Choice schemes for Kelley-Morse set theory

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Colloquium Logicum

September 4, 2014

This is joint work with Joel David Hamkins and Thomas Johnstone (CUNY).

Classes in first-order set theory

The objects of first-order set theory are sets.

Suppose that $\mathcal{M} = \langle M, \in \rangle \models \mathrm{ZFC}$.

- The classes of $\mathcal{M}$ are the definable sub-collections of M:
  $A \subseteq M$ is a class of if there is a formula $\varphi(x, y)$ and $a \in M$ such that

$$A = \{x \mid \mathcal{M} \models \varphi(x, a)\}.$$

- Properties of $\mathcal{M}$ in relation to a non-definable collection can be studied by adding a unary predicate for it.
- Does ZFC continue to hold in the language with a predicate for a given non-definable collection?

Non-definable collections: two case studies

Tarski's undefinability of truth

Suppose $\mathcal{M} = \langle M, \in \rangle \models \mathrm{ZFC}$.

- The collection of all Gödel codes of true formulas in $\mathcal{M}$
$$T = \{\ulcorner \varphi(\bar{a}) \urcorner \mid \mathcal{M} \models \varphi(\bar{a})\}$$
is never definable.
- Is it possible that $\langle M, \in, T \rangle \models \mathrm{ZFC}$? YES!

Reinhardt Axiom

Suppose $\mathcal{M} = \langle M, \in \rangle \models \mathrm{ZFC}$.

- The ultimate large cardinal axiom: there exists an elementary embedding
$$j : M \to M.$$
- An elementary embedding $j : M \to M$ is never definable. (easy proof)
- Is it possible that $\langle M, \in, j \rangle \models \mathrm{ZFC}$? NO! (the famous Kunen's Inconsistency)

Question: What is a general setting for considering non-definable collections?

Second-order set theory

Second-order set theory has two sorts of objects: sets and classes.

Syntax: two-sorted logic

- separate sorts for sets and classes
- corresponding sorts for quantifiers
- convention: uppercase letters for classes, lowercase letters for sets

Semantics:

A model is a triple $\mathcal{M} = \langle M, \in, \mathcal{S} \rangle$, where M is the sets and $\mathcal{S}$ is the classes.

Alternatively, we can formalize second-order set theory in first-order logic:

- objects are classes,
- define that a set is a class that is an element of some other class.

Axiomatizing second-order set theory

Standard axiomatizations

- Gödel-Bernays (GBC)
- Kelley-Morse (KM)

Commonalities:

- First-order axioms: ZFC.
- Extensionality.
- Replacement: if F is a class function and a is a set, then $F \upharpoonright a$ is a set.
- There is a global choice function (there exists a global well-ordering).

Difference: class existence principle - comprehension axiom schemes

- GBC: Comprehension for first-order formulas.
  If $\varphi(x, Y)$ is a first-order formula and A is a class, then
  $$\{x \mid \varphi(x, A)\} \text{ is a class.}$$
- KM: Comprehension for second-order formulas.
  If $\varphi(x, Y)$ is a second-order formula and A is a class, then
  $$\{x \mid \varphi(x, A)\} \text{ is a class.}$$

Strength of GBC

Observation: If $M \models \mathrm{ZFC}$ has a definable global well-ordering, then

$$\langle M, \in, \mathcal{S}\rangle \models \mathrm{GBC},$$

where $\mathcal{S}$ is the definable classes of M.

- models of $V = L$
- models of $V = \mathrm{HOD}$

Theorem: (Solovay) Every (countable) model of ZFC can be extended to a model of GBC without adding sets.

Proof: Force to add a global well-ordering. Forcing conditions are set well-orders, ordered by extension. The forcing extension has no new sets (because the forcing is $<\kappa$-closed for every cardinal κ) and the new classes satisfy GBC. $\square$

- GBC is equiconsistent with ZFC.
- GBC is conservative over ZFC:
  any property of sets provable in GBC is already provable in ZFC.

Strength of KM: existence of truth predicate

Suppose $\mathcal{M} = \langle M, \in, \mathcal{S}\rangle \models \mathrm{KM}$.

Definition: A class $T \in \mathcal{S}$ is a truth predicate for $\langle M, \in\rangle$ if it satisfies Tarksi's truth conditions: for every $\ulcorner\varphi\urcorner \in M$ (φ possibly nonstandard),

- if φ is atomic, $M \models \varphi(\bar{a})$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$,
- $\ulcorner\neg\varphi(\bar{a})\urcorner \in T$ iff $\ulcorner\varphi(\bar{a})\urcorner \notin T$,
- $\ulcorner\varphi(\bar{a}) \wedge \psi(\bar{a})\urcorner \in T$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$ and $\ulcorner\psi(\bar{a})\urcorner \in T$,
- $\ulcorner\exists x\varphi(x, \bar{a})\urcorner \in T$ iff $\exists b\, \ulcorner\varphi(b, \bar{a})\urcorner \in T$.

Observation: If T is a truth predicate, then $M \models \varphi(\bar{a})$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$.

Observation: If T is a truth predicate and $\ulcorner\varphi\urcorner \in \mathrm{ZFC}^{\mathrm{M}}$ (φ possibly nonstandard), then $\varphi \in T$.

Proof:

- Is every (potentially nonstandard) instance of replacement in T?
- Suppose $\ulcorner\forall a \in d \exists! b\, \varphi(a, b)\urcorner \in T$ ("!" means unique) .
- $\mathcal{M} \models \forall a \in d \exists! b \ulcorner\varphi(a, b)\urcorner \in T$.
- $r = \{b \mid \ulcorner\varphi(a, b)\urcorner \in T \text{ and } a \in d\}$ exists by replacement. $\square$

Strength of KM: existence of truth predicate (continued)

Theorem: If $\mathcal{M} = \langle M, \in, \mathcal{S}\rangle \models \mathrm{KM}$, then $\mathcal{S}$ has a truth predicate for $\langle M, \in\rangle$.

Proof:

- $\mathcal{M}$ satisfies the second-order assertion:

  $\exists\,\Sigma^0_0$-truth predicate and $\forall n \in \omega\,(\exists\,\Sigma^0_n$-truth predicate $\rightarrow \exists\,\Sigma^0_{n+1}$-truth predicate$)$.
- (by induction) $\mathcal{M}$ satisfies the second-order assertion:

  $$\forall n \in \omega\,\exists\,\Sigma_n\text{-truth predicate.}$$
- (by comprehension) $\mathcal{M}$ satisfies:

  $$\exists C\,\forall n \in \omega\; C_n \text{ is the } \Sigma_n\text{-truth predicate.}$$

$C_n := \{x \mid (n, x) \in C\}$ is the slice on coordinate n of C. The Σ_n-truth predicate is unique in $\mathcal{M}$. $\square$

Strength of KM: existence of transitive models of ZFC

Theorem: If $\mathcal{M} = \langle M, \in, \mathcal{S}\rangle \models \mathrm{KM}$, then M is the union of an elementary chain of its rank initial segments V_α^M:

$$V_{\alpha_0}^M \prec V_{\alpha_1}^M \prec \cdots \prec V_{\alpha_\xi}^M \prec \cdots \prec M,$$

and $\mathcal{M}$ thinks that each $V_{\alpha_\xi}^M \models \mathrm{ZFC}$.

Proof:

- Let $T \in \mathcal{S}$ be the truth predicate for $\langle M, \in\rangle$.
- $\mathrm{ZFC}^{\mathrm{M}} \subseteq \mathrm{T}$.
- Let $\langle V_\alpha^M, \in, T \cap V_\alpha^M\rangle \prec_{\Sigma_2} \langle M, \in, T\rangle$ (using reflection).
- $\langle V_\alpha^M, \in, T \cap V_\alpha^M\rangle \models$ "$T \cap V_\alpha^M$ is a truth predicate".
- $V_\alpha^M \prec M$ (Tarski-Vaught criteria).
- $\mathcal{M}$ satisfies that $V_\alpha^M \models \mathrm{ZFC}^{\mathrm{M}}$. $\square$

Note: Full reflection holds in models of KM!

Strength of KM: below an inaccessible cardinal

Corollary: KM is stronger than the iterated Con-hierarchy of Con(ZFC).

Theorem: KM is weaker than the assertion that there exists an inaccessible cardinal.

Proof:
Suppose $V \models \mathrm{ZFC}$ and κ is inaccessible in V.

- $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}$.
- There are many KM models in V_κ (take a countable elementary substructure of $V_{\kappa+1}$).
- Con(Con(KM)) holds. $\square$

Note: It suffices to assume $V \models \mathrm{ZF}$, κ is inaccessible, $V_\kappa \models \mathrm{ZFC}$.

Choice schemes for second-order set theory

The choice scheme and its variants are choice/collection principles for classes.

Choice scheme: If φ is a second-order formula and A is a class, then

$$\forall x \exists X \varphi(x, X, A) \to \exists Z \forall x\, \varphi(x, Z_x, A).$$

$Z_x := \{z \mid (x, z) \in Z\}$ is the slice of Z on coordinate x.

- If for every set, there is a class witnessing a given property, then there is a choice/collecting class of witnesses.
- First studied by Marek and Mostowski in the 1970s?

Set-sized choice scheme: If φ is a second-order formula, a is a set and A is a class, then

$$\forall x \in a \exists X \varphi(x, X, A) \to \exists Z \forall x \in a\, \varphi(x, Z_x, A).$$

Σ^1_n-choice scheme: bounds the complexity of φ.

Parameterless choice scheme: no parameters in φ.

Choice schemes and GBC

Observation: The Σ^1_0-choice scheme for ω-many choices can fail in a model of GBC.

Proof:

- Suppose M is a transitive model of $V = L$.
- $\mathcal{M} = \langle M, \in, \mathcal{S}\rangle \models \mathrm{GBC}$, where $\mathcal{S}$ is the definable classes of M.
- $\mathcal{M} \models \forall n \in \omega \exists\, \Sigma_n$-truth predicate (because M is standard).
- If there is Z such that Z_n is the Σ_n-truth predicate, then truth is definable from Z! $\square$

Choice schemes and KM

A model of KM + choice scheme

If $V \models \mathrm{ZFC}$ and κ is inaccessible in V, then

$$\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM} + \text{choice scheme}.$$

Question: Does (some fragment of) the choice scheme follow from KM?

Question: Does the set-sized choice scheme imply the (full) choice scheme over KM?

Candidate counterexamples: $\langle V_\kappa, \in, V_{\kappa+1}\rangle$, where

- $V \models \mathrm{ZF}$ and κ is inaccessible in V,
- choice holds in V_κ,
- choice fails for objects of size κ.

Choice schemes for second-order arithmetic

There are deep analogies between second-order set theory and second-order arithmetic.

Primer:

- Two sorts of objects: numbers and sets of numbers.
- Syntax: two-sorted logic.
- Semantics: A model of second-order arithmetic is: $\langle M, +, \times, <, 0, 1, \mathcal{S}\rangle$, where M is numbers and $\mathcal{S}$ is sets.
- Analogues of GBC and KM are ACA_0 and Z_2.
- First-order axioms: Peano Arithmetic (PA).
- Differ in set existence principles (and restrictions on induction).
- ACA_0: Comprehension (and induction) for first-order formulas with set parameters.
- Z_2: Comprehension (and induction) for second-order formulas with set parameters.
- If $V \models \mathrm{ZF}$, then $\langle \omega, +, \times, <, 0, 1, P(\omega)\rangle \models \mathrm{Z}_2$.
- The choice scheme: if φ is a second-order formula and A is a set, then

$$\forall n \exists X \varphi(n, X, A) \to \exists Z \forall n\, \varphi(n, Z_n, A).$$

Choice schemes and Z_2

Question: Does (some fragment of) the choice scheme follow from Z_2?

Theorem: Z_2 proves the Σ^1_2-choice scheme.

Proof: Suppose $\mathcal{M} = \langle M, +, \times, <, 0, 1, \mathcal{S}\rangle \models \mathrm{Z}_2$ and $\mathcal{M} \models \forall n \exists X \varphi(n, X)$, where φ is Σ^1_2.

(We ignore parameters for simplicity.)

- If α is an ordinal coded in $\mathcal{S}$, then $\mathcal{S}$ has a set coding L_α.
- $\mathcal{M}$ has its own constructible universe $L^{\mathcal{M}}$!
- $\mathcal{M}$ satisfies Shoenfield Absoluteness:
  If ψ is a Σ^1_2-assertion, then
  $$\mathcal{M} \models \psi \text{ iff } L^{\mathcal{M}} \models \psi.$$
  In $L^{\mathcal{M}}$, ψ is interpreted as an assertion about numbers and sets of numbers.
- $L^{\mathcal{M}}$ has a witness to every Σ^1_2-assertion $\exists X \varphi(n, X)$.
- Choose the $L^{\mathcal{M}}$-least X and use comprehension to collect! $\square$

The Feferman-Lévy model

Properties:

- $N \models \mathrm{ZF}$,
- each $\aleph_n^L$ is countable ($\aleph_n^L$ is the n^{th} uncountable cardinal of L),
- $\aleph_\omega^L = \aleph_1^N$ is the first uncountable cardinal.

Construction:

- Force with finite-support product $\mathbb{P} = \Pi_{n<\omega} \mathrm{Coll}(\omega, \aleph_n)$ over L to collapse the first ω-many successor cardinals of ω to ω.
- Let $G \subseteq \mathbb{P}$ be L-generic and $G_m = G \upharpoonright \Pi_{n<m} \mathrm{Coll}(\omega, \aleph_n)$.
- $N \subseteq L[G]$ is a symmetric model of ZF with the property: A is a subset of ordinals in N iff $A \in V[G_m]$ for some m.

Failure of Π^1_2-choice for Z_2

Theorem: (Feferman, Lévy) Π^1_2-choice scheme can fail in a model of Z_2.

Proof: Consider $\mathcal{M} = \langle \omega, +, \times, <, 0, 1, P(\omega)\rangle^N \models \mathrm{Z}_2$.

- Every $L_{\aleph_n}$ is coded in $\mathcal{M}$, but $L_{\aleph_\omega}$ is not coded in $\mathcal{M}$.
- We cannot collect the (codes of) $L_{\aleph_n}$.
- The assertion
$$\forall n \exists X = L_{\aleph_n} \to \exists Z \forall n\, Z_n = L_{\aleph_n}$$
fails in $\mathcal{M}$.
- The assertion $X = L_{\aleph_n}$ is Π^1_2 (it is Π^1_1 whether a set of numbers codes an ordinal). $\square$

Generalization of the Feferman-Lévy model

Properties

- $N \models \mathrm{ZF}$, κ is inaccessible in N, $V_\kappa^N \models \mathrm{ZFC}$,
- each $(\kappa^{+n})^L$ has size κ ($(\kappa^{+n})^L$ is the n^{th} successor cardinal of κ in L),
- $(\kappa^{+\omega})^L = (\kappa^+)^N$.

Construction:

- Suppose κ is inaccessible in L.
- Force with finite-support product $\mathbb{P} = \Pi_{n<\omega} \mathrm{Coll}(\kappa, \kappa^{+n})$ over L to collapse the first ω-many successor cardinals of κ to κ.
- Let $G \subseteq \mathbb{P}$ be L-generic and $G_m = G \upharpoonright \Pi_{n<m} \mathrm{Coll}(\omega, \kappa^{+n})$.
- $N \subseteq L[G]$ is a symmetric model of ZF with the property: A is a subset of ordinals in N iff $A \in V[G_m]$ for some m.

Failure of Π^1_1-choice for KM

Theorem: Π^1_1-choice scheme for ω-many choices can fail in a model of KM.

Proof: Consider $\mathcal{M} = \langle V_\kappa, \in, V_{\kappa+1}\rangle^N \models \mathrm{KM}$.

- Every $L_{\kappa+n}$ is coded in $\mathcal{M}$, but $L_{\kappa+\omega}$ is not coded in $\mathcal{M}$.
- We cannot collect the (codes of) $L_{\kappa+n}$.
- The assertion
$$\forall n \in \omega \exists X = L_{\kappa+n} \to \exists Z \forall n \in \omega\, X = L_{\kappa+n}$$
fails in $\mathcal{M}$.
- The assertion $X = L_{\kappa+n}$ is Π^1_1 (it is Π^1_0 whether a class codes an ordinal). $\square$

Failure of Π^1_0-choice for KM

Question: Does KM prove Π^1_0-choice?

Theorem: (G., Johnstone, Hamkins) A model of KM can fail to make ω-many class choices for a Π^1_0 property. A model of KM can satisfy the set-sized choice scheme, but fail to make class many class choices for a Π^1_0-property.

- There is a model $\mathcal{M} \models \mathrm{KM}$ and a Π^1_0-formula φ such that the assertion
$$\forall n \in \omega \exists X\, \varphi(n, X) \to \exists Z \forall n \in \omega\, \varphi(n, Z_n)$$
fails in $\mathcal{M}$.
- There is a model $\mathcal{M} \models \mathrm{KM}$ in which the set-sized choice scheme holds, but the assertion
$$\forall x \exists X\, \varphi(x, X) \to \exists Z \forall x\, \varphi(x, Z_x),$$
fails in $\mathcal{M}$ for some Π^1_0-formula φ.

Proof idea: $\mathcal{M}$ has a collection of class trees, each of whom has branches, but we cannot choose a single branch from every tree.

Homogeneous κ-Souslin trees

Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal in V.

Definition:

- If T is a tree, a branch of T is a cofinal path through T.
- A κ-tree is a sub-tree of ${}^{<\kappa}2$ of height κ.
- A κ-tree is Souslin if it has no antichains of size κ and no branches.
- A κ-tree T is homogeneous if for any nodes t and s on the same level of T, there is an automorphism that maps t to s.

Homogeneous κ-Souslin trees:

- A κ-Souslin tree T is a partial order and forcing with T adds a branch through T.
- T is a weakly homogeneous partial order.

  A poset $\mathbb{P}$ is weakly homogeneous if for any conditions $p, q \in \mathbb{P}$, there is an automorphism π such that $\pi(p)$ and q are compatible.
- Weak homogeneity yields key properties of symmetric models of ZF.

Failure of Π^1_0-choice for KM: symmetric model 1

Properties:

- $N \models \mathrm{ZF}$, κ is inaccessible in N, $V_\kappa^N \models \mathrm{ZFC}$,
- N has a sequence $\langle T_n \mid n \in \omega\rangle$ of κ-trees such that
  - each T_n has branches,
  - there is no choice set of branches.

Construction:

Theorem: There is a model $V \models \mathrm{ZFC}$ with an inaccessible cardinal κ having a sequence $\langle T_n \mid n < \omega\rangle$ of homogeneous κ-Souslin trees such that:

- The product forcing $\Pi_{n<m} T_n$ is $<\kappa$-distributive (and κ-cc).
- The forcing $\Pi_{n<m} T_n$ does not add branches to any T_k with $k \geq n$.

Proof: Force with full-support product $\Pi_{n\in\omega}\mathbb{Q}_n$, where $\mathbb{Q}_n = \mathbb{Q}$ adds a homogeneous κ-Souslin tree. $\square$

- Force with finite-support product $\mathbb{P} = \Pi_{n<\omega} T_n$ to add branches to all T_n.
- Let $G \subseteq \mathbb{P}$ be V-generic and $G_m = G \restriction \Pi_{n<m} T_n$.
- $N \subseteq V[G]$ is a symmetric model of ZF with the property: A is a subset of ordinals in N iff $A \in V[G_m]$ for some m.

Failure of Π^1_0-choice for KM: ω-many choices

Theorem: (G., Johnstone, Hamkins) Π^1_0-choice scheme for ω-many choices can fail in a model of KM.

Proof: Consider $\mathcal{M} = \langle V_\kappa, \in, V_{\kappa+1}\rangle^N \models \mathrm{KM}$.

- The assertion

$$\psi(\overrightarrow{T}) := \forall n \in \omega \exists B \text{ branch of } T_n \rightarrow \exists Z \forall n \in \omega\, Z_n \text{ is a branch of } T_n$$

  fails in $\mathcal{M}$.
- The assertion B is a branch of T_n is Π^1_0 (in the parameter $\overrightarrow{T}$). $\square$

Note: A more complicated construction (starting with a Mahlo κ) eliminates the parameter $\overrightarrow{T}$ by forcing to code it into V_κ before forcing with $\Pi_{n<\omega} T_n$.

Failure of Π^1_0-choice: symmetric model 2

Properties:

- $N \models \mathrm{ZF}$, κ is inaccessible in N, $V^N_\kappa \models \mathrm{ZFC}$,
- N has a sequence $\langle T_\xi \mid \xi < \kappa\rangle$ of κ-trees such that
  - each T_ξ has branches,
  - there is no choice set of branches.

Construction:

Theorem: There is a model $V \models \mathrm{ZFC}$ with an inaccessible cardinal κ having a sequence $\langle T_\xi \mid \xi < \kappa\rangle$ of homogeneous κ-Souslin trees such that:

- The product forcing $\Pi_{\xi<\alpha} T_\xi$ is $<\kappa$-distributive (and κ-cc)for every $\alpha < \kappa$.
- The forcing $\Pi_{\xi<\alpha} T_\xi$ does not add branches to any T_β with $\beta \geq \alpha$.

Proof: Force with bounded support product $\Pi_{\xi<\kappa}\mathbb{Q}_\xi$, where $\mathbb{Q}_\xi = \mathbb{Q}$ adds a homogeneous κ-Souslin tree. □

- Force with bounded support product $\mathbb{P} = \Pi_{\xi<\kappa} T_\xi$ to add branches to all T_ξ.
- Let $G \subseteq \mathbb{P}$ be V-generic and $G_\alpha = G \restriction \Pi_{\xi<\alpha} T_\xi$.
- $N \subseteq V[G]$ is a symmetric model of ZF with the property:
  A is a subset of ordinals in N iff $A \in V[G_\alpha]$ for some α.

Failure of Π^1_0-choice for KM: ORD-many choices

Theorem: (G., Johstone, Hamkins) There is a model of KM in which the set-sized choice scheme holds, but Π^1_0-choice scheme fails.

Proof: Consider $\mathcal{M} = \langle V_\kappa, \in, V_{\kappa+1}\rangle^N \models \mathrm{KM}$.

- $\mathcal{M}$ satisfies the set-sized choice scheme!
- The assertion

$$\psi(\overrightarrow{T}) := \forall\xi\exists B \text{ branch of } T_\xi \to \exists Z\forall\xi\, Z_\xi \text{ is a branch of } T_\xi$$

  fails in $\mathcal{M}$.

Note: A more complicated construction (starting with a Mahlo κ) eliminates the parameter $\overrightarrow{T}$ by forcing to code it into V_κ before forcing with $\Pi_{\xi<\kappa} T_\xi$.

The theory KM^+

Definition: The theory KM^+ consists of KM together with the (full) choice scheme.

Abstract applications: KM^+ proves that:

- The Łoś theorem holds for internal (and external) ultrapowers of models of KM^+.
  - Suppose $\mathcal{M} = \langle M, \in, \mathcal{S}\rangle \models \mathrm{KM}^+$, $I \in M$ and $U \in M$ is an ultrafilter on I.
  - Classes in $\mathrm{Ult}(\mathcal{M}, U)$ are represented by coded functionals $\mathcal{F} = \langle A_i \mid i \in I\rangle$ in $\mathcal{S}$.
- Second-order complexity classes are closed under first-order quantification. Suppose that $\varphi(x)$ is equivalent to a Σ^1_n-assertion, then so are
  - $\forall x \varphi(x)$,
  - $\exists x \varphi(x)$.

Concrete applications:

- nonstandard set theory (with infinitesimals)
- properties of class forcing extensions

Weaknesses in KM

Absorption of first-order quantifiers

Theorem: (G., Johnstone, Hamkins) There is a model of KM whose second-order complexity classes are not closed under (even bounded) first-order quantification.

The Łoś Theorem

Theorem: The Łoś theorem for internal ultrapowers is equivalent (over KM) to the set-sized choice scheme.

Question: How badly does the Łoś theorem fail for ultrapowers of KM models?

Theorem: (G., Johnstone, Hamkins) There is a model of KM whose internal ultrapower by an ultrafilter on ω is not a model of KM.

Questions

Question: Is KM^+ (consistency-wise) stronger than KM?

Dependent choice scheme: If φ is a second-order formula and A is a class, then

$$\forall\beta\forall X : \beta \to \mathcal{S}\ \exists Y\, \varphi(X, Y, A) \to \exists Z : \mathrm{ORD} \to \mathcal{S}\ \forall\beta\, \varphi(Z \upharpoonright \beta, Z_\beta, A).$$

ω-**dependent choice scheme**: If φ is a second-order formula and A is a class, then

$$\forall X \exists Y\, \varphi(X, Y, A) \to \exists Z : \omega \to \mathcal{S}\ \forall n \in \omega\, \varphi(Z_n, Z_{n+1}, A).$$

Question: Does KM^+ prove the ω-dependent choice scheme/dependent choice scheme?

Thank you!