Indestructibility for Ramsey cardinals

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The indestructibility template

Goal: Show that a large cardinal is indestructible by a given forcing.

Suppose that κ is a large cardinal characterized by the existence of elementary embeddings $j : M \to N$ satisfying a list of properties $\mathscr{L}$.

Fix a forcing notion $\mathbb{P}$ and a V-generic filter $G \subseteq \mathbb{P}$.

In $V[G]$:

- Fix an elementary embedding $j : M \to N$ from V satisfying $\mathscr{L}$.
- Lift (extend) j to an elementary embedding $j : M[G] \to N[H]$, where $H = j(G)$ is N-generic for $j(\mathbb{P})$.
- Verify that the lift j satisfies $\mathscr{L}$ in $V[G]$.

Weak κ-models M and M-ultrafilters

Suppose that κ is a cardinal.

A weak κ-model is a transitive $M \models \mathrm{ZFC}^-$ (no powerset) of size κ with $\kappa \in M$.

A κ-model is a weak κ-model M such that $M^{<\kappa} \subseteq M$.

- If $M \prec H_{\kappa^+}$ has size κ, then M is a weak κ-model.
- If $\kappa^{<\kappa} = \kappa$, then there are κ-models $M \prec H_{\kappa^+}$.

Suppose that M is a weak κ-model.

A set $U \subseteq P(\kappa)^M$ is an M-ultrafilter if $\langle M, \in, U \rangle \models U$ is a normal ultrafilter on κ.

- U is closed under diagonal intersections $\Delta_{\xi<\kappa} A_\xi$ for sequences $\vec{A} = \langle A_\xi \mid \xi < \kappa \rangle \in M$
- In most interesting cases, $U \notin M$.
- The Łoś-Theorem holds for ultrapowers by an M-ultrafilter.

Weak κ-models M and M-ultrafilters (continued)

Suppose that M is a weak κ-model.

An M-ultrafilter U is:

good if the ultrapower of M by U is well-founded.

weakly amenable if for every $A \in M$ with $|A|^M = \kappa$, $U \cap A \in M$.

- Required to iterate the ultrapower construction along the ordinals.

countably complete if for every $\langle A_n \mid n \in \omega \rangle$, with $A_n \in U$, $\bigcap_{n<\omega} A_n \neq \emptyset$.

- **Theorem**: (Kunen) All iterated ultrapowers by a weakly amenable countably complete M-ultrafilter are well-founded.

M-ultrafilters and elementary embeddings

Suppose that M is a weak κ-model.

Proposition: (Folklore)

- If U is a good M-ultrafilter, then the ultrapower map $j : M \to N$ is an elementary embedding with $\text{crit}(j) = \kappa$.
- If $j : M \to N$ is an elementary embedding with $\text{crit}(j) = \kappa$, then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is a good M-ultrafilter.

Proposition: (Folklore)

- If U is a good weakly amenable M-ultrafilter, then the ultrapower map $j : M \to N$ is κ-powerset preserving: $P(\kappa)^M = P(\kappa)^N$.
- If $j : M \to N$ is a κ-powerset preserving elementary embedding, then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is weakly amenable.

Elementary embeddings characterizations of smaller large cardinals

Theorem: (Folklore) TFAE for a cardinal κ with $\kappa^{<\kappa} = \kappa$.

- κ is weakly compact.
- Every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a good M-ultrafilter on κ.
- Every $A \subseteq \kappa$ is an element of a κ-model M for which there is a good M-ultrafilter on κ.
- Every weak κ-model M has a good M-ultrafilter on κ.

(G.) A cardinal κ is 1-iterable if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a good weakly amenable M-ultrafilter on κ.

Proposition (G.) A 1-iterable cardinal is a limit of ineffable cardinals.

Theorem: (Mitchell) A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a weakly amenable countably complete M-ultrafilter on κ.

(G.) A cardinal κ is strongly Ramsey if every $A \subseteq \kappa$ is an element of a κ-model M for which there is a weakly amenable M-ultrafilter on κ.

Proposition (G.) A strongly Ramsey cardinal is a limit of Ramsey cardinals.

Proposition: (G.) The assertion that every κ-model M has a weakly amenable M-ultrafilter is inconsistent.

A toolbox for indestructibility

Lifting criterion: Suppose that

- M is a weak κ-model,
- $j : M \to N$ is an elementary embedding,
- $\mathbb{P} \in M$ is a forcing notion,
- $G \subseteq \mathbb{P}$ is M-generic,
- $H \subseteq j(\mathbb{P})$ is N-generic.

Then j lifts to $j : M[G] \to N[H]$ if and only if $j\,``\,G \subseteq H$.

Diagonalization criterion: Suppose that N is a κ-model and $\mathbb{P} \in N$ is a forcing notion that is $<\kappa$-closed in N. Then there is an N-generic filter $H \subseteq \mathbb{P}$.

Proposition: Suppose that M is a κ-model and U is an M-ultrafilter on κ. Then the ultrapower N of M by U is a κ-model.

Ground closure criterion: Suppose that N is a κ-model, $\mathbb{P} \in N$ is a forcing notion, and $H \subseteq \mathbb{P}$ is N-generic. Then $N[H]$ is a κ-model.

A prototypical lifting argument

Suppose that $\mathbb{P}_\kappa$ is the Easton support iteration adding a Cohen subset to every regular cardinal below κ.

Theorem: (Folklore) Suppose that κ is weakly compact and $G \subseteq \mathbb{P}_\kappa$ is V-generic. Then κ is weakly compact in $V[G]$.

Proof: Fix $A \subseteq \kappa$ in $V[G]$.

- Fix a nice $\mathbb{P}_\kappa$-name $\dot{A}$ for A and observe that $\dot{A} \in H_{\kappa^+}$ ($\mathbb{P}_\kappa$ has κ-cc).
- Fix a κ-model M with $V_\kappa, \dot{A} \in M$ for which there is a good M-ultrafilter U on κ.
- Let $j : M \to N$ be the ultrapower map by U.
- By the lifting criterion, we need an N-generic filter $H \subseteq j(\mathbb{P}_\kappa)$ with $j\,"\,G = G \subseteq H$.
- In N, $j(\mathbb{P}_\kappa) = \mathbb{P}^N_{j(\kappa)} \cong \mathbb{P}_\kappa * \dot{\mathbb{P}}_{\text{tail}}$.
- Use G for $\mathbb{P}_\kappa$ to satisfy the lifting criterion.
- By the ground closure criterion, $N[G]$ is a κ-model.
- $\mathbb{P}_{\text{tail}} = (\dot{\mathbb{P}}_{\text{tail}})_G$ is $<\kappa$-closed in $N[G]$.
- By the diagonalization criterion, there is an $N[G]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$. $\square$

Ramsey cardinal difficulties

We cannot use the diagonalization criterion because the Ramsey embeddings are on weak κ-models.

We need to verify that a lift of an ultrapower map by a weakly amenable countably complete M-ultrafilter is:

- weakly amenable (usually easy)
- countably complete

The Ramsey indestructibility toolbox

(G., Johnstone) A weak κ-model M is special if there is a sequence $M_0 \prec M_1 \prec \cdots \prec M_n \prec \cdots \prec M$ of length ω such that:

- $M_n \in M$
- M_n is a weak κ-model.

A weak κ-model M is almost special if the M_n are not required to be transitive.

Proposition: (G., Johnstone) A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a special weak κ-model M for which there is a weakly amenable countably complete M-ultrafilter on κ.

Proposition: (G. Johnstone) If M is a special weak κ-model, U is a good M-ultrafilter on κ, and $j : M \to N$ is the ultrapower map by U, then N is almost special.

Special diagonalization criterion: (G., Johnstone) Suppose that N is an almost special weak κ-model and $\mathbb{P} \in N$ is a $\leq \kappa$-forcing notion in N. Then there is an N-generic filter $H \subseteq \mathbb{P}$.

The Ramsey indestructibility toolbox (continued)

Theorem: (G., Johnstone) Suppose that

- M is a weak κ-model,
- U is a good M-ultrafilter on κ,
- $j : M \to N$ is the ultrapower map by U,
- $\mathbb{P} \in M$ a countably complete forcing notion in V,
- $G \subseteq \mathbb{P}$ is M-generic.

If j lifts to $j : M[G] \to N[H]$, then the lift is the ultrapower by a countably complete $M[G]$-ultrafilter.

The Ramsey indestructibility properties

Theorem: A Ramsey cardinal is indestructible by:

- small forcing
- (G., Johnstone) canonical forcing of the GCH
- (G., Johnstone) fast function forcing

Theorem: (G., Johnstone) A Ramsey cardinal κ can be made indestructible by the forcing notions:

- $\mathrm{Add}(\kappa, \theta)$ for any cardinal θ
- club shooting forcing on κ

Corollary: It is consistent relative to a Ramsey cardinal, that the GCH fails for the first time at a Ramsey cardinal.

Extensions that cannot create new Ramsey cardinals

(Hamkins, 2003) Suppose that $V \subseteq W$ are transitive models of (some fragment of) ZFC and δ is a cardinal in W.

The pair $V \subseteq W$ satisfies the δ-cover property if for every $A \in W$ with $A \subseteq V$ and $|A|^W < \delta$, there is $B \in V$ with $A \subseteq B$ and $|B|^V < \delta$.

"Every set of size less than δ in W that is contained in V can be covered by a set of size less than δ in V."

The pair $V \subseteq W$ satisfies the δ-approximation property if whenever $A \in W$ with $A \subseteq V$ and $A \cap a \in V$ for every a of size less than δ in V, then $A \in V$.

"If a set in W is contained in V and V has all pieces of it of size less than δ, then V has the set."

Theorem: (G.) Suppose that $V \subseteq W$ satisfies the δ-cover and δ-approximation properties, $V^\omega \subseteq V$ in W, and $\kappa > \delta$ is Ramsey in W, then κ is Ramsey in V.