A natural model of the Multiverse axioms

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MIT Logic Seminar

This work is joint with Joel David Hamkins
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- (1934) Skolem uses the Compactness Theorem to construct a model of ZFC having an ill-founded ω - an ω-nonstandard model of ZFC.

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CH is independent of ZFC (even with large cardinal axioms!).

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- Set theorists build generic extensions satisfying various properties or their negations.
  - Diamond Principle - guessing principle for subsets of ω_1
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- Set theorists build generic extensions exhibiting nearly any wish list of properties.
- Set theorists study relationships between a universe and its generic extensions.
  - does a universe satisfying φ have a generic extension satisfying ψ?
  - how similar to the original universe can we make a generic extension satisfying φ?

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- Many large cardinal axioms imply the existence of well-founded (iterated) ultrapowers of the universe.
  - a cardinal κ is measurable if there is a non-principal κ-complete ultrafilter on κ
  - there is a measurable cardinal if and only if there is a elementary embedding $j : V \to M$ of the universe into an inner model M
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- Connections with forcing:
  - set theorists studied generic extensions of universes with large cardinals
  - a universe V may not have definable well-founded ultrapowers, but its generic extension $V[G]$ will have (iterated) well-founded ultrapowers of V

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Summary: Set theorists are investigating a multitude of possible universes and their interrelationships.

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- Similarly, we study the nonstandard models of arithmetic but the natural numbers will always be the model of arithmetic.

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- Some arguments against the Universe View:
  - starting from any universe, we can force CH or $\neg$CH without changing much of its structure
  - large cardinals can be destroyed by forcing
  - PFA is destroyed by Cohen forcing

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- Wellfoundedness Mirage axiom: Every universe M in $\mathcal{M}$ is a set in another universe N in $\mathcal{M}$, which thinks M is ω-nonstandard.
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The Multiverse Axioms (continued)

- Reverse Ultrapower Axiom: For every universe M_1 in $\mathcal{M}$ and every ultrafilter U_1 in M_1, there is M_0 in $\mathcal{M}$, with an ultrafilter U_0, such that M_1 is the internal ultrapower of M_0 by U_0, sending U_0 to U_1.

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- Reverse Embedding Axiom: For every embedding $j_1 : M_1 \to M_2$ between two universes M_1 and M_2 in $\mathcal{M}$ that is definable in another universe N in $\mathcal{M}$ and thought by N to be elementary, there is M_0 in $\mathcal{M}$ and a similarly definable $j_0 : M_0 \to M_1$ in N such that j_1 is the iterate of j_0, meaning $j_1 = j_0(j_0)$.
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Main Theorem (G. and Hamkins, 2010)

If ZFC is consistent, then the Multiverse of countable computably saturated models satisfies all the Multiverse axioms.

Introduction to computably saturated models

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- A model M of a computable language $\mathcal{L}$ is computably saturated if for every tuple $\bar{a} \in M$, every finitely realizable computable type $p(\bar{x}, \bar{a})$ is already realized in M.

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- A model M of a computable language $\mathcal{L}$ is computably saturated if for every tuple $\bar{a} \in M$, every finitely realizable computable type $p(\bar{x}, \bar{a})$ is already realized in M.

Theorem (?Wilmers, 1975)

Every consistent theory T in a computable language $\mathcal{L}$ having infinite models has a computably saturated model in every infinite cardinality.

Countable computably saturated models of ZFC

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- Not all countable ω-nonstandard models are computably saturated.
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  - start with any countable ω-nonstandard model M of ZFC
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- If ZFC is consistent, then there are $2^{\aleph_0}$ non-isomorphic countable computably saturated models of ZFC.

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- If M is computably saturated and $p(a)$ is a type of $a \in M$, then $p(x)$ is in $SSy(M)$.
  Proof: The type $q(y, a) = \{\ulcorner \varphi(x) \urcorner \in y \leftrightarrow \varphi(a) \mid \varphi(x) \text{ formula}\}$ must be realized.

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  Proof: The type $q(y, a) = \{\ulcorner\varphi(x)\urcorner \in y \leftrightarrow \varphi(a) \mid \varphi(x) \text{ formula}\}$ must be realized.
- If M is computably saturated, then $Th(M)$ - the theory of M - is in $SSy(M)$.

Standard System saturated models of ZFC

Definition (Wilmers, 1975)

A model M of ZFC is *$SSy(M)$-saturated* if for every $a \in M$, every finitely realizable type $p(x, a)$ in $SSy(M)$ is already realized in M.

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Theorem (Wilmers, 1975)

A model M of ZFC is computably saturated if and only if it is $SSy(M)$-saturated.

Characterization of countable computably saturated models of ZFC

Theorem (Folklore, 1970's)

Any two countable computably saturated models of ZFC with the same theory and same standard system are isomorphic.

Proof:
A back and forth construction using standard system saturation together with the fact that the type of every element is in the standard system.

Multiverse axioms and computable saturation

Key Observation:
Every model of ZFC that is an element of an ω-nonstandard model of ZFC is computably saturated.
Proof:

- N is an ω-nonstandard model of ZFC and $M \in N$ is a model of ZFC
- $p(x,b)$ is a computable type finitely realizable over M
- $p(x,b)$ is in $SSy(N)$ since all computable sets are in $SSy(N)$
- there is $p \in N$ such that $p \cap \omega = p(x,y)$
- for every $n \in \omega$, there is $w \in N$ such that
  $N \models$ "For every $\ulcorner\varphi(x,y)\urcorner \in p \cap n$ $M \models \varphi(w,b)$"
  "N thinks that p is finitely realizable"
- there is a nonstandard $a \in N$ and $W \in N$ such that
  $N \models$ "For every $\ulcorner\varphi(x,y)\urcorner \in p \cap a$ $M \models \varphi(W,b)$" by undefinability of ω in N
  "N thinks that p is realizable up to a nonstandard natural number"
- W realizes $p(x,b)$ in M since $p \cap a$ includes all of $p(x,y)$

Multiverse axioms and computable saturation (continued)

Well-foundeness Mirage Lemma

Every countable computably saturated model of ZFC contains an isomorphic copy of itself as an element that it thinks is a countable ω-nonstandard model of a fragment of ZFC.

Proof:

- M is countable computably saturated model of ZFC
- $Th(M)$ is in $SSy(M)$ and hence there is $t \in M$ such that $t \cap \omega = Th(M)$
- for every $n \in \omega$, $M \models Con(t \cap n)$
  "M thinks that every finite fragment t is consistent"
- there is a nonstandard $a \in M$ such that $M \models Con(t \cap a)$
  "M thinks that t is consistent up to a nonstandard natural number"
- M has a model K of the theory $t \cap a$ that it thinks
  - is countable
  - is ω-nonstandard
  - satisfies a fragment of ZFC

  by compactness theorem applied inside M
- $K \models Th(M)$ since $t \cap a$ includes all of $Th(M)$
- ω^M is an initial segment ω^K and hence $SSy(K) = SSy(M)$
- $K \cong M$ by the characterization

A natural model of the Multiverse axioms

Main Theorem

If ZFC is consistent, then the Multiverse $\mathcal{M}$ of countable computably saturated models satisfies all the Multiverse axioms.

Sketch of Proof:

- Countability axiom and Well-Foundedness Mirage axiom
  - by the Well-foundedness Mirage Lemma (WfML)
- Realizability axiom
  - N is definable in a universe M in $\mathcal{M}$
  - M is countable in a universe K in $\mathcal{M}$ by the WfML
  - N is in K
  - N is computably saturated by the Key Observation and hence N is in $\mathcal{M}$
- Forcing extension axiom
  - M is in $\mathcal{M}$
  - M is countable in a universe N in $\mathcal{M}$
  - for every partial order of M, there is a generic filter G in N and hence $M[G]$ is in N
  - $M[G]$ is computably saturated by the Key Observation and hence $M[G]$ is in $\mathcal{M}$
- Reverse Ultrapower Axiom and Reverse Embedding Axiom
  - use the WfML

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Question: Must a Multiverse satisfying the Multiverse axioms consist of countable models?

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Observation:

- If M is a model of ZFC+Con(ZFC), then the Multiverse of all countable computably saturated models of ZFC from the perspective of M satisfies the Multiverse axioms.
- If ZFC+Con(ZFC) is consistent, then there is a model M of ZFC+$Con(ZFC)$ with $|\omega^M| = \kappa$ for any infinite cardinal κ.

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Question: Must a Multiverse satisfying the Multiverse axioms consist of countable models?

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- If *M* is a model of ZFC+*Con*(ZFC), then the Multiverse of all countable computably saturated models of ZFC from the perspective of *M* satisfies the Multiverse axioms.
- If ZFC+*Con*(ZFC) is consistent, then there is a model *M* of ZFC+*Con*(*ZFC*) with $|\omega^M| = \kappa$ for any infinite cardinal κ.

Theorem (G. and Hamkins)

If there are saturated models of ZFC of cardinality κ, then the Multiverse of these satisfies the Multiverse axioms.