INNER MODELS WITH LARGE CARDINAL FEATURES USUALLY OBTAINED BY FORCING

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ABSTRACT. We construct a variety of inner models exhibiting large cardinal features usually obtained by forcing. For example, if there is a supercompact cardinal, then there is an inner model with a Laver indestructible supercompact cardinal. If there is a supercompact cardinal, then there is an inner model with a supercompact cardinal κ for which $2^\kappa = \kappa^+$, another for which $2^\kappa = \kappa^{++}$ and another in which the least strongly compact cardinal is supercompact. If there is a strongly compact cardinal, then there is an inner model with a strongly compact cardinal, for which the measurable cardinals are bounded below it and another inner model W with a strongly compact cardinal, such that $H_{\kappa^+}^V \subseteq \mathrm{HOD}^W$. Similar facts hold for supercompact, measurable and strongly Ramsey cardinals. If a cardinal is supercompact up to a weakly iterable cardinal, then there is an inner model of the Proper Forcing Axiom and another inner model with a supercompact cardinal in which GCH + $V =$ HOD holds. Under the same hypothesis, there is an inner model with level by level equivalence between strong compactness and supercompactness, and indeed, another in which there is level by level inequivalence between strong compactness and supercompactness. If a cardinal is strongly compact up to a weakly iterable cardinal, then there is an inner model in which the least measurable cardinal is strongly compact. If there is a weakly iterable limit δ of $<\delta$-supercompact cardinals, then there is an inner model with a proper class of Laver-indestructible supercompact cardinals. We describe three general proof methods, which can be used to prove many similar results.

1. INTRODUCTION

The theme of this article is to inquire the extent to which several set-theoretic large cardinal properties obtainable by forcing must also already be found in an inner model. We find this interesting in the case of supercompact and other large cardinals that seem to lay beyond the current reach of the fine-structural inner model program. For example, one reason we know that the GCH is relatively consistent with many large cardinals, especially the smaller large cardinals, is that the fine-structural inner models that have been constructed for these large cardinals satisfy the GCH; another reason is that the canonical forcing of the GCH preserves all the standard large cardinals. In the case of supercompact and other very large large cardinals, we currently lack such fine-structural inner models and therefore have relied on the forcing argument alone when showing relative consistency with

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the GCH. It seems quite natural to inquire, without insisting on fine structure, whether these cardinals nevertheless have an inner model with the GCH.

Test Question 1. If there is a supercompact cardinal, then must there be an inner model with a supercompact cardinal in which the GCH also holds?

Test Question 2. If there is a supercompact cardinal, then must there be an inner model with a supercompact cardinal κ such that $2^\kappa = \kappa^+$?

Test Question 3. If there is a supercompact cardinal, then must there be an inner model with a supercompact cardinal κ such that $2^\kappa > \kappa^+$?

We regard these test questions and the others we are about to introduce as stand-ins for many more variant questions, asking of a particular large cardinal feature known to be forceable, whether it must hold already in an inner model. Sy Friedman [Fri06] defines that a theory T is *internally consistent* if there is an inner model of T. Thus, our questions are inquiring about the internal consistency of these forceable large cardinal properties. To be precise, we define that a class W is an *inner model* if it is a transitive class model of ZFC, containing all ordinals. The inner-model consistency strength of a theory is the consistency strength of the assertion that the theory is internally consistent. Interestingly, this can sometimes exceed the ordinary consistency strength of the assertion. In this terminology, Test Questions 1, 2 and 3 inquire whether the inner model consistency strength of a supercompact cardinal plus the corresponding amount of the GCH is the same as just a supercompact cardinal.

In several cases we shall be able to go beyond mere internal consistency by finding especially attractive inner models with the desired property. For example, in some of our answers below, we shall produce for every cardinal θ an inner model W with the desired feature, such that also $W^\theta \subseteq W$, meaning that W contains all θ-sequences over W. This appears to be an especially strong form of internal consistency, and it would be interesting to investigate the extent to which it is indeed stronger than mere internal consistency.

Let us continue with a few more test questions that we shall use to frame our later discussion. Forcing, of course, can also achieve large cardinal features that we do not expect to hold in the fine-structural inner models. For example, Laver [Lav78] famously proved that after his forcing preparation, any supercompact cardinal κ is made *(Laver) indestructible*, meaning that it remains supercompact after any further $<\kappa$-directed closed forcing. In contrast, large cardinals are almost universally destructible over their fine-structural inner models (e.g. see [Ham94, Theorem 1.1]). Nevertheless, giving up the fine-structure, we may still ask for indestructibility in an inner model.

Test Question 4. If there is a supercompact cardinal, then must there be an inner model with an indestructible supercompact cardinal?

For another example, recall that Baumgartner [Bau84] proved that if κ is a supercompact cardinal, then there is a forcing extension with the Proper Forcing Axiom (PFA). We inquire whether there must in fact be an inner model with the PFA:

Test Question 5. If there is a supercompact cardinal, then must there be an inner model with the Proper Forcing Axiom?

Next, we inquire the extent to which there must be inner models W having a very rich HOD^W. With class forcing, one can easily force $V = \text{HOD}$ while preserving all of the most well-known large cardinal notions, and of course, one finds $V = \text{HOD}$ in the canonical inner models of large cardinals. Must there also be such inner models for the very large large cardinals?

Test Question 6. If there is a supercompact cardinal, then must there be an inner model with a supercompact cardinal satisfying $V = \text{HOD}$?

In addition, one may easily force to make any particular set A definable in a forcing extension by forcing that preserves all the usual large cardinals, and so another version of this question inquires:

Test Question 7. If there is a supercompact cardinal, then for every set A, must there be an inner model W with a supercompact cardinal such that $A \in \text{HOD}^W$?

Similarly, if there is a measurable cardinal, then does every set A have an inner model with a measurable cardinal in which $A \in \text{HOD}^W$? What of other large cardinal notions? What if one restricts to $A \in H_{\kappa^+}$? There is an enormous family of such questions surrounding the HODs of inner models. Furthermore, apart from large cardinals, for which sets A is there an inner model W with $A \in \text{HOD}^W$? There are numerous variants of this question.

More generally, whenever a feature is provably forceable in the presence of a certain large cardinal, then we ask: is there already an inner model with that feature? How robust can these inner models be?

Before continuing, we fix some terminology. Suppose κ is a regular cardinal. A forcing notion is *<κ-directed closed* when any directed subset of it of size less than κ has a lower bound. (This is what Laver in [Lav78] refers to as κ-*directed closed*.) A forcing notion is *≤κ-closed* if any decreasing chain of length less than or equal to κ has a lower bound. A forcing notion is *≤κ-strategically closed* if in the game of length $\kappa + 1$ in which two players alternately select conditions from it to construct a descending $(\kappa + 1)$-sequence, with the second player playing at limit stages, the second player has a strategy that allows her always to continue playing. A forcing notion is *<κ-strategically closed* if in the game of length κ in which two players alternately select conditions from it to construct a descending κ-sequence, with the second player playing at limit stages, the second player has a strategy that allows her always to continue playing. If a poset $\mathbb{P}$ is ≤κ-closed, then it is also ≤κ-strategically closed. If λ is an ordinal, then $\text{Add}(\kappa, \lambda)$ is the standard poset for adding λ many Cohen subsets to κ. A Boolean algebra $\mathbb{B}$ is $(\kappa, 2)$-*distributive* if every $f : \kappa \to 2$ in the generic extension by $\mathbb{B}$ is in the ground model. A transitive set $M \models \text{ZFC}^-$ is a κ-*model* if $|M| = \kappa$, $\kappa \in M$ and $M^{<\kappa} \subseteq M$. An elementary embedding $j : M \to N$ is said to *lift* to another elementary embedding $j^* : M^* \to N^*$, where $M \subseteq M^*$ and $N \subseteq N^*$, if the two embeddings agree on the smaller domain $j^* \restriction M = j$. An elementary embedding $j : M \to N$ having critical point κ is κ-*powerset preserving* if M and N have the same subsets of κ. A cardinal κ is *strongly Ramsey* if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a κ-powerset preserving elementary embedding $j : M \to N$.

2. THREE PROOF METHODS

In order best to introduce our methods, which we view as the main contribution of this article, we shall begin with Test Question 4, which is answered by Theorem 8

below. We shall give three different arguments with this conclusion, using different proof methods (our third method will prove a slightly weaker result, because it requires a slightly stronger hypothesis). These methods are robust enough directly to answer many variants of the test questions. In Sections 4 and 5, we describe how some further modifications of the methods enable them to prove additional related results.

Theorem 8. *If there is a supercompact cardinal, then there is an inner model with an indestructible supercompact cardinal.*

The first proof makes use of an observation of Hamkins and Seabold involving Boolean ultrapowers (see [HS]), which is essentially encapsulated in Theorems 10 and 11.

Definition 9. A forcing notion $\mathbb{P}$ is $<\kappa$-*friendly* if for every $\gamma < \kappa$, there is a condition $p \in \mathbb{P}$ below which the restricted forcing $\mathbb{P} \restriction p$ adds no subsets to γ.

Theorem 10 (Hamkins, Seabold [HS])**.** *If κ is a strongly compact cardinal and $\mathbb{P}$ is a $<\kappa$-friendly notion of forcing, then there is an inner model W satisfying every sentence forced by $\mathbb{P}$ over V.*

Proof. The proof uses Boolean ultrapowers (see [HS] for a full account). To make this paper self-contained, we shall review the method. Suppose that $\mathbb{B}$ is the complete Boolean algebra corresponding to the forcing notion $\mathbb{P}$. Let $V^{\mathbb{B}}$ be the usual class of $\mathbb{B}$-names, endowed as a Boolean-valued structure by the usual recursive definition of the Boolean values $[\![\varphi]\!]$ for every assertion φ in the forcing language. Now suppose that $U \subseteq \mathbb{B}$ is an ultrafilter, not necessarily generic in any sense, and define the equivalence relation $\sigma =_U \tau \iff [\![\sigma = \tau]\!] \in U$. When U is not V-generic, this relation is not the same as $\mathrm{val}(\sigma, U) = \mathrm{val}(\tau, U)$. Nevertheless, the relation $\sigma \in_U \tau \iff [\![\sigma \in \tau]\!] \in U$ is well-defined with respect to $=_U$, and we may form the quotient structure $V^{\mathbb{B}}/U$ as the collection of (Scott's trick reduced) equivalence classes $[\tau]_U$. The relation $\in_U$ is set-like, because whenever $\sigma \in_U \tau$, then σ is $=_U$ equivalent to a mixture of the names in the domain of τ, and there are only set many such mixtures. One can easily establish Łos' theorem that $V^{\mathbb{B}}/U \models \varphi[[\tau]_U] \iff [\![\varphi(\tau)]\!] \in U$. In particular, any statement φ that is forced by $\mathbb{1}$ will be true in $V^{\mathbb{B}}/U$. Thus, since U is in V, we have produced in V a class model $V^{\mathbb{B}}/U$ satisfying the desired theory; but there is no reason so far to suppose that this model is well-founded.

In order to find an ultrafilter U for which $V^{\mathbb{B}}/U$ is well-founded, we shall make use of our assumption that $\mathbb{P}$ and hence also $\mathbb{B}$ is $<\kappa$-friendly for a strongly compact cardinal κ. Just as with classical powerset ultrapowers, the structure $V^{\mathbb{B}}/U$ is well-founded if and only if U is countably complete (see [HS]). Next, consider any $\theta \geq |\mathbb{B}|$ and let $j : V \to M$ be a θ-strong compactness embedding, so that $j \text{ "} \mathbb{B} \subseteq s \in M$ for some $s \in M$ with $|s|^M < j(\kappa)$. Since $j(\mathbb{B})$ is $<j(\kappa)$-friendly, there is a condition $p \in j(\mathbb{B})$ such that $j(\mathbb{B}) \restriction p$ adds no new subsets to $\lambda = |s|^M$. Thus, $j(\mathbb{B}) \restriction p$ is $(\lambda, 2)$-distributive in M. Applying this, it follows in M that

$$p = p \wedge \mathbb{1} = p \wedge \bigwedge_{b \in s} (b \vee \neg b) = \bigwedge_{b \in s} (p \wedge b) \vee (p \wedge \neg b) = \bigvee_{f \in 2^s} \bigwedge_{b \in s} (p \wedge (\neg)^{f(b)} b),$$

where $(\neg)^0 b = b$ and $(\neg)^1 b = \neg b$, and where we use distributivity to deduce the final equality. Since p is not 0, it follows that there must be some f with $q =$

$\bigwedge_{b \in s} p \wedge (\neg)^{f(b)} b \neq 0$. Note that $f(b)$ and $f(\neg b)$ must have opposite values. Now we use q as a seed to define the ultrafilter $U = \{ a \in \mathbb{B} \mid q \leq j(a) \}$, which is the same as $\{ a \in \mathbb{B} \mid f(j(a)) = 0 \}$. This is easily seen to be a κ-complete filter using the fact that $cp(j) = \kappa$ (just as in the power-set ultrafilter cases known classically). It is an ultrafilter precisely because s covers j " $\mathbb{B}$, so either $f(j(a)) = 0$ or $f(\neg j(a)) = 0$, and so either $a \in U$ or $\neg a \in U$, as desired. In summary, using this ultrafilter U, the structure $V^{\mathbb{B}}/U$ is a well-founded set-like model of the desired theory. The corresponding Mostowski collapse is the desired inner model W. □

The metamathematical reader will observe that Theorem 10 is more properly described as a theorem scheme, since we defined a certain inner model, using $\mathbb{P}$ as a parameter, and then proved of each sentence forceable by $\mathbb{P}$ over V, that this sentence also holds in the inner model. By Tarski's theorem on the non-definability of truth, it does not seem possible to state the conclusion of Theorem 10 in a single first order statement. Several similar theorems in this article will also be theorem schemes.

The following account of the Boolean ultrapower may be somewhat more illuminating.

Theorem 11 ([HS])**.** *If κ is strongly compact and $\mathbb{P}$ is $<\kappa$-friendly, then there is an elementary embedding $j : V \to \overline{V}$ into an inner model $\overline{V}$ and a $\overline{V}$-generic filter $G \subseteq j(\mathbb{P})$ with $G \in V$. In particular, $W = \overline{V}[G]$ fulfills Theorem 10.*

Proof. This is actually what is going on in the Boolean quotient. We may define the canonical predicate for the ground model of $V^{\mathbb{B}}$ by $[\![\tau \in \check{V}]\!] = \bigvee_{x \in V} [\![\tau = \check{x}]\!]$, and let $\overline{V} = \{ [\tau]_U \mid [\![\tau \in \check{V}]\!] \in U \}$, which is actually the same as $\{ [\tau]_U \mid [\![\tau \in \check{V}]\!] = \mathbb{1} \}$. An easy induction on formulas shows that the map $j : x \mapsto [\check{x}]_U$ is an elementary embedding $j : V \to \overline{V}$, and this is the map known as the Boolean ultrapower. If $\dot{G}$ is the (usual) canonical name for the generic filter, then $[\![\dot{G}$ is $\check{V}$-generic for $\check{\mathbb{B}}]\!] = 1$, and so the corresponding equivalence class $G = [\dot{G}]_U$ is $\overline{V}$-generic for $[\check{\mathbb{B}}]_U = j(\mathbb{B})$. Since these embeddings and equivalence classes all exist in V, we have the entire Boolean ultrapower

$$j : V \to \overline{V} \subseteq \overline{V}[G]$$

existing in V, as desired. The structure $\overline{V}[G]$ is isomorphic to the quotient $V^{\mathbb{B}}/U$ by the map associating $[\tau]_U = [\dot{val}(\check{\tau}, \dot{G})]_U$ in $V^{\mathbb{B}}/U$ with $val([\check{\tau}]_U, G)$ in $\overline{V}[G]$. □

Certain instances of this phenomenon are already well known. For example, consider Prikry forcing with respect to a normal measure μ on a measurable cardinal κ, which is $<\kappa$-friendly because it adds no bounded subsets to κ. If $V \to M_1 \to M_2 \to \cdots$ is the usual iteration of μ, with a direct limit to $j_\omega : V \to M_\omega$, then the critical sequence $\kappa_0, \kappa_1, \kappa_2, \ldots$ is well known to be M_ω-generic for the corresponding Prikry forcing at $j_\omega(\kappa)$ using $j_\omega(\mu)$. This is precisely the situation occurring in Theorem 11, where we have an embedding $j : V \to \overline{V}$ and a $\overline{V}$-generic filter $G \subseteq j(\mathbb{P})$ all inside V. Thus, Theorem 11 generalizes this classical aspect about Prikry forcing to all friendly forcing under the stronger assumption of strong compactness.

We now derive Theorem 8 as a corollary.

Proof of Theorem 8. We shall apply Theorem 10 by finding a $<\kappa$-friendly version of the Laver preparation. The original Laver preparation of [Lav78] is not friendly, because there are many stages $\gamma < \kappa$ at which it definitely adds, for example, a

Cohen subset to γ. But a relatively simple modification will make it $<\kappa$-friendly. Suppose that $\ell : \kappa \to V_\kappa$ is a Laver function. It follows easily that the restriction $\ell \restriction (\gamma, \kappa)$ to any final segment (γ, κ) of κ is also a Laver function, and the corresponding Laver preparation $\mathbb{P}_{\ell \restriction (\gamma, \kappa)}$ is $\leq\gamma$-closed, hence adding no new subsets to γ, while still forcing indestructibility for κ. Let $\mathbb{P} = \oplus\{\mathbb{P}_{\ell \restriction (\gamma, \kappa)} \mid \gamma < \kappa\}$ be the lottery sum of all these various preparations[1], so that the generic filter in effect selects a single γ and then forces with $\mathbb{P}_{\ell \restriction (\gamma, \kappa)}$. This poset is $<\kappa$-friendly, since a condition could opt in the lottery to use a preparation with γ as large below κ as desired. The point is that the Laver preparation works fine for indestructibility even if we allow it to delay the start of the forcing as long as desired, and such a modification makes it $<\kappa$-friendly. So Theorem 10 applies, and Theorem 8 now follows as a corollary. $\square$

After realizing that Theorem 8 could be proved via Boolean ultrapowers, we searched for a direct proof. We arrived at the following stronger result, which produces more robust inner models W, satisfying a closure condition $W^\theta \subseteq W$.

Theorem 12. *If there is a supercompact cardinal, then for every cardinal θ there is an inner model W with an indestructible supercompact cardinal, such that $W^\theta \subseteq W$.*

Proof. Suppose that κ is supercompact. By a result of Solovay [Sol74], the SCH holds above κ, and so if θ is any singular strong limit cardinal of cofinality at least κ, then $2^{\theta^{<\kappa}} = \theta^+$. Consider any such θ as large as desired above κ, and let $j : V \to M$ be a θ-supercompactness embedding, the ultrapower by a normal fine measure on $P_\kappa(\theta)$. Thus, $\theta < j(\kappa)$ and $M^\theta \subseteq M$. By elementarity, $j(\kappa)$ is supercompact in M. Let $\mathbb{P}$ be the Laver preparation of $j(\kappa)$ in M, with nontrivial forcing only in the interval $(\theta, j(\kappa))$. That is, we put off the start of the Laver preparation until beyond θ, and this is exactly what corresponds to the use of friendliness in the earlier proof. Notice that $\mathbb{P}$ is $\leq\theta$-closed in M, and therefore also $\leq\theta$-closed in V. But also, $\mathbb{P}$ has size $j(\kappa)$ in M, and has at most $j(2^\kappa)$ many dense subsets in M. Observe in V that $|j(2^\kappa)| \leq (2^\kappa)^{\theta^{<\kappa}} \leq (2^{\theta^{<\kappa}})^{\theta^{<\kappa}} = 2^{\theta^{<\kappa}} = \theta^+$. In V we may therefore enumerate the dense subsets of $\mathbb{P}$ in M in a θ^+ sequence, and using the fact that $\mathbb{P}$ is $\leq\theta$-closed, diagonalize to meet them all. So there is in V an M-generic filter $G \subseteq \mathbb{P}$. Thus, $M[G]$ is an inner model of V, in which $j(\kappa)$ is an indestructible supercompact cardinal. Since $M^\theta \subseteq M$, it follows that $M[G]$ contains all θ-sequences of ordinals in V, and so also $M[G]^\theta \subseteq M[G]$. So $W = M[G]$ is as desired. $\square$

This second method of proof can be generalized to the following, where we define that $\mathbb{P}$ is $<\kappa$-*superfriendly*, if for every $\gamma < \kappa$ there is a condition $p \in \mathbb{P}$ such that $\mathbb{P} \restriction p$ is $\leq\gamma$-strategically closed. It was the superfriendliness of the Laver preparation that figured in the proof of Theorem 12 and the proof generalizes in a straightforward way to obtain the theorem below.

Theorem 13. *If κ is supercompact and $\mathbb{P}$ is $<\kappa$-superfriendly, then for every θ there is an inner model W satisfying every statement forced by $\mathbb{P}$ over V and for which $W^\theta \subseteq W$.*

Proof. Indeed, in the style of Theorem 11, we show that there is an embedding $j : V \to M$ into a transitive class M, such that in V there is an M-generic filter

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[1]If $\mathcal{A} = \{\mathbb{P}_i \mid i \in I\}$, then the *lottery sum* $\oplus\mathcal{A}$ is the partial order with underlying set $\{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathcal{A} \text{ and } p \in \mathbb{P}\} \cup \{\mathbf{1}\}$, ordered by $\langle \mathbb{P}, p \rangle \leq \langle \mathbb{Q}, q \rangle$ if and only if $\mathbb{P} = \mathbb{Q}$ and $p \leq q$, with $\mathbf{1}$ above everything. The lottery preparation of [Ham00] employs long iterations of such sums.

$G \subseteq j(\mathbb{P})$, such that $M[G]^\theta \subseteq M[G]$, so that $W = M[G]$ is the desired inner model. As in the proof of Theorem 12, using Solovay's theorem we may choose θ a singular strong limit cardinal of cofinality at least κ as large as desired, such that $2^{\theta^{<\kappa}} = \theta^+$, and also $|\mathbb{P}| \leq \theta$. Let $j : V \to M$ be a θ-supercompactness embedding, the ultrapower by a normal fine measure on $P_\kappa(\theta)$. In particular, $M^\theta \subseteq M$. Using the fact that $j(\mathbb{P})$ is $<j(\kappa)$-superfriendly in M, there is a condition $p \in j(\mathbb{P})$ such that $j(\mathbb{P}) \restriction p$ is $\leq\theta$-strategically closed. Since $\mathbb{P}$ has size at most θ, the number of dense subsets of $\mathbb{P}$ is at most $2^\theta = \theta^+$. Thus, the number of dense subsets of $j(\mathbb{P})$ in M, counted in V, is at most $|j(\theta^+)| \leq (\theta^+)^{\theta^{<\kappa}} = (2^{\theta^{<\kappa}})^{\theta^{<\kappa}} = 2^{\theta^{<\kappa}} = \theta^+$, by our choice of θ. Thus, in V we may line up into a θ^+ sequence the dense subsets of $j(\mathbb{P})$ that are in M. Since $j(\mathbb{P})$ is $<\theta^+$-strategically closed in M and $M^\theta \subseteq M$, we may diagonalize against this sequence and build in V an M-generic filter $G \subseteq j(\mathbb{P})$. The model $W = M[G]$ is as desired. Note that $W^\theta \subseteq W$ since every θ-sequence of ordinals from V is already in M, and W can code sets with ordinals. $\qquad\square$

We can now solve several more of the test questions as corollaries.

Theorem 14. *If there is a supercompact cardinal, then there is an inner model with an indestructible supercompact cardinal κ such that $2^\kappa = \kappa^+$ and another inner model with an indestructible supercompact cardinal κ such that $2^\kappa = \kappa^{++}$. Thus, the answers to Test Questions 2, 3 and 4 are yes. Indeed, for any cardinal θ, such inner models W can be found for which also $W^\theta \subseteq W$.*

Proof. Let $\mathbb{P}$ be the $<\kappa$-friendly version of the Laver preparation used in Theorem 8, which is easily seen to be $<\kappa$-superfriendly, and let $\dot{\mathbb{Q}} = \dot{\text{Add}}(\kappa^+, 1)$ be the subsequent forcing to ensure $2^\kappa = \kappa^+$. The combination $\mathbb{P} * \dot{\mathbb{Q}}$ remains $<\kappa$-superfriendly, forces $2^\kappa = \kappa^+$ and preserves the indestructible supercompactness of κ. Thus, by either Theorem 10 or 13, there is an inner model satisfying this theory. Similarly, if $\dot{\mathbb{R}} = \dot{\text{Add}}(\kappa, \kappa^{++})$, then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is $<\kappa$-superfriendly, preserves the indestructible supercompactness of κ and forces $2^\kappa = \kappa^{++}$, so again there is an inner model of the desired theory. The method of Theorem 13 will ensure in each case, for any desired cardinal θ, that the inner model W satisfies $W^\theta \subseteq W$. $\qquad\square$

The proof admits myriad alternatives. For example, we could have just as easily forced $2^\kappa = \kappa^{+++}$, or GCH on a long block of cardinals at κ and above, or failures of this, in any definable pattern above κ. If $\mathbb{Q}$ is any $<\kappa$-directed closed forcing, to be performed after the (superfriendly) Laver preparation, then the combination $\mathbb{P} * \dot{\mathbb{Q}}$ is $<\kappa$-superfriendly and preserves the indestructible supercompactness of κ. Thus, any statement forced by $\mathbb{P} * \dot{\mathbb{Q}}$, using any parameter in V_κ, will be true in the inner models W arising in Theorems 10 and 13. See also Theorem 15 for an intriguing application of this method.

We now apply these methods to the family of questions surrounding Questions 6 and 7. The following theorem answers Question 7 and several of its variants, but not Question 6. With our third proof method in a later section, we will deduce the full conclusion of Question 6 using a slightly stronger hypothesis.

Theorem 15.

(1) *If κ is strongly compact, then there is an inner model W with a strongly compact cardinal, such that $H_{\kappa^+}^V \subseteq \text{HOD}^W$. If the GCH holds below κ, then for any $A \in H_{\kappa^+}^V$, one can arrange that A is definable in W without parameters.*

(2) *If κ is measurable and $2^\kappa = \kappa^+$, then there is an inner model W with a measurable cardinal, such that $H^V_{\kappa^+} \subseteq \text{HOD}^W$. If the GCH holds below κ, then for any $A \in H^V_{\kappa^+}$, one can arrange that A is definable in W without parameters.*

(3) *If κ is supercompact, then for every set $A \in H^V_{\kappa^+}$ and every cardinal θ, there is an inner model W with a supercompact cardinal in which $A \in \text{HOD}^W$ and $W^\theta \subseteq W$. If the GCH holds below κ, then one can arrange that A is definable in W without parameters.*

In particular, by Theorem 15(3), the answer to Question 7 is yes.

Proof. For statement (1), we use the Boolean ultrapower method of Theorems 10 and 11. For $\gamma < \kappa$, let $\mathbb{Q}_\gamma$ be the poset that codes $P(\gamma)$ into the GCH pattern on a block of cardinals above γ, using $\leq\gamma$-closed forcing. Let $\mathbb{P}$ be the lottery sum $\oplus\{\mathbb{Q}_\gamma \mid \gamma < \kappa\}$, which is $<\kappa$-superfriendly and therefore $<\kappa$-friendly. Since each $\mathbb{Q}_\gamma$ is small relative to κ, it follows by the results of [LS67] that κ remains strongly compact after forcing with $\mathbb{P}$. By Theorem 11, there is an embedding $j : V \to \overline{V}$ into an inner model $\overline{V}$, and in V, there is a $\overline{V}$-generic filter $G \subseteq j(\mathbb{P})$. Since $\kappa \leq \text{cp}(j)$, $V_\kappa = \overline{V}_\kappa$. Consequently, the ordinal γ selected by G in the lottery $j(\mathbb{P})$ must be at least κ. Thus, G is coding $P(\gamma)$ and hence also $P(\kappa)$ into the continuum function in some interval between κ and $j(\kappa)$. So in the inner model $W = \overline{V}[G]$, we have a strongly compact cardinal $j(\kappa)$, as well as $P(\kappa)^V \subseteq \text{HOD}^W$ and hence $H^V_{\kappa^+} \subseteq \text{HOD}^W$, as desired.

For the second part of statement (1), in the case that the GCH holds below κ in V, consider any $A \in H_{\kappa^+}$. Let $\hat{A} \subseteq \kappa$ be a subset of κ coding A in some canonical way. Let $\mathbb{P}$ be the forcing as in the previous paragraph, but modified so that the lottery also may choose the order in which the sets in $P(\gamma)$ are coded. By inspecting the proof of Theorem 11, i.e., by forcing below the appropriate condition, we may assume that G opts for a poset in $j(\mathbb{P})$ that begins coding at γ, which is the successor of a cardinal of cofinality κ, and that $\hat{A}$ is the first set to be coded. Note that in $W = \overline{V}[G]$, the cardinal γ is definable as the place where the GCH coding starts, and so κ is definable as the cofinality of the predecessor of γ, and so $\hat{A}$ and hence A are definable in W without parameters. As in the previous paragraph, we also have a strongly compact cardinal in W and $H^V_{\kappa^+} \subseteq W$.

For statement (2), suppose that κ is measurable and $2^\kappa = \kappa^+$. We follow the method of Theorem 12. Let $j : V \to M$ be the ultrapower by any normal measure on κ. Let $\mathbb{P}$ be the forcing used to prove the second part of statement (1), which by lottery selects some $\gamma < \kappa$ and an enumeration of $P(\gamma)$, which is then coded into the GCH pattern above γ. In the forcing $j(\mathbb{P})$, consider a condition p that opts to code $P(\kappa)$. Thus, $j(\mathbb{P}) \restriction p$ is $\leq\kappa$-closed and has size less than $j(\kappa)$. Since $2^\kappa = \kappa^+$, the number of subsets of $j(\mathbb{P})$ in M, counted in V, is bounded by $|j(2^\kappa)|^V \leq (2^\kappa)^\kappa = \kappa^+$. Thus, by diagonalization, we may construct in V an M-generic filter $G \subseteq j(\mathbb{P})$ below p. Let $W = M[G]$. By the results of [LS67], the cardinal $j(\kappa)$ remains measurable in W, since below p the forcing was small relative to $j(p)$. In addition, every set in $P(\kappa)^V = P(\kappa)^M$ is coded into the continuum function of W, so $H^V_{\kappa^+} \subseteq \text{HOD}^W$, as desired. If the GCH holds below κ, then we can define κ in W as the place where the coding begins, and we can define the first set that is coded (among others), without parameters.

For statement (3), where κ is supercompact, we may use the same argument as in statements (1) and (2), but employing the method of Theorem 12. The resulting inner model $W = M[G]$ will have $H_{\kappa^+}^V \subseteq \mathrm{HOD}^W$ and $W^\theta \subseteq W$, as desired. And again, when the GCH holds below κ in V, then we may arrange that any particular $A \in H_{\kappa^+}^V$ is definable in W without parameters. □

Let us highlight the consequences of this theorem with the following striking example. Namely, suppose that κ is strongly compact in V and the GCH holds. Both of these statements remain true in the forcing extension $V[c]$ obtained by adding a single V-generic Cohen real c. Since this forcing is almost homogeneous, we know c is not in $\mathrm{HOD}^{V[c]}$. (See [Rei06] and [Rei07] for the definition of almost homogeneous and a discussion of this property of almost homogeneous forcing.) Nevertheless, by Theorem 15, there are inner models $W \subseteq V[c]$ such that $H_{\kappa^+}^V \subseteq \mathrm{HOD}^W$, with a strongly compact cardinal, in which c is definable without parameters!

Our first two proof methods are attractive in that they are able to answer several of the test questions with the provably optimal hypothesis and, moreover, while also producing inner models with especially desirable features, such as $W^\theta \subseteq W$ for any desired θ. Nevertheless, and perhaps as a consequence, these methods seem unable to produce inner models in which the full GCH holds, say, if the CH fails in V, because the resulting inner models for those methods will agree with V up to V_κ and beyond, where κ is the initial supercompact cardinal. Similarly, neither method seems able to produce an inner model in which PFA holds, since the only known forcing to attain this—a long countable support iteration of proper forcing—adds Cohen reals unboundedly often and is therefore highly non-friendly. Furthermore, the methods seem not easily to accommodate class forcing, and allow us only to put particular sets A into HOD^W for an inner model W, without having W fully satisfy $V = \mathrm{HOD}$. Therefore, these methods seem unable to answer Questions 1, 5 and 6. (With our third proof method, we shall give partial solutions of these questions in Theorems 23 and 25, by using a stronger hypothesis.) hypothesis.) Another unusual feature of our first two methods, as used in Theorems 8, 12 and 14, is that it is not the same supercompact cardinal κ that is found to be supercompact in the desired inner model. Rather, it is in each case the ordinal $j(\kappa)$ that is found to be supercompact (and indestructible or with fragments or failures of the GCH) in an inner model. A modified version of Question 4 could ask, after all, whether every supercompact cardinal κ is itself indestructibly supercompact in an inner model. For precisely this question, we don't know, but if κ is supercompact up to a weakly iterable cardinal above κ, then the answer is yes by Theorem 22. (See Section 3 for the definition of weakly iterable cardinal.)

So let us now turn to the third method of proof, which will address these concerns, at the price of an additional large cardinal hypothesis. We shall use this method to produce an inner model with a supercompact cardinal and the full GCH, an inner model with the PFA and an inner model where κ itself is indestructibly supercompact, among other possibilities. The method is very similar to the methods introduced and fruitfully applied by Sy Friedman [Fri06] and by Sy Friedman and Natasha Dobrinen [DF08], [DF10], where they construct class generic filters in V over an inner model W. Also Ralf Schindler, in a personal communication with the third author, used a version of the method to provide an answer to Test Question 5, observing that if there is a supercompact cardinal with a measurable cardinal above it, then there is an inner model of the PFA.

Theorem 16 (Schindler). *If there is a supercompact cardinal with a measurable cardinal above it, then there is an inner model with the PFA.*

The basic idea is that if κ is supercompact and $\kappa < \delta$ for some measurable cardinal δ, then one finds a countable elementary substructure $X \prec V_\theta$, with $\delta \ll \theta$, whose Mostowski collapse is a countable iterable structure with a supercompact cardinal κ_0 below a measurable cardinal δ_0. By iterating the measurable cardinal δ_0 of this structure out of the universe, one arrives at a full inner model M, and because κ_0 was below the critical point of the iteration, which is δ_0, it follows that both κ_0 and even $P(\kappa_0)^M$ are countable in V. Thus, by the usual diagonalization in V, there is an M-generic filter G for the Baumgartner PFA forcing (or whatever other forcing was desired), and so $M[G]$ is the desired inner model. This method generalizes to *any* forcing notion below a measurable cardinal.

In the subsequent sections of this article, we shall elaborate on the details of this argument, while also explaining how to reduce the hypothesis from a measurable cardinal above the supercompact cardinal to merely a weakly iterable cardinal above. The construction encounters a few complications in the class-length forcing iterations, since (unlike the argument above) these iterations will be stretched to proper class size during the iteration, and so one cannot quite so easily produce the desired M-generic filter. Nevertheless, the new method remains fundamentally similar to the argument we described in the previous paragraph. Finally, we shall give several additional applications of the method.

3. Iterable Structures

We now develop some basic facts about iterable structures, which shall be sufficient to carry out the third proof method. In particular, we shall review the fact that any structure elementarily embedding into an iterable structure is itself iterable, and for a special class of forcing required in later arguments, we shall give sufficient conditions for a forcing extension of a countable iterable structure to remain iterable.

Consider structures of the form $\langle M, \delta, U \rangle$ where $M \models \mathrm{ZFC}^-$ is transitive, δ is a cardinal in M, and $U \subseteq \mathcal{P}(\delta)^M$. The set U is an *M-ultrafilter*, if $\langle M, \delta, U \rangle \models$ "U is a normal ultrafilter". An M-ultrafilter U is *weakly anemable* if $U \cap A \in M$ for every set A of size δ in M. By using only the equivalence classes of functions in M, an M-ultrafilter suffices for the usual ultrapower construction. It is easy to see that U is weakly amenable exactly when M and the ultrapower of M by U have the same subsets of δ, that is the ultrapower embedding is δ-powerset preserving. In this case, it turns out that one can define the iterated ultrapower of M by U to any desired ordinal length. We say that $\langle M, \delta, U \rangle$ is *iterable* if U is a weakly amenable M-ultrafilter and all of these resulting iterated ultrapowers are well-founded.

Definition 17. A cardinal δ is *weakly iterable* if there is an iterable structure $\langle M, \delta, U \rangle$ containing V_δ as an element.

It is clear that measurable cardinals are weakly iterable. Ramsey cardinals also are weakly iterable, since if δ is Ramsey, every $A \subseteq \delta$ is an element of an iterable structure $\langle M, \delta, U \rangle$ (see [Mit79]). On the other hand, a weakly iterable cardinal need not even be regular. For example, every measurable cardinal remains weakly iterable after Prikry forcing, because the ground model iterable structures still exist. More generally, we claim that the least weakly iterable cardinal must have

cofinality ω. To see this, suppose that δ is a weakly iterable cardinal of uncountable cofinality with the iterable structure $\langle M, \delta, U \rangle$. We shall argue that there is a smaller weakly iterable cardinal of cofinality ω. Choose $X_0 \prec M$ for some countable X_0 containing δ, and let $\gamma_0 = \sup(X_0 \cap \delta)$. Inductively define $X_{n+1} \prec_{\Sigma_n} M$ with $\gamma_{n+1} = \sup(X_{n+1} \cap \delta) < \delta$ satisfying $V_{\gamma_{n+1}} \subseteq X_{n+1}$ and $|X_{n+1}| < \delta$. This is possible since δ is inaccessible in M, so the witnesses we need to add to X_{n+1} below δ will be bounded below δ, even if δ may be singular in V. Observe that if $X_\omega = \bigcup_{n \in \omega} X_n$ and $\langle N, \gamma, W \rangle$ is the collapse of the structure $\langle X_\omega, \delta, U \cap X_\omega \rangle$, then δ collapses to $\gamma = \sup_n \gamma_n$ and so $V_\gamma \in N$. The iterability of $\langle N, \gamma, W \rangle$ will follow from Lemma 18 below, completing the argument that γ is a weakly iterable cardinal of cofinality ω below δ.

If δ is weakly iterable with the iterable structure $\langle M, \delta, U \rangle$, then δ is at least ineffable in M and therefore, the existence of weakly iterable cardinals carries at least this large cardinal strength (see [Git09]). In fact, weakly iterable cardinals cannot exist in L (see [GW10]), but it follows from [Wel04] that they are weaker than an ω_1-Erdős cardinal. Note that the inaccessibility of δ in the domain of the iterable structure witnessing its weak iterability implies that it is a $\beth$-fixed point and $V_\delta \models$ ZFC, by the absoluteness of satisfaction.

Lemma 18. *Suppose $\langle M, \delta, U \rangle$ is iterable. Suppose further that $\langle N, \gamma, W \rangle$ is a structure for which there exists an elementary embedding $\rho : N \to M$ in the language with $\in$ only with the additional properties that $\rho(\gamma) = \delta$, and whenever $x \in N$ is such that $x \subseteq W$, then $\rho(x) \subseteq U$. Then $\langle N, \gamma, W \rangle$ is iterable as well.*

Proof. This is a standard idea. We shall demonstrate the iterability of $\langle N, \gamma, W \rangle$ by embedding the iterated ultrapowers of N by W into the iterated ultrapowers of M by U. Let $\{j_{\xi\gamma} : M_\xi \to M_\gamma \mid \xi < \gamma \in \mathrm{Ord}\}$ be the directed system of iterated ultrapowers of $M = M_0$ with the associated sequence of ultrafilters $\{U_\xi \mid \xi \in \mathrm{Ord}\}$ and the critical sequence $\{\kappa_\xi \mid \xi \in \mathrm{Ord}\}$. Also, let $\{h_{\xi\gamma} : N_\xi \to N_\gamma \mid \xi < \gamma < \alpha\}$ be the not necessarily well-founded directed system of iterated ultrapowers of $N = N_0$ with the associated sequence of ultrafilters $\{W_\xi \mid \xi \in \mathrm{Ord}\}$. Let $\{W_0^i : i \in I\}$ be an enumeration of all subsets of W_0 that are elements of N_0, and define $W_\xi^i = h_{0\xi}(W_0^i)$. It is easy to see that if $A \subseteq W_\xi$ is an element of N_ξ, then $A \subseteq W_\xi^i$ for some $i \in I$. We shall show that the following diagram commutes:

$$
\begin{array}{ccccccccccc}
M_0 & \xrightarrow{j_{01}} & M_1 & \xrightarrow{j_{12}} & M_2 & \xrightarrow{j_{23}} & \ldots & \xrightarrow{j_{\xi\xi+1}} & M_{\xi+1} & \xrightarrow{j_{\xi+1\xi+2}} & \ldots \\
\uparrow{\scriptstyle\rho_0} & & \uparrow{\scriptstyle\rho_1} & & \uparrow{\scriptstyle\rho_2} & & & & \uparrow{\scriptstyle\rho_\xi} & & \\
N_0 & \xrightarrow{h_{01}} & N_1 & \xrightarrow{h_{12}} & N_2 & \xrightarrow{h_{23}} & \ldots & \xrightarrow{h_{\xi\xi+1}} & N_{\xi+1} & \xrightarrow{h_{\xi+1\xi+2}} & \ldots
\end{array}
$$

where

(1) $\rho_{\xi+1}([f]_{W_\xi}) = j_{\xi\xi+1}(\rho_\xi(f))(\kappa_\xi)$,

(2) if λ is a limit ordinal and t is a thread in the direct limit N_λ with domain $[\beta, \lambda)$, then $\rho_\lambda(t) = j_{\beta\lambda}(\rho_\beta(t(\beta)))$, and

(3) $\rho_\xi(W_\xi^i) \subseteq U_\xi$.

We shall argue that the ρ_ξ exist by induction on ξ. Let $\rho_0 = \rho$, and note that ρ_0 satisfies condition (3) by hypothesis. Suppose inductively that ρ_ξ has the desired properties. Define $\rho_{\xi+1}$ as in (1) above. Since $\rho_\xi(W_\xi^i) \subseteq U_\xi$ by the inductive assumption, it follows that $\rho_{\xi+1}$ is a well-defined elementary embedding. The commutativity of the diagram is also clear. It remains to verify that $\rho_{\xi+1}(W_{\xi+1}^i) \subseteq U_{\xi+1}$.

Recall that

$$W_{\xi+1}^i = h_{\xi\xi+1}(W_\xi^i) = [c_{W_\xi^i}]_{W_\xi}.$$

Let $\rho_\xi(W_\xi^i) = v$. Then by the inductive assumption, $v \subseteq U_\xi$. Thus,

$$\rho_{\xi+1}(W_{\xi+1}^i) = j_{\xi\xi+1}(c_v)(\kappa_\xi) = c_{j_{\xi\xi+1}(v)}(\kappa_\xi) = j_{\xi\xi+1}(v) \subseteq U_{\xi+1}.$$

This completes the inductive step. The limit case also follows easily.   □

Note that if ρ is an elementary embedding of the whole structure $\langle N, \gamma, W \rangle$, then the hypotheses of Lemma 18 follow easily. This is how Lemma 18 will be used in most applications below.

In the next section, we shall build inner models by iterating out these countable iterable structures and forcing over the limit model inside the universe, just as we explained in the proof sketch for Theorem 16. In other arguments, however, the desired forcing will be stretched to proper class length, and so we shall proceed instead by first forcing over the countable structure and then iterating the extended structure. For these arguments, therefore, we need to understand when a forcing extension of an iterable countable structure remains iterable. For a certain general class of forcing notions and embeddings, we shall show in Theorem 19 that indeed the lift of an iterable embedding to a forcing extension remains iterable, and what is more, lifting just the first step of the iteration to the forcing extension can lead to a lift of the entire iteration. In rather general circumstances, therefore, the iteration of a lift is a lift of the iteration.

This argument will rely on the following characterization of when an ultrapower of a forcing extension is a lift of the ultrapower of the ground model. Suppose that M is a transitive model of ZFC$^-$, that $\mathbb{P}$ is a poset in M and that $G \subseteq \mathbb{P}$ is M-generic. Suppose further that U is an M-ultrafilter on a cardinal δ in M and U^* is an $M[G]$-ultrafilter extending U, both with well-founded ultrapowers. Then the ultrapower by U^* lifts the ultrapower by U if and only if every $f : \delta \to M$ in $M[G]$ is U^*-equivalent to some $g : \delta \to M$ in M. To see why this no-new-functions characterization is true, note that the ultrapower by U^* must have the form of a forcing extension by the image of $\mathbb{P}$, with the ground model consisting exactly of the equivalence classes of functions $f : \delta \to M$ in $M[G]$. Applying this characterization, if we lift the first embedding in the iteration, then the ultrafilter derived from the lift will have the above property. The key to the argument will be to capture this property as a schema of first-order statements over the forcing extension and propagate it along the iteration using elementarity.

Let us now discuss a class of posets for which this strategy proves successful. Suppose $j : M \to N$ is an elementary embedding with critical point δ. We define that a poset $\mathbb{P} \in M$ is *j-useful* if $\mathbb{P}$ is δ-c.c. in M and $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{P}}_{\text{tail}}$, where $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{P}}_{\text{tail}}$ is $\leq\delta$-strategically closed" in N. There are numerous examples of such posets arising in the context of forcing with large cardinals, and we shall mention several in Sections 4 and 5. We presently explain how the property of j-usefulness allows us to find lifts of an ultrapower embedding to the forcing extension, so that the iteration of the lift is the lift of the iteration. If $\mathbb{Q}$ is any poset and X is a set, not necessarily transitive, define as usual that a condition $q \in \mathbb{Q}$ is *X-generic for* $\mathbb{Q}$ if for every V-generic filter $G \subseteq \mathbb{Q}$ containing q and every maximal antichain $A \subseteq \mathbb{Q}$ with $A \in X$, the intersection $G \cap A \cap X \neq \emptyset$; in other words, q forces over V that the generic filter meets the maximal antichains of X inside X. Suppose

$j : M \to N$ and $\mathbb{P}$ is j-useful. Our key observation about j-usefulness is that if $X \in N$ is sufficiently elementary in N with $X^{<\delta} \subseteq X$ and $|X| = \delta$ in N, then every condition $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{P}}_{\text{tail}} \cap X$ can be strengthened to an X-generic condition. First, observe that every condition in $\mathbb{P}$ is X-generic for $\mathbb{P}$, since maximal antichains of $\mathbb{P}$ have size less than δ and so if X contains such an antichain as an element, it must be a subset as well. Thus, for the pair $(p, \dot{q})$ to be X-generic for $j(\mathbb{P})$, it suffices for p to force that $\dot{q}$ sits below some element of every dense subset of $\mathbb{P}_{\text{tail}}$ in $X[\dot{G}]$. Such a $\dot{q}$ is found by a simple diagonalization argument, using the facts that $|X| = \delta$, $X^{<\delta} \subseteq X$ and $\dot{\mathbb{P}}_{\text{tail}}$ is forced to be $\leq\delta$-strategically closed.

Let us use the notation $\langle M, \delta, U \rangle \models$ "I am H_{δ^+}" to mean that M believes every set has size at most δ. We now prove that if a certain external genericity condition is met, then the iteration of a lift is a lift of the iteration.

Theorem 19. *Suppose that $\langle M_0, \delta, U_0 \rangle \models$ "I am H_{δ^+}" is iterable, and that the first step of the iteration $j_{01} : M_0 \to M_1$ lifts to an embedding $j_{01}^* : M_0[G_0] \to M_1[G_1]$ on the forcing extension, where $G_0 \subseteq \mathbb{P}$ is M_0-generic, $j_{01}^*(G_0) = G_1 \subseteq j_{01}(\mathbb{P})$ is M_1-generic and $\mathbb{P}$ is j_{01}-useful.*

$$
\begin{array}{ccc}
M_0[G_0] & \xrightarrow{\;j_{01}^*\;} & M_1[G_1] \\
\cup & & \cup \\
M_0 \xrightarrow{\;j_{01}\;} & M_1 \xrightarrow{\;j_{12}\;} \cdots \longrightarrow M_\xi \xrightarrow{\;j_{\xi\xi+1}\;} & \cdots
\end{array}
$$

Then j_{01}^* is the ultrapower by a weakly amenable $M_0[G_0]$-ultrafilter U_0^* extending U_0. Furthermore, if G_1 meets certain external dense sets $D_a \subseteq j_{01}(\mathbb{P})$ for $a \in M_0$ described in the proof below, then $\langle M_0[G_0], \delta, U_0^* \rangle$ is iterable, and the entire iteration of $\langle M_0[G_0], \delta, U_0^* \rangle$ lifts the iteration of $\langle M_0, \delta, U_0 \rangle$ step-by-step.

$$
\begin{array}{ccccc}
M_0[G_0] & \xrightarrow{\;j_{01}^*\;} & M_1[G_1] \xrightarrow{\;j_{12}^*\;} \cdots \longrightarrow & M_\xi[G_\xi] & \xrightarrow{\;j_{\xi\xi+1}^*\;} \cdots \\
\cup & & \cup & \cup & \\
M_0 & \xrightarrow{\;j_{01}\;} & M_1 \xrightarrow{\;j_{12}\;} \cdots \longrightarrow & M_\xi & \xrightarrow{\;j_{\xi\xi+1}\;} \cdots
\end{array}
$$

Thus, the iteration of the lift is a lift of the iteration.

Proof. Suppose that the ultrapower $j_{01} : M_0 \to M_1$ by U_0 lifts to $j_{01}^* : M_0[G_0] \to M_1[G_1]$, with $j_{01}^*(G_0) = G_1$. By the normality of U_0, it follows that every element of M_1 has the form $j_{01}(f)(\delta)$ for some $f \in M_0^\delta \cap M_0$. Every element of $M_1[G_1]$ is τ_{G_1} for some $j_{01}(\mathbb{P})$-name $\tau \in M_1$, and so $\tau = j_{01}(t)(\delta)$ for some function $t \in M_0$. Define a function f in $M_0[G_0]$ by $f(\alpha) = t(\alpha)_{G_0}$, and observe that $j_{01}^*(f)(\delta) = j_{01}(t)(\delta)_{j_{01}^*(G_0)} = \tau_{G_1}$. Thus, every element of $M_1[G_1]$ has the form $j_{01}^*(f)(\delta)$ for some $f \in M_0[G_0]^\delta \cap M_0[G_0]$. It follows that j_{01}^* is the ultrapower of $M_0[G_0]$ by the $M[G_0]$-ultrafilter $U_0^* = \{ X \subseteq \delta \mid X \in M_0[G], \delta \in j_{01}^*(X) \}$, which extends U_0. Note that since $\mathbb{P}$ is j_{01}-useful, it follows that $j_{01}(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{P}}_{\text{tail}}$, where $\dot{\mathbb{P}}_{\text{tail}}$ adds no new subsets of δ and $\mathbb{P}$ is δ-c.c. From this, we obtain that $P(\delta)^{M_1[G_1]} = P(\delta)^{M_1[G_0]} = P(\delta)^{M_0[G_0]}$, and so U_0^* is weakly amenable to $M_0[G_0]$. It therefore makes sense to speak of the iterated ultrapowers of $\langle M_0[G_0], \delta, U_0^* \rangle$, apart from the question of whether these iterates are well-founded.

The fact that the ultrapower $j_{01} : M_0 \to M_1$ by U_0 lifts to the ultrapower $j_{01}^* : M_0[G_0] \to M_1[j_{01}^*(G_0)]$ by U_0^* is exactly equivalent to the assertion that for every function $f \in M_0^\delta \cap M_0[G_0]$ there is a function $g \in M_0^\delta \cap M_0$ such that f and g agree on a set in U_0^*. In slogan form: Every new function agrees with an old function. This

property is first order expressible in the expanded structure $\langle M_0[G_0], \delta, U_0^*, M \rangle$, by a statement with complexity at least Π_2. If j_{01}^* were sufficiently elementary on this structure, then it would preserve the truth of this statement and we could deduce easily that the iterates of U_0^* are step-by-step lifts of the corresponding iterates of U_0, completing the proof. Unfortunately, in the general case we cannot be sure that j_{01}^* is sufficiently elementary on this expanded structure. Similarly, although the original embedding $j_{01} : M_0 \to M_1$ is fully elementary, it may not be fully elementary on the corresponding expanded structure $j_{01} : \langle M_0, \delta, U_0 \rangle \to \langle M_1, \delta_1, U_1 \rangle$. The rest of this argument, therefore, will be about getting around this difficulty by showing that if G_1 satisfies an extra genericity criterion, then the iteration of U_0^* does indeed lift the iteration of U_0.

Specifically, through this extra requirement on G_1, we will arrange that for every $a \in M_0$, there is a set $m_a \in M_0$ such that

(1) m_a is a transitive model of ZFC^- containing $\mathbb{P}$ and a, and
(2) every $f : \delta \to m_a$ in $m_a[G_0]$ is u_a^*-equivalent to some $g : \delta \to m_a$ in m_a,

where $u_a^* = m_a \cap U_0^*$, which is an element of $M_0[G_0]$ by the weak amenability of U_0^* to $M_0[G_0]$.

Let us first suppose that we have already attained (1) and (2) for every a and explain next how this leads to the conclusion of the theorem. Suppose inductively that the iteration of U_0^* on $M_0[G_0]$ is a step-by-step lift of the iteration of U_0 on M_0 up to stage ξ. Note that limit stages come for free, because if every successor stage before a limit is a lift, then the limit stage is also a lift. Thus, we assume that the diagram in the statement of the theorem is accurate through stage ξ, so that in particular the ξth iteration $j_{0\xi}^* : M_0[G_0] \to M_\xi[G_\xi]$ of U_0^* is a lift of the ξth iteration $j_{0\xi} : M_0 \to M_\xi$ of U_0, and we consider the next step $M_\xi[G_\xi] \to \text{Ult}(M_\xi[G_\xi], U_\xi^*)$. Since any given instance of (1) and (2), for fixed a, is expressible in $M_0[G_0]$ as a statement about $(m_a, G_0, u_a^*, a, \mathbb{P})$, it follows by elementarity that $j_{0\xi}^*(m_a)$ is a transitive model of ZFC^- containing $j_{0\xi}(\mathbb{P})$, and that every $f : j_{0\xi}^*(\delta) \to j_{0\xi}^*(m_a)$ in $j_{0\xi}^*(m_a)[G_\xi]$ is $j_{0\xi}^*(u_a^*)$-equivalent to a function $g : j_{0\xi}^*(\delta) \to j_{0\xi}^*(m_a)$ in $j_{0\xi}(m_a)$. Note that since $u_a^* \subseteq U_0^*$, it follows by an easy argument that $j_{0\xi}^*(u_a^*) \subseteq U_\xi^*$. Thus, as far as $j_{0\xi}^*(m_a)$ and $j_{0\xi}^*(m_a)[G_\xi]$ are concerned, every new function agrees with an old function. But now the key point is that the $j_{0\xi}(m_a)$ exhaust M_ξ, since every object in M_ξ has the form $j_{0\xi}(f)(s)$ for some finite $s \subseteq \delta_\xi$, and thus once we put f into m_a by a suitable choice of a, then $j_{0\xi}(f)(s)$ will be in $j_{0\xi}(m_a)$. From this, it follows that the $j_{0\xi}^*(m_a[G_0])$ exhaust $M_\xi[G_\xi]$, since every element of $M_\xi[G_\xi]$ has a name in M_ξ. Therefore, every new function in $M_\xi[G_\xi]$ agrees on a set in U_ξ^* with an old function in M_ξ, and so the ultrapower of $M_\xi[G_\xi]$ by U_ξ^* is a lift of $j_{\xi\xi+1}$. Thus, we have continued the step-by-step lifting one additional step, and so by induction, the entire iteration lifts step-by-step as claimed.

It remains to explain how we achieve (1) and (2) for every $a \in M_0$. First, we observe that M_0 is the union of transitive models m of ZFC. This is because any set $A \subseteq \delta$ in M_0 is also in M_1 and therefore in $H_{j_{01}(\delta)}^{M_1}$, which is a model of ZFC since $j_{01}(\delta)$ is inaccessible in M_1. By collapsing an elementary substructure of this structure in M_1, therefore, we find a size δ transitive model $m \models \text{ZFC}$ with $A \in m \in M_1$. Since m has size δ and $M_0 = H_{\delta^+}^{M_1}$ by weak amenability, it follows that $m \in M_0$ as well. Thus, for any $a \in M_0$ there are numerous models m as in statement (1), even with full ZFC.

For any such m, let $X_m = \{\, j_{01}(f)(\delta) \mid f \in m \,\}$. It is not difficult to check that $X_m \prec j_{01}(m)$, by verifying the Tarski-Vaught criterion. Also, since $j_{01} \upharpoonright m \in M_1$, it follows that $X_m \in M_1$, although by Replacement the map $m \mapsto X_m$ cannot exist in M_1, since M_1 is the union of all X_m. For any $a \in M_0$, let

$$D_a = \left\{\, q \in j_{01}(\mathbb{P}) \mid q \text{ is } X_m\text{-generic for some transitive } m \models \text{ZFC}^- \text{ with } a \in m \in M_0 \,\right\}.$$

Recall that a condition q is X_m-generic for $j_{01}(\mathbb{P})$ if every M_1-generic filter $G \subseteq j_{01}(\mathbb{P})$ has $G \cap D \cap X_m \neq \emptyset$ for every dense set $D \subseteq j_{01}(\mathbb{P})$ in M_1. Because the definition of D_a refers to the various X_m, there is little reason to expect that D_a is a set in M_1. Nevertheless, we shall argue anyway that it is a dense subset of $j_{01}(\mathbb{P})$.

To see this, fix a and any condition $p \in j_{01}(\mathbb{P})$. Since $p = j(\vec{p})(\delta)$ for some function $\vec{p} \in M_0$, we may find as we explained above a transitive set $m \in M_0$ with $a, \vec{p}, \mathbb{P} \in m \models \text{ZFC}$. We may also ensure in that argument that $m^{<\delta} \subseteq m$ in M_0. It follows that $X_m^{<\delta} \subseteq X_m$ in M_1, and since $\vec{p} \in m$, we also know that $p = j(\vec{p})(\delta) \in X_m$. The forcing $j_{01}(\mathbb{P})$ is in X_m and factors as $\mathbb{P} * \dot{\mathbb{P}}_{\text{tail}}$, where $\mathbb{P}$ is δ-c.c. and $\dot{\mathbb{P}}_{\text{tail}}$ is forced to be $\leq\delta$-strategically closed. Since M_1 knows that X_m has size δ, it can perform a diagonalization below p of the dense sets for the tail forcing, and thereby produce a $\mathbb{P}$-name for a condition in $\dot{\mathbb{P}}_{\text{tail}}$ meeting all those dense sets. (This is where we have used the key property of j-usefulness mentioned before the theorem.) Thus, M_1 can build an X_m-generic condition q for $j_{01}(\mathbb{P})$ below p. This establishes that D_a is dense, as we claimed.

We now suppose that G_1 meets all the dense sets D_a, and use this to establish (1) and (2). For any $a \in M_0$, we have a condition $q \in G_1$ that is X_m-generic for some transitive $m \models \text{ZFC}^-$ in M_0 containing $\langle a, \mathbb{P} \rangle$, thereby satisfying (1). From this, it follows that $X_m[G_1] \cap M_1 = X_m$, since for any name in X_m for an object in M_1, X_m has a dense set of conditions deciding its value, and since G_1 meets this dense set inside X_m, the decided value must also be in X_m. Now, suppose that $f : \delta \to m$ is a function in $m[G_0]$, so that $f = \dot{f}_{G_0}$ for some name $\dot{f} \in m$. Since $\dot{f} \in m$, it follows that $j_{01}(\dot{f}) \in X_m$, and so $j_{01}(f)(\delta) \in X_m[G_1]$. Since $\text{ran}(f) \subseteq m$, it follows that $\text{ran}(j_{01}(f)) \subseteq j_{01}(m)$, which is contained in M_1. Thus, $j_{01}(f)(\delta) \in X_m[G_1] \cap M_1$, which is equal to X_m. But every element of X_m has the form $j_{01}(g)(\delta)$ for some function $g \in m$, and so $j_{01}(f)(\delta) = j_{01}(g)(\delta)$ for such a function g. It follows that f and g agree on a set in U_0^* and we have established (2), completing the argument. $\square$

A special case of the theorem occurs when $\mathbb{P}$ has size smaller than δ in M_0. In this case, $\dot{\mathbb{P}}_{\text{tail}}$ is trivial and the extra genericity condition is automatically satisfied, since the dense sets D_a would be elements of M_1. The nontrivial case of the theorem occurs when the forcing $\mathbb{P}$ has size δ, and its image is therefore stretched on the ultrapower side. We are unsure about the extent to which it could be true generally that the iteration of a lift is a lift of the iteration. Surely some hypotheses are needed on the forcing, since if $\mathbb{P}$ is an iteration of length δ and $j(\mathbb{P})$ adds new subsets to δ at stage δ, for example, then the lift j_{01}^* will not be weakly amenable, making it impossible to iterate. Our j-usefulness hypothesis avoids this issue, but we are not sure whether it is possible to omit the external genericity assumption we made on G_1. Nevertheless, this extra genericity assumption appears to be no more difficult to attain in practice than ordinary M_1-genericity. For example, in the case of countable structures:

Corollary 20. *If $\langle M, \delta, U \rangle \models$ "I am H_{δ^+}" is a countable iterable structure and $\mathbb{P} \in M$ is useful for the ultrapower of M by U, then there is an M-generic filter $G \subseteq \mathbb{P}$ and $M[G]$-ultrafilter U^* extending U such that $\langle M[G], \delta, U^* \rangle$ is iterable, and the iteration of $M[G]$ by U^* is a step-by-step lift of the iteration of M by U.*

Proof. This is simply a special case of the previous theorem. When M is countable, then there is no trouble in finding an M-generic filter G and M_1-generic filter G_1 satisfying the extra genericity requirement, since there are altogether only countably many dense sets to meet. □

4. The third proof method

In this section, for the third proof method, we generalize the proof sketch of Theorem 16 given at the end of Section 2. For the arguments here, we shall use the hypothesis of having a weakly iterable cardinal δ with V_δ a model containing large cardinals. We shall use the structure $\langle M, \delta, U \rangle$ witnessing the weak iterability of δ to produce a countable iterable structure and build the inner model out of the iterates of this structure or the iterates of its forcing extension.

Theorem 21. *If $\langle M, \delta, U \rangle$ is iterable with a poset $\mathbb{P} \in V_\delta^M$, then there is an inner model satisfying every sentence forced by $\mathbb{P}$ over V_δ^M.*

Proof. Let $\langle M_0, \delta_0, U_0 \rangle$ be obtained by collapsing a countable elementary substructure of $\langle M, \delta, U \rangle$ containing $\mathbb{P}$. By Lemma 18, $\langle M_0, \delta_0, U_0 \rangle$ is iterable. Also, if $\mathbb{Q}$ is the collapse of the poset $\mathbb{P}$, then by elementarity $\mathbb{Q}$ forces the same sentences over $V_{\delta_0}^{M_0}$ that $\mathbb{P}$ forces over V_δ^M. Let $\{ j_{\xi\eta} : M_\xi \to M_\eta \mid \xi < \eta \in \mathrm{Ord} \}$ be the corresponding directed system of iterated ultrapowers of M_0, and consider the inner model $W = \bigcup_{\xi \in \mathrm{Ord}} j_{0\xi}(V_{\delta_0}^{M_0})$, which is the cumulative part of the iteration lying below the critical sequence. Since $V_{\delta_0}^{M_0} \prec W$ and $V_{\delta_0}^W = V_{\delta_0}^{M_0}$, it follows that $\mathbb{Q}$ forces the same sentences over $V_{\delta_0}^{M_0}$ as over W, and these are the same as forced by $\mathbb{P}$ over V_δ^M. Since $\mathbb{Q}$ lies below the critical point δ_0 of the iteration, the model W contains only countably many dense subsets of $\mathbb{Q}$ and so we can build a W-generic filter G directly. Thus, the model $W[G]$, an inner model of V, satisfies the requirement of the theorem. □

Let us now apply this theorem to the case of an indestructible supercompact cardinal.

Theorem 22. *If κ is $<\delta$-supercompact for a weakly iterable cardinal δ above κ, then there is an inner model in which κ is an indestructible supercompact cardinal.*

Proof. Suppose that κ is $<\delta$-supercompact for a weakly iterable cardinal δ above κ and the weak iterability of δ is witnessed by an iterable structure $\langle M, \delta, U \rangle$, with $V_\delta \in M$. In particular, κ is $<\delta$-supercompact in M. Note that the Laver preparation $\mathbb{P}$ of κ is small relative to δ in M. Thus, by Theorem 21, there is an inner model W_0 satisfying the theory forced by $\mathbb{P}$ over V_δ. The forcing $\mathbb{P}$, of course, makes κ indestructibly supercompact in $V_\delta^{\mathbb{P}}$, and so the inner model W_0 has an indestructible supercompact cardinal κ_0.

In order to prove the full claim, we must find a W in which κ itself is indestructibly supercompact. For this, let us look more closely at how the inner model W_0 arises from the proof of Theorem 21. Specifically, the indestructible supercompact cardinal κ_0 of W_0 arises inside a countable iterable structure M_0, obtained

by a Mostowski collapse of a countable structure containing κ, and κ_0 is below the critical point δ_0 of the iteration. Thus, κ_0 is not moved by the iteration and is therefore a countable ordinal in V, even though it is indestructibly supercompact in W_0. Since in particular κ_0 is measurable in W_0, we may consider the internal system of embeddings obtained by iterating a normal measure on κ_0 in W_0. The successive images of κ_0 lead to the critical sequence $\{\kappa_\alpha \mid \alpha \in \mathrm{Ord}\}$, which is a closed unbounded class of ordinals, containing all cardinals of V. It follows that κ itself appears on this critical sequence, as the κth element $\kappa = \kappa_\kappa$. In particular, if $j : W_0 \to W_\kappa$ is the κth iteration of the normal measure, then $j(\kappa_0) = \kappa$, and so by elementarity, W_κ is an inner model in which κ itself is an indestructible supercompact cardinal. $\qquad\square$

It should be clear that once there is an inner model W containing an indestructible supercompact cardinal, and this cardinal is a mere countable ordinal in V, then in fact it can be arranged that any desired cardinal of V is an indestructible supercompact cardinal in an inner model. For example, this argument shows that if there is a cardinal that is supercompact up to a weakly iterable cardinal, then there are inner models W in which $\aleph_1^V$ is indestructibly supercompact, or $\aleph_2^V$ or $\aleph_\omega^V$ is indestructibly supercompact, and so on, as desired.

The method also provides an answer to Test Question 5.

Theorem 23. *If κ is $<\delta$-supercompact for a weakly iterable cardinal δ above κ, then there is an inner model of the* PFA.

Proof. Let $\langle M, \delta, U \rangle$ be an iterable structure containg V_δ. Then κ is supercompact in V_δ, and so the Baumgartner forcing $\mathbb{P} \in V_\delta$ forces the PFA over V_δ. Thus, by Theorem 21, there is an inner model of the PFA. $\qquad\square$

Let us return to Test Question 1, where we aim to produce an inner model with a supercompact cardinal and the full GCH. In Theorem 14, we approached this, by finding inner models of a supercompact cardinal such that $2^\kappa = \kappa^+$ or such that $2^\kappa = \kappa^{++}$, and the proof generalized to get various GCH patterns at or above κ. The proofs of those theorems, however, relied on the friendliness of the iteration up to κ, and so seem unable to attain the full GCH. For example, if CH fails in V, then there can be no friendly forcing of the GCH. The third proof method, however, does work to produce such an inner model. We cannot apply Theorem 21 directly to the case of the poset forcing the GCH, since it is a class forcing over V_δ. Following the proof of Theorem 21, we would need at the last step to obtain a generic for a class forcing over the inner model W, and there is no obvious reason to suppose that such a W-generic can be constructed. Instead, using Theorem 19, we shall follow the modified strategy of forcing over the countable iterable structure first and then iterating out to produce the inner model. Note that if the GCH fails in V, then for large θ one cannot expect to find the GCH in the robust type of inner models W for which $W^\theta \subseteq W$, since such a property would inject the GCH violations from V into W.

The following theorem generalizes Theorem 21 to the case of class forcing with respect to V_δ.

Theorem 24. *If $\langle M, \delta, U \rangle$ is iterable and $\mathbb{P} \subseteq V_\delta^M$ is a poset in M and useful for the ultrapower by U, then there is an inner model satisfying every sentence forced by $\mathbb{P}$ over V_δ^M.*

Proof. We may assume without loss of generality that $\langle M, \delta, U \rangle \models$ "I am H_{δ^+}". (If not, replace M with $H_{\delta^+}^M$ and observe that the structure $\langle H_{\delta^+}^M, \delta, U \rangle$ remains iterable since it has all the same functions $f : \delta \to H_{\delta^+}^M$ as M.) As in Theorem 21, let $\langle M_0, \delta_0, U_0 \rangle$ be a countable iterable structure obtained by collapsing a countable elementary substructure of $\langle M, \delta, U \rangle$ containing $\mathbb{P}$, and let $\mathbb{Q}$ be the image of $\mathbb{P}$ under the collapse. Since M_0 is countable, there is by Corollary 20 an M_0-generic filter $G_0 \subseteq \mathbb{Q}$ and an $M_0[G_0]$-ultrafilter U_0^* extending U_0 such that $\langle M_0[G_0], \delta_0, U_0^* \rangle$ is iterable, and such that the iteration of $M_0[G_0]$ by U_0^* is a step-by-step lifting of the iteration of M_0 by U_0. Note that $V_{\delta_0}^{M_0[G_0]} = V_{\delta_0}^{M_0}[G_0]$ satisfies the theory forced by $\mathbb{P}$ over V_δ^M. Let $\{j_{\xi\eta} : M_\xi[G_\xi] \to M_\eta[G_\eta] \mid \xi < \eta \in \mathrm{Ord}\}$ be the directed system of iterated ultrapowers of $M_0[G_0]$, and consider $W = \bigcup_{\xi \in \mathrm{Ord}} j_{0\xi}(V_{\delta_0}^{M_0[G_0]})$. Since the iteration of U_0^* lifts the iteration of U_0 on M_0 step-by-step, it follows that $W = \bar{W}[H]$, where $\bar{W} = \bigcup_{\xi \in \mathrm{Ord}} j_{0\xi}(V_{\delta_0}^{M_0})$ and H is the $\bar{W}$-generic filter arising from $\bigcup_\xi j_{0\xi}(G_0)$ for the class forcing obtained by $\bigcup_\xi j_{0\xi}(\mathbb{Q})$. By elementarity, W satisfies the same sentences that are forced to hold over $V_{\delta_0}^{M_0}$ by $\mathbb{Q}$, and these are the same as those forced to hold over V_δ^M by $\mathbb{P}$. $\qquad\square$

We may now apply Theorem 24 to provide answers to Questions 1 and 6, from a stronger hypothesis.

Theorem 25. *If κ is $<\delta$-supercompact for a weakly iterable cardinal δ, then there is an inner model in which κ is supercompact and the* GCH *plus* $V = $ HOD *holds.*

Proof. Let $\langle M, \delta, U \rangle$ be an iterable structure containing V_δ, and as before, assume without loss of generality that $M \models$ "I am H_{δ^+}". Observe that the canonical class forcing of the GCH is definable over V_δ and useful for the ultrapower embedding. Note that although δ may be singular in V, it is Mahlo (and more) in M, and so the forcing is δ-c.c. inside M. By Theorem 24, there is an inner model with a supercompact cardinal and the GCH. To obtain an inner model where κ itself is supercompact, simply follow the second part of the proof of Theorem 22. One can similarly obtain an inner model satisfying $V = $ HOD without the GCH by coding sets into the continuum function, making essentially the same argument. (See, e.g., the coding method used in [Rei06, Theorem 11] or [Rei07, Theorem 11].) If GCH$+V = $ HOD is desired, as in the statement of the theorem, then one should use a coding method compatible with the GCH. For example, the $\diamondsuit_\gamma^*$ coding method used in [BT09], in conjunction with the proof of [Rei06, Theorem 11] or [Rei07, Theorem 11], forces GCH $+ V = $ HOD while preserving supercompactness, and has the desired closure properties for this argument. $\qquad\square$

The hypotheses of Theorems 22, 23 and 25 can be improved slightly, since it is not required that δ is weakly iterable, but rather only that

(*) κ is $<\delta$-supercompact inside an iterable structure $\langle M, \delta, U \rangle$.

It is irrelevant assuming (*) whether $V_\delta \subseteq M$ or not, since the only use of that in our argument was to ensure that κ was $<\delta$-supercompact in M.

Next, we improve the iteration method to find more robust inner models, which not only satisfy the desired theory, but which also agree with V up to δ. This sort of additional feature cannot be attained by iterating a countable model out of

the universe, which is ultimately how our earlier instances of the iteration method proceeded.

Suppose as usual that $\langle M, \delta, U \rangle$ is a structure where $M \models \mathrm{ZFC}^-$, δ is a cardinal in M, and U is a weakly amenable M-ultrafilter. As a shorthand, let us refer to these structures as *weakly amenable*. A weakly amenable structure that is closed under $<\delta$-sequences is automatically iterable. This is because it will be correct about the countable completeness of the ultrafilter, which suffices for iterability (see [Kun70]). Moreover, closure under $<\delta$-sequences implies that $V_\delta^M = V_\delta$. Thus, if there exists a weakly amenable structure with $M^{<\delta} \subseteq M$, then δ is weakly iterable. The existence of these structures, however, has a significantly larger consistency strength than the existence of a weakly iterable cardinal that is between Ramsey and measurable cardinals (see [Git09]).

Theorem 26. *Suppose $\langle M, \delta, U \rangle$ is weakly amenable with $M^{<\delta} \subseteq M$. Suppose that $\mathbb{P} \subseteq V_\delta$ is a poset in M such that for every $\gamma < \delta$, there is a condition $p \in \mathbb{P}$ such that $\mathbb{P} \restriction p$ is $\leq\gamma$-strategically closed and useful for the ultrapower of M by U. Then there is an inner model W of V satisfying every sentence forced by $\mathbb{P}$ over V_δ and with $V_\delta^W = V_\delta$.*

Proof. As usual, without loss of generality, we assume that $M \models$ "I am H_{δ^+}". The hypothesis on $\mathbb{P}$ is a superfriendly version of usefulness. Consider the first two steps of the iteration

$$M = M_0 \xrightarrow{\ j_{01}\ } M_1 \xrightarrow{\ j_{12}\ } M_2.$$

Our strategy will be to lift the *second* step of the iteration. We shall produce in V a lift $j_{12}^* : M_1[G_1] \to M_2[G_2]$, where $G_1 \subseteq j_{01}(\mathbb{P})$ is M_1-generic and $j_{12}^*(G_1) = G_2$ is M_2-generic for $j_{02}(\mathbb{P})$, while also satisfying the extra genericity requirement of Theorem 19. By that theorem, therefore, the lift will be iterable and the desired inner model will be obtained by iterating it out of the universe.

To begin, note that the structure $\langle M_1, \delta_1, U_1 \rangle$ arising from the ultrapower of $\langle M, \delta, U \rangle$ is certainly iterable, since it was obtained after one step of the iterable structure $\langle M, \delta, U \rangle$. In addition, the closure assumptions on M_0 ensure that $M_1^{<\delta} \subseteq M_1$ and $|M_1| = \delta$, and also that $M_2^{<\delta} \subseteq M_2$ and $|M_2| = \delta$. By the useful and superfriendly assumption on $\mathbb{P}$, and using elementarity, we may find a condition $q \in j_{01}(\mathbb{P})$ below which $j_{01}(\mathbb{P})$ is $<\delta$-strategically closed in M_1 and useful for the ultrapower of M_1 by U_1. Thus, below the appropriate condition p, $j_{02}(\mathbb{P})$ factors in M_2 as $j_{01}(\mathbb{P}) * \dot{\mathbb{P}}_{\mathrm{tail}}$, where $j_{01}(\mathbb{P})$ is $j_{01}(\delta)$-c.c. and $\dot{\mathbb{P}}_{\mathrm{tail}}$ is forced to be $\leq j_{01}(\delta)$-strategically closed. Since there are only δ many dense subsets of $j_{02}(\mathbb{P})$ in M_2 and $M_2^{<\delta} \subseteq M_2$, we may diagonalize to find an M_2-generic filter $G_2 \subseteq j_{02}(\mathbb{P})$ below p in V. It follows that $G_1 = G_2 \restriction j_{01}(\delta)$ is M_1-generic for $j_{01}(\mathbb{P})$, and we may lift the embedding j_{12} to $j_{12}^* : M_1[G_1] \to M_2[G_2]$. We may furthermore arrange in the diagonalization that G_2 also meets all the external dense sets D_a arising in Theorem 19, since there are only δ many such additional sets, and they can simply be folded into the diagonalization. Thus, by Theorem 19, the lift j_{12}^* is iterable. Let $\{j_{1\xi}^* : M_1[G_1] \to M_\xi[G_\xi]\}$ be the corresponding iteration, and let $W = \bigcup_\xi V_{j_{1\xi}(\delta_1)}^{M_\xi[G_\xi]}$ be the resulting inner model. This is the union of an elementary chain, and so W is an elementary extension of $M_1[G_1]$, which satisfies all sentences forced by $\mathbb{P}$ over V_δ and includes $H_{\delta^+}^M$. In particular, $V_\delta \subseteq W$ and so $V_\delta^W = V_\delta$, completing the proof. $\square$

Theorem 27. *If κ is indestructibly $<\delta$-supercompact in a weakly amenable $\langle M, U, \delta \rangle$ with $M^{<\delta} \subseteq M$, then there is an inner model W satisfying $V = \mathrm{HOD}$ in which κ is indestructibly supercompact and for which $V_\delta^W = V_\delta$.*

Proof. Let $\langle M, \delta, U \rangle$ be weakly amenable with $M^{<\delta} \subseteq M$. It follows that κ is indestructibly supercompact in V_δ. Let $\mathbb{P}$ be the forcing notion that first generically chooses (via a lottery sum) an ordinal γ_0 in the interval $[\kappa, \delta)$, and then performs an Easton support iteration of length δ. $\mathbb{P}$ does nontrivial forcing at regular cardinals γ in the interval $[\gamma_0, \delta)$, with forcing that either forces the GCH to hold at γ or to fail at γ, using the lottery sum $\oplus \{\mathrm{Add}(\gamma^+, 1), \mathrm{Add}(\gamma, \gamma^{++})\}$. An easy density argument (see the proof of [Fri09, Lemma 13.1]) shows that any particular set of ordinals below δ added by this forcing will be coded into the GCH pattern below δ, and so $\mathbb{P}$ forces $V = \mathrm{HOD}$ over V_δ. By indestructibility, the forcing $\mathbb{P}$ preserves the indestructible supercompactness of κ. Furthermore, the forcing $\mathbb{P}$ is definable in V_δ, and the choice of γ_0 makes the forcing as closed as desired below δ, as well as useful for the ultrapower of M by U. Thus, the hypotheses of Theorem 26 are satisfied. So by that theorem, there is an inner model W satisfying $V = \mathrm{HOD}$ and having $V_\delta^W = V_\delta$. Since κ is below δ, the critical point of the iteration of M by U, it is not moved by that iteration, and so κ is indestructibly supercompact in W. $\square$

Next, we consider a variant of one of the questions mentioned after Test Question 7, asking the extent to which sets can be placed into the HOD of an inner model.

Theorem 28. *If κ is strongly Ramsey, then for any $A \in H_{\kappa^+}$, there is an inner model W containing A and satisfying $V = \mathrm{HOD}$. If the GCH holds below κ, then one can arrange that A is definable in W without parameters.*

Proof. From our earlier discussion, we know that κ is strongly Ramsey if every $A \in H_{\kappa^+}$ can be placed into a weakly amenable structure $\langle M, \kappa, U \rangle$ with $M^{<\kappa} \subseteq M$.

Starting with a weakly amenable $\langle M, \kappa, U \rangle$ with $M^{<\kappa} \subseteq M$ and $A \in M$, we use the same forcing as in the proof of Theorem 27 and appeal to Theorem 26 to obtain an inner model W satisfying $V = \mathrm{HOD}$ and having $A \in W$, as desired.

Lastly, if the GCH holds below κ, then as in Theorem 15, we may arrange the coding to begin with coding A, and thereby make A definable in W without parameters. $\square$

Corollary 29. *If there is a proper class of strongly Ramsey cardinals, then every set A is an element of some inner model W satisfying $V = \mathrm{HOD}$.*

Proof. Under this hypothesis, every set A is in H_{δ^+} for some strongly Ramsey cardinal δ, and so is in an inner model W satisfying $V = \mathrm{HOD}$ by Theorem 28. $\square$

To summarize the situation with our test questions, we have provided definite affirmative answers to Test Questions 2, 3, 4 and 7, along with several variants, but have only provided the affirmative conclusion of Test Questions 1, 5 and 6 from the (consistency-wise) stronger hypothesis that there is a cardinal supercompact up to a weakly iterable cardinal (or at least supercompact inside an iterable structure). We do not know if this hypothesis can be weakened for these results to merely a supercompact cardinal. Perhaps either Woodin's new approach to building non-fine-structural inner models of a supercompact cardinal, or Foreman's approach of [For09] for constructing inner models of very large cardinals, will provide the answers to these questions.

5. Further Applications

We shall now describe how variants of our methods can be used to obtain a further variety of inner models. First, using the methods of Theorem 25 and a stronger hypothesis, we can obtain:

Theorem 30. *Suppose that δ is a weakly iterable cardinal and a limit of cardinals that are $<\delta$-supercompact. Then:*

(1) *There is an inner model with a proper class of supercompact cardinals, all Laver indestructible.*

(2) *There is an inner model with a proper class of supercompact cardinals, where the GCH holds.*

(3) *There is an inner model with a proper class of supercompact cardinals, where $V = \mathrm{HOD}$ and the Ground Axiom holds.*

Of course, there are numerous other possibilities, for any of the usual forcing iterations; we mention only these three as representative. In each case, the natural forcing has the same closure properties needed to support the argument of Theorem 25. In the case of statement (2), for example, one uses the canonical Easton support forcing of the GCH, and in statement (3), one uses any of the usual iterations that force every set to be coded into the GCH pattern of the continuum function, a state of affairs that implies both $V = \mathrm{HOD}$ and the Ground Axiom, the assertion that the universe was not obtained by set forcing over any inner model (see [Ham05], [Rei06] and [Rei07]).

For the next application of our methods, we show that there are inner models witnessing versions of classical results of Magidor [Mag76].

Theorem 31.

(1) *If κ is $<\delta$-strongly compact for a weakly iterable cardinal δ, then there is an inner model in which κ is both the least strongly compact and the least measurable cardinal.*

(2) *If there is a strongly compact cardinal κ, then there is an inner model in which the least strongly compact cardinal has only boundedly many measurable cardinals below it.*

(3) *If there is a supercompact cardinal κ, then for every cardinal θ, there is an inner model W in which the least strongly compact cardinal is the least supercompact cardinal and for which $W^\theta \subseteq W$.*

(4) *If κ is $<\delta$-supercompact for a weakly iterable cardinal δ, then there is an inner model in which κ is both the least strongly compact and least supercompact cardinal.*

Proof. For (1), let $\mathbb{P}$ be Magidor's notion of iterated Prikry forcing from [Mag76], which adds a Prikry sequence to every measurable cardinal below κ. Since $|\mathbb{P}| = \kappa$, it is small with respect to δ. By the arguments of [Mag76], the cardinal κ becomes both the least strongly compact and the least measurable cardinal in $V_\delta^{\mathbb{P}}$. Thus, by Theorem 21, there is an inner model in which the least strongly compact cardinal is the least measurable cardinal, and by the methods from the second part of the proof of Theorem 22, this cardinal may be taken as κ itself.

For (2), we begin by noting that the partial ordering $\mathbb{P}$ mentioned in the preceding paragraph is not $<\kappa$-friendly. However, in analogy to the first proof given for Theorem 8, for every $\gamma < \kappa$, let $\mathbb{P}_\gamma$ be Magidor's notion of iterated Prikry forcing

from [Mag76] which adds a Prikry sequence to every measurable cardinal in the open interval (γ, κ). By [Mag76], forcing with $\mathbb{P}_\gamma$ adds no subsets to γ. Let $\mathbb{P}^* = \oplus\{\mathbb{P}_\gamma \mid \gamma < \kappa\}$ be their lottery sum. Magidor's arguments of [Mag76] together with $\mathbb{P}^*$'s definition as a lottery sum show that after forcing with $\mathbb{P}^*$, for some $\gamma < \kappa$, κ is the least strongly compact cardinal, and there are no measurable cardinals in the open interval (γ, κ). Since $\mathbb{P}^*$ is $<\kappa$-friendly, (2) now follows by Theorem 10.

For (3), let $\gamma < \kappa$, and let $\mathbb{P}_\gamma$ be the reverse Easton iteration of length κ which adds a non-reflecting stationary set of ordinals of cofinality ω to every non-measurable regular limit of strong cardinals in the open interval (γ, κ). (In other words, $\mathbb{P}_\gamma$ does trivial forcing except at those $\delta \in (\gamma, \kappa)$ which are non-measurable regular limits of strong cardinals, where it adds a non-reflecting stationary set of ordinals of cofinality ω to δ.) By the remarks in [Apt05, Section 2], after forcing with $\mathbb{P}_\gamma$, κ becomes both the least strongly compact and least supercompact cardinal. Let $\mathbb{P} = \oplus\{\mathbb{P}_\gamma \mid \gamma < \kappa\}$ be their lottery sum. Since $\mathbb{P}$ is $<\kappa$-superfriendly, (3) now follows by Theorem 13.

Finally, for (4), we note that for any $\gamma < \kappa$, $\mathbb{P}_\gamma$ of the preceding paragraph is $<\kappa$-friendly, since it is $<\kappa$-superfriendly. Because $|\mathbb{P}_\gamma| = \kappa$, (4) now follows by Theorem 21 and the methods from the second part of the proof of Theorem 22. □

Before stating our next application, we briefly recall some definitions. Say that a model V of ZFC containing supercompact cardinals satisfies *level by level equivalence between strong compactness and supercompactness* if for every $\kappa < \lambda$ regular cardinals, κ is λ strongly compact if and only if κ is λ supercompact. Say that a model V of ZFC containing supercompact cardinals satisfies *level by level inequivalence between strong compactness and supercompactness* if for every non-supercompact measurable cardinal κ, there is some $\lambda > \kappa$ such that κ is λ strongly compact yet κ is not λ supercompact. Models satisfying level by level equivalence between strong compactness and supercompactness were first constructed in [AS97], and models satisfying level by level inequivalence between strong compactness and supercompactness have been constructed in [Apt02], [Apt10b] and [Apt10a].

Theorem 32.

(1) *If the GCH holds and κ is $<\delta$-supercompact for a weakly iterable cardinal δ, then there is an inner model in which κ is supercompact and the GCH and level by level equivalence between strong compactness and supercompactness hold.*

(2) *If the GCH holds and κ is δ-supercompact for a weakly iterable cardinal δ, then there is an inner model in which κ is supercompact and the GCH and level by level inequivalence between strong compactness and supercompactness hold.*

Proof. For (1), assume that κ and δ are least such that κ is $<\delta$-supercompact and δ is a weakly iterable cardinal. Let $\mathbb{P}$ be the class forcing from [AS97] defined over V_δ such that $V_\delta^{\mathbb{P}} \models$ "κ is supercompact, and the GCH and level by level equivalence between strong compactness and supercompactness hold". We refer readers to [AS97, Section 3] for the exact definition of $\mathbb{P}$, which is rather complicated. We do note, however, that if $\langle M, \delta, U \rangle$ is an iterable structure containing V_δ, then because κ and δ are least such that κ is $<\delta$-supercompact and δ is weakly iterable, $\mathbb{P}$ is useful for the ultrapower embedding. Therefore, by Theorem 24, there is an inner model containing a supercompact cardinal in which the GCH and level by level

equivalence between strong compactness and supercompactness hold, and by the methods from the second part of the proof of Theorem 22, this cardinal may be taken as κ itself.

For (2), assume that κ and δ are least such that κ is δ-supercompact and δ is a weakly iterable cardinal. It is a general fact that if γ is ρ-supercompact, ρ is weakly iterable, and $j : V \to M$ is an elementary embedding witnessing the ρ-supercompactness of γ, then $M \models$ "ρ is weakly iterable". Therefore, by the proof of [Apt10b, Theorem 2], there are cardinals $\kappa_0 < \delta_0 < \kappa$ and a partial ordering $\mathbb{P} \in V_{\delta_0}$ such that $V_{\delta_0}^{\mathbb{P}} \models$ "κ_0 is supercompact, and the GCH and level by level inequivalence between strong compactness and supercompactness hold". Thus, by Theorem 21, there is an inner model containing a supercompact cardinal in which the GCH and level by level inequivalence between strong compactness and supercompactness hold, and by the methods from the second part of the proof of Theorem 22, this cardinal may be taken as κ itself. □

We mentioned at the opening of this article that we take our Test Questions as representative of the many more similar questions one could ask, inquiring about the existence of inner models realizing various large cardinal properties usually obtained by forcing. We would similarly like to take our answers—and in particular, the three proof methods we have described—as providing a key to answering many of them. Indeed, we encourage the reader to go ahead and formulate similar interesting questions and see if these methods are able to provide an answer. Going forward, we are especially keen to find or learn of generalizations of our first two methods, in Theorems 10, 11 and 13, which might allow us to find the more robust inner models provided by these methods for a greater variety of situations.

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