Kelley-Morse set theory and choice principles for classes

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Why second-order set theory?

Proper classes are collections of sets that are too “big” to be sets themselves. Naive set theories which treated them as sets ran into paradoxes.

- the universe $V$ of all sets
- the collection $\text{ORD}$ of all ordinals

In first-order set theory, classes are informally defined as the definable (with parameters) sub-collections of the model.

- We cannot study proper classes within the formal framework of first-order set theory, but only in the meta-theory.
- The notion of class as a definable sub-collection is too restrictive!

Second-order set theory is a formal framework for studying the properties of sets as well as classes.

- Classes are objects in the model.
- We can axiomatize properties of classes.
- We can quantify over classes.
- Non-definable classes are allowed.
Why formally study classes?

**Proper class forcing**: we can obtain models
- with desired continuum functions,
- in which every set is coded into continuum pattern,
- in which there is no definable linear ordering of sets.

**Reinhardt axiom**: there exists an elementary embedding $j: V \rightarrow V$.
- It is not expressible in first-order set theory.
- It is easy to see that if $V \models ZF$, there no definable elementary $j: V \rightarrow V$.
- (**Kunen Inconsistency**) There is no elementary $j: V \rightarrow V$ in any model of a “reasonable” second-order set theory with AC.
- (**Open Problem**) Can there be an elementary $j: V \rightarrow V$ in a model of a “reasonable” second-order set theory without AC?

**Model theoretic constructions with ultrafilters on classes**
- Models of first-order set theory with interesting model theoretic properties are obtained as ultrapowers of models of second-order set theory.
- Ultrafilter measures classes.
- Elements of ultrapower are equivalence classes of class functions.
Primer on second-order set theory

Structures have two types of objects: sets and classes.

**Syntax**: two-sorted logic
- separate variables and quantifiers for sets and classes
- Relations and functions must specify sort for each coordinate.
- Convention: uppercase letters for classes and lowercase letters for sets.

**Language of set theory**:
- $\in$ relation: sets,
- $\in$ relation: sets $\times$ classes.

**Semantics**: A model is $\mathcal{M} = \langle M, \in, S \rangle$, where $\langle M, \in \rangle$ is a model of first-order set theory and $S$ consists of subsets of $M$.

**Alternative formalization**: first-order logic
- objects are classes,
- sets are defined to be those classes that are elements of other classes.
Foundations of second-orders set theory: basic requirements

**Bold Claim**: A reasonable foundation should imply the basic properties of a ZFC model together with its definable sub-collections.

- The class of sets $V$ is a model of ZFC.
  (Axioms) $\text{ZFC for sets}$.  

- $V$ together with predicates for finitely many classes is a model of ZFC.
  (Axioms) **class replacement**: the restriction of a class function to a set is a set.

- (Class existence principle) Every first-order definable sub-collection of $V$ is a class.
  (Axioms) **first-order** class comprehension scheme
  **class comprehension** scheme for second-order formulas in $\Gamma$:
  if $\varphi(x, X)$ is in $\Gamma$ and $A$ is a class, then $\{x \mid \varphi(x, A)\}$ is a class.
Foundations of second-order set theory: GBC

First foundation is developed by Bernays, Gödel, and von Neumann in the 1930s. It codifies the \textit{“basic requirements”}.

\textbf{GBC: modern formulation}

- set axioms: ZFC
- class extensionality
- class replacement
- \textbf{global choice}: there exists a global choice function class (\textit{not basic} and equivalent to the existence of a \textit{well-ordering of }\mathcal{V})
- first-order class comprehension

\textbf{GBC is predicative}: definitions of classes don’t quantify over classes.
Foundations of second-order set theory: GBC (continued)

Observation:
- There are models of ZFC that don't have a definable well-ordering.
- The constructible universe \( L \) has a definable well-ordering.
- \( L \) together with its definable sub-collections is a model of GBC.
- GBC is equiconsistent with ZFC.

Theorem: (Solovay) Every countable model of ZFC can be extended to a model of GBC without adding sets.

Proof sketch: Force to add a global well-ordering with a forcing that doesn’t add sets because it is \(<\kappa\)-closed for every cardinal \( \kappa \). □

Corollary: GBC is conservative over ZFC: any property of sets provable in GBC is already provable in ZFC.
Foundations of second-order set theory: KM

A second-order theory is **impredicative** if it has comprehension for second-order formulas: definitions of classes can quantify over classes.

Impredicative set theories were first studied by Wang and Morse in the 1940s.

The **Kelley-Morse** axioms first appeared in Kelley’s *General Topology* textbook in 1955.

**KM:** modern formulation

- GBC
- (full) *second-order* comprehension
Strength of KM

**Theorem:** (Marek, Mostowski?) If $\langle V, \in, S \rangle \models \text{KM}$, then $V$ is the union of an elementary chain of its rank initial segments $V_\alpha$:

$$V_{\alpha_0} \prec V_{\alpha_1} \prec \cdots \prec V_{\alpha_\xi} \prec \cdots \prec V,$$

and $V$ thinks that each $V_{\alpha_\xi} \models \text{ZFC}$.

**Proof sketch:**

- Every model of KM has a truth predicate class $\text{Tr}$ coding first-order truth.
- Gödel-codes of all (even nonstandard) ZFC axioms are in $\text{Tr}$.
- If $\langle V_\alpha, \in, \text{Tr} \rangle \prec_\Sigma_2 \langle V, \in, \text{Tr} \rangle$, then $V_\alpha \prec V$. □

**Corollary:**

- Full reflection holds in models of KM!
- KM proves $\text{Con}(\text{ZFC})$ (and much more).
- The consistency strength of KM is greater than the strength of the theory: $\text{ZFC} + \text{there is a transitive model of ZFC}$. 
Strength of KM (continued)

**Theorem:** KM is weaker than the theory ZFC + there is an inaccessible cardinal.

**Proof sketch:**
Suppose $V \models ZFC$ and $\kappa$ is inaccessible in $V$.

- $\langle V_\kappa, \in, V_{\kappa+1} \rangle \models KM$ (and more).
- Countable elementary substructures of $V_{\kappa+1}$ give models of KM in $V_\kappa$.
- $\text{Con}(\text{Con}(\text{KM}))$ holds.
This is a choice/collection principle for classes.

It was first studied by Marek and Mostowski in the 1970s.

“Every definable $V$-indexed family of collections of classes has a choice function.”

**Coded functions $F : V \to S$:**

- If $Z$ is a class and $x$ is a set, then $Z_x := \{y \mid (x, y) \in Z\}$ is the class coded on the $x^{th}$-slice of $Z$.
- $Z$ codes $F : V \to S$ with $F(x) = Z_x$.

**Definition:** The choice scheme consists of assertions

$$\forall x \exists X \varphi(x, X, A) \rightarrow \exists Z \forall x \varphi(x, Z_x, A)$$

for every second-order $\varphi(x, X, Y)$ and class $A$. 
Some fragments of the choice scheme:

- $\Sigma^1_n$ (or $\Pi^1_n$) choice scheme: bounds the complexity of $\varphi$
- parameterless choice scheme: no parameters allowed in $\varphi$
- set-sized choice scheme: consists of assertions

$$\forall x \in a \exists X \varphi(x, X, A) \rightarrow \exists Z \forall x \in a \varphi(x, Z_x, A).$$

for every second-order $\varphi(x, X, Y)$, class $A$, and set $a$.

Some applications of the choice scheme: (more on this later)

- (Folklore) The Łoś Theorem for internal second-order ultrapowers of models of KM is equivalent (over KM) to the set-sized choice scheme.
- KM + choice scheme proves that first-order quantifiers don’t affect the second-order complexity of an assertion.
- (Mostowski?) The theory KM + choice scheme is bi-interpretable with the theory ZFC$^- +$ there exists an inaccessible cardinal.
Does KM imply (some fragment of) the choice scheme?

**Observation:** If $V \models ZFC$ and $\kappa$ is inaccessible in $V$, then

$$\langle V_\kappa, \in, V_{\kappa+1} \rangle \models KM + \text{choice scheme}. $$

**Proof sketch:** Use choice in $V$ for families of subsets of $V_{\kappa+1}$. □

If we want a model of KM in which the choice scheme fails, choice should fail in $V$ for some easily describable family of subsets of $V_{\kappa+1}$.

**Observation:** Suppose $V \models ZF$, $\kappa$ is regular in $V$, $V_\kappa \models ZFC$, $V_\kappa$ is well-orderable in $V$, then $\langle V_\kappa, \in, V_{\kappa+1} \rangle \models KM$.

**Independence strategy:**

- Start in $V$ having an inaccessible cardinal $\kappa$.
- Perform a clever forcing and move to $V[G]$.
- Find a symmetric submodel $N \models ZF$ of $V[G]$ such that $\kappa$ is regular in $N$ and $V_\kappa^N = V_\kappa$, but choice fails for some easily describable family of subsets of $V_{\kappa+1}$. 
A detour into second-order arithmetic

There are two types of objects: **numbers** and **sets** of numbers (reals).

**Syntax**: two-sorted logic

**Language of second-order arithmetic**: 
- language of first-order arithmetic: \( \mathcal{L}_A = \{+ , \cdot , < , 0 , 1 \} \)
- \( \in \) relation: numbers \( \times \) sets

**Semantics**: A model is \( \mathcal{M} = \langle M , + , \cdot , < , 0 , 1 , S \rangle \), where \( \langle M , + , \cdot , < , 0 , 1 \rangle \) is a model of first-order arithmetic and \( S \) consists of subsets of \( M \).

**Basic requirements**: 
- \( \langle M , + , \cdot , < , 0 , 1 \rangle \models \text{PA} \) (Peano axioms),
- **Induction axiom scheme**: consists of assertions

\[
(\varphi(0, m, A) \land \forall n (\varphi(n, m, A) \rightarrow \varphi(n + 1, m, A))) \rightarrow \forall n \varphi(n, m, A)
\]

for every first-order \( \varphi(n, m, X) \), set \( A \), and number \( m \).
Foundations of second-order arithmetic

**ACA₀**: arithmetical comprehension

- analogue of **GBC**
- basic requirements + **first-order** comprehension
- If $V \models ZF$, then $\langle \omega, +, \cdot, <, 0, 1, S \rangle \models ACA₀$, where $S$ is the collection of all definable subsets of $\langle \omega, +, \cdot, <, 0, 1 \rangle$.

**Z₂**: full second-order arithmetic

- analogue of **KM**
- basic requirements + **second-order** comprehension
- If $V \models ZF$, then $\langle \omega, +, \cdot, <, 0, 1, P(\omega) \rangle \models Z₂$. 
The choice scheme in second-order arithmetic

Definition: The choice scheme consists of assertions

\[ \forall n \exists X \varphi(n, X, A) \rightarrow \exists Z \forall n \varphi(n, Z_n, A) \]

for every second-order \( \varphi(n, X, Y) \) and set \( A \).

Choice scheme fragments:
- \( \Sigma^1_n \) (or \( \Pi^1_n \)) choice scheme
- Parameterless choice scheme

Observation: If \( V \models ZF + AC_\omega \), then \( \langle \omega, +, \cdot, <, 0, 1, P(\omega) \rangle \models \mathbb{Z}_2 + \text{choice scheme} \).

Independence strategy:
- Perform a clever forcing and move to \( V[G] \).
- Find a symmetric submodel \( N \models ZF \) of \( V[G] \) such that choice fails for some easily describable family of subsets of \( P(\omega) \).
Unexpected free choice

**Theorem:** $\mathbb{Z}_2$ proves the $\Sigma^1_2$-choice scheme.

**Proof sketch:** Suppose:
- $\mathcal{M} = \langle \mathcal{M}, +, \times, <, 0, 1, S \rangle \models \mathbb{Z}_2$
- $\mathcal{M} \models \forall n \exists X \varphi(n, X)$, where $\varphi$ is $\Sigma^1_2$.

Observe:
- If $\alpha$ is an ordinal coded in $S$, then $S$ has a set coding $L_\alpha$.
- $\mathcal{M}$ has its own constructible universe $L^\mathcal{M}$!
- $\mathcal{M}$ satisfies Shoenfield Absoluteness with respect to $L^\mathcal{M}$: If $\psi$ is $\Sigma^1_2$, then $\mathcal{M} \models \psi$ iff $L^\mathcal{M} \models \psi$.
- $L^\mathcal{M} \models \exists X \varphi(n, X)$ for every $n$.
- Choose the $L^\mathcal{M}$-least $X$ and use comprehension to collect!

If $\varphi$ has a set parameter $A$, replace $L^\mathcal{M}$ with $L[A]^\mathcal{M}$. □

**Question:** What about $\Pi^1_2$-choice scheme?
Independence of $\Pi^1_2$-choice scheme from $Z_2$

**The Feferman-Lévy model**

Classic symmetric model $N \models ZF$ in which $\aleph_1$ is a countable union of countable sets.

**Symmetric model properties:**

- $N \models ZF$,
- all $\aleph_n^L$ (the $n^{th}$ cardinal of $L$) are countable in $N$,
- $\aleph_\omega^L = \aleph_1^N$ is the first uncountable cardinal of $N$.

**Construction:**

- Force with finite-support product $\mathbb{P} = \prod_{n<\omega} \text{Coll}(\omega, \aleph_n)$ over $L$ to collapse the first $\omega$-many successor cardinals of $\omega$ to $\omega$.
- Let $G \subseteq \mathbb{P}$ be $L$-generic and $G_m = G \upharpoonright \prod_{n<m} \text{Coll}(\omega, \aleph_n)$.
- $N \subseteq L[G]$ is a symmetric model of $ZF$ with the property:
  - $A$ is a subset of ordinals in $N$ iff $A \in L[G_m]$ for some $m$. 

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Independence $\Pi_2^1$-choice scheme from $Z_2$ (continued)

**Theorem:** (Feferman, Lévy) $\Pi_2^1$-choice scheme can fail in a model of $Z_2$.

**Proof sketch:** Consider $\mathcal{M} = \langle \omega, +, \times, <, 0, 1, P(\omega) \rangle^N \models Z_2$.

- $P(\omega)^N = \bigcup_{n<\omega} P(\omega)^{L[G_n]}$.
- Every $L_{\aleph_n}$ is coded in $\mathcal{M}$, but $L_{\aleph_\omega}$ is not coded in $\mathcal{M}$.
- We cannot collect the codes of $L_{\aleph_n}$.
- The assertion
  $$\forall n \exists X \text{ codes } L_{\aleph_n} \rightarrow \exists Z \forall n Z_n \text{ codes } L_{\aleph_n}$$
  fails in $\mathcal{M}$.
- The assertion “$X$ codes $L_{\aleph_n}$” is $\Pi_2^1$:
  $\langle \Pi_1^1 \land \forall Y (Y \text{ codes } L_\beta \text{ with } \beta > \alpha \rightarrow L_\beta \text{ thinks } \alpha = \aleph_n) \rangle$. $\square$
Independence of $\Pi^1_1$-choice scheme from $\text{KM}$

**Natural strategy:** Do the Feferman-Lévy construction above an inaccessible cardinal $\kappa$.

**Symmetric model properties:**
- $N \models \text{ZF}$, $\kappa$ is regular in $N$, $V^N_\kappa = V_\kappa$. (This gives $\langle V_\kappa, \in, V_{\kappa+1} \rangle^N \models \text{KM}$.)
- All $(\kappa^{n+})^L$ (the $n^{th}$ successor cardinal of $\kappa$ in $L$) have size $\kappa$.
- $(\kappa^+)^L = (\kappa^+)^N$.

**Construction:**
- Suppose $\kappa$ is inaccessible in $L$.
- Force with finite-support product $\mathbb{P} = \prod_{n<\omega} \text{Coll}(\kappa, \kappa^{n+})$ over $L$ to collapse the first $\omega$-many successor cardinals of $\kappa$ to $\kappa$.
- Let $G \subseteq \mathbb{P}$ be $L$-generic and $G_m = G \upharpoonright \prod_{n<m} \text{Coll}(\omega, \kappa^{n+})$.
- $N \subseteq L[G]$ is a symmetric model of $\text{ZF}$ with the property: $A$ is a subset of ordinals in $N$ iff $A \in L[G_m]$ for some $m$. 
Independence of $\Pi_1^1$-choice scheme from KM (continued)

**Theorem:** $\Pi_1^1$-choice scheme can fail in a model of KM.

**Proof:** Consider $\mathcal{M} = \langle V_\kappa, \in, V_{\kappa+1} \rangle^N \models \text{KM}$.

- $V_{\kappa+1}^N = \bigcup_{n<\omega} V_{\kappa+1}^{L[G_n]}$.
- Every $L_{\kappa+n}$ is coded in $\mathcal{M}$, but $L_{\kappa+\omega}$ is not coded in $\mathcal{M}$.
- We cannot collect the (codes of) $L_{\kappa+n}$.
- The assertion

$$\forall n \in \omega \exists X \text{ codes } L_{\kappa+n} \rightarrow \exists Z \forall n \in \omega X \text{ codes } L_{\kappa+n}$$

fails in $\mathcal{M}$.

- The assertion "$X$ codes $L_{\kappa+n}$" is $\Pi_1^1$: "$X$ codes $L_\alpha$" is $\Pi_0^1$.
- $\Pi_1^1$-choice already fails for $\omega$-many choices!

**Question:** What about $\Pi_0^1$-choice scheme?
Is $\Pi^1_0$-choice scheme independent from KM?

**Independence strategy:**
- Start in $V$ having:
  - an inaccessible cardinal $\kappa$,
  - $\omega$-many normal $\kappa$-Souslin trees $\langle T_n \mid n < \omega \rangle$ with the property:
    forcing with $\prod_{n < m} T_n$ doesn’t add branches to any $T_n$ with $n \geq m$.
- Forcing with a normal $\kappa$-Souslin tree adds a cofinal branch.
- Move to a forcing extension $V[G]$ by $P = \prod_{n < \omega} T_n$.
- Let $G_m = G \upharpoonright \prod_{n < m} T_n$.
- Find a symmetric model $N \subseteq V[G]$ with the property:
  $A$ is a subset of ordinals in $N$ iff $A \in V[G_m]$ for some $m$.
- Each $T_n$ has a branch in $N$, but there is no collecting set of branches!
A detour into homogeneous $\kappa$-Souslin trees

Definition: Suppose $\kappa$ is a cardinal.
- A normal $\kappa$-tree is subtree of $^{<\kappa}2$ of height $\kappa$ whose every node has
  - 2 immediate successors,
  - a successor at every higher level.
- A normal $\kappa$-Souslin tree has no branches of size $\kappa$.
- A homogeneous tree has for any two nodes on the same level an automorphism that moves one to the other.
  - A homogeneous tree is a weakly homogeneous poset.

Theorem: There is a universe $V$ with an inaccessible (or Mahlo) cardinal $\kappa$ having $\kappa$-many homogeneous normal $\kappa$-Souslin trees $\langle T_\xi \mid \xi < \kappa \rangle$ with the property: forcing with $\prod_{\xi < \delta} T_\xi$ ($\delta < \kappa$) doesn’t add branches to any $T_\alpha$ with $\alpha \geq \delta$.

Proof sketch: Suppose $\kappa$ is inaccessible (or Mahlo).
- Let $Q$ be the forcing to add a homogeneous $\kappa$-Souslin tree.
- Force with bounded-support product $P = \prod_{\xi < \kappa} Q_\xi$, where $Q_\xi = Q$. □
Independence of $\Pi^1_0$-choice from KM

Symmetric model properties:

- $N \models ZF$, $\kappa$ is regular in $N$, $V^N_{\kappa} = V_{\kappa}$. (This gives $\langle V_{\kappa}, \in, V_{\kappa+1} \rangle^N \models KM$.)
- $N$ has a sequence $\vec{T} = \langle T_n \mid n \in \omega \rangle$ of $\kappa$-trees such that
  - each $T_n$ has a branch,
  - there is no collecting set of branches.

Construction:

- Suppose:
  - $\kappa$ is inaccessible,
  - $\langle T_n \mid n < \omega \rangle$ are homogeneous normal $\kappa$-Souslin trees with the property: forcing with $\prod_{n < m} T_n$ doesn’t add branches to $T_n$ for $n \geq m$.
- Force with finite-support product $\mathbb{P} = \prod_{n < \omega} T_n$ to add branches to all $T_n$.
- Let $G \subseteq \mathbb{P}$ be $V$-generic and $G_m = G \restriction \prod_{n < m} T_n$.
- $N \subseteq V[G]$ is a symmetric model of ZF with the property: $A$ is a subset of ordinals in $N$ iff $A \in V[G_m]$ for some $m$.
  (This uses weak homogeneity of the trees.)
Independence of $\Pi^1_0$-choice from $\text{KM}$ (continued)

**Theorem:** (G., Johnstone, Hamkins) $\Pi^1_0$-choice scheme can fail in a model of $\text{KM}$.

**Proof sketch:** Consider $\mathcal{M} = \langle V_\kappa, \in, V_{\kappa+1} \rangle^N \models \text{KM}$.

- The assertion
  \[ \Psi(\vec{T}) := \forall n \in \omega \exists B \text{ branch of } T_n \rightarrow \exists Z \forall n \in \omega Z_n \text{ is a branch of } T_n \]
  fails in $\mathcal{M}$.
- The assertion “$B$ is a branch of $T_n$” is $\Pi^1_0$ (in the parameter $\vec{T}$).
- $\Pi^1_0$-choice already fails for $\omega$-many choices! □

**Theorem:** (G., Johnstone, Hamkins) Parameterless $\Pi^1_0$-choice scheme can fail in a model of $\text{KM}$.

**Proof sketch:** Suppose $\kappa$ is Mahlo and $\vec{T}$ has the required property.

- Code $\vec{T}$ into the continuum function below $\kappa$ using Easton product forcing.
- In the Easton product extension $V[G]$:
  - $\kappa$ is inaccessible (even Mahlo),
  - $\vec{T}$ is definable in $V_\kappa$ and continues to have the required property. □
Separating fragments of the choice scheme

**Theorem:** (G., Johnstone, Hamkins) There is a model of $\mathsf{KM}$ in which set-sized choice scheme holds, but $\Pi^1_0$-choice scheme fails.

**Proof sketch:**
- Suppose $\kappa$ is inaccessible,
  - $\langle T_\xi \mid \xi < \kappa \rangle$ are homogeneous normal $\kappa$-Souslin trees with the property: forcing with $\prod_{\xi < \delta} T_\xi$ doesn’t add branches to $T_\xi$ for $\xi \geq \delta$.
- Force with bounded-support product $\mathbb{P} = \prod_{\xi < \kappa} T_\xi$.
- Let $G \subseteq \mathbb{P}$ be $V$-generic and $G_\delta = G \restriction \prod_{\xi < \delta} T_\xi$.
- Construct symmetric model $N \subseteq V[G]$ of $\mathsf{ZF}$ with the property: $A$ is a subset of ordinals in $N$ iff $A \in V[G_\delta]$ for some $\delta$. □

**Theorem:** (G., Johnstone, Hamkins) There is a model of $\mathsf{KM}$ in which set-sized choice scheme holds, but parameterless $\Pi^1_0$-choice scheme fails.

**Proof sketch:** Suppose $\kappa$ is Mahlo and code $\overrightarrow{T}$ into the continuum function below $\kappa$. □
Separating fragments of the choice scheme (continued)

**Theorem**: (G., Johnstone, Hamkins) There is a model of KM in which parameterless choice scheme holds, but $\Pi^1_1$-choice scheme fails.

**Proof sketch**: 

- We use ideas of Guzicki who proved a similar result for $Z_2$.
- Suppose $\kappa$ is inaccessible in $L$.
- Force with bounded-support product $\prod_{\xi<\kappa^+} \text{Coll}(\kappa, \kappa^{(+\xi)})$ over $L$.
- Let $G \subseteq \mathbb{P}$ be $L$-generic and $G_\delta = G \upharpoonright \prod_{\xi<\delta} \text{Coll}(\kappa, \kappa^{(+\xi)})$.
- Construct symmetric model $N \subseteq L[G]$ of ZF with the property: $A$ is a subset of ordinals in $N$ iff $A \in L[G_\delta]$ for some $\delta$.
- Consider $M = \langle V_\kappa, \in, V_{\kappa+1} \rangle^N$.
- Let $A$ code $(\kappa^+)^L$ in $M$ and let $\alpha_\xi$ be the ordinal corresponding to $\xi$ in $A$.
- The assertion 

$$\Psi(A) := \forall \xi \exists X \text{ codes } L_{\alpha_\xi} \rightarrow \exists Z \forall \xi X \text{ codes } L_{\alpha_\xi}$$

fails in $M$.
- Parameterless choice holds in $M$. □
The theory \( \text{KM}^+ \)

**Definition:** The theory \( \text{KM}^+ \) consists of \( \text{KM} \) together with the choice scheme.

**Bold claim:** \( \text{KM}^+ \) is a better foundation for second-order set theory than \( \text{KM} \).

- \( \text{KM}^+ \) proves Łoś Theorem for internal second-order ultrapowers.
  - Suppose \( \mathcal{M} = \langle M, \in, S \rangle \models \text{KM}^+ \), \( I \in M \) and \( U \in M \) is an ultrafilter on \( I \).
  - Classes in \( \text{Ult}(\mathcal{M}, U) \) are represented by coded functions \( \mathcal{F} = \langle A_i | i \in I \rangle \) in \( S \).

- \( \text{KM}^+ \) proves that first-order quantifiers don’t affect second-order complexity: suppose \( \varphi(x) \) is equivalent to a \( \Sigma^1_n \)-formula, then so are
  - \( \forall x \varphi(x) \),
  - \( \exists x \varphi(x) \).

- \( \text{KM}^+ \) is bi-interpretable with the theory \( \text{ZFC}^- \) + there is an inaccessible.

**Open question:** What is the consistency strength of \( \text{KM}^+ \) compared to \( \text{KM} \)?
Weaknesses of KM

**Theorem:** (Folklore) Łoś Theorem for internal second-order ultrapowers is equivalent (over KM) to the set-sized choice scheme.

**Question:** How badly does Łoś Theorem fail for ultrapowers of KM models?

**Theorem:** (G., Johnstone, Hamkins) There is a model of KM whose internal second-order ultrapower by an ultrafilter on ω is not a model of KM.

**Theorem:** (G., Johnstone, Hamkins) There is a $\Sigma^1_1$-formula $\varphi(x)$ and a model of KM in which $\forall x \varphi(x)$ is not equivalent to a $\Sigma^1_1$-formula.
Stronger choice principles for classes?

**Definition:** The $\omega$-dependent choice scheme consists of assertions

$$\forall X \exists Y \varphi(X, Y, A) \rightarrow \exists Z : \omega \rightarrow S \ \forall n \in \omega \varphi(Z_n, Z_{n+1})$$

for every second-order formula $\varphi$ and class $A$.

“Every definable relation on classes with no terminal nodes has an $\omega$-branch.”

**Observation:** $\text{KM}^+ + \omega$-dependent choice scheme proves reflection for second-order assertions: every second-order formula is reflected by some coded collection of classes.

**Definition:** The $\text{ORD}$-dependent choice scheme consists of assertions

$$\forall \beta \forall X : \beta \rightarrow S \ \exists Y \varphi(X, Y, A) \rightarrow \exists Z : \text{ORD} \rightarrow S \ \forall \beta \varphi(Z \upharpoonright \beta, Z_\beta, A)$$

for every second-order $\varphi$ and class $A$. 
Dependent choice scheme

**Theorem:** (Simpson, unpublished) There is a model of $Z_2$ in which choice scheme holds but $\Pi^1_2$-dependent choice scheme fails.

**Theorem** (Antos, Friedman) $K\text{M}^++\text{ORD}$-dependent choice scheme is preserved by all definable tame hyperclass forcing.

**Open question:** Can we separate the choice scheme, $\omega$-dependent choice scheme, and ORD-dependent choice scheme?
Thank you!