $\langle \dot{b}_\xi \mid \xi < \delta \rangle \in X$ . Thus,  $X$  can construct a  $\mathbb{P}$ -name  $\tau$  such that  $\tau_G = \langle a_\xi \mid \xi < \delta \rangle$ . Clearly the same argument works in the case where  $X^\gamma \subseteq X$  and  $\mathbb{P}$  has the  $\gamma^+$ -cc.  $\square$

Next, we define a class of weak  $\kappa$ -models whose embeddings will have the properties required for lifting using diagonalization criterion (2).

**Definition 3.9.** A weak  $\kappa$ -model  $M$  is  $\alpha$ -special if it is the union of a continuous elementary chain of transitive substructures  $\kappa \in m_0 \prec m_1 \prec \dots \prec m_\xi \prec \dots$  for  $\xi < \alpha$  such that each  $m_\xi \in M$ ,  $|m_\xi|^M = \kappa$ , and for non-limit  $\xi$ ,  $m_\xi^{<\kappa} \subseteq m_\xi$  in  $M$ . A weak  $\kappa$ -model  $M$  is  $\alpha$ -almost special if the  $m_\xi$  are not required to be transitive.

**Lemma 3.10.** *If  $M$  is an  $\alpha$ -special weak  $\kappa$ -model and  $j : M \rightarrow N$  is the ultrapower map by a weakly amenable  $M$ -ultrafilter on  $\kappa$ , then  $N$  is  $\alpha$ -almost special. Indeed, if a sequence  $\langle m_\xi \mid \xi < \alpha \rangle$  witnesses that  $M$  is  $\alpha$ -special, then the sequence  $\langle x_\xi \mid \xi < \alpha \rangle$ , where  $x_\xi = \{j(f)(\kappa) \mid f : \kappa \rightarrow m_\xi, f \in m_\xi\}$ , witnesses that  $N$  is  $\alpha$ -almost special.*

*Proof.* By Remarks 2.1 (1), the ultrapower  $N = \{j(f)(\kappa) \mid f : \kappa \rightarrow M, f \in M\}$  and so it is clear that  $N = \bigcup_{\xi < \alpha} x_\xi$ . Note that each  $x_\xi \cong \text{Ult}(m_\xi, U \cap m_\xi)$  via the map  $\varphi : [f]_U \mapsto j(f)(\kappa)$ . Thus,

$$\begin{aligned} x_\xi \models \varphi(j(f)(\kappa)) &\leftrightarrow \{\nu < \kappa \mid m_\xi \models \varphi(f(\nu))\} \in U \\ &\leftrightarrow \{\nu < \kappa \mid M \models \varphi(f(\nu))\} \in U \text{ (since } m_\xi \prec M\text{)} \\ &\leftrightarrow N \models \varphi(j(f)(\kappa)). \end{aligned}$$

It follows that each  $x_\xi \prec N$  and hence for  $\xi < \mu$ , we have  $x_\xi \prec x_\mu$ . Suppose that  $\xi$  is not a limit ordinal. To verify that  $x_\xi$  is closed under  $<\kappa$ -sequences in  $N$ , we fix some  $\langle a_\nu \mid \nu < \delta \rangle$  where  $\delta < \kappa$  and each  $a_\nu \in x_\xi$ . Since  $m_\xi$  is transitive, it is coded by a binary relation on  $\kappa$  that Mostowski collapses to it, and hence  $m_\xi \in N$ . Moreover,  $j \upharpoonright m_\xi$  is in  $N$  as well, since given a bijection  $g : \kappa \rightarrow m_\xi$  in  $M$ , we have that for  $a \in m_\xi$ ,  $j(a) = j(g)(\nu)$  where  $g(\nu) = a$ . Thus, there is a sequence  $\vec{f} = \langle f_\nu \mid \nu < \delta \rangle \in N$  such that  $f_\nu \in m_\xi$  and  $a_\nu = j(f_\nu)(\kappa)$ . The sequence  $\vec{f}$  is an element of  $M$  by  $\kappa$ -powerset preservation, and hence  $\vec{f} \in m_\xi$  by closure. Thus, there is  $F \in m_\xi$  such that  $F(\eta)(\nu) = f_\nu(\eta)$  for all  $\nu < \delta$  and  $\eta < \kappa$ . It is now easy to see that  $j(F)(\kappa) = \langle a_\nu \mid \nu < \delta \rangle$ . Finally,  $\kappa \in x_0$  since  $\kappa = j(\text{id})(\kappa)$ , where  $\text{id} : \kappa \rightarrow \kappa$  such that  $\text{id}(\alpha) = \alpha$  is in  $m_0$ .  $\square$

**Remark 3.11.** If  $M$  is an  $\omega$ -special weak  $\kappa$ -model,  $j : M \rightarrow N$  is the ultrapower map by a weakly amenable  $M$ -ultrafilter on  $\kappa$ , and the  $x_i$  are defined as in Lemma 3.10, then  $j \upharpoonright m_i \subseteq x_i$  and so in particular,  $m_i \cap V_\kappa \subseteq x_i$ .

For proof, observe that if  $a \in m_i$ , then  $j(a) = j(f)(c_a)$ , where  $c_a : \kappa \rightarrow \{a\}$ , is in  $x_i$  and if  $a$  is as well in  $V_\kappa$ , then  $j(a) = a$ .

Now we argue that Ramsey and  $\alpha$ -iterable cardinals are characterized, as well, by the existence of elementary embeddings for  $\omega$ -special weak  $\kappa$ -models. For simplicity, we start with 1-iterable cardinals.

**Lemma 3.12.** *If  $\kappa$  is 1-iterable, then every  $A \subseteq \kappa$  is contained in an  $\omega$ -special weak  $\kappa$ -model  $M$  for which there exists a 1-good  $M$ -ultrafilter on  $\kappa$ .*

*Proof.* Fix a weak  $\kappa$ -model  $\overline{M}$  containing  $V_\kappa \cup \{A\}$  for which there exists a 1-good  $\overline{M}$ -ultrafilter  $\overline{U}$  on  $\kappa$ , and let  $\vec{j} : \overline{M} \rightarrow \overline{N}$  be the ultrapower map. First, note that  $\overline{U}$  remains a 1-good ultrafilter for the substructure  $H_{\kappa^+}^{\overline{M}} = \{B \in \overline{M} \mid |\text{Trcl}(B)|^{\overline{M}} \leq \kappa\}$ , which is itself a weak  $\kappa$ -model. So we will assume without loss that  $\overline{M} = H_{\kappa^+}^{\overline{M}}$ . Since  $\overline{M}$  satisfies that  $H_{\alpha^+}$  exists for all  $\alpha < \kappa$  by virtue of containing  $V_\kappa$ , it follows by elementarity that  $H_{\kappa^+}$  exists in  $\overline{N}$ . Since  $j$  is  $\kappa$ -powerset preserving by Lemma 2.2, it must be that  $\overline{M} = H_{\kappa^+}^{\overline{N}} \in \overline{N}$ . Working inside  $\overline{N}$ , we build a transitive elementary substructure  $m_0 \prec \overline{M}$  of size  $\kappa$  such that  $A \in m_0$  and  $m_0^{<\kappa} \subseteq m_0$ . Note that  $m_0 \in M$  since it is of hereditary size  $\kappa$ . Now suppose inductively that we are given  $m_i \prec \overline{M}$ , for some  $i < \omega$ , such that  $m_i \in \overline{M}$ . Note that  $\overline{U} \cap m_i \in \overline{M}$  by weak amenability. Working inside  $\overline{N}$ , we build a transitive elementary substructure  $m_{i+1} \prec \overline{M}$  of size  $\kappa$ , such that  $m_i, \overline{U} \cap m_i \in m_{i+1}$  and  $m_{i+1}^{<\kappa} \subseteq m_{i+1}$ . By construction, the union model  $M = \bigcup_{i \in \omega} m_i$  is an  $\omega$ -special weak- $\kappa$  model and  $U = \overline{U} \cap M$  is a weakly amenable  $M$ -ultrafilter. The ultrapower of  $M$  by  $U$  is well-founded as it embeds into  $\overline{N}$ . Thus, we found an  $\omega$ -special weak  $\kappa$ -model containing  $A$  for which there exists a 1-good  $M$ -ultrafilter on  $\kappa$ .  $\square$

**Lemma 3.13.** *If  $\kappa$  is Ramsey, then every  $A \subseteq \kappa$  is contained in an  $\omega$ -special weak  $\kappa$ -model  $M$  for which there exists a weakly amenable  $\kappa$ -intersecting  $M$ -ultrafilter on  $\kappa$ .*

*Proof.* Fix a weak  $\kappa$ -model  $\overline{M}$  containing  $V_\kappa \cup \{A\}$  for which there exists a weakly amenable  $\kappa$ -intersecting  $\overline{M}$ -ultrafilter  $\overline{U}$  on  $\kappa$ . We construct an  $\omega$ -special  $M \subseteq \overline{M}$  as in the proof of Lemma 3.12 and note that  $U = \overline{U} \cap M$  clearly remains  $\kappa$ -intersecting.  $\square$

Next, we argue that if  $\alpha > 1$  and every  $A$  is contained in a weak  $\kappa$ -model  $M$  for which there exists an  $\alpha$ -good  $M$ -ultrafilter on  $\kappa$ , then every  $A$  is in fact contained in an  $\omega$ -special such weak  $\kappa$ -model. More specifically, we will show that if  $\overline{M}$  is a weak  $\kappa$ -model with an  $\alpha$ -good  $\overline{M}$ -ultrafilter  $\overline{U}$  on  $\kappa$  and  $M \subseteq \overline{M}$  is constructed as in the proof of Lemma 3.12, then  $U = \overline{U} \cap M$  remains  $\alpha$ -good. The argument relies chiefly on the following lemma from [GW11] and we include the proof here for completeness of presentation.

**Lemma 3.14.** *Suppose that  $M_0$  is a weak  $\kappa_0$ -model with an  $\alpha$ -good  $M_0$ -ultrafilter  $U_0$  on  $\kappa_0$  and  $N_0 \prec M_0$  is another weak  $\kappa_0$ -model for which  $W_0 = U_0 \cap N_0$  remains weakly amenable, then  $W_0$  is  $\alpha$ -good.*

*Proof.* The strategy will be to verify that the iterated ultrapowers of  $N_0$  by  $W_0$  are well-founded by embedding the iteration by  $W_0$  into the iteration by  $U_0$ . Let

$$\{j_{\xi\gamma} : M_\xi \rightarrow M_\gamma \mid \xi < \gamma < \alpha\}$$

be the directed system of iterated ultrapowers of  $M_0$  with the associated sequence of ultrafilters  $\{U_\xi \mid \xi < \alpha\}$ . Also, let

$$\{h_{\xi\gamma} : N_\xi \rightarrow N_\gamma \mid \xi < \gamma < \alpha\}$$

be the (not necessarily well-founded) directed system of iterated ultrapowers of  $N_0$  with the associated sequence of ultrafilters  $\{W_\xi \mid \xi < \alpha\}$ . Let

$$S_0 = \{w \in N_0 \mid w \subseteq W_0\},$$

be the collection of all subsets of  $N_0$  that are elements of  $W_0$ , and define

$$S_\xi = \{h_{0\xi}(w) \mid w \in S_0\}.$$

We shall show that the following is a commutative diagram of elementary embeddings between transitive structures:

$$\begin{array}{ccccccccccc} M_0 & \xrightarrow{j_{01}} & M_1 & \xrightarrow{j_{12}} & M_2 & \xrightarrow{j_{23}} & \dots & \xrightarrow{j_{\xi\xi+1}} & M_{\xi+1} & \xrightarrow{j_{\xi+1\xi+2}} & \dots \\ \uparrow \rho_0 & & \uparrow \rho_1 & & \uparrow \rho_2 & & & & \uparrow \rho_{\xi+1} & & \\ N_0 & \xrightarrow{h_{01}} & N_1 & \xrightarrow{h_{12}} & N_2 & \xrightarrow{h_{23}} & \dots & \xrightarrow{h_{\xi\xi+1}} & N_{\xi+1} & \xrightarrow{h_{\xi+1\xi+2}} & \dots \end{array}$$

where

- (1)  $\rho_0$  is the identity map,
- (2)  $\rho_{\xi+1}([f]_{W_\xi}) = [\rho_\xi(f)]_{U_\xi}$ ,
- (3) if  $\lambda$  is a limit ordinal and  $t$  is a thread in the direct limit  $N_\lambda$  with domain  $[\beta, \lambda)$ , then  $\rho_\lambda(t) = j_{\beta\lambda}(\rho_\beta(t(\beta)))$ ,
- (4)  $W_\xi = \bigcup S_\xi$ , and
- (5)  $\rho_\xi(w) \subseteq U_\xi$  for all  $w \in S_\xi$ .

The argument is by induction on  $\xi$ . For the base case, note that  $\rho_0$  satisfies condition (5) and  $W_0$  satisfies condition (4) trivially, and  $N_0$  is transitive by assumption. Suppose inductively that  $\rho_\eta$  are elementary for  $\eta \leq \xi$ ,  $\rho_\xi : N_\xi \rightarrow M_\xi$  satisfies condition (5), and  $W_\xi = \bigcup S_\xi$ . A basic argument then shows that  $W_{\xi+1} = \bigcup S_{\xi+1}$ . Define  $\rho_{\xi+1}$  as in (2) above. To see that  $\rho_{\xi+1}$  is well-defined and elementary, note that if  $A \in W_\xi$ , then  $A \in w$  for some  $w \in S_\xi$  and so, in particular,  $\rho_\xi(A) \in U_\xi$ . The commutativity of the diagram is also clear. It remains to verify that  $\rho_{\xi+1}(w) \subseteq U_{\xi+1}$  for all  $w \in S_{\xi+1}$ . Fix  $w \in S_{\xi+1}$  and  $\bar{w} \in S_\xi$  such that  $w = h_{\xi\xi+1}(\bar{w}) = [c_{\bar{w}}]_{W_\xi}$ . Let  $\rho_\xi(\bar{w}) = v \subseteq U_\xi$  by the inductive assumption. Thus,

$$\rho_{\xi+1}(w) = [c_v]_{U_\xi} = j_{\xi\xi+1}(v) \subseteq U_{\xi+1}.$$

The last relation follows since  $v \subseteq U_\xi$ . This completes the inductive step. The limit case also follows easily.  $\square$

**Lemma 3.15.** *If  $\kappa$  is  $\alpha$ -iterable, then every  $A \subseteq \kappa$  is contained in an  $\omega$ -special weak  $\kappa$ -model  $M$  for which there exists an  $\alpha$ -good  $M$ -ultrafilter on  $\kappa$ .*

*Proof.* Fix  $A \subseteq \kappa$  and a weak  $\kappa$ -model  $\bar{M}$  containing  $V_\kappa \cup \{A\}$  for which there exists an  $\alpha$ -good  $\bar{M}$ -ultrafilter  $\bar{U}$  on  $\kappa$ . Let  $\bar{M} = H_{\kappa^+}^{\bar{M}}$  and  $\bar{U} = \bar{M} \cap \bar{U}$ . It is clear that  $\bar{U}$  is a weakly amenable  $\bar{M}$ -ultrafilter. Observe also that  $\bar{U}$  is  $\alpha$ -good since  $\bar{M}$  has all the same functions  $f : \kappa \rightarrow \bar{M}$  as  $\bar{M}$ , and therefore the iterated ultrapowers of  $\bar{M}$  by  $\bar{U}$  are restrictions of the corresponding iterated ultrapowers of  $\bar{M}$  by  $\bar{U}$ . We construct a special weak  $\kappa$ -model  $M \subseteq \bar{M}$  as in the proof of Lemma 3.12 with the weakly amenable  $M$ -ultrafilter  $U = M \cap \bar{U}$ , and observe that  $U$  is  $\alpha$ -good by Lemma 3.14. Thus, we found an  $\omega$ -special weak  $\kappa$ -model containing  $A$  for which there exists an  $\alpha$ -good  $M$ -ultrafilter on  $\kappa$ .  $\square$

For one of the lifting arguments, we will need that super Ramsey cardinals can be characterized by existence of embeddings for  $\kappa$ -special  $\kappa$ -models.

**Lemma 3.16.** *If  $\kappa$  is super Ramsey, then every  $A \subseteq \kappa$  is contained in a  $\kappa$ -special  $\kappa$ -model  $M \prec H_{\kappa^+}$  for which there exists a weakly amenable  $M$ -ultrafilter on  $\kappa$ .*

*Proof.* Fix a  $\kappa$ -model  $\bar{M} \prec H_{\kappa^+}$  containing  $A$  for which there exists a weakly amenable  $\bar{M}$ -ultrafilter  $U$  on  $\kappa$  and let  $j : \bar{M} \rightarrow \bar{N}$  be the ultrapower map. Construct  $M \prec \bar{M} = H_{\kappa^+}^{\bar{M}}$  in  $\bar{N}$ .  $\square$

In [AGH12], techniques were introduced for lifting entire iterations of embeddings, and we adopt these techniques here to iterations characterizing  $\alpha$ -iterable cardinals. Suppose that  $j : M \rightarrow N$  is the ultrapower by an  $\alpha$ -good  $M$ -ultrafilter  $U$  on  $\kappa$ ,  $\mathbb{P} \in M$  is a forcing notion and  $G \subseteq \mathbb{P}$  is  $M$ -generic. Suppose also that we are able to lift  $j$  to  $j : M[G] \rightarrow N[H]$  and the resulting lift is the ultrapower by an  $M[G]$ -ultrafilter  $W$  that is again weakly amenable. We shall show that if the  $N$ -generic filter  $H$  satisfies an additional technical condition (see Theorem 3.18), then the embeddings in the rest of the iteration lift automatically. The iteration composed of the lifts will turn out to be precisely the  $\alpha$ -iteration by  $W$ , confirming that  $W$  is  $\alpha$ -good. The argument relies on the following standard fact about lifts of ultrapower embeddings adapted here to models of  $\text{ZFC}^-$ .

**Lemma 3.17.** *Suppose that  $M$  is a weak  $\kappa$ -model,  $U$  is a 0-good  $M$ -ultrafilter on  $\kappa$ ,  $\mathbb{P} \in M$  is a forcing notion and  $G \subseteq \mathbb{P}$  is  $M$ -generic. If  $W \supseteq U$  is a 0-good  $M[G]$ -ultrafilter on  $\kappa$ , then the ultrapower by  $W$  lifts the ultrapower by  $U$  if and*

only if every function  $f : \kappa \rightarrow M$  in  $M[G]$  is  $W$ -equivalent to some  $g : \kappa \rightarrow M$  in  $M$ .

*Proof.* Suppose that  $j : M \rightarrow N$  is the ultrapower map by  $U$  and  $h : M[G] \rightarrow K$  is the ultrapower map by  $W$ . For the forward direction, suppose that  $h \upharpoonright M = j$  and  $\tau_G = f : \kappa \rightarrow M$  is a function in  $M[G]$ . Note that the range of  $f$  is contained in

$$A = \{a \in M \mid \exists p \in \mathbb{P} \exists \xi \in \kappa p \Vdash \tau(\check{\xi}) = \check{a}\}$$

which is a set in  $M$  by replacement. Thus,  $h(f)(\kappa) \in h(A) = j(A) \subseteq N$  and so  $h(f)(\kappa) = j(g)(\kappa) = h(g)(\kappa)$  for some  $g \in M$ , from which it follows that  $f$  is  $W$ -equivalent to  $g$ . For the backward direction, suppose that every  $f : \kappa \rightarrow M$  in  $M[G]$  is  $W$ -equivalent to some  $g : \kappa \rightarrow M$  in  $M$ . Observe that the map  $\varphi$  sending  $[f]_U \in N$  to  $[f]_W \in K$  is well-defined and an isomorphism of  $N$  with a substructure of  $K$ . It follows from our assumption that this substructure must be transitive and hence is equal to  $N$ , making  $\varphi$  the identity map. Thus,  $N \subseteq K$  and  $j(a) = [c_a]_U = [c_a]_W = h(a)$  for  $a \in M$ . We will now argue that  $h(G)$  is an  $N$ -generic filter for  $j(\mathbb{P})$  and  $K = N[h(G)]$ . Fix a dense set  $D \subseteq j(\mathbb{P})$  in  $N$ . Then  $D = j(f)(\kappa)$  for some  $f : \kappa \rightarrow M$  in  $M$  and we can assume that all  $f(\xi) = D_\xi$  are dense in  $\mathbb{P}$ . Since  $M[G]$  satisfies that  $G$  meets all dense sets in the range of  $f$ , by elementarity,  $K$  satisfies that  $h(G)$  meets all dense sets in the range of  $h(f) = j(f)$ , and so in particular, it meets  $h(f)(\kappa) = j(f)(\kappa) = D$ . Thus,  $h(G)$  is  $N$ -generic. Clearly  $N[h(G)] \subseteq K$  and so it remains to check that  $K \subseteq N[h(G)]$ . Fix  $a \in K$  and let  $a = h(f)(\kappa)$  for some  $f : \kappa \rightarrow M[G]$  in  $M[G]$ . Let  $t \in M[G]$  be some transitive set containing  $f$  and  $\kappa$  and let  $t = \tau_G$  for some  $\mathbb{P}$ -name  $\tau \in M$ . Let  $T = \{\sigma \mid \langle \sigma, p \rangle \in \tau\}$ , so that  $T[G] \supseteq t$ . By elementarity, in  $K$ ,  $h(t)$  is a transitive set containing  $h(f)$  and  $\kappa$ , and therefore  $a = h(f)(\kappa)$ . Also, by elementarity, in  $K$ ,  $h(t) \subseteq h(T)[h(G)] = j(T)[h(G)] \subseteq N[h(G)]$ .  $\square$

**Theorem 3.18.** *Suppose that  $M_0$  is an  $\omega$ -special weak  $\kappa_0$ -model as witnessed by the sequence  $\langle m_i \mid i < \omega \rangle$ ,  $\mathbb{P} \in m_0$  is a forcing notion and  $G_0 \subseteq \mathbb{P}$  is  $M_0$ -generic. Suppose further that the ultrapower  $j_{01} : M_0 \rightarrow M_1$  by an  $\alpha$ -good  $M_0$ -ultrafilter  $U_0$  on  $\kappa_0$  lifts to the ultrapower  $j_{01} : M_0[G_0] \rightarrow M_1[G_1]$  by a weakly amenable  $M_0[G_0]$ -ultrafilter  $W_0$  and that the  $N_0$ -generic filter  $G_1$  is additionally  $x_i$ -generic for all  $i \in \omega$  where  $x_i = \{j_{01}(f)(\kappa_0) \mid f : \kappa_0 \rightarrow m_i, f \in m_i\}$ . Then  $W_0$  is  $\alpha$ -good.*

*Proof.* By Lemma 3.17, since  $j_{01} : M_0[G_0] \rightarrow M_1[G_1]$  is a lift of  $j_{01} : M_0 \rightarrow M_1$ , it follows that every  $f : \kappa_0 \rightarrow M_0$  in  $M_0[G_0]$  is  $W_0$ -equivalent to some  $g : \kappa_0 \rightarrow M_0$  in  $M_0$ . The strategy will be to use elementarity to propagate this property that every new function is equivalent to an old function along the entire  $W_0$ -iteration, thereby showing that every embedding in the  $W_0$ -iteration is a lift of the corresponding embedding in the  $U_0$ -iteration and thus well-founded. It will follow that by lifting just the first step, we have already lifted the entire iteration. The difficulty we must surmount is that elementarity is guaranteed only for the language  $\mathcal{L} = \{\in\}$ , and it is not immediately clear how to express the “no new functions” property as a first-order statement without the predicate for the ultrafilter. This is where we will use the extra genericity of  $G_1$ , which will allow us to capture the property that every new function is equivalent to an old function as a schema of first-order statements in the language  $\mathcal{L} = \{\in\}$ .

Let  $w_i = W_0 \cap m_i[G_0]$ , and note that it is an element of  $M_0[G_0]$  by the weak amenability of  $W_0$ . The critical consequence of the hypothesis that the  $M_1$ -generic

filter  $G_1$  is  $x_i$ -generic for  $x_i = \{j_{01}(f)(\kappa_0) \mid f : \kappa_0 \rightarrow m_i, f \in m_i\}$  is that  $M_0[G_0]$  satisfies the following schema of first-order statements  $(^*i)$  for  $i \in \omega$ .

$(^*i)$  Every  $f : \kappa_0 \rightarrow m_i$  in  $m_i[G_0]$  is  $w_i$ -equivalent to some  $g : \kappa_0 \rightarrow m_i$  in  $m_i$ .

Let us prove this. First, recall that  $\kappa_0 \in x_i$  and  $j_{01} \upharpoonright m_i \subseteq x_i$  (Remark 3.11). Since  $G_1$  is  $x_i$ -generic, it follows by Remarks 3.5 (1) that  $x_i[G_1] \cap M_1 = x_i$ . Now suppose that  $f : \kappa_0 \rightarrow m_i$  is any function in  $m_i[G_0]$  and  $\dot{f} \in m_i$  is a  $\mathbb{P}$ -name such that  $(\dot{f})_{G_0} = f$ . Since  $\dot{f} \in m_i$ , it follows that  $j_{01}(\dot{f}) \in x_i$  and hence  $j_{01}(f)(\kappa_0) \in x_i[G_1]$ . Since  $M_0[G_0]$  satisfies that  $\text{ran}(f) \subseteq m_i$ , we have that  $\text{ran}(j_{01}(f)) \subseteq j_{01}(m_i) \subseteq M_1$ . Thus  $j_{01}(f)(\kappa_0) \in x_i[G_1] \cap M_1 = x_i$  and hence  $j_{01}(f)(\kappa_0) = j_{01}(g)(\kappa_0)$  for some  $g : \kappa_0 \rightarrow m_i$  in  $m_i$  completing the proof that  $(^*i)$  holds in  $M_0[G_0]$ . It is this schema of statements  $(^*i)$  that we will propagate along the iteration using elementarity.

Let  $\{j_{\xi\gamma} : M_\xi \rightarrow M_\gamma \mid \xi < \gamma < \alpha\}$  be the directed system of iterated ultrapowers of  $M_0$  with the associated sequence of ultrafilters  $\{U_\xi \mid \xi < \alpha\}$  and the critical sequence  $\{\kappa_\xi \mid \xi < \alpha\}$ . Clearly, each  $M_\xi = \bigcup_{i \in \omega} j_{0\xi}(m_i)$ .

By assumption, the first step of the  $W_0$ -iteration  $j_{01} : M_0[G_0] \rightarrow M_1[G_1]$  is the lift of the first step of the  $U_0$ -iteration  $j_{01} : M_0 \rightarrow M_1$ . Now assume inductively that every step of the  $W_0$ -iteration up to  $\xi$  is a lift of the corresponding step of the  $U_0$ -iteration:

$$M_0[G_0] \xrightarrow{j_{01}} M_1[G_1] \xrightarrow{j_{12}} \dots \longrightarrow M_\xi[G_\xi].$$

Let  $W_\xi$  be the  $M_\xi$ -ultrafilter associated to stage  $\xi$  of the iteration above. A standard argument shows that  $W_\xi = \bigcup_{i \in \omega} j_{0\xi}(w_i)$ . By applying elementarity to the schema of statements  $(^*i)$ , we shall argue that  $M_\xi[G_\xi]$  and  $W_\xi$  satisfies the characterization of Lemma 3.17 and therefore the next embedding in the  $W_0$ -iteration has the form  $j_{\xi\xi+1} : M_\xi[G_\xi] \rightarrow M_{\xi+1}[G_{\xi+1}]$ .

Applying  $j_{0\xi}$  to each statement  $(^*i)$ , we obtain the corresponding statement:

$(^*i)_\xi$  Every  $f : \kappa_\xi \rightarrow j_{0\xi}(m_i)$  in  $j_{0\xi}(m_i)[G_\xi]$  is  $j_{0\xi}(w_i)$ -equivalent to some  $g : \kappa_\xi \rightarrow j_{0\xi}(m_i)$  in  $j_{0\xi}(m_i)$ .

Since  $M_\xi = \bigcup_{i \in \omega} j_{01}(m_i)$ , we have that every  $f : \kappa_\xi \rightarrow M_\xi$  is in one such  $j_{0\xi}(m_i)$  and since  $W_\xi = \bigcup_{i \in \omega} j_{01}(w_i)$ , we have that  $j_{0\xi}(w_i) \subseteq W_\xi$ . It follows that  $M_\xi[G_\xi]$  and  $W_\xi$  satisfy the characterization of Lemma 3.17, and so the next embedding in the  $W_0$ -iteration has the form  $j_{\xi\xi+1} : M_\xi[G_\xi] \rightarrow M_{\xi+1}[G_{\xi+1}]$ . The limit stages follow easily.  $\square$

Using the original definition of Ramsey cardinals, it is not difficult to see that Ramsey cardinals are preserved by small forcing. The next lemma will provide a proof of this using embeddings. First, we need to make the following easy observation. We shall say that a sequence  $\vec{y} = \langle y_\eta \mid \eta < \alpha \rangle$  covers another sequence  $\vec{x} = \langle x_\xi \mid \xi < \beta \rangle$  if for every  $\xi < \alpha$ , there is  $\eta < \beta$  such that  $x_\xi = y_\eta$ .

**Remark 3.19.** Suppose that  $\mathbb{P}$  is a forcing notion with the  $\kappa$ -cc for an inaccessible  $\kappa$  and  $G \subseteq \mathbb{P}$  is  $V$ -generic. If  $\vec{x}$  is a sequence of elements of  $V$  of length less than  $\kappa$  in  $V[G]$ , then it is covered by a sequence  $\vec{y}$  of length less than  $\kappa$  in  $V$ .

**Lemma 3.20.** *Suppose that  $\kappa$  is inaccessible,  $M$  is a weak  $\kappa$ -model, and  $j : M \rightarrow N$  is the ultrapower map by a weakly amenable  $\kappa$ -intersecting  $M$ -ultrafilter on  $\kappa$ . If  $\mathbb{P} \in V_\kappa \cap M$  is a poset and  $G \subseteq \mathbb{P}$  is  $V$ -generic, then the lift  $j : M[G] \rightarrow N[G]$  is the ultrapower map by a weakly amenable  $\omega_1$ -intersecting  $M[G]$ -ultrafilter in  $V[G]$ .*

*Proof.* Suppose that  $j$  is the ultrapower map by an  $M$ -ultrafilter  $U$ . First, we argue that the lift  $j : M[G] \rightarrow N[G]$  is  $\kappa$ -powerset preserving, and so by Lemma 2.2, it will follow that it is the ultrapower map by a weakly amenable  $M[G]$ -ultrafilter, call it  $W$ . If  $B \subseteq \kappa$  in  $N[G]$ , then  $N$  has a nice  $\mathbb{P}$ -name  $\dot{B} \in H_{\kappa^+}^N$  such that  $(\dot{B})_G = B$ . It follows that  $\dot{B} \in M$  and hence  $B \in M[G]$ . Now we argue that  $W$  is  $\omega_1$ -intersecting. For this, it suffices to show that whenever  $\langle A_n \mid n \in \omega \rangle$  is a sequence of subsets of  $\kappa$  in  $V[G]$  such that each  $A_n \in M[G]$  and  $\kappa \in j(A_n)$ , then  $\bigcap_{n \in \omega} A_n \neq \emptyset$ . We start by fixing such a sequence  $\langle A_n \mid n \in \omega \rangle$ . For each  $n$ , we fix a  $\mathbb{P}$ -name  $\dot{A}_n \in M$  with  $(\dot{A}_n)_G = A_n$  and fix a condition  $p_n \in G$  such that  $p_n \Vdash \check{\kappa} \in j(\dot{A}_n)$  over  $N$ . For each  $n \in \omega$ , we let

$$S_n = \{\alpha < \kappa \mid p_n \Vdash \check{\alpha} \in \dot{A}_n \text{ over } M\}.$$

Individually, each  $S_n \in M$ , and, indeed,  $S_n \in U$  since  $\kappa \in j(S_n)$ . The sequence  $\langle S_n \mid n < \omega \rangle$  is an element of  $V[G]$ , but not necessarily an element of  $V$ . Nevertheless, by Remark 3.19, there is a sequence  $\langle T_\xi \mid \xi < \beta \rangle$  in  $V$  for some  $\beta < \kappa$  covering the sequence  $\langle S_n \mid n < \omega \rangle$  of  $V[G]$ . By thinning out if necessary, we may assume that all  $T_\xi$  are elements of  $U$ . Since  $U$  is  $\kappa$ -intersecting, we have some  $\gamma \in \bigcap_{\xi < \beta} T_\xi$ . Hence  $\gamma \in \bigcap_{n \in \omega} S_n$ . It follows that each  $p_n \Vdash \check{\gamma} \in \dot{A}_n$  over  $M$  and so  $\gamma \in (\dot{A}_n)_G = A_n$ . Thus,  $\bigcap_{n \in \omega} A_n \neq \emptyset$ , concluding the proof that  $W$  is  $\omega_1$ -intersecting.  $\square$

The next lemma states that for countably closed posets, the lift of the ultrapower map by an  $\omega_1$ -intersecting ultrafilter to the generic extension is again the ultrapower map by an  $\omega_1$ -intersecting ultrafilter.

**Lemma 3.21.** *Suppose that  $M$  is a weak  $\kappa$ -model and  $j : M \rightarrow N$  is the ultrapower map by an  $\omega_1$ -intersecting  $M$ -ultrafilter on  $\kappa$ . Suppose further that  $\mathbb{P} \in M$  is a countably closed forcing notion and  $G \subseteq \mathbb{P}$  is  $M$ -generic. If the ultrapower map  $j$  lifts to an elementary embedding  $j : M[G] \rightarrow N[j(G)]$ , then the lift  $j$  is the ultrapower map by an  $\omega_1$ -intersecting  $M[G]$ -ultrafilter in  $V[G]$ .*

*Proof.* Let  $j : M \rightarrow N$  be the ultrapower map by an  $M$ -ultrafilter  $U$ . We need to verify that whenever  $\langle A_n \mid n \in \omega \rangle$  is a sequence of subsets of  $\kappa$  in  $V[G]$  such that each  $A_n \in M[G]$  and  $\kappa \in j(A_n)$ , then  $\bigcap_{n \in \omega} A_n \neq \emptyset$ . We fix such a sequence  $\langle A_n \mid n \in \omega \rangle$ . We also fix  $\mathbb{P}$ -names  $\dot{A}_n \in M$  with  $(\dot{A}_n)_G = A_n$ , and note that the sequence of names  $\langle \dot{A}_n \mid n \in \omega \rangle \in V$  by countable closure of  $\mathbb{P}$ . Next, we choose a  $\mathbb{P}$ -name  $\dot{S}$  such that  $\mathbb{1} \Vdash \text{“}\dot{S} \text{ is an } \omega\text{-sequence”}$  and for all  $n$ ,  $\mathbb{1} \Vdash \dot{S}(\check{n}) = \dot{A}_n$  over  $V$  (the name is constructed from the sequence  $\langle \dot{A}_n \mid n \in \omega \rangle$ ). Now we suppose towards a contradiction that  $\bigcap_{n \in \omega} A_n = \emptyset$  and choose a condition  $p \in G$  such that  $p \Vdash \bigcap \dot{S} = \emptyset$  over  $V$ .

Since  $p \in G$ , we have that  $j(p) \in j(G)$ . In the filter  $j(G)$ , we choose an  $\omega$ -descending sequence of conditions below  $j(p)$ ,

$$j(p) \geq p_0 \geq p_1 \geq \cdots \geq p_n \geq \cdots,$$

such that  $p_n \Vdash \check{\kappa} \in j(\dot{A}_n)$  over  $N$ . Since  $j : M \rightarrow N$  is the ultrapower by  $U$ , we may fix for each  $n \in \omega$ , a function  $f_n : \kappa \rightarrow \mathbb{P}$  such that  $p_n = [f_n]_U$ . By countable closure of  $\mathbb{P}$ , the sequence of functions  $\langle f_n \mid n \in \omega \rangle \in V$ . Now observe that the following sets are in  $U$ :

- (1)  $S_n = \{\xi < \kappa \mid f_n(\xi) \Vdash \check{\xi} \in \dot{A}_n \text{ over } M\}$  for  $n \in \omega$ ,
- (2)  $T_n = \{\xi < \kappa \mid f_{n+1}(\xi) \leq f_n(\xi)\}$  for  $n \in \omega$ ,

$$(3) S = \{\xi < \kappa \mid f_0(\xi) \leq p\}.$$

Note that the sequences  $\langle S_n \mid n < \omega \rangle$  and  $\langle T_n \mid n < \omega \rangle$  are themselves elements of  $V$  by countable closure of  $\mathbb{P}$ . Thus, since  $U$  is  $\omega_1$ -intersecting in  $V$ , we may intersect all these sets to obtain an ordinal  $\alpha < \kappa$  such that:

- (1) for all  $n < \omega$ ,  $f_n(\alpha) \Vdash \check{\alpha} \in \dot{A}_n$  over  $M$ ,
- (2) for all  $n < \omega$ ,  $f_{n+1}(\alpha) \leq f_n(\alpha)$ ,
- (3)  $f_0(\alpha) \leq p$ .

Once again by closure of  $\mathbb{P}$ , we may fix a condition  $q$  below the descending  $\omega$ -sequence

$$p \geq f_0(\alpha) \geq f_1(\alpha) \geq \cdots \geq f_n(\alpha) \geq \cdots,$$

which consequently has the following properties:

- (1) for all  $n \in \omega$ ,  $q \Vdash \check{\alpha} \in \dot{A}_n$  over  $M$ ,
- (2)  $q \Vdash \bigcap \dot{S} = \emptyset$  over  $V$ .

Suppose that  $\bar{G} \subseteq \mathbb{P}$  is any  $V$ -generic filter containing  $q$ . Since for all  $n \in \omega$ ,  $q \Vdash \check{\alpha} \in \dot{A}_n$  over  $M$ , we have that  $\alpha \in (\dot{A}_n)_{\bar{G}}$  for all  $n \in \omega$  in  $V[\bar{G}]$ . On the other hand, since  $q \Vdash \bigcap \dot{S} = \emptyset$  over  $V$  and for all  $n \in \omega$ ,  $\mathbb{1} \Vdash \dot{S}(\check{n}) = \dot{A}_n$  over  $V$ , we have that  $\bigcap (\dot{S})_{\bar{G}} = \bigcap_{n \in \omega} (\dot{A}_n)_{\bar{G}} \neq \emptyset$  in  $V[\bar{G}]$ . Thus, we have reached a contradiction showing  $j : M[G] \rightarrow N[j(G)]$  is the ultrapower by an  $\omega_1$ -intersecting  $M[G]$ -ultrafilter.  $\square$

We end this section with a standard lemma to be used in later arguments.

**Lemma 3.22.** *Suppose that  $\mathbb{P}$  is an iteration of inaccessible length  $\kappa$  such that for all  $\alpha < \kappa$ ,  $\Vdash_{\alpha} \dot{Q}_{\alpha} \in \check{V}_{\kappa}$  and a direct limit is taken on a stationary set of stages below  $\kappa$ , then  $\mathbb{P}$  has size  $\kappa$  and the  $\kappa$ -cc.*

A more detailed exposition of standard lifting techniques, including Lemma 3.22, can be found in [Cum10].

#### 4. INDESTRUCTIBLE RAMSEY AND RAMSEY-LIKE CARDINALS

**4.1. Small forcing.** We start by showing that Ramsey and Ramsey-like cardinals cannot be destroyed by small forcing. Suppose  $\kappa$  is Ramsey or Ramsey-like and  $\mathbb{P}$  is small relative to  $\kappa$ . By replacing  $\mathbb{P}$  with an isomorphic copy, we can assume that  $\mathbb{P} \in V_{\kappa}$ . Suppose  $G \subseteq \mathbb{P}$  is  $V$ -generic. Note that if  $A \subseteq \kappa$  in  $V[G]$ , then it has a  $\mathbb{P}$ -name in  $H_{\kappa^+}$ . Thus, to verify that  $\kappa$  retains its large cardinal property in  $V[G]$ , we will show that every embedding  $j : M \rightarrow N$  of the type characterizing  $\kappa$  can be lifted to  $j : M[G] \rightarrow N[j(G)]$  and the lift retains the relevant properties. The same strategy will be employed in other indestructibility argument provided that every subset of  $\kappa$  in the forcing extension has a name in  $H_{\kappa^+}$ .

**Theorem 4.1.** *Ramsey and Ramsey-like cardinals are preserved by small forcing.*

*Proof.* First, suppose that  $\kappa$  is strongly Ramsey. Suppose that  $\mathbb{P}$  is small relative to  $\kappa$  and  $G \subseteq \mathbb{P}$  is  $V[G]$ . Let  $j : M \rightarrow N$  be a  $\kappa$ -powerset preserving embedding of  $\kappa$ -models. Since  $\mathbb{P} \in V_{\kappa}$ , we have that  $j(\mathbb{P}) = \mathbb{P}$  and  $j \restriction G = G$  and so  $j$  lifts to  $j : M[G] \rightarrow N[G]$  in  $V[G]$  by the lifting criterion (Lemma 3.1). The model  $M[G]$  is a  $\kappa$ -model in  $V[G]$  by the generic closure criterion (Lemma 3.8) since  $\mathbb{P}$  has the  $\kappa$ -cc. The argument that the lift is  $\kappa$ -powerset preserving is given in the proof of Lemma 3.20. Note that if  $M \prec H_{\kappa^+}$ , then  $M[G] \prec H_{\kappa^+}[G] = H_{\kappa^+}^{V[G]}$ ,



which gives the argument for super Ramsey cardinals. In the argument for Ramsey cardinals, we start with  $j : M \rightarrow N$  that is the ultrapower by a weakly amenable  $\kappa$ -intersecting  $M$ -ultrafilter and use Lemma 3.20 to conclude that the lift is the ultrapower by an  $\omega_1$ -intersecting  $M[G]$ -ultrafilter in  $V[G]$ .

In the argument for  $\alpha$ -iterable cardinals, we start with  $j : M \rightarrow N$  that is the ultrapower of an  $\omega$ -special weak- $\kappa$  model  $M$  by a weakly amenable  $\alpha$ -good  $M$ -ultrafilter  $U$ . Let  $\langle m_i \mid i < \omega \rangle$  witness that  $M$  is  $\omega$ -special and assume with loss that  $\mathbb{P} \in m_0$ . Let  $\langle x_i \mid i < \omega \rangle$ , defined as in Lemma 3.10, witness that  $N$  is  $\omega$ -almost special. To argue that the lift is the ultrapower by an  $\alpha$ -good  $M[G]$ -ultrafilter it suffices, by Theorem 3.18, to show that the  $V$ -generic filter  $G$  is  $x_i$ -generic for all  $i \in \omega$ . Recall that  $V_\kappa \cap m_i \subseteq x_i$  and so we have  $\mathbb{P} \in x_i$  and  $\mathbb{P} \subseteq x_i$ , from which it follows that any  $V$ -generic filter is  $x_i$ -generic.  $\square$

**4.2. Canonical forcing of the GCH.** Next, we show that Ramsey and Ramsey-like cardinals are indestructible by the canonical forcing of the GCH. Recall that if  $\kappa$  and  $\theta$  are cardinals, we call  $\text{Add}(\kappa, \theta)$  the poset, generalizing the Cohen poset, which adds  $\theta$  many subsets to  $\kappa$  with conditions of size less than  $\kappa$ . If  $\kappa$  is regular, then  $\text{Add}(\kappa, \theta)$  is  $<\kappa$ -closed and if  $2^{<\kappa} = \kappa$ , then it has the  $\kappa^+$ -cc. A forcing iteration  $\mathbb{P}$  is said to have *Easton support* if direct limits are taken at inaccessible cardinals and inverse limits are taken everywhere else. The canonical forcing of the GCH is an ORD-length Easton support iteration  $\mathbb{P}$  where at stage  $\alpha$  if  $\alpha$  is an infinite cardinal in  $V^{\mathbb{P}_\alpha}$ , we force with  $\text{Add}(\alpha^+, 1)$ , and with the trivial poset otherwise. To show that a Ramsey or a Ramsey-like cardinal  $\kappa$  is indestructible by the canonical forcing of the GCH, it suffices to argue that it is indestructible by the set forcing  $\mathbb{P}_\kappa$ , the iteration  $\mathbb{P}$  up to  $\kappa$ , since the tail of the forcing is  $\leq \kappa$ -closed and therefore cannot destroy these large cardinals. Since  $\kappa$  is a limit of inaccessible cardinals, by Lemma 3.22, the iteration  $\mathbb{P}_\kappa$  has size  $\kappa$  and the  $\kappa$ -cc. Note also that  $\mathbb{P}_\kappa \subseteq V_\kappa$  and therefore every subset of  $\kappa$  in an extension by  $\mathbb{P}_\kappa$  has a name in  $H_{\kappa^+}$ .

**Theorem 4.2.** *Ramsey and Ramsey-like cardinals are indestructible by the canonical forcing of the GCH.*

*Proof.* First, suppose  $\kappa$  is strongly Ramsey. Let  $G \subseteq \mathbb{P}_\kappa$  be  $V$ -generic. Let  $j : M \rightarrow N$  be a  $\kappa$ -powerset preserving embedding of  $\kappa$ -models. The poset  $\mathbb{P}_\kappa$  is an element of  $M$  automatically since it is a definable subset of  $V_\kappa$  which is in every  $\kappa$ -model.

In order to lift  $j$  to  $M[G]$ , by the lifting criterion, we require an  $N$ -generic filter for the poset  $j(\mathbb{P}_\kappa) = \mathbb{P}_{j(\kappa)}^N \cong \mathbb{P}_\kappa * \dot{\mathbb{P}}_{\text{tail}}$ , the canonical GCH iteration up to  $j(\kappa)$  of  $N$ , containing  $j \restriction G = G$ . The lifting criterion is satisfied by using the  $V$ -generic filter  $G$  for the  $\mathbb{P}_\kappa$  portion of  $\mathbb{P}_{j(\kappa)}^N$ . Let  $(\dot{\mathbb{P}}_{\text{tail}})_G = \mathbb{P}_{\text{tail}}$ , and note that it is  $\leq \kappa$ -closed in  $N[G]$  since the first poset in the iteration is  $\text{Add}(\kappa^+, 1)^{N[G]}$ . By the generic closure criterion,  $N[G]^{<\kappa} \subseteq N[G]$  in  $V[G]$ . Thus, by diagonalization criterion (1),  $V[G]$  has an  $N[G]$ -generic filter  $G_{\text{tail}}$  for  $\mathbb{P}_{\text{tail}}$ , and so we can lift  $j$  to  $j : M[G] \rightarrow N[G][G_{\text{tail}}]$ . Since  $\mathbb{P}_\kappa$  has the  $\kappa$ -cc,  $M[G]$  is a  $\kappa$ -model in  $V[G]$  by the generic closure criterion. The model  $N[G]$  satisfies that  $\mathbb{P}_{\text{tail}}$  is  $\leq \kappa$ -closed and therefore has the same subsets of  $\kappa$  as  $N[G][G_{\text{tail}}]$  and the models  $M[G]$  and  $N[G]$  have the same subsets of  $\kappa$  by the argument from the proof of Lemma 3.20. Note that if  $M \prec H_{\kappa^+}$ , then  $M[G] \prec H_{\kappa^+}[G] = H_{\kappa^+}^{V[G]}$ , which gives the argument for super Ramsey cardinals.

In the argument for Ramsey cardinals, we start with an  $\omega$ -special weak  $\kappa$ -model  $M$  such that  $V_\kappa \in M$ , and  $j : M \rightarrow N$  that is the ultrapower by a weakly amenable  $\omega_1$ -intersecting  $M$ -ultrafilter on  $\kappa$ . Let  $\langle m_i \mid i < \omega \rangle$  witness that  $M$  is  $\omega$ -special

and  $\langle x_i \mid i < \omega \rangle$  witness that  $N$  is  $\omega$ -almost special. Then  $N[G]$  together with the sequence  $\langle x_i[G] \mid i < \omega \rangle$  and the poset  $\mathbb{P}_{\text{tail}}$  satisfies the requirements of diagonalization criterion 2 (Lemma 3.6) in  $V[G]$ . Thus,  $V[G]$  has an  $N[G]$ -generic filter  $G_{\text{tail}}$  for  $\mathbb{P}_{\text{tail}}$ , and so we can lift  $j$  to  $j : M[G] \rightarrow N[G][G_{\text{tail}}]$ . The iteration  $\mathbb{P}_\kappa$  is countably closed since the first poset in it is  $\text{Add}(\omega_1, 1)$ , and so, by Lemma 3.21, the lift of  $j$  is the ultrapower by an  $\omega_1$ -intersecting  $M[G]$ -ultrafilter in  $V[G]$ .

In the argument for  $\alpha$ -iterable cardinals, we start with an  $\omega$ -special weak  $\kappa$ -model  $M$  such that  $V_\kappa \in M$ , and  $j : M \rightarrow N$  that is the ultrapower by a weakly amenable  $\alpha$ -good  $M$ -ultrafilter on  $\kappa$ . Let  $\langle m_i \mid i < \omega \rangle$  witness that  $M$  is  $\omega$ -special and let  $\langle x_i \mid i < \omega \rangle$  witness that  $N$  is almost  $\omega$ -special. Since  $V_\kappa \subseteq m_i$ , it follows that  $V_\kappa \subseteq x_i$ , and so in particular,  $\mathbb{P}_\kappa \subseteq x_i$ . Also, since  $\kappa \in x_i$  and  $\mathbb{P}_\kappa$  is definable from  $\kappa$ , it is in  $x_i$ . Thus, any  $V$ -generic filter, and  $G$  in particular, is  $x_i$ -generic for all  $i$ . Using that  $\mathbb{P}_\kappa$  has the  $\kappa$ -cc, by the generic closure criterion, we have that all  $x_i[G]^{<\kappa} \subseteq x_i[G]$  in  $N[G]$ . Finally, each  $x_i[G] \prec N[G]$  by Remarks 3.5(3). Thus, we can use diagonalization criterion (2) to obtain an  $N[G]$ -generic filter  $G_{\text{tail}}$  for  $\mathbb{P}_{\text{tail}}$  that is  $x_i[G]$ -generic for all  $i$ . By Remarks 3.5(2), the  $N$ -generic filter  $G * G_{\text{tail}}$  is then  $x_i$ -generic for all  $i$ . Thus, by Theorem 3.18,  $j : M[G] \rightarrow N[G][G_{\text{tail}}]$  is the ultrapower by an  $\alpha$ -good  $M[G]$ -ultrafilter.  $\square$

**4.3. The forcing  $\text{Add}(\kappa, \theta)$ .** To produce a forcing extension in which an  $\alpha$ -iterable, Ramsey, or strongly Ramsey  $\kappa$  is indestructible by  $\text{Add}(\kappa, \theta)$  for every cardinal  $\theta$ , we use the standard preparatory forcing to produce a forcing extension in which the large cardinal property of  $\kappa$  becomes indestructible by  $\text{Add}(\kappa, 1)$ , and argue that it is, in fact, already indestructible by  $\text{Add}(\kappa, \theta)$  for every  $\theta$ . A different argument for super Ramsey cardinals will be given later in the section. Let  $\mathbb{P}_\kappa$  be the Easton support forcing iteration of length  $\kappa$  where at stage  $\alpha$  if  $\alpha$  is a cardinal in  $V^{\mathbb{P}_\alpha}$ , we force with  $\text{Add}(\alpha, 1)$ , and with the trivial poset otherwise. We will start  $\mathbb{P}_\kappa$  with  $\text{Add}(\omega_1, 1)$  to ensure that it is countably closed. Note that, by Lemma 3.22,  $\mathbb{P}_\kappa$  has size  $\kappa$  and the  $\kappa$ -cc, and also that  $\mathbb{P}_\kappa \subseteq V_\kappa$ . The preparatory forcing will be  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1)$ . Let us argue that every  $\mathbb{P}_\kappa$ -name for an element of  $\text{Add}(\kappa, 1)$  is equivalent to one of at most  $\kappa$ -many  $\mathbb{P}_\kappa$ -names and therefore we may assume that  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1)$  is a subset of  $V_\kappa$  as well. Every element of  $\text{Add}(\kappa, 1)$  is a function  $f : \kappa \rightarrow 2$  with bounded support, and since  $\mathbb{P}_\kappa$  has the  $\kappa$ -cc, there is  $\gamma < \kappa$  such that  $\mathbb{P}_\kappa$  forces the support to be a subset of  $\gamma$ . It follows that we can associate every  $\mathbb{P}_\kappa$ -name for an element of  $\text{Add}(\kappa, 1)$  with a nice  $\mathbb{P}_\kappa$ -name for a subset of  $\gamma$  for some  $\gamma < \kappa$ , and there are only  $\kappa$  many of these. Thus, every subset of  $\kappa$  in an extension by  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1)$  has a name in  $H_{\kappa^+}$ .

**Theorem 4.3.** *If  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension in which it becomes indestructible by the forcing  $\text{Add}(\kappa, 1)$ .*

*Proof.* First, suppose that  $\kappa$  is strongly Ramsey. We start by arguing that  $\kappa$  remains strongly Ramsey after the preparatory forcing  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1)$ . Let  $G * g \subseteq \mathbb{P}_\kappa * \text{Add}(\kappa, 1)$  be  $V$ -generic. Let  $j : M \rightarrow N$  be a  $\kappa$ -powerset preserving embedding of  $\kappa$ -models. The poset  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1)$  is an element of  $M$  as it is a definable subset of  $V_\kappa \in M$ . We will lift  $j$  in two steps in  $V[G][g]$ , first to  $M[G]$ , and then to  $M[G][g]$ .

To lift  $j$  to  $M[G]$ , we require an  $N$ -generic filter for the poset  $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \text{Add}(\kappa, 1) * \dot{\mathbb{P}}_{\text{tail}}$  containing  $j''G = G$ . We satisfy the requirement of diagonalization criterion (1) by using the  $V$ -generic filter  $G * g$  for the  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1)$  portion of  $j(\mathbb{P}_\kappa)$ . Let  $(\dot{\mathbb{P}}_{\text{tail}})_{G * g} = \mathbb{P}_{\text{tail}}$ , and note that it is  $\leq \kappa$ -closed in  $N[G][g]$ . Since  $\mathbb{P}_\kappa$

has the  $\kappa$ -cc, by the generic closure criterion,  $N[G]^{<\kappa} \subseteq N[G]$  in  $V[G]$ . The poset  $\text{Add}(\kappa, 1)$  is  $<\kappa$ -closed in  $V[G]$  and so  $N[G]^{<\kappa} \subseteq N[G]$  in  $V[G][g]$ . Finally, by the ground closure criterion (Lemma 3.7), we have that  $N[G][g]^{<\kappa} \subseteq N[G][g]$  in  $V[G][g]$ . Thus, by diagonalization criterion (1),  $V[G][g]$  has an  $N[G][g]$ -generic filter  $G_{\text{tail}}$  and so we can lift  $j$  to  $j : M[G] \rightarrow N[j(G)]$ , with  $j(G) = G * g * G_{\text{tail}}$ .

Next, we lift  $j$  to  $M[G][g]$  in  $V[G][g]$ . For this portion of the lift, we require an  $N[j(G)]$ -generic filter for the poset  $j(\text{Add}(\kappa, 1)) = \text{Add}(j(\kappa), 1)^{N[j(G)]}$  containing  $j * g = g$ . By construction, we have ensured that  $g \in N[j(G)]$ , from which it follows that  $B = \bigcup g$  is a condition in  $\text{Add}(j(\kappa), 1)^{N[j(G)]}$ . Note that any  $N[j(G)]$ -generic filter containing  $B$  satisfies the lifting criterion, making  $B$  a *master condition* for the lift. The poset  $\text{Add}(j(\kappa), 1)^{N[j(G)]}$  is  $\leq\kappa$ -closed in  $N[j(G)]$ . We argued previously that  $N[G][g]^{<\kappa} \subseteq N[G][g]$  in  $V[G][g]$  and so, by the ground closure criterion,  $N[G][g][G_{\text{tail}}]^{<\kappa} \subseteq N[G][g][G_{\text{tail}}]$ . Thus, by diagonalization criterion (1) applied below  $B$ ,  $V[G][g]$  has an  $N[j(G)]$ -generic filter  $\bar{g}$  containing  $B$ , and so we are able to lift  $j$  to  $j : M[G][g] \rightarrow N[j(G * g)]$ , with  $j(G * g) = G * g * G_{\text{tail}} * \bar{g}$ , in  $V[G][g]$ .

By an identical argument as for  $N[G][g]$ , we conclude that  $M[G][g]$  is a  $\kappa$ -model in  $V[G][g]$ . A set  $C \subseteq \kappa$  in  $N[j(G * g)]$  could not have been added by the  $\leq\kappa$ -closed  $\mathbb{P}_{\text{tail}} * \text{Add}(j(\kappa), 1)$  and hence it is already in  $N[G][g]$  and the rest of the argument to verify  $\kappa$ -powerset preservation is as before. Note that if  $M \prec H_{\kappa^+}$ , then  $M[G][g] \prec H_{\kappa^+}[G][g] = H_{\kappa^+}^{V[G][g]}$ , which gives the argument for super Ramsey cardinals.

Finally, we argue that  $\kappa$  remains strongly Ramsey after forcing with  $\text{Add}(\kappa, 1)$  over  $V[G][g]$ . Here it remains to observe that the poset  $\text{Add}(\kappa, 1)$  is unchanged in any forcing extension by the  $<\kappa$ -closed  $\text{Add}(\kappa, 1)$  and so forcing with  $\text{Add}(\kappa, 1)$  over  $V[G]$  followed by forcing with  $\text{Add}(\kappa, 1)$  again is equivalent to forcing with  $\text{Add}(\kappa, 1) \times \text{Add}(\kappa, 1) \cong \text{Add}(\kappa, 1)$ .

In the argument for Ramsey cardinals, we start with an  $\omega$ -special weak  $\kappa$ -model  $M$  such that  $V_\kappa \in M$  and  $j : M \rightarrow N$  that is the ultrapower by a weakly amenable  $\omega_1$ -intersecting  $M$ -ultrafilter on  $\kappa$ . Let  $\langle m_i \mid i < \omega \rangle$  witness that  $M$  is  $\omega$ -special and  $\langle x_i \mid i < \omega \rangle$  witness that  $N$  is  $\omega$ -almost special. Then  $N[G][g]$  together with the sequence  $\langle x_i[G][g] \mid i < \omega \rangle$  and the poset  $\mathbb{P}_{\text{tail}}$  satisfies the requirements of diagonalization criterion (2) in  $V[G][g]$ . Thus,  $V[G][g]$  has an  $N[G]$ -generic filter  $G_{\text{tail}}$  for  $\mathbb{P}_{\text{tail}}$ , and so we can lift  $j$  to  $j : M[G] \rightarrow N[j(G * g)]$  with  $j(G) = G * g * G_{\text{tail}}$ . Next, we use diagonalization criterion (2) with the model  $N[j(G)]$  and the sequence  $\langle x_i[j(G)] \mid i < \omega \rangle$  to obtain an  $N[j(G)]$ -generic filter  $\bar{g}$  for  $\text{Add}(j(\kappa), 1)^{N[j(G)]}$  containing  $B$ . Finally, it remains to observe that the two-step iteration  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1)$  is countably closed, and so by Lemma 3.21, the lift  $j : M[G][g] \rightarrow N[j(G)][\bar{g}]$  is the ultrapower by an  $\omega_1$ -intersecting  $M[G][g]$ -ultrafilter in  $V[G][g]$ .

In the argument for  $\alpha$ -iterable cardinals, we start with an  $\omega$ -special weak  $\kappa$ -model  $M$  such that  $V_\kappa \in M$ , and  $j : M \rightarrow N$  that is the ultrapower by a weakly amenable  $\alpha$ -good  $M$ -ultrafilter on  $\kappa$ . Let  $\langle m_i \mid i < \omega \rangle$  witness that  $M$  is  $\omega$ -special and let  $\langle x_i \mid i < \omega \rangle$  witness that  $N$  is almost  $\omega$ -special. Since  $V_\kappa \subseteq m_i$ , it follows that  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1) \subseteq x_i$ . The poset  $\mathbb{P}_\kappa * \text{Add}(\kappa, 1)$  is definable from  $\kappa$  and therefore is in  $x_i$  as well. Thus,  $G * g$  is  $x_i$ -generic for all  $i$ . Exactly following the argument above, which showed  $N[G][g]^{<\kappa} \subseteq N[G][g]$  in  $V[G][g]$ , we conclude that each  $x_i[G][g]^{<\kappa} \subseteq x_i[G][g]$  in  $N[G][g]$ . Each  $x_i[G][g] \prec N[G][g]$  by Remarks 3.5(3). Thus, we can use diagonalization criterion (2) to obtain  $N[G][g]$ -generic filter  $G_{\text{tail}}$

for  $\mathbb{P}_{\text{tail}}$  that is  $x_i[G][g]$ -generic for all  $i$ . By Remarks 3.5(2), the  $N$ -generic filter  $j(G) = G * g * G_{\text{tail}}$  is  $x_i$ -generic for all  $i$ . It now follows, by the ground closure criterion, that each  $x_i[j(G)]^{<\kappa} \subseteq x_i[j(G)]$  in  $N[j(G)]$ . Since  $j(\kappa) \in x_i$  (Remark 3.11),  $\text{Add}(j(\kappa), 1)^{N[j(G)]} \in x_i[j(G)]$ . So we can use diagonalization criterion 2 with the model  $N[j(G)]$  and the sequence  $\langle x_i[j(G)] \mid i < \omega \rangle$  to obtain an  $N[j(G)]$ -generic filter  $\bar{g}$  for  $\text{Add}(j(\kappa), 1)^{N[j(G)]}$  containing  $B$  that is  $x_i[j(G)]$ -generic for all  $i$ . The combined  $N$ -generic filter  $j(G * g) = G * g * G_{\text{tail}} * \bar{g}$  is therefore  $x_i$ -generic for all  $i$ , and hence, by Theorem 3.18, the final lift  $j : M[G][g] \rightarrow N[j(G * g)]$  is the ultrapower by an  $\alpha$ -good  $M[G][g]$ -ultrafilter.  $\square$

If  $p \in \text{Add}(\kappa, \theta)$ , define that the support,  $\text{supp}(p)$ , of  $p$  is the collection of slices  $\xi < \theta$  mentioned in  $p$ . If  $S \subseteq \theta$ , we can factor  $\text{Add}(\kappa, \theta)$  as  $\text{Add}(\kappa, \theta) \cong \mathbb{Q}_S \times \mathbb{Q}_{\theta \setminus S}$ , where

$$\mathbb{Q}_S = \{p \in \text{Add}(\kappa, \theta) \mid \text{supp}(p) \subseteq S\}$$

and

$$\mathbb{Q}_{\theta \setminus S} = \{p \in \text{Add}(\kappa, \theta) \mid \text{supp}(p) \subseteq \theta \setminus S\},$$

and if  $G \subseteq \text{Add}(\kappa, \theta)$  is  $V$ -generic, we can correspondingly factor  $G \cong G_S \times G_{\theta \setminus S}$ .

**Theorem 4.4.** *If  $\kappa$  is  $\alpha$ -iterable, Ramsey, or strongly Ramsey, then there is a forcing extension in which its large cardinal property becomes indestructible by the forcing  $\text{Add}(\kappa, \theta)$  for every cardinal  $\theta$ .*

*Proof.* First, suppose that  $\kappa$  is strongly Ramsey. Using Theorem 4.3, we may assume that  $\kappa$  is indestructible by  $\text{Add}(\kappa, 1)$ . We shall argue that, in this case, it is also indestructible by  $\text{Add}(\kappa, \theta)$  for every cardinal  $\theta$ . It is immediate that  $\kappa$  is indestructible as well by  $\text{Add}(\kappa, \kappa) \cong \text{Add}(\kappa, 1)$ . Let  $G \subseteq \text{Add}(\kappa, \theta)$  be  $V$ -generic and fix  $A \subseteq \kappa$  in  $V[G]$  and a nice  $\text{Add}(\kappa, \theta)$ -name  $\dot{A} = \bigcup_{\alpha \in \kappa} \{\dot{\alpha}\} \times \mathcal{A}_\alpha$  for  $A$ , where  $\mathcal{A}_\alpha$  are antichains of  $\text{Add}(\kappa, \theta)$ . Since the poset  $\text{Add}(\kappa, \theta)$  has the  $\kappa^+$ -cc,  $\bigcup_{\alpha < \kappa} \mathcal{A}_\alpha$  has size at most  $\kappa$ . Let  $S$  be the union of all  $\text{supp}(p)$  for  $p \in \bigcup_{\alpha < \kappa} \mathcal{A}_\alpha$ . Observe that  $S$  has size at most  $\kappa$ . It follows that  $(\dot{A})_G = (\dot{A})_{G_S}$  is an element of  $V[G_S]$  (as defined above) where  $\kappa$  remains strongly Ramsey since  $\mathbb{Q}_S \cong \text{Add}(\kappa, \kappa)$ . Thus, in  $V[G_S] \subseteq V[G]$ , the set  $A$  is contained a  $\kappa$ -model  $M$  for which there exists a  $\kappa$ -powerset preserving embedding  $j : M \rightarrow N$ . Finally, we observe that  $M$  continues to be a  $\kappa$ -model in  $V[G] = V[G_S][G_{\theta \setminus S}]$  since forcing with  $\mathbb{Q}_S$  does not add new  $< \kappa$ -sequences.

In the argument for Ramsey cardinals, note that if  $U$  is a weakly amenable  $\omega_1$ -intersecting  $M$ -ultrafilter for a weak  $\kappa$ -model  $M$  in  $V[G_S]$ , then it continues to be so in  $V[G_S][G_{\theta \setminus S}]$  because the further extension doesn't add new  $\omega$ -sequences. In the argument for  $\alpha$ -iterable cardinals, note that being a weakly amenable  $\alpha$ -good  $M$ -ultrafilter for a weak  $\kappa$ -model  $M$  is absolute for transitive models.  $\square$

The above argument does not readily generalize to the case of super Ramsey cardinals. If  $\kappa$  was a super Ramsey cardinal indestructible by  $\text{Add}(\kappa, 1)$  in  $V$ , then  $\kappa$  will be super Ramsey in  $V[G_S]$ , but it may not be super Ramsey in the further  $V[G]$  because  $H_{\kappa^+}^{V[G]}$  is different from  $H_{\kappa^+}^{V[G_S]}$ . The argument we give to show that super Ramsey cardinals can be made indestructible by all forcing  $\text{Add}(\kappa, \theta)$  is more complicated. We start with a lemma.

**Lemma 4.5.** *Suppose  $\kappa$  is super Ramsey,  $2^\kappa = \kappa^+$ , and  $\theta > \kappa^+$  is a cardinal. If  $G \subseteq \text{Add}(\kappa, \theta)$  is  $V$ -generic and  $s \in H_{\kappa^+}^{V[G]}$ , then there is  $S \subseteq \theta$  of size  $\kappa^+$  such that  $s \in H_{\kappa^+}^{V[G_S]} \prec H_{\kappa^+}^{V[G]}$ .*

*Proof.* We work in  $V[G]$ . Let  $M_0$  be an elementary substructure of  $H_{\kappa^+}^{V[G]}$  of size  $\kappa^+$  such that  $H_{\kappa^+}^V \subseteq M_0$  and  $s \in M_0$ . If  $A \subseteq \kappa$  is in  $M_0$ , then  $A = \sigma_G$  for some nice  $\text{Add}(\kappa, \theta)$ -name  $\sigma \in V$ . Using a well-ordering of a large enough rank initial segment of  $V$ , we can choose a unique such  $\sigma_A$  to correspond to every  $A \subseteq \kappa$  in  $M_0$ . Since  $\text{Add}(\kappa, \theta)$  has the  $\kappa^+$ -cc, the union,  $\text{supp}(\sigma_A)$ , of  $\text{supp}(p)$  over  $p$  appearing in  $\sigma_A$  has size at most  $\kappa$ . Let  $S_0$  be the union of  $\text{supp}(\sigma_A)$  over all  $A \subseteq \kappa$  in  $M_0$ . We can assume without loss that  $S_0 \in V$  since otherwise, we can cover it with a set of size  $\kappa^+$  that is in  $V$  (using that  $\text{Add}(\kappa, \theta)$  has the  $\kappa^+$ -cc). Let  $M_1$  be an elementary substructure of  $H_{\kappa^+}^{V[G]}$  extending  $M_0$  and  $H_{\kappa^+}^{V[G_{S_0}]}$ . Define  $S_1$  for  $M_1$  analogously to  $S_0$  and continue this process to build a sequence  $\{\langle M_\xi, S_\xi \rangle \mid \xi < \kappa^+\}$  as follows. Given  $\langle M_\xi, S_\xi \rangle$ , we construct  $M_{\xi+1}$  and  $S_{\xi+1}$ , by applying the same procedure we used to obtain  $\langle M_1, S_1 \rangle$  from  $\langle M_0, S_0 \rangle$ . At limit stages  $\lambda$ , we let  $M_\lambda = \bigcup_{\xi < \lambda} M_\xi$  and  $S_\lambda = \bigcup_{\xi < \lambda} S_\xi$ . Let  $M = \bigcup_{\xi < \kappa^+} M_\xi$  and  $S = \bigcup_{\xi < \kappa^+} S_\xi$ .

Consider  $V[G_S]$ . We would like to argue that  $M$  is precisely  $H_{\kappa^+}^{V[G_S]}$ . In one direction, suppose that  $A \subseteq \kappa$  is in  $V[G_S]$ . Note that there must be  $\xi < \kappa^+$  such that  $A \in V[G_{S_\xi}]$  and hence  $A \in H_{\kappa^+}^{V[G_{S_\xi}]} \subseteq M$ . For the other direction, it suffices to show that  $M \subseteq V[G_S]$ . So suppose  $A \subseteq \kappa$  is in  $M$ . Then at the stage  $\xi$  when  $A$  was added, we added the coordinates to  $S_\xi$  such that  $A = \sigma_{G_{S_\xi}}$  for some  $\text{Add}(\kappa, \theta)$ -name  $\sigma$  in  $V$ . Thus,  $A \in V[G_S]$ .  $\square$

It follows from Lemma 4.5, using an argument analogous to the proof of Theorem 4.4, that to show that a super Ramsey  $\kappa$  can be made indestructible by all  $\text{Add}(\kappa, \theta)$ , it suffices to show that it can be made indestructible by  $\text{Add}(\kappa, \kappa^+)$ . Note that if  $\kappa$  is super Ramsey and  $2^\kappa = \delta > \kappa^+$ , we can always force to collapse  $\delta$  to  $\kappa^+$  while preserving that  $\kappa$  is super Ramsey because  $\text{Coll}(\kappa^+, \delta)$  is  $\leq \kappa$ -closed.

Let  $\mathbb{P}_\kappa$  be the Easton support forcing iteration of length  $\kappa$  where at stage  $\alpha$  if  $\alpha$  is a cardinal in  $V^{\mathbb{P}_\alpha}$ , we force with  $\text{Add}(\alpha, \alpha^+)$ , and with the trivial poset otherwise. Note that, by Lemma 3.22,  $\mathbb{P}_\kappa$  has size  $\kappa$  and the  $\kappa$ -cc, and also that  $\mathbb{P}_\kappa \subseteq V_\kappa$ . The preparatory forcing for super Ramsey cardinals  $\kappa$  will be  $\mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^+)$ . The argument that follows uses ideas from [Lev95] for building a generic filter from a sequence of increasingly powerful master conditions.

**Theorem 4.6.** *If  $\kappa$  is super Ramsey, then there is a forcing extension in which it becomes indestructible by the forcing  $\text{Add}(\kappa, \kappa^+)$ .*

*Proof.* It suffices to argue that  $\kappa$  remains super Ramsey after the preparatory forcing  $\mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^+)$ . Let  $G * H \subseteq \mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^+)$  be  $V$ -generic. Fix  $A \subseteq \kappa$  in  $V[G]$  and a  $\mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^+)$ -name  $\dot{A}$  for  $A$ . The union of all  $\text{supp}(p)$  over  $p$  appearing in  $\dot{A}$  must be bounded below  $\kappa^+$ , and therefore without loss (using an automorphism argument) we can assume that all conditions in  $\dot{A}$  come from the first coordinate of  $\text{Add}(\kappa, \kappa^+)$ . In other words,  $\dot{A}$  is an  $\text{Add}(\kappa, 1)$ -name. So we can fix a  $\kappa$ -model  $M \prec H_{\kappa^+}$  containing  $\dot{A}$  for which there exists a  $\kappa$ -powerset preserving embedding  $j : M \rightarrow N$  of  $\kappa$ -models. The poset  $\mathbb{P}_\kappa$  is an element of  $M$  as it is a definable subset of  $V_\kappa \in M$ . For the argument below, we need to additionally assume that  $M$  is  $\kappa$ -special, using Lemma 3.16.

First, we lift  $j$  to  $M[G]$ . As usual, it suffices to find an  $N[G]$ -generic filter for the poset  $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \text{Add}(\kappa, \kappa^+) * \mathbb{P}_{\text{tail}}$  containing  $j \restriction G = G$ . We will use here the filter  $G * h * G_{\text{tail}}$ , where  $h$  is the restriction of  $H$  to  $(\kappa^+)^N$  and  $G_{\text{tail}}$  comes from diagonalization criterion (1). Note that  $h$  is  $N[G]$ -generic for  $\text{Add}(\kappa, \kappa^+)^{N[G]}$  because  $N[G]$  is a  $\kappa$ -model in  $V[G]$  and therefore  $\text{Add}(\kappa, \kappa^+)^{N[G]}$  is precisely the restriction of  $\text{Add}(\kappa, \kappa^+)^{V[G]}$  to coordinates below  $(\kappa^+)^N$ . Thus, we have lifted  $j$  to  $j : M[G] \rightarrow N[j(G)]$  with  $j(G) = G * h * G_{\text{tail}}$ .

The poset  $\mathbb{Q} = \text{Add}(\kappa, \kappa^+)^{N[G]}$  is not an element of  $M[G]$ , but it is definable there as the collection of all partial functions  $p : \kappa \times \alpha \rightarrow 2$  for some  $\alpha \in \text{ORD}^M = (\kappa^+)^N$  with domain of size less than  $\kappa$ . From the point of view of  $M[G]$ ,  $\mathbb{Q}$  is a class partial order with set-sized antichains. Because the antichains are all elements of  $M[G]$ , the definability and truth lemmas hold for  $\mathbb{Q}$  in  $M[G]$ . This conclusion needs a combination of the following two results. The first result is that if  $M \models \text{ZFC}^-$  and  $\mathbb{P}$  is a class partial order of  $M$  having the property that every sub-class  $A \subseteq \mathbb{P}$  has a subset  $a \subseteq A$  which is pre-dense in  $A$ , then  $M$  has a class set-complete Boolean algebra into which  $\mathbb{P}$  densely embeds. Here is how we construct the Boolean algebra  $\mathbb{B}$ . Given  $X \subseteq \mathbb{P}$ , we define that  $X^\perp = \{p \in \mathbb{P} \mid p \perp X\}$  (where the notation  $p \perp X$  means that  $p$  is incompatible to all elements of  $X$ ). If  $x$  and  $y$  are subsets of  $\mathbb{P}$  in  $M$ , we define that  $x \sim y$  whenever  $x^\perp = y^\perp$  (note that  $x^\perp$  and  $y^\perp$  may be classes). It is easily seen that this is an equivalence relation and its equivalence classes will be the elements of  $\mathbb{B}$ . For the Boolean operations on  $\mathbb{B}$ , we define that  $\neg[x] = [a]$ , where  $[a]$  is a set pre-dense in  $x^\perp$  and  $[x] \wedge [y] = [x \cup y]$ . It is not difficult but tedious to check that the result is a set-complete Boolean algebra definable in  $M$  and  $\mathbb{P}$  densely embeds into  $\mathbb{B}$  by mapping  $p$  to the class  $[a_p]$ , where  $a_p$  is pre-dense in  $\{q \in \mathbb{P} \mid q \leq p\}$ . Now we use a recent result of [PH15] (Theorem 1.3) showing that the existence of a set-complete Boolean completion implies that definability and truth lemmas hold.<sup>7</sup>

Similarly the poset  $\text{Add}(\kappa, \kappa^+)^{V[G]}$  is a class partial order of  $H_{\kappa^+}[G]$  for which the definability and truth lemmas hold. Now using the fact that  $h$  is the restriction of  $H$  to  $\mathbb{Q}$ , it follows that  $M[G][h] \prec H_{\kappa^+}[G][H] = H_{\kappa^+}^{V[G][H]}$ . Thus, it remains to show that we can lift  $j$  to  $M[G][h]$  and argue that the lift is  $\kappa$ -powerset preserving.

Let  $\bar{\mathbb{Q}}$  be the class partial in  $N[j(G)]$  consisting of all partial functions  $p : j(\kappa) \times \alpha \rightarrow 2$  for some  $\alpha \in \text{ORD}^N$  with domain of size less than  $j(\kappa)$ . The poset  $\bar{\mathbb{Q}}$  has only set-sized antichains. Diagonalization criterion (1) holds for the pair  $\mathbb{Q}$  and  $\bar{\mathbb{Q}}$  because definability and truth lemmas hold for both. Thus, to lift  $j$  to  $M[G][h]$ , we need an  $N[j(G)]$ -generic filter  $\bar{h}$  for  $\bar{\mathbb{Q}}$  such that  $j \restriction h \subseteq \bar{h}$ . Let  $h(\xi)$  be the subset of  $\kappa$  on coordinate  $\xi$  of  $h$ . Observe that  $j \restriction h$  consists of the same subsets of  $\kappa$  as  $h$ , but now  $h(\xi)$  sits on coordinate  $j(\xi)$ . Let  $\langle m_\xi \mid \xi < \kappa \rangle$  witness that  $M$  is  $\kappa$ -special. Let  $\delta_\xi = m_\xi \cap (\kappa^+)^N$ . Observe that each  $\delta_\xi$  is an initial segment of  $(\kappa^+)^N$  since  $\kappa \in m_\xi$  and  $m_\xi \prec M$ , and  $(\kappa^+)^N$  is the union of the increasing sequence of the  $\delta_\xi$ . Let  $h_\xi = h \cap m_\xi$  be  $h$  restricted to coordinates in  $\delta_\xi$ , and let  $p_\xi = j \restriction h_\xi$ . Next, observe that  $p_\xi$  can be constructed from  $h$  and  $j \restriction \delta_\xi$ , both of which are elements of  $N[j(G)]$ , and so, in particular, each  $p_\xi$  is an element of  $\bar{\mathbb{Q}}$ . The conditions  $p_\xi$  will be the increasingly more powerful master conditions for the lift.

<sup>7</sup>This argument was worked out jointly with Joel David Hamkins and Philip Schlicht.

We are now going to construct an  $N[j(G)]$ -generic filter  $\bar{h}$  compatible with  $j \restriction h$  in  $\kappa$ -many steps. Because  $j$  is an ultrapower map, the ordinals  $j(\delta_\xi)$  are unbounded in  $\text{ORD}^N$ . Thus, every maximal antichain  $A$  of  $\mathbb{Q}$  is already contained in some  $\text{Add}(j(\kappa), j(\delta_\xi))^{N[G]}$ . In  $V[G][H]$ , enumerate all maximal antichains of  $\mathbb{Q}$  in  $N[j(G)]$  in a  $\kappa$ -sequence  $\langle A_\xi \mid \xi < \kappa \rangle$ . We start with  $A_0$ , which must be contained in some  $\text{Add}(j(\kappa), j(\delta_{\xi_0}))^{N[G]}$ . The condition  $p_{\xi_0}$  is an element of  $\text{Add}(j(\kappa), j(\delta_{\xi_0}))$ , and hence we can choose  $q_0 \in \text{Add}(j(\kappa), j(\delta_{\xi_0}))$  below  $p_{\xi_0}$  and some element of  $A_0$ . Note that  $q_0$  is compatible with  $j \restriction h$ . Now we suppose inductively that we have defined an increasing sequence of conditions  $\langle q_\xi \mid \xi < \alpha \rangle$  such that each  $q_\xi$  is compatible to  $j \restriction h$  and  $q_\xi$  has above it an element of  $A_\xi$ . Note that the sequence  $\langle q_\xi \mid \xi < \alpha \rangle$  is an element of the  $\kappa$ -model  $N[j(G)]$ . Let  $q_\alpha^*$  be the union of the  $q_\xi$ . Choose  $\text{Add}(j(\kappa), j(\delta_{\xi_\alpha}))$  containing  $A_\alpha$  and  $q_\alpha^*$ . Let  $q_\alpha$  be any condition below  $q_\alpha^*$ ,  $p_\alpha$ , and some element of  $A_\alpha$ . It should be clear that the sequence  $\langle q_\xi \mid \xi < \kappa \rangle$  generates an  $N[j(G)]$ -generic filter  $\bar{h}$  that is compatible with  $j \restriction h$  and therefore contains all elements of  $j \restriction h$ . Finally, we note that the argument to show that the lift is  $\kappa$ -powerset preserving is standard.  $\square$

**Corollary 4.7.** *If  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension preserving this in which  $2^\kappa > \kappa^+$ . Indeed, there is such a forcing extension in which  $\kappa$  is the first cardinal at which the GCH fails.*

*Proof.* For the moreover part, first force the GCH to hold, then perform the appropriate preparatory forcing to make the cardinal indestructible by all  $\text{Add}(\kappa, \theta)$ , which preserves the GCH (this is a name counting argument), followed by  $\text{Add}(\kappa, \kappa^{++})$ .  $\square$

Thus, unlike a measurable cardinal, a Ramsey or Ramsey-like cardinal at which the GCH fails does not have higher consistency strength and the failure of the GCH need not reflect below. In fact, in [GC15], Cody and Gitman showed that if  $\kappa$  is a strongly Ramsey or a Ramsey cardinal and  $F$  is any Easton function<sup>8</sup> such that  $F \restriction \kappa \subseteq \kappa$ , then there is a forcing extension in which  $\kappa$  retains the large cardinal property, and for all regular cardinals  $\delta$ , we have  $2^\delta = F(\delta)$ .

**4.4. Fast function forcing.** Recall that a *fast function* is a generically added ordinal-guessing function on a large cardinal that mimics a Laver function on a supercompact cardinal. If a large cardinal  $\kappa$  is characterized by the existence of a certain type of elementary embeddings, then a fast function  $f : \kappa \rightarrow \kappa$  has the property that for every reasonably chosen ordinal  $\theta$ ,<sup>9</sup> there is an embedding  $j$  of the type characterizing the cardinal such that  $j(f)(\kappa) = \theta$ . The fast function grows “fast enough” to create an arbitrarily large gap between  $\kappa$  and  $j(f)(\kappa)$ , a useful property for lifting the embedding to a forcing extension. For an inaccessible cardinal  $\kappa$ , the fast function forcing  $\mathbb{F}_\kappa$ , invented by Woodin, consists of conditions that are partial functions  $p : \kappa \rightarrow \kappa$ , ordered by inclusion, such that  $\gamma \in \text{dom}(p)$  implies  $p \restriction \gamma \subseteq \gamma$ , and for every inaccessible cardinal  $\gamma \leq \kappa$ , we have  $|\text{dom}(p \restriction \gamma)| <$

<sup>8</sup>An *Easton function*  $F$  is a class function on the regular cardinals such that  $F(\delta) \leq F(\gamma)$  whenever  $\delta < \gamma$ , and  $\text{cof}(\delta) < F(\delta)$ . Easton showed that if the GCH holds and  $F$  is an Easton function, then there is a generic extension in which  $2^\delta = F(\delta)$  for all regular cardinals  $\delta$  [Eas70].

<sup>9</sup>The ordinals  $\theta$  that can reasonably be made the value of  $j(f)(\kappa)$  are restricted by the properties of the embedding characterization. For example, if a cardinal is characterized by the existence of well-founded ultrapowers of weak  $\kappa$ -models, as are  $\alpha$ -iterable cardinals, then the target of the embedding has size  $\kappa$  and hence  $\theta$  must be below  $\kappa^+$ .

$\gamma$ . The fast function  $f : \kappa \rightarrow \kappa$  is the union of the generic filter for  $\mathbb{F}_\kappa$ . Note that, since  $\kappa$  is inaccessible, the poset  $\mathbb{F}_\kappa \subseteq V_\kappa$ .

For  $\gamma < \kappa$ , let  $\mathbb{F}_\gamma$  be the subposet of  $\mathbb{F}_\kappa$  consisting of all  $p$  whose domain is contained in  $\gamma$ , let  $\mathbb{F}_{[\gamma, \kappa]}$  be the subposet of  $\mathbb{F}_\kappa$  consisting of all  $p$  whose domain is contained in  $[\gamma, \kappa)$ , and define  $\mathbb{F}_{(\gamma, \kappa)}$  analogously. For any  $p \in \mathbb{F}_\kappa$  and  $\gamma \in \text{dom}(p)$ , the poset  $\mathbb{F}_\kappa \restriction p$  (consisting of all  $q \leq p$  in  $\mathbb{F}_\kappa$ ) factors as the product

$$\mathbb{F}_\gamma \restriction (p \restriction \gamma) \times \mathbb{F}_{[\gamma, \kappa]} \restriction (p \restriction [\gamma, \kappa)).$$

In particular, if  $p = \{\langle \gamma, \delta \rangle\}$  for some  $\gamma \leq \delta$ , then  $\mathbb{F}_\kappa \restriction p$  factors as  $\mathbb{F}_\gamma \times \mathbb{F}_{(\delta, \kappa)}$ , and the second factor  $\mathbb{F}_{(\delta, \kappa)}$  is  $\leq \delta$ -closed. For a more detailed exposition on fast functions, see [Ham00].

**Theorem 4.8.** *If  $\kappa$  is Ramsey or Ramsey-like, then this is preserved to a forcing extension  $V[f]$  by the fast function forcing  $\mathbb{F}_\kappa$ . Specifically,*

- (1) *if  $j : M \rightarrow N$  is a  $\kappa$ -powerset preserving embedding of  $\kappa$ -models, or*
- (2)  *$j : M \rightarrow N$  is the ultrapower of an  $\omega$ -special weak  $\kappa$ -model by an  $\omega_1$ -intersecting weakly amenable  $M$ -ultrafilter, or*
- (3)  *$j : M \rightarrow N$  is the ultrapower of an  $\omega$ -special weak  $\kappa$ -model by an  $\omega_1$ -intersecting weakly amenable  $M$ -ultrafilter,*

*then there is a lift  $j : M[f] \rightarrow N[j(f)]$  satisfying the same properties with  $j(f)(\kappa) = \theta$ .*

*Proof.* First, suppose that  $\kappa$  is strongly Ramsey. The poset  $\mathbb{F}_\kappa$  is an element of  $M$  since it is a definable subset of  $V_\kappa \in M$ . First, we verify that  $M[f]^{<\kappa} \subseteq M[f]$  in  $V[f]$ . We cannot apply the generic closure criterion since  $\mathbb{F}_\kappa$  does not have the  $\kappa$ -cc. Instead, our strategy will be to show that for arbitrarily large inaccessible cardinals  $\alpha < \kappa$ , we have  $M[f]^\alpha \subseteq M[f]$  in  $V[f]$ . Observe that there is an inaccessible  $\alpha > \beta$  and  $\gamma \in \text{dom}(f)$  such that  $\gamma < \alpha < f(\gamma) = \delta$  because conditions with this property are dense in  $\mathbb{F}_\kappa$ . Thus, the condition  $p = \{\langle \gamma, \delta \rangle\}$  is in the generic filter. Below  $p$ , the poset  $\mathbb{F}_\kappa$  factors as  $\mathbb{F}_\gamma \times \mathbb{F}_{(\delta, \kappa)}$  and  $f$  factors correspondingly as  $f_\gamma \times f_{(\delta, \kappa)}$ . We argue that  $M[f_\gamma][f_{(\delta, \kappa)}]^\alpha \subseteq M[f_\gamma][f_{(\delta, \kappa)}]$  in  $V[f]$ . Since  $\mathbb{F}_\gamma$  clearly has the  $\alpha^+$ -cc, by the generic closure criterion,  $M[f_\gamma]^\alpha \subseteq M[f_\gamma]$  in  $V[f_\gamma]$ . Also, since  $\mathbb{F}_{(\delta, \kappa)}$  is  $\leq \alpha$ -closed,  $M[f_\gamma]^\alpha \subseteq M[f_\gamma]$  in  $V[f]$ . Finally, by the ground closure criterion,  $M[f_\gamma][f_{(\delta, \kappa)}]^\alpha \subseteq M[f_\gamma][f_{(\delta, \kappa)}]$  in  $V[f]$ . Thus,  $M[f]^\alpha \subseteq M[f]$  in  $V[f]$  for arbitrarily large  $\alpha$  and so it follows that  $M[f]^{<\kappa} \subseteq M[f]$  in  $V[f]$ .

Next, we shall lift the embedding  $j$  to  $M[f]$  so that  $j(f)(\kappa) = \theta$ . Consider  $j(\mathbb{F}_\kappa) = \mathbb{F}_{j(\kappa)}^N$ , the poset to add a fast function on  $j(\kappa)$  from the perspective of  $N$ . Let  $p = \{\langle \kappa, \theta \rangle\}$  and factor  $\mathbb{F}_{j(\kappa)}^N \restriction p \cong \mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}}$ , where  $\mathbb{F}_{\text{tail}} = \mathbb{F}_{(\kappa, j(\kappa))}^N \restriction p$  is  $\leq \kappa$ -closed in  $N$ . We will build an  $N$ -generic filter for  $\mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}}$  containing  $j \restriction f = f$  and take its upward closure, which contains  $p$ , thus ensuring that  $j(f)(\kappa) = \theta$ . Unlike, in the previous arguments, we cannot apply diagonalization criterion (1) to  $N[f]$  and  $\mathbb{F}_{\text{tail}}$  in  $V[f]$  since  $\mathbb{F}_{\text{tail}}$  is not  $<\kappa$ -closed in  $V[f]$ . Instead, we exploit the fact that  $\mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}} \cong \mathbb{F}_{\text{tail}} \times \mathbb{F}_\kappa$ . We use diagonalization criterion (1) to obtain an  $N$ -generic filter  $f_{\text{tail}}$  for  $\mathbb{F}_{\text{tail}}$  in  $V$ , where the poset is  $<\kappa$ -closed. Since  $f$  is  $V$ -generic for  $\mathbb{F}_\kappa$ , it is  $N[f_{\text{tail}}]$ -generic and so  $f_{\text{tail}} \times f$  is  $N$ -generic for  $\mathbb{F}_{\text{tail}} \times \mathbb{F}_\kappa$ . Thus, we are able to lift  $j : M \rightarrow N$  to  $j : M[f] \rightarrow N[f][f_{\text{tail}}]$  in  $V[f]$  with  $j(f)(\kappa) = \theta$ .

We already showed that  $M[f]$  is a  $\kappa$ -model and so it remains to verify that the lift  $j : M[f] \rightarrow N[f][f_{\text{tail}}]$  is  $\kappa$ -powerset preserving. It suffices to observe that since  $\mathbb{F}_{\text{tail}}$  is  $\leq \kappa$ -closed in  $N$ , there are no new subsets of  $\kappa$  in  $N[f_{\text{tail}}]$ , and so  $M$  and  $N[f_{\text{tail}}]$



have the same subsets of  $\kappa$ . Note that if  $M \prec H_{\kappa^+}$ , then  $M[f] \prec H_{\kappa^+}[f] = H_{\kappa^+}^{V[f]}$ , which gives the argument for super Ramsey cardinals.

The argument is modified to Ramsey and  $\alpha$ -iterable cardinals precisely in the same way as the previous arguments.  $\square$

We can reformulate the elementary embedding characterization of Ramsey cardinals using embeddings instead of  $M$ -ultrafilters. Having a weakly amenable  $\omega_1$ -intersecting  $M$ -ultrafilter is equivalent to having an embedding  $j : M \rightarrow N$  with critical point  $\kappa$ ,  $P^M(\kappa) = P^N(\kappa)$ , and the property that whenever  $\langle A_n \mid n < \omega \rangle$  is a sequence of subsets of  $\kappa$  with  $A_n \in M$  and  $\kappa \in A_n$ , then  $\bigcap_{n < \omega} A_n \neq \emptyset$ . Let's say that such embeddings have the *Ramsey property*. The advantage of this reformulation is that we are no longer restricted to ultrapower embeddings whose target has size  $\kappa$  and we can get the following generalization of the fast function property.

**Theorem 4.9.** *If  $\kappa$  is Ramsey and  $f$  is a  $V$ -generic fast function, then for every  $A \subseteq \kappa$  and every ordinal  $\theta$ , there is an  $\omega$ -special weak  $\kappa$ -model  $M$  containing  $A$  and an embedding  $j : M \rightarrow N$  with the Ramsey property that lifts to  $j : M[f] \rightarrow N[j(f)]$  having the Ramsey property and  $j(f)(\kappa) = \theta$ .*

*Proof.* We fix an  $\omega$ -special weak  $\kappa$ -model  $M$  containing  $A$  and  $V_\kappa$  for which there exists a weakly amenable  $\omega_1$ -intersecting  $M$ -ultrafilter  $U$ . Let  $\{j_{\xi\gamma} : M_\xi \rightarrow M_\gamma \mid \xi < \gamma \in \text{ORD}\}$ , where  $M_0 = M$ , be the associated sequence of iterated ultrapowers and  $\{\kappa_\xi \mid \xi \in \text{ORD}\}$ , where  $\kappa_0 = \kappa$ , be the critical sequence. First, we argue that every iterate embedding  $j_{0\gamma} : M \rightarrow M_\gamma$  has the Ramsey property. This follows by assumption for  $\gamma = 1$ , so suppose  $\gamma > 1$  and observe that if  $\kappa \in j_{0\gamma}(A)$ , then  $j_{1\gamma}(\kappa) \in j_{0\gamma}(A)$ , and so by elementarity  $\kappa \in j_{01}(A)$ .

Now we fix an ordinal  $\theta$  and choose an ordinal  $\gamma$  such that  $\kappa_\gamma \leq \theta < \kappa_{\gamma+1}$ . We shall lift the iterate embedding  $j_{0\gamma}$  to  $j_{0\gamma} : M[f] \rightarrow M_\gamma[j(f)]$  having the Ramsey property and  $j_{0\gamma}(f)(\kappa) = \theta$ . This will require appropriate analogs of Lemma 3.10 and Lemma 3.21 for iterates of an ultrapower map and some standard facts about iterations (see [Kan09] Section 19 for details).

- (1) Every element  $x$  of  $M_\gamma$  has the form  $x = j_{0\gamma}(g)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n})$  for some  $n \in \omega$ , function  $g : [\kappa]^{n_\gamma} \rightarrow M$ , and  $\gamma_i < \gamma$ .
- (2) Given  $x \in M$ , we have that  $M_\gamma \models \varphi(j_{0\gamma}(x), \kappa_{\gamma_1}, \dots, \kappa_{\gamma_n})$  if and only if

$$\{(\xi_1, \dots, \xi_n) \in [\kappa]^{n_\gamma} \mid M \models \varphi(x, \xi_1, \dots, \xi_n)\} \in U^{n_\gamma},$$

where  $U^{n_\gamma}$  is the  $n_\gamma$ -fold product ultrafilter of  $U$ .

Sets in  $U^{n_\gamma}$  and  $U$  are connected by the following useful fact:

- (3) For every  $A \in U^{n_\gamma}$ , there is  $\bar{A} \in U$  such that if  $\alpha_1 < \dots < \alpha_n$  are in  $\bar{A}$ , then  $(\alpha_1, \dots, \alpha_n) \in A$ .

For ease of notation, let  $j_{0\gamma} = h$ . For the analog of Lemma 3.10, we define

$$y_i = \{h(f)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n}) \mid n \in \omega, f : [\kappa]^{n_\gamma} \rightarrow m_i, f \in m_i, \text{ and } \gamma_i < \gamma\}.$$

Clearly,  $M_\gamma = \bigcup_{i \in \omega} y_i$ . Moreover, each  $y_i$  is an element of  $M_\gamma$  of size  $\kappa_\gamma$ , since it is definable from  $h$  "  $m_i$ , which is in  $M_\gamma$ . Now following the proof of Theorem 4.8, we factor  $\mathbb{F}_{h(\kappa)}^{M_\gamma}$  below the condition  $p = \{(\kappa, \theta)\}$  as  $\mathbb{F}_\kappa \times \mathbb{F}_{\text{tail}}$ , where  $\mathbb{F}_{\text{tail}} = \mathbb{F}_{(\theta, h(\kappa))}^{M_\gamma} \upharpoonright p$  is  $\leq \theta$ -closed and hence  $\leq \kappa_\gamma$ -closed. The model  $M_\gamma$  together with the sequence  $\langle y_i \mid i < \omega \rangle$  and the poset  $\mathbb{F}_{\text{tail}}$  satisfies the requirements of diagonalization criterion (2) for the cardinal  $\kappa_\gamma$ . Thus, we are able to lift  $h$  to the  $\kappa$ -powerset

preserving  $h : M[f] \rightarrow M_\gamma[h(f)]$  with  $h(f)(\kappa) = \theta$ . It remains to verify that  $h$  has the rest of the Ramsey property.

Following the proof of Lemma 3.21, we fix a sequence  $\langle A_n \mid n < \omega \rangle$  of subsets of  $\kappa$  in  $V[f]$  such that each  $A_n$  is an element of  $M[f]$  with  $\kappa \in h(A_n)$ , and we also fix  $\mathbb{F}_\kappa$ -names  $\dot{A}_n \in M$  such that  $A_n = (\dot{A}_n)_f$ . As in that proof, we choose a descending  $\omega$ -sequence of conditions  $p_n \in j(f)$  below  $j(p)$ , where  $p \Vdash \bigcap \dot{S} = \emptyset$  over  $V$ , such that  $p_n \Vdash \dot{\kappa} \in h(\dot{A}_n)$  over  $M_\gamma$ . Using fact (1), each

$$p_n = h(g_n)(\kappa_{\gamma_0^{(n)}}, \dots, \kappa_{\gamma_{i_n}^{(n)}}),$$

and we shall assume for convenience that  $\gamma_0^{(n)} = 0$ , meaning that the first element in the sequence is  $\kappa$ . Define for  $n < \omega$ , the sets

$$S_n = \{\vec{\xi} \in [\kappa]^{i_n} \mid g_n(\vec{\xi}) \Vdash \xi_0 \in \dot{A}_n \text{ over } M\}.$$

and the sets

$$T_n = \{\vec{\xi} \in [\kappa]^{j_n} \mid g_{n+1}(\vec{\alpha}) \leq g_n(\vec{\beta})\},$$

where  $j_n = |\{\gamma_0^{(n)}, \dots, \gamma_{i_n}^{(n)}, \gamma_0^{(n+1)}, \dots, \gamma_{i_{n+1}}^{(n+1)}\}|$  and  $\vec{\alpha}$  and  $\vec{\beta}$  are subsequences of  $\vec{\xi}$  corresponding to how elements of  $\{\gamma_0^{(n)}, \dots, \gamma_{i_n}^{(n)}, \gamma_0^{(n+1)}, \dots, \gamma_{i_{n+1}}^{(n+1)}\}$  intertwine. Finally, define

$$T = \{\vec{\xi} \in [\kappa]^{i_0} \mid g_0(\vec{\xi}) \leq p\}.$$

Using fact (2), each  $S_n$ ,  $T_n$  and  $T$  is an element of some product ultrafilter  $U^m$ , and so, by fact (3), there are corresponding sets  $\bar{S}_n$ ,  $\bar{T}_n$ , and  $\bar{T}$  in  $U$ . Let  $I$  be the intersection of all  $\bar{S}_n$ ,  $\bar{T}_n$ , and  $\bar{T}$ . Since the  $M$ -ultrafilter  $U$  is  $\omega_1$ -intersecting, the set  $I$  is non-empty, but, in fact, it has size  $\kappa$  (if  $I$  was bounded by  $\alpha < \kappa$ , then we could add  $\kappa \setminus \alpha$  to the sets we are intersecting, thus contradicting that  $U$  is  $\omega_1$ -intersecting). Let  $\alpha$  be the countable order-type of the union of all  $\{\kappa_{\gamma_0^{(n)}}, \dots, \kappa_{\gamma_{i_n}^{(n)}}\}$ , and let  $I_\alpha$  be an initial segment of  $I$  of order-type  $\alpha$ . Let  $\vec{\gamma}_n$  be a subsequence of  $I_\alpha$  of length  $i_n$  that corresponds to where  $\{\kappa_{\gamma_0^{(n)}}, \dots, \kappa_{\gamma_{i_n}^{(n)}}\}$  sits inside  $I$ , and note that by the earlier assumption that  $\kappa$  is the first element of every  $\{\kappa_{\gamma_0^{(n)}}, \dots, \kappa_{\gamma_{i_n}^{(n)}}\}$ , we have that  $\delta$ , the first element of  $I_\alpha$ , is the first element of every  $\vec{\gamma}_n$ . It follows that:

- (1) for all  $n < \omega$ ,  $g_n(\vec{\gamma}_n) \Vdash \delta \in \dot{A}_n$  over  $M$ ,
- (2) for all  $n < \omega$ ,  $g_{n+1}(\vec{\gamma}_{n+1}) \leq g_n(\vec{\gamma}_n)$ ,
- (3)  $g_0(\vec{\gamma}_0) \leq p$ .

The contradiction, showing that  $\bigcap_{n < \omega} A_n \neq \emptyset$ , is now obtained exactly as in the proof of Lemma 3.21.  $\square$

**4.5. Virtually Ramsey cardinals.** Virtually Ramsey cardinals were introduced in [SW11] in order to provide an upper bound for the consistency strength of a variant of Chang's conjecture. They are defined by weakening the characterization of Ramsey cardinals in terms of the existence of good sets of indiscernibles (see Definition 4.10). In [GW11], it was shown that if  $\kappa$  is strongly Ramsey, then there is a forcing extension in which  $\kappa$  remains virtually Ramsey but is no longer Ramsey, thus separating the two notions. Here, we improve this result to show that we can obtain such a forcing extension by starting with just a Ramsey cardinal. This answers positively a question of Welch posed in [Git11]. We start with the definition of *good sets of indiscernibles*.

**Definition 4.10.** Suppose  $\kappa$  is a cardinal and  $A \subseteq \kappa$ . A set  $I^A \subseteq \kappa$  is a *good set of indiscernibles* for the structure  $\langle L_\kappa[A], A \rangle$  if for all  $\gamma \in I^A$ , we have:

- (1)  $\langle L_\gamma[A \cap \gamma], A \cap \gamma \rangle \prec \langle L_\kappa[A], A \rangle$ ,
- (2)  $I^A \setminus \gamma$  is a set of indiscernibles for the structure  $\langle L_\kappa[A], A, \xi \rangle_{\xi \in \gamma}$ .

**Theorem 4.11.** A cardinal  $\kappa$  is Ramsey if and only if for every  $A \subseteq \kappa$  there is a good set of indiscernibles  $I^A$  for  $\langle L_\kappa[A], A \rangle$  such that  $|I^A| = \kappa$ .

See [Dod82] (Section 17) for proof.

For  $A \subseteq \kappa$ , we define that  $\mathcal{S}^A = \{\alpha < \kappa \mid \text{there is an unbounded good set of indiscernibles } I_\alpha^A \subseteq \alpha \text{ for } \langle L_\kappa[A], A \rangle\}$ .

**Definition 4.12.** A cardinal  $\kappa$  is *virtually Ramsey* if for every  $A \subseteq \kappa$ , the set  $\mathcal{S}^A$  contains a club of  $\kappa$ .

It is easily seen that Ramsey cardinals are virtually Ramsey by noting that if  $I^A$  is a good set of indiscernibles for  $\langle L_\kappa[A], A \rangle$  of size  $\kappa$ , then the club of all its limit points is a subset of  $\mathcal{S}^A$ . A virtually Ramsey cardinal is Mahlo and one that is weakly compact is already Ramsey. Thus, a natural strategy to separate virtually Ramsey and Ramsey cardinals is to destroy the weak compactness of a Ramsey cardinal while preserving that it is virtually Ramsey. This strategy was implemented in [GW11], using a two-step iteration of Kunen from [Kun78] to destroy and then resurrect a weakly compact cardinal. The argument started with a strongly Ramsey  $\kappa$  and produced a forcing extension in which  $\kappa$  remained virtually Ramsey but was no longer weakly compact. Here we improve on this result by starting with a Ramsey cardinal.

**Theorem 4.13.** If  $\kappa$  is Ramsey, then there is a forcing extension in which  $\kappa$  is virtually Ramsey but not Ramsey.

*Sketch of proof.* By using Theorem 4.3, we may assume that  $\kappa$  is indestructible by  $\text{Add}(\kappa, 1)$ . Let  $\mathbb{Q}$  be the poset to add a  $\kappa$ -Souslin tree having a group of automorphisms with a transitive action, described in [GW11]. If  $\mathbb{T}$  is the  $\kappa$ -Souslin tree added by  $\mathbb{Q}$  in the forcing extension, then viewing it as a forcing notion and forcing with it adds a branch through  $\mathbb{T}$ . Letting  $\dot{\mathbb{T}}$  be the  $\mathbb{Q}$ -name for the  $\kappa$ -Souslin tree it adds, we shall force with the iteration  $\mathbb{Q} * \dot{\mathbb{T}}$ . The critical observation about the iteration  $\mathbb{Q} * \dot{\mathbb{T}}$  is that it has size  $\kappa$  and a dense subset that is  $<\kappa$ -closed, from which it follows that it is forcing equivalent to  $\text{Add}(\kappa, 1)$ . Let  $T * B \subseteq \mathbb{Q} * \dot{\mathbb{T}}$  be  $V$ -generic and consider the forcing extensions  $V[T]$  and  $V[T][B]$ . It is clear that  $\kappa$  is no longer weakly compact  $V[T]$  since it contains a  $\kappa$ -tree, namely  $T$ , without a branch. However,  $\kappa$  is again Ramsey in  $V[T][B]$  since, as we observed,  $\mathbb{Q} * \dot{\mathbb{T}}$  is forcing equivalent to  $\text{Add}(\kappa, 1)$ , which preserves that  $\kappa$  is Ramsey by assumption. But this implies that  $\kappa$  must have already been virtually Ramsey in  $V[T]$  as the forcing  $\mathbb{T}$  is  $<\kappa$ -closed and stationary preserving, and therefore cannot create new virtually Ramsey cardinals. To summarize, the cardinal  $\kappa$  is virtually Ramsey in the forcing extension  $V[T]$ , but it cannot be Ramsey since, in particular, it is not weakly compact.  $\square$

For complete details of the argument, see [GW11]. It is interesting to note that Welch very recently showed that every virtually Ramsey cardinal is Ramsey in

the core model  $K^{10}$ , and so virtually Ramsey cardinals and Ramsey cardinals are equiconsistent.

## 5. LOSING THE LARGE CARDINAL PROPERTY IN HOD

In [CFH15], techniques were developed to show that certain large cardinals can lose their large cardinal property in HOD. Using the indestructibility we obtained for Ramsey and Ramsey-like cardinals, we can apply the techniques of [CFH15] to show that a Ramsey or a Ramsey-like cardinal can lose its large cardinal property in HOD, and indeed need not even be weakly compact there.

**Theorem 5.1.** *If  $\kappa$  is Ramsey or Ramsey-like, then there is a forcing extension in which  $\kappa$  retains its large cardinal property, but it is not even weakly compact in HOD.*

*Sketch of proof.* By using Theorem 4.3, we may assume that  $\kappa$  is indestructible by  $\text{Add}(\kappa, 1)$ . Let  $\mathbb{Q} * \mathbb{T}$  be Kunen's two-step iteration from the proof of Theorem 4.13 for killing and then resurrecting a weakly compact cardinal. Let  $T * B \subseteq \mathbb{Q} * \mathbb{T}$  be  $V$ -generic and consider the forcing extensions  $V[T]$  and  $V[T][B]$ . In  $V[T]$ , we force with the standard GCH coding forcing  $\mathbb{R}$  to code  $T$  into the continuum pattern above  $\kappa$  and let  $H \subseteq \mathbb{R}$  be  $V[T]$ -generic. Note that the  $V[T]$ -generic  $B$  is also  $V[T][H]$ -generic because  $T$  is still a  $\kappa$ -Souslin tree in  $V[T][H]$ . The tree  $T$  remains  $\kappa$ -Souslin because  $\mathbb{R}$  is  $\leq \kappa$ -closed and every branch of a  $\kappa$ -Souslin tree is automatically generic. Thus,  $H$  and  $B$  are mutually generic, giving us that  $V[T][H][b] = V[T][b][H]$ . Also, since  $T$  has size  $\kappa$ , and so obviously has the  $\kappa^+$ -cc,  $\mathbb{R}$  is  $\leq \kappa$ -distributive in  $V[T][b]$  by Easton's Lemma (if  $\mathbb{P}$  has the  $\kappa^+$ -cc and  $\mathbb{Q}$  is  $\leq \kappa$ -closed, then  $\mathbb{Q}$  remains  $\leq \kappa$ -distributive after forcing with  $\mathbb{P}$ ). By our indestructibility assumption and the fact that  $\leq \kappa$ -distributive forcing cannot destroy Ramsey or Ramsey-like cardinals, we get that  $\kappa$  retains its large cardinal property in  $V[T][B][H]$ . But we will now argue that  $\kappa$  is not weakly compact in  $\text{HOD}^{V[T][B][H]}$ . Since  $V[T][B][H] = V[T][H][B]$  and the forcing  $T$  is weakly homogeneous<sup>11</sup> in  $V[T][H]$ ,  $\text{HOD}^{V[T][B][H]} \subseteq V[T][H]$ . The tree  $T$  is an element of  $\text{HOD}^{V[T][B][H]}$  because it was coded into the continuum pattern, and it is  $\kappa$ -Souslin there because it is  $\kappa$ -Souslin in  $V[T][H]$ . Thus,  $\kappa$  is not weakly compact in  $\text{HOD}^{V[T][B][H]}$ .  $\square$

In particular, it follows that none of the Ramsey-like cardinals are downward absolute to transitive inner models. In contrast, it was shown in [GW11] that the  $\alpha$ -iterable cardinals are downward absolute to  $L$  and that strongly Ramsey and super Ramsey cardinals are downward absolute to the core model  $K$ .

## 6. QUESTIONS

We have shown that Ramsey and Ramsey-like cardinals are indestructible by small forcing. A parallel question is whether these cardinals can be created by small forcing. It is easy to see, using the original characterization of Ramsey cardinals

<sup>10</sup>Personal communication.

<sup>11</sup>A poset  $\mathbb{P}$  is said to be *weakly homogeneous* if given any two conditions  $p$  and  $q$ , there is an automorphism that maps  $p$  to a condition compatible with  $q$ . If a poset is weakly homogeneous, then every statement with check name parameters that is forced by some condition is already forced by  $\mathbf{1}$ .

that they cannot be created by small forcing. If  $\kappa$  is Ramsey in a small forcing extension  $V[g]$ , then it be inaccessible in  $V$ . By the smallness of the forcing, every set of size  $\kappa$  in  $V[g]$  has a subset of size  $\kappa$  in  $V$ . Thus, every  $f : [\kappa]^{<\omega} \rightarrow 2$  in  $V$  has a homogeneous set  $\overline{H}$  of size  $\kappa$  in  $V$  that is a subset of the homogeneous set  $H$  of  $V[g]$ . We do not know how to obtain the same result using the embedding characterization.

**Question 6.1.** If  $\kappa$  is Ramsey-like in a generic extension by small forcing, does it follow that it had the same large cardinal property in the ground model?

Hamkins and Johnstone, for instance, showed in [HJ10] that a strongly unfoldable cardinal  $\kappa$  can be made indestructible by all  $<\kappa$ -closed  $\kappa^+$ -preserving forcing.

**Question 6.2.** Are Ramsey cardinals indestructible by some wide class of forcing notions?

## REFERENCES

- [AGH12] Arthur Apter, Victoria Gitman, and Joel David Hamkins. Inner models with large cardinal features usually obtained by forcing. *Archive for Mathematical Logic*, 51:257–283, 2012.
- [CFH15] Yong Cheng, Sy-David Friedman, and Joel David Hamkins. Large cardinals need not be large in HOD. *Annals of Pure and Applied Logic*, 166(11):1186 – 1198, 2015.
- [Cum10] James Cummings. Iterated forcing and elementary embeddings. In *Handbook of set theory. Vols. 1, 2, 3*, pages 775–883. Springer, Dordrecht, 2010.
- [Dod82] A. J. Dodd. *The core model*, volume 61 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1982.
- [Eas70] William Bigelow Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.
- [EH62] P. Erdős and A. Hajnal. Some remarks concerning our paper “On the structure of set-mappings”. Non-existence of a two-valued  $\sigma$ -measure for the first uncountable inaccessible cardinal. *Acta Math. Acad. Sci. Hungar.*, 13:223–226, 1962.
- [Gai74] Haim Gaifman. Elementary embeddings of models of set-theory and certain subtheories. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)*, pages 33–101. Amer. Math. Soc., Providence R.I., 1974.
- [GC15] Victoria Gitman and Brent Cody. Easton’s theorem for Ramsey and strongly Ramsey cardinals. *Annals of Pure and Applied Logic*, 166(9):934–952, 2015.
- [GHJ] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone. What is the theory ZFC without power set? To appear in *Mathematical Logic Quarterly*.
- [Git07] Victoria Gitman. *Applications of the proper forcing axiom to models of Peano arithmetic*. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–City University of New York.
- [Git11] Victoria Gitman. Ramsey-like cardinals. *The Journal of Symbolic Logic*, 76(2):519–540, 2011.
- [GW11] Victoria Gitman and Philip D. Welch. Ramsey-like cardinals II. *The Journal of Symbolic Logic*, 76(2):541–560, 2011.
- [Ham] Joel David Hamkins (<http://mathoverflow.net/users/1946/joel-david-hamkins>). Forcing with nontransitive models. <http://mathoverflow.net/questions/162888/forcing-with-nontransitive-models> (version: 2015-06-29).
- [Ham00] Joel David Hamkins. The lottery preparation. *Ann. Pure Appl. Logic*, 101(2-3):103–146, 2000.
- [HJ10] Joel David Hamkins and Thomas A. Johnstone. Indestructible strong unfoldability. *Notre Dame J. Form. Log.*, 51(3):291–321, 2010.
- [Jen74] Ronald Björn Jensen. Measurable cardinals and the GCH. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)*, pages 175–178. Amer. Math. Soc., Providence, R.I., 1974.

- [Kan09] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2009. Large cardinals in set theory from their beginnings, Paperback reprint of the 2003 edition.
- [Kun70] Kenneth Kunen. Some applications of iterated ultrapowers in set theory. *Ann. Math. Logic*, 1:179–227, 1970.
- [Kun78] Kenneth Kunen. Saturated ideals. *J. Symbolic Logic*, 43(1):65–76, 1978.
- [Lev95] Jean-Pierre Levinski. Filters and large cardinals. *Ann. Pure Appl. Logic*, 72(2):177–212, 1995.
- [LS67] Azriel Lévy and Robert M. Solovay. Measurable cardinals and the continuum hypothesis. *Israel J. Math.*, 5:234–248, 1967.
- [Mit79] William Mitchell. Ramsey cardinals and constructibility. *J. Symbolic Logic*, 44(2):260–266, 1979.
- [PH15] Philipp Lucke Ana Njgomig Philipp Schlicht Peter Holy, Regula Krapf. Class forcing, the forcing theorem, and boolean completions. *Preprint*, 2015.
- [She98] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.
- [SW11] Ian Sharpe and Philip Welch. Greatly Erdős cardinals with some generalizations to the Chang and Ramsey properties. *Ann. Pure Appl. Logic*, 162(11):863–902, 2011.
- [Wel88] P. D. Welch. Coding that preserves Ramseyness. *Fund. Math.*, 129(1):1–7, 1988.

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