Kelley-Morse set theory and choice principles for classes

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Why second-order set theory?

Proper classes are collections of sets that are too "big" to be sets themselves. Naive set theories which treated them as sets ran into paradoxes.

- the universe V of all sets
- the collection ORD of all ordinals

In first-order set theory, classes are informally defined as the definable (with parameters) sub-collections of the model.

- We cannot study proper classes within the formal framework of first-order set theory, but only in the meta-theory.
- The notion of class as a definable sub-collection is too restrictive!

Second-order set theory is a formal framework for studying the properties of sets as well as classes.

- Classes are objects in the model.
- We can axiomatize properties of classes.
- We can quantify over classes.
- Non-definable classes are allowed.

Why formally study classes?

Proper class forcing: we can obtain models

- with desired continuum functions,
- in which every set is coded into continuum pattern,
- in which there is no definable linear ordering of sets.

Reinhardt axiom: there exists an elementary embedding $j : V \to V$.

- It is not expressible in first-order set theory.
- It is easy to see that if $V \models \mathrm{ZF}$, there no definable elementary $j : V \to V$.
- (Kunen Inconsistency) There is no elementary $j : V \to V$ in any model of a "reasonable" second-order set theory with AC.
- (Open Problem) Can there be an elementary $j : V \to V$ in a model of a "reasonable" second-order set theory without AC?

Model theoretic constructions with ultrafilters on classes

- Models of first-order set theory with interesting model theoretic properties are obtained as ultrapowers of models of second-order set theory.
- Ultrafilter measures classes.
- Elements of ultrapower are equivalence classes of class functions.

Primer on second-order set theory

Structures have two types of objects: sets and classes.

Syntax: two-sorted logic

- separate variables and quantifiers for sets and classes
- Relations and functions must specify sort for each coordinate.
- Convention: uppercase letters for classes and lowercase letters for sets.

Language of set theory:

- $\in$ relation: sets,
- $\in$ relation: sets $\times$ classes.

Semantics: A model is $\mathcal{M} = \langle M, \in, \mathcal{S}\rangle$, where $\langle M, \in\rangle$ is a model of first-order set theory and $\mathcal{S}$ consists of subsets of M.

Alternative formalization: first-order logic

- objects are classes,
- sets are defined to be those classes that are elements of other classes.

Foundations of second-orders set theory: basic requirements

Bold Claim: A reasonable foundation should imply the basic properties of a ZFC model together with its definable sub-collections.

- The class of sets V is a model of ZFC.
  (Axioms) ZFC for sets.
- V together with predicates for finitely many classes is a model of ZFC.
  (Axioms) class replacement: the restriction of a class function to a set is a set.
- (Class existence principle) Every first-order definable sub-collection of V is a class.
  (Axioms) first-order class comprehension scheme
  class comprehension scheme for second-order formulas in Γ:
  if $\varphi(x, X)$ is in Γ and A is a class, then $\{x \mid \varphi(x, A)\}$ is a class.

Foundations of second-order set theory: GBC

First foundation is developed by Bernays, Gödel, and von Neumann in the 1930s. It codifies the "basic requirements".

GBC: modern formulation

- set axioms: ZFC
- class extensionality
- class replacement
- global choice: there exists a global choice function class (not basic and equivalent to the existence of a well-ordering of V)
- first-order class comprehension

GBC is predicative: definitions of classes don't quantify over classes.

Foundations of second-order set theory: GBC (continued)

Observation:

- There are models of ZFC that don't have a definable well-ordering.
- The constructible universe L has a definable well-ordering.
- L together with its definable sub-collections is a model of GBC.
- GBC is equiconsistent with ZFC.

Theorem: (Solovay) Every countable model of ZFC can be extended to a model of GBC without adding sets.

Proof sketch: Force to add a global well-ordering with a forcing that doesn't add sets because it is $<\kappa$-closed for every cardinal κ. $\square$

Corollary: GBC is conservative over ZFC: any property of sets provable in GBC is already provable in ZFC.

Foundations of second-order set theory: KM

A second-order theory is impredicative if it has comprehension for second-order formulas: definitions of classes can quantify over classes.

Impredicative set theories were first studied by Wang and Morse in the 1940s.

The Kelley-Morse axioms first appeared in Kelley's General Topology textbook in 1955.

KM: modern formulation

- GBC
- (full) second-order comprehension

Strength of KM

Theorem: (Marek, Mostowski?) If $\langle V, \in, \mathcal{S}\rangle \models \mathrm{KM}$, then V is the union of an elementary chain of its rank initial segments V_α:

$$V_{\alpha_0} \prec V_{\alpha_1} \prec \cdots \prec V_{\alpha_\xi} \prec \cdots \prec V,$$

and V thinks that each $V_{\alpha_\xi} \models \mathrm{ZFC}$.

Proof sketch:

- Every model of KM has a truth predicate class Tr coding first-order truth.
- Gödel-codes of all (even nonstandard) ZFC axioms are in Tr.
- If $\langle V_\alpha, \in, \mathrm{Tr}\rangle \prec_{\Sigma_2} \langle V, \in, \mathrm{Tr}\rangle$, then $V_\alpha \prec V$. $\square$

Corollary:

- Full reflection holds in models of KM!
- KM proves $\mathrm{Con(ZFC)}$ (and much more).
- The consistency strength of KM is greater than the strength of the theory: ZFC + there is a transitive model of ZFC.

Strength of KM (continued)

Theorem: KM is weaker than the theory ZFC + there is an inaccessible cardinal.

Proof sketch:
Suppose $V \models \mathrm{ZFC}$ and κ is inaccessible in V.

- $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}$ (and more).
- Countable elementary substructures of $V_{\kappa+1}$ give models of KM in V_κ.
- $\mathrm{Con}(\mathrm{Con}(\mathrm{KM}))$ holds.

The choice scheme in second-order set theory

- This is a choice/collection principle for classes.
- It was first studied by Marek and Mostowski in the 1970s.
- "Every definable V-indexed family of collections of classes has a choice function."

Coded functions $\mathcal{F}: V \to \mathcal{S}$:

- If Z is a class and x is a set, then $Z_x := \{y \mid (x, y) \in Z\}$ is the class coded on the x^{th}-slice of Z.
- Z codes $\mathcal{F}: V \to \mathcal{S}$ with $F(x) = Z_x$.

Definition: The choice scheme consists of assertions

$$\forall x\, \exists X\, \varphi(x, X, A) \to \exists Z\, \forall x\, \varphi(x, Z_x, A)$$

for every second-order $\varphi(x, X, Y)$ and class A.

The choice scheme in second-order set theory (continued)

Some fragments of the choice scheme:

- Σ^1_n (or Π^1_n) choice scheme: bounds the complexity of φ
- parameterless choice scheme: no parameters allowed in φ
- set-sized choice scheme: consists of assertions

$$\forall x \in a\, \exists X\, \varphi(x, X, A) \to \exists Z\, \forall x \in a\, \varphi(x, Z_x, A).$$

for every second-order $\varphi(x, X, Y)$, class A, and set a.

Some applications of the choice scheme: (more on this later)

- (Folklore) The Łoś Theorem for internal second-order ultrapowers of models of KM is equivalent (over KM) to the set-sized choice scheme.
- KM + choice scheme proves that first-order quantifiers don't affect the second-order complexity of an assertion.
- (Mostowski?) The theory KM + choice scheme is bi-interpretable with the theory ZFC^- + there exists an inaccessible cardinal.

Does KM imply (some fragment of) the choice scheme?

Observation: If $V \models \mathrm{ZFC}$ and κ is inaccessible in V, then

$$\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM} + \text{choice scheme}.$$

Proof sketch: Use choice in V for families of subsets of $V_{\kappa+1}$. $\square$

If we want a model of KM in which the choice scheme fails, choice should fail in V for some easily describable family of subsets of $V_{\kappa+1}$.

Observation: Suppose $V \models \mathrm{ZF}$, κ is regular in V, $V_\kappa \models \mathrm{ZFC}$, V_κ is well-orderable in V, then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}$.

Independence strategy:

- Start in V having an inaccessible cardinal κ.
- Perform a clever forcing and move to $V[G]$.
- Find a symmetric submodel $N \models \mathrm{ZF}$ of $V[G]$ such that κ is regular in N and $V_\kappa^N = V_\kappa$, but choice fails for some easily describable family of subsets of $V_{\kappa+1}^N$.

A detour into second-order arithmetic

There are two types of objects: numbers and sets of numbers (reals).

Syntax: two-sorted logic

Language of second-order arithmetic:

- language of first-order arithmetic: $\mathscr{L}_A = \{+, \cdot, <, 0, 1\}$
- $\in$ relation: numbers $\times$ sets

Semantics: A model is $\mathcal{M} = \langle M, +, \cdot, <, 0, 1, \mathcal{S}\rangle$, where $\langle M, +, \cdot, <, 0, 1\rangle$ is a model of first-order arithmetic and $\mathcal{S}$ consists of subsets of M.

Basic requirements:

- $\langle M, +, \cdot, <, 0, 1\rangle \models \mathrm{PA}$ (Peano axioms),
- Induction axiom scheme: consists of assertions

$$(\varphi(0, m, A) \wedge \forall n\, (\varphi(n, m, A) \to \varphi(n+1, m, A))) \to \forall n\, \varphi(n, m, A)$$

for every first-order $\varphi(n, m, X)$, set A, and number m.

Foundations of second-order arithmetic

ACA_0: arithmetical comprehension

- analogue of GBC
- basic requirements + first-order comprehension
- If $V \models \mathrm{ZF}$, then $\langle \omega, +, \cdot, <, 0, 1, \mathcal{S} \rangle \models \mathrm{ACA}_0$, where $\mathcal{S}$ is the collection of all definable subsets of $\langle \omega, +, \cdot, <, 0, 1 \rangle$.

Z_2: full second-order arithmetic

- analogue of KM
- basic requirements + second-order comprehension
- If $V \models \mathrm{ZF}$, then $\langle \omega, +, \cdot, <, 0, 1, P(\omega) \rangle \models \mathrm{Z}_2$.

The choice scheme in second-order arithmetic

Definition: The choice scheme consists of assertions

$$\forall n\,\exists X\,\varphi(n,X,A)\to\exists Z\,\forall n\,\varphi(n,Z_n,A)$$

for every second-order $\varphi(n,X,Y)$ and set A.

Choice scheme fragments:

- Σ^1_n (or Π^1_n) choice scheme
- parameterless choice scheme

Observation: If $V\models \mathrm{ZF}+\mathrm{AC}_\omega$, then $\langle\omega,+,\cdot,<,0,1,P(\omega)\rangle\models \mathrm{Z}_2+$ choice scheme.

Independence strategy:

- Perform a clever forcing and move to $V[G]$.
- Find a symmetric submodel $N\models \mathrm{ZF}$ of $V[G]$ such that choice fails for some easily describable family of subsets of $P(\omega)$.

Unexpected free choice

Theorem: Z_2 proves the Σ^1_2-choice scheme.

Proof sketch: Suppose:

- $\mathcal{M} = \langle M, +, \times, <, 0, 1, \mathcal{S}\rangle \models \mathrm{Z}_2$
- $\mathcal{M} \models \forall n \exists X \varphi(n, X)$, where φ is Σ^1_2.

Observe:

- If α is an ordinal coded in $\mathcal{S}$, then $\mathcal{S}$ has a set coding L_α.
- $\mathcal{M}$ has its own constructible universe $L^{\mathcal{M}}$!
- $\mathcal{M}$ satisfies Shoenfield Absoluteness with respect to $L^{\mathcal{M}}$:
  If ψ is Σ^1_2, then $\mathcal{M} \models \psi$ iff $L^{\mathcal{M}} \models \psi$.
- $L^{\mathcal{M}} \models \exists X \varphi(n, X)$ for every n.
- Choose the $L^{\mathcal{M}}$-least X and use comprehension to collect!

If φ has a set parameter A, replace $L^{\mathcal{M}}$ with $L[A]^{\mathcal{M}}$. $\square$

Question: What about Π^1_2-choice scheme?

Independence of Π^1_2-choice scheme from Z_2

The Feferman-Lévy model

Classic symmetric model $N \models \mathrm{ZF}$ in which $\aleph_1$ is a countable union of countable sets.

Symmetric model properties:

- $N \models \mathrm{ZF}$,
- all $\aleph_n^L$ (the n^{th} cardinal of L) are countable in N,
- $\aleph_\omega^L = \aleph_1^N$ is the first uncountable cardinal of N.

Construction:

- Force with finite-support product $\mathbb{P} = \prod_{n<\omega} \mathrm{Coll}(\omega, \aleph_n)$ over L to collapse the first ω-many successor cardinals of ω to ω.
- Let $G \subseteq \mathbb{P}$ be L-generic and $G_m = G \restriction \prod_{n<m} \mathrm{Coll}(\omega, \aleph_n)$.
- $N \subseteq L[G]$ is a symmetric model of ZF with the property: A is a subset of ordinals in N iff $A \in L[G_m]$ for some m.

Independence Π^1_2-choice scheme from Z_2 (continued)

Theorem: (Feferman, Lévy) Π^1_2-choice scheme can fail in a model of Z_2.

Proof sketch: Consider $\mathcal{M} = \langle \omega, +, \times, <, 0, 1, P(\omega)\rangle^N \models \mathrm{Z}_2$.

- $P(\omega)^N = \bigcup_{n<\omega} P(\omega)^{L[G_n]}$.
- Every $L_{\aleph_n}$ is coded in $\mathcal{M}$, but $L_{\aleph_\omega}$ is not coded in $\mathcal{M}$.
- We cannot collect the codes of $L_{\aleph_n}$.
- The assertion
$$\forall n\, \exists X \text{ codes } L_{\aleph_n} \rightarrow \exists Z\, \forall n\, Z_n \text{ codes } L_{\aleph_n}$$
fails in $\mathcal{M}$.
- The assertion "X codes $L_{\aleph_n}$" is Π^1_2:
$$\underbrace{X \text{ codes } L_\alpha}_{\Pi^1_1} \wedge \forall Y (\underbrace{Y \text{ codes } L_\beta \text{ with } \beta > \alpha}_{\Pi^1_1} \longrightarrow \underbrace{L_\beta \text{ thinks } \alpha = \aleph_n}_{\Pi^1_0}). \ \square$$

Independence of Π^1_1-choice scheme from KM

Natural strategy: Do the Feferman-Lévy construction above an inaccessible cardinal κ.

Symmetric model properties:

- $N \models \mathrm{ZF}$, κ is regular in N, $V_\kappa^N = V_\kappa$. (This gives $\langle V_\kappa, \in, V_{\kappa+1}\rangle^N \models \mathrm{KM}$.)
- all $(\kappa^{+n})^L$ (the n^{th} successor cardinal of κ in L) have size κ,
- $(\kappa^{+\omega})^L = (\kappa^+)^N$.

Construction:

- Suppose κ is inaccessible in L.
- Force with finite-support product $\mathbb{P} = \prod_{n<\omega} \mathrm{Coll}(\kappa, \kappa^{+n})$ over L to collapse the first ω-many successor cardinals of κ to κ.
- Let $G \subseteq \mathbb{P}$ be L-generic and $G_m = G \upharpoonright \prod_{n<m} \mathrm{Coll}(\omega, \kappa^{+n})$.
- $N \subseteq L[G]$ is a symmetric model of ZF with the property: A is a subset of ordinals in N iff $A \in L[G_m]$ for some m.

Independence of Π^1_1-choice scheme from KM (continued)

Theorem: Π^1_1-choice scheme can fail in a model of KM.

Proof: Consider $\mathcal{M} = \langle V_\kappa, \in, V_{\kappa+1}\rangle^N \models \mathrm{KM}$.

- $V^N_{\kappa+1} = \bigcup_{n<\omega} V^{L[G_n]}_{\kappa+1}$.
- Every $L_{\kappa+n}$ is coded in $\mathcal{M}$, but $L_{\kappa+\omega}$ is not coded in $\mathcal{M}$.
- We cannot collect the (codes of) $L_{\kappa+n}$.
- The assertion

$$\forall n \in \omega\, \exists X \text{ codes } L_{\kappa+n} \to \exists Z\, \forall n \in \omega\, X \text{ codes } L_{\kappa+n}$$

  fails in $\mathcal{M}$.
- The assertion "X codes $L_{\kappa+n}$" is Π^1_1: "X codes L_α" is Π^1_0.
- Π^1_1-choice already fails for ω-many choices!

Question: What about Π^1_0-choice scheme?

Is Π^1_0-choice scheme independent from KM?

Independence strategy:

- Start in V having:
  - an inaccessible cardinal κ,
  - ω-many normal κ-Souslin trees $\langle T_n \mid n < \omega\rangle$ with the property: forcing with $\prod_{n<m} T_n$ doesn't add branches to any T_n with $n \geq m$.
- Forcing with a normal κ-Souslin tree adds a cofinal branch.
- Move to a forcing extension $V[G]$ by $\mathbb{P} = \prod_{n<\omega} T_n$.
- Let $G_m = G \restriction \prod_{n<m} T_n$.
- Find a symmetric model $N \subseteq V[G]$ with the property: A is a subset of ordinals in N iff $A \in V[G_m]$ for some m.
- Each T_n has a branch in N, but there is no collecting set of branches!

A detour into homogeneous κ-Souslin trees

Definition: Suppose κ is a cardinal.

- A normal κ-tree is subtree of ${}^{<\kappa}2$ of height κ whose every node has
  - 2 immediate successors,
  - a successor at every higher level.
- A normal κ-Souslin tree has no branches of size κ.
- A homogeneous tree has for any two nodes on the same level an automorphism that moves one to the other.
  - A homogeneous tree is a weakly homogeneous poset.

Theorem: There is a universe V with an inaccessible (or Mahlo) cardinal κ having κ-many homogeneous normal κ-Souslin trees $\langle T_\xi \mid \xi < \kappa\rangle$ with the property: forcing with $\prod_{\xi<\delta} T_\xi$ ($\delta < \kappa$) doesn't add branches to any T_α with $\alpha \geq \delta$.

Proof sketch: Suppose κ is inaccessible (or Mahlo).

- Let $\mathbb{Q}$ be the forcing to add a homogeneous κ-Souslin tree.
- Force with bounded-support product $\mathbb{P} = \prod_{\xi<\kappa} \mathbb{Q}_\xi$, where $\mathbb{Q}_\xi = \mathbb{Q}$. $\square$

Independence of Π^1_0-choice from KM

Symmetric model properties:

- $N \models \mathrm{ZF}$, κ is regular in N, $V_\kappa^N = V_\kappa$. (This gives $\langle V_\kappa, \in, V_{\kappa+1}\rangle^N \models \mathrm{KM}$.)
- N has a sequence $\overrightarrow{T} = \langle T_n \mid n \in \omega\rangle$ of κ-trees such that
  - each T_n has a branch,
  - there is no collecting set of branches.

Construction:

- Suppose:
  - κ is inaccessible,
  - $\langle T_n \mid n < \omega\rangle$ are homogeneous normal κ-Souslin trees with the property: forcing with $\prod_{n<m} T_n$ doesn't add branches to T_n for $n \geq m$.
- Force with finite-support product $\mathbb{P} = \prod_{n<\omega} T_n$ to add branches to all T_n.
- Let $G \subseteq \mathbb{P}$ be V-generic and $G_m = G \restriction \prod_{n<m} T_n$.
- $N \subseteq V[G]$ is a symmetric model of ZF with the property: A is a subset of ordinals in N iff $A \in V[G_m]$ for some m. (This uses weak homogeneity of the trees.)

Independence of Π^1_0-choice from KM (continued)

Theorem: (G., Johnstone, Hamkins) Π^1_0-choice scheme can fail in a model of KM.

Proof sketch: Consider $\mathcal{M} = \langle V_\kappa, \in, V_{\kappa+1}\rangle^N \models \mathrm{KM}$.

- The assertion

$$\Psi(\overrightarrow{T}) := \forall n \in \omega\, \exists B \text{ branch of } T_n \to \exists Z\, \forall n \in \omega\, Z_n \text{ is a branch of } T_n$$

  fails in $\mathcal{M}$.
- The assertion "B is a branch of T_n" is Π^1_0 (in the parameter $\overrightarrow{T}$).
- Π^1_0-choice already fails for ω-many choices! □

Theorem: (G., Johnstone, Hamkins) Parameterless Π^1_0-choice scheme can fail in a model of KM.

Proof sketch: Suppose κ is Mahlo and $\overrightarrow{T}$ has the required property.

- Code $\overrightarrow{T}$ into the continuum function below κ using Easton product forcing.
- In the Easton product extension $V[G]$:
  - κ is inaccessible (even Mahlo),
  - $\overrightarrow{T}$ is definable in V_κ and continues to have the required property. □

Separating fragments of the choice scheme

Theorem: (G., Johnstone, Hamkins) There is a model of KM in which set-sized choice scheme holds, but Π^1_0-choice scheme fails.

Proof sketch:

- Suppose
  - κ is inaccessible,
  - $\vec{T} = \langle T_\xi \mid \xi < \kappa\rangle$ are homogeneous normal κ-Souslin trees with the property: forcing with $\prod_{\xi<\delta} T_\xi$ doesn't add branches to T_ξ for $\xi \geq \delta$.
- Force with bounded-support product $\mathbb{P} = \prod_{\xi<\kappa} T_\xi$.
- Let $G \subseteq \mathbb{P}$ be V-generic and $G_\delta = G \upharpoonright \prod_{\xi<\delta} T_\xi$.
- Construct symmetric model $N \subseteq V[G]$ of ZF with the property: A is a subset of ordinals in N iff $A \in V[G_\delta]$ for some δ. $\square$

Theorem: (G., Johnstone, Hamkins) There is a model of KM in which set-sized choice scheme holds, but parameterless Π^1_0-choice scheme fails.

Proof sketch: Suppose κ is Mahlo and code $\vec{T}$ into the continuum function below κ. $\square$

Separating fragments of the choice scheme (continued)

Theorem: (G., Johnstone, Hamkins) There is a model of KM in which parameterless choice scheme holds, but Π^1_1-choice scheme fails.

Proof sketch:

- We use ideas of Guzicki who proved a similar result for Z_2.
- Suppose κ is inaccessible in L.
- Force with bounded-support product $\prod_{\xi<\kappa^+} \mathrm{Coll}(\kappa, \kappa^{(+\xi)})$ over L.
- Let $G \subseteq \mathbb{P}$ be L-generic and $G_\delta = G \restriction \prod_{\xi<\delta} \mathrm{Coll}(\kappa, \kappa^{(+\xi)})$.
- Construct symmetric model $N \subseteq L[G]$ of ZF with the property: A is a subset of ordinals in N iff $A \in L[G_\delta]$ for some δ.
- Consider $\mathcal{M} = \langle V_\kappa, \in, V_{\kappa+1}\rangle^N$.
- Let A code $(\kappa^+)^L$ in $\mathcal{M}$ and let α_ξ be the ordinal corresponding to ξ in A.
- The assertion

$$\Psi(A) := \forall\xi\, \exists X \text{ codes } L_{\alpha_\xi} \to \exists Z\, \forall\xi\, X \text{ codes } L_{\alpha_\xi}$$

  fails in $\mathcal{M}$.
- Parameterless choice holds in $\mathcal{M}$. $\square$

The theory KM^+

Definition: The theory KM^+ consists of KM together with the choice scheme.

Bold claim: KM^+ is a better foundation for second-order set theory than KM.

- KM^+ proves Łoś Theorem for internal second-order ultrapowers.
  - Suppose $\mathcal{M} = \langle M, \in, \mathcal{S}\rangle \models \mathrm{KM}^+$, $I \in M$ and $U \in M$ is an ultrafilter on I.
  - Classes in $\mathrm{Ult}(\mathcal{M}, U)$ are represented by coded functions $\mathcal{F} = \langle A_i \mid i \in I\rangle$ in $\mathcal{S}$.
- KM^+ proves that first-order quantifiers don't affect second-order complexity: suppose $\varphi(x)$ is equivalent to a Σ^1_n-formula, then so are
  - $\forall x \varphi(x)$,
  - $\exists x \varphi(x)$.
- KM^+ is bi-interpretable with the theory $\mathrm{ZFC}^- +$ there is an inaccessible.

Open question: What is the consistency strength of KM^+ compared to KM?

Weaknesses of KM

Theorem: (Folklore) Łoś Theorem for internal second-order ultrapowers is equivalent (over KM) to the set-sized choice scheme.

Question: How badly does Łoś Theorem fail for ultrapowers of KM models?

Theorem: (G., Johnstone, Hamkins) There is a model of KM whose internal second-order ultrapower by an ultrafilter on ω is not a model of KM.

Theorem: (G., Johnstone, Hamkins) There is a Σ^1_1-formula $\varphi(x)$ and a model of KM in which $\forall x\varphi(x)$ is not equivalent to a Σ^1_1-formula.

Stronger choice principles for classes?

Definition: The ω-dependent choice scheme consists of assertions

$$\forall X \exists Y\, \varphi(X, Y, A) \to \exists Z : \omega \to S\ \forall n \in \omega\, \varphi(Z_n, Z_{n+1})$$

for every second-order formula φ and class A.
"Every definable relation on classes with no terminal nodes has an ω-branch."

Observation: KM^+ + ω-dependent choice scheme proves reflection for second-order assertions: every second-order formula is reflected by some coded collection of classes.

Definition: The ORD-dependent choice scheme consists of assertions

$$\forall \beta \forall X : \beta \to \mathcal{S}\ \exists Y\, \varphi(X, Y, A) \to \exists Z : \mathrm{ORD} \to \mathcal{S}\ \forall \beta\, \varphi(Z \restriction \beta, Z_\beta, A)$$

for every second-order φ and class A.

Dependent choice scheme

Theorem: (Simpson, unpublished) There is a model of Z_2 in which choice scheme holds but Π^1_2-dependent choice scheme fails.

Theorem (Antos, Friedman) $\mathrm{KM}^+ + \mathrm{ORD}$-dependent choice scheme is preserved by all definable tame hyperclass forcing.

Open question: Can we separate the choice scheme, ω-dependent choice scheme, and ORD-dependent choice scheme?

Thank you!