Incomparable $\omega_1$-like models of set theory

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The $\omega_1$-like models of Peano Arithmetic and set theory

“$\omega_1$-like models bridge the gap between countable and uncountable structures.”
–Roman Kossak

**Definition:** A model $\mathcal{M} = \langle M, +, \cdot, <, 0, 1 \rangle \models \text{PA}$ is $\omega_1$-like if:
- $M$ has size $\omega_1$,
- every proper initial segment is countable: for every $b \in M$, the initial segment $b^\mathcal{M} = \{ a \in M \mid a < b \}$ is countable.

**Definition:** A model $\mathcal{M} = \langle M, \in \rangle \models \text{ZFC}$ is $\omega_1$-like if:
- $M$ has size $\omega_1$,
- every proper initial segment of the ordinals is countable: for every $\beta \in \text{ORD}^\mathcal{M}$, the initial segment $\beta^\mathcal{M} = \{ \alpha \in M \mid \alpha \in^\mathcal{M} \beta \}$ is countable (it follows that every rank initial segment $V^\mathcal{M}_\alpha$ is countable).

The $\omega_1$-like models are locally countable, yet they inherit the set-theoretic structure of $\omega_1$.

This allows us to build exotic $\omega_1$-like models with surprising properties.

Given any cardinal $\kappa$, we similarly define the notion of a $\kappa$-like model.
Existence of $\omega_1$-like models of PA

**Definition:** Suppose $\mathcal{M} \prec \mathcal{N} \models \text{PA}$. Then $\mathcal{N}$ is an **end-extension** of $\mathcal{M}$, $\mathcal{M} \prec_e \mathcal{N}$, if for every $b \in N \setminus M$, we have $b > a$ for all $a \in M$.

**Tagline:** “New elements are added only on top.”

**Theorem:** (MacDowell, Specker, 1961) *Every* model of PA has a proper end-extension.
**Existence of $\omega_1$-like models of PA (continued)**

**Corollary:** There is an $\omega_1$-like model of PA.

**Proof:** We construct an elementary chain of models of PA of length $\omega_1$

$$M_0 \prec M_1 \prec \cdots \prec M_\omega \prec \cdots M_\xi \prec M_{\xi+1} \prec \cdots$$

using transfinite recursion:

- $M_0$ is any countable model of PA,
- $M_{\xi+1}$ is a proper countable end-extension of $M_\xi$,
- $M_\lambda = \bigcup_{\xi<\lambda} M_\xi$ for limit ordinals $\lambda$.

Clearly $M = \bigcup_{\xi<\omega_1} M_\xi$ is $\omega_1$-like. □

**Corollary:** For every cardinal $\kappa$, there is a $\kappa$-like model of PA.
Models of set theory

A model $\mathcal{M} = \langle M, \in \rangle \models ZFC$ is the union of its von Neumann hierarchy.

Inside $\mathcal{M}$, we define:

- $V_0^\mathcal{M} = \emptyset$,
- $V_{\alpha+1}^\mathcal{M} = P(V_\alpha^\mathcal{M})$ (powerset of $V_\alpha^\mathcal{M}$),
- $V_\lambda^\mathcal{M} = \bigcup_{\alpha < \lambda} V_\alpha^\mathcal{M}$ for limit ordinals $\lambda$.
- The rank of $a \in M$ is the least $\alpha$ such that $a \in V_{\alpha+1}^\mathcal{M} \setminus V_\alpha^\mathcal{M}$.

Note: The set membership relation $\in^\mathcal{M}$ is (externally) not necessarily well-founded. So the order $\langle \text{ORD}^\mathcal{M}, \in^\mathcal{M} \rangle$ is not necessarily well-founded.
End-extensions of models of set theory

**Definition:** Suppose $\mathcal{M} \prec \mathcal{N} \models \text{ZFC}$. Then $\mathcal{N}$ is an end-extension of $\mathcal{M}$, $\mathcal{M} \prec_e \mathcal{N}$, if for every $b \in N \setminus M$, the rank $\beta$ of $b$ is greater than all $\alpha \in \text{ORD}^M$.

**Tagline:** “New elements are added only on top.”
End-extensions of models of set theory (continued)

**Sneak Preview:** It is not true that every model of ZFC has a proper end-extension!

**Theorem:** (Keisler, Morley, 1968) Every countable model of ZFC has a proper end-extension.

- Original proof is an omitting types argument.
- Alternative proof uses an ultrapower construction.
- The ultrapower argument is flexible: build desired properties into end-extensions.
Skolem ultrapowers

Suppose $\mathcal{M} = \langle M, \in \rangle \models \text{ZFC}$.

$A \subseteq M$ is a class of $\mathcal{M}$ if there is a formula $\varphi(x, y)$ and $s \in M$ such that

$$A = \{ a \in M \mid \mathcal{M} \models \varphi(a, s) \}.$$ 

**Skolem ultrapower:**
- Class functions $F : \text{ORD}^\mathcal{M} \to M$
- Ultrafilter $U$ on $O$ - the collection of classes of ordinals

**Question:** Do Skolem ultrapowers satisfy the Łoś Theorem?
Skolem ultrapowers: Łoś Theorem

Inductive assumption:

\[ \prod_{\mathcal{O}} \mathcal{M}/U \models \phi([G]_U, [F]_U) \iff \{ \alpha \in \text{ORD}^\mathcal{M} \mid \mathcal{M} \models \phi(G(\alpha), F(\alpha)) \} \in U. \]

Existential quantifier: \( \exists x \phi(x, [F]_U) \)

Suppose \( \{ \alpha \in \text{ORD}^\mathcal{M} \mid \mathcal{M} \models \exists x \phi(x, F(\alpha)) \} \in U. \)

Is there a class function \( G : \text{ORD}^\mathcal{M} \to \mathcal{M} \) such that

\[ \mathcal{M} \models \exists x \phi(x, F(\alpha)) \Rightarrow \mathcal{M} \models \phi(G(\alpha), F(\alpha))? \]

We must choose one witness for every \( \phi(x, F(\alpha)) \).

This needs a definable global choice function:

\[ C : \mathcal{M} \setminus \emptyset \to \mathcal{M} \text{ such that } C(x) \in x. \]

Remark: There are models of set theory without a definable global choice function.

Moral: To take Skolem ultrapowers, we may need to “add classes” to \( \mathcal{M} \).
Gödel-Bernays set theory - GBC

To allow non-definable classes, we need a foundation for set theory with both set and class objects.

**Formalization:**

- **Two-sorted logic:** separate variables and quantifiers for sets and classes.
- **Typical model:** $\mathcal{M} = \langle M, \in, S \rangle$, where $M$ is the sets and $S$ is the classes.

  We can also formalize in first-order logic: take classes as elements of the model, define sets as any class that is an element of some class.

**GBC axioms:**

**Sets:** Extensionality, Regularity, Pairing, Infinity, Union, Powerset.

**Classes:**

- **Class comprehension** limited to formulas with set quantifiers:
  if $\varphi(x, y, Y)$ is a formula with set quantifiers, then $\forall a \in M \forall A \in S \\{x \mid \varphi(x, a, A)\}$ is a class.

- **Replacement:** if $F$ is a class function and $a$ is a set, then $F \upharpoonright a$ is a set.

- There is a global choice function.

If we allow comprehension for any formula in the two-sorted logic, we get the **Kelley-Morse set theory** - KM.
Models of GBC

- Suppose $\langle M, \in \rangle \models \text{ZFC}$ and $D$ is its collection of (definable) classes. If $D$ has a global choice function, then $\langle M, \in, D \rangle \models \text{GBC}$.

**Theorem:** (Solovay) Every countable model $\langle M, \in \rangle \models \text{ZFC}$ can be extended to a countable model $\langle M, \in, S \rangle \models \text{GBC}$.

- The definable classes can be expanded to include a global choice function.
- We force to add a global well-order, using a variant of forcing which adds classes without adding sets.
  
  The class partial order $\mathbb{P}$ consists of all set well-orders ordered by extension.

- GBC is conservative over ZFC: any property of sets provable in GBC is already provable in ZFC.

  Kelley-Morse set theory is not conservative over ZFC: $\text{Con}(\text{ZFC})$ is provable in KM.

**Theorem:** If $\langle M, \in, S \rangle \models \text{GBC}$, then a Skolem ultrapower of $\langle M, \in \rangle$ satisfies the Łoś Theorem.

**Moral:** Forcing to add classes without adding sets can create rich collections of classes for Skolem ultrapowers of countable models.
Skolem ultrapower end-extensions

Suppose $\langle M, \in \rangle \models \text{ZFC}$ is countable. Extend to a countable $\mathcal{M} = \langle M, \in, S \rangle \models \text{GBC}$. Let $U$ be some ultrafilter on $O$ - the collection of all classes of ordinals.

Observation:

- If $[F]_U$ has rank $\beta \in \text{ORD}^\mathcal{M}$ in $\prod O \mathcal{M}/U$, then we can assume $F : \text{ORD}^\mathcal{M} \to V_{\beta+1}^\mathcal{M}$.
- If $\mathcal{M} \prec_e \prod O \mathcal{M}/U$, then every class $F : \text{ORD}^\mathcal{M} \to V_{\alpha}^\mathcal{M}$ is constant on a set in $U$.

Goal: Construct $U$ so that every class $F : \text{ORD}^\mathcal{M} \to V_{\alpha}^\mathcal{M}$ is constant on a set in $U$. 
Theorem: There is an ultrafilter $U$ on $O$ such that $\mathcal{M} \preceq_e \prod_{\mathcal{M}} \mathcal{M}/U$.

Proof: $U$ is generated by a descending $\omega$-sequence of proper classes in $O$:

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

- Enumerate all class $F : \text{ORD}^\mathcal{M} \rightarrow V^\mathcal{M}_\alpha$ as $\langle F_n \mid n < \omega \rangle$.
- $F_0$ is constant on some proper class $A_0$.
- $F_{n+1}$ is constant on some proper class $A_{n+1} \subseteq A_n$.

This generates an ultrafilter because for every $A \in O$, we considered $F$ such that

- $F(\alpha) = 1$ for all $\alpha \in A$,
- $F(\alpha) = 0$ for all $\alpha \notin A$. $\square$
Existence of $\omega_1$-like models of set theory

**Theorem:** (Keisler, Morley, 1968) If ZFC is consistent, then there is an $\omega_1$-like model of ZFC.

**Proof:** We construct an elementary chain of models of ZFC of length $\omega_1$:

\[ M_0 \prec M_1 \prec \cdots \prec M_\omega \prec \cdots M_\xi \prec M_{\xi+1} \prec \]

- $M_0$ is any countable model of ZFC,
- if $M_{\xi+1}$ is a countable end-extension of $M_\xi$,
- $M_\lambda = \bigcup_{\xi < \lambda} M_\xi$ for limit ordinals $\lambda$.

Clearly $M = \bigcup_{\xi < \omega_1} M_\xi$ is $\omega_1$-like. □

The existence of a well-founded $\omega_1$-like model is equiconsistent with an inaccessible cardinal.

**Theorem:** (Enayat, 2001) It is consistent relative to a weakly compact cardinal that there are no $\omega_2$-like models of ZFC.
Notable $\omega_1$-like models

**Theorem:** (Kaufmann, 1983) There is an $\omega_1$-like model of ZFC without a proper end-extension.

**Definition:** A model of set theory is **Leibnizian** if every element has a unique type.

**Theorem:** (Enayat, 2003) There is an $\omega_1$-like Leibnizian model of ZFC.

**Definition:** A model of cardinality $\kappa$ is **Jónsson** if it has no elementary substructures of cardinality $\kappa$.

**Theorem:** (Knight, 1976) There is an $\omega_1$-like Jónsson model of ZFC (PA).
Embeddings of models of PA

**Definition:** Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of PA. A map $j : M \to N$ is an embedding of $\mathcal{M}$ into $\mathcal{N}$ if

- $j(0) = 0$ and $j(1) = 1$,
- $j(a + b) = j(a) + j(b)$ and $j(a \cdot b) = j(a) \cdot j(b)$,
- $a < b \rightarrow j(a) < j(b)$.

**Definition:** The standard system of a model $\mathcal{M} \models \text{PA}$ is the collection:

$$\text{SSy}(\mathcal{M}) = \{ A \cap \mathbb{N} \mid A \text{ is definable (with parameters) over } \mathcal{M} \}.$$  

**Theorem:** Every embedding $j : M \to N$ of models $\mathcal{M}, \mathcal{N}$ of PA is $\Delta_0$-elementary.

**Proof:** By the MRDP Theorem. □

MRDP (Matiyasevich, Robinson, Davis, Putnam) Theorem: Over $\text{PA}$, every $\Sigma_1$-formula is equivalent to a formula with a single existential quantifier (a set of integers is Diophantine iff it is computably enumerable).
Embeddings of models of PA (continued)

**Theorem:** (Friedman, 1973) There is an embedding $j : M \to N$ between countable models $M$ and $N$ of PA iff

- $\text{SSy}(M) \subseteq \text{SSy}(N)$,
- $N$ satisfies the $\Sigma_1$-theory of $M$.

**Proof:**

$(\Rightarrow)$ Every embedding is $\Delta_0$-elementary.

$(\Leftarrow)$ Back and forth argument. □

**Theorem:** (Kossak, 1985) There are $\omega_1$-like models $M$ and $N$ of PA such that

- $\text{SSy}(M) = \text{SSy}(N)$,
- $M$ and $N$ have the same theory,

but there is no embedding between $M$ and $N$. 
Embeddings of models of set theory

**Definition:** Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of ZFC. A map $j : M \to N$ is an embedding if $a \in b \to j(a) \in j(b)$.

**Theorem:** (Hamkins, 2012) For any two countable models $\mathcal{M}$ and $\mathcal{N}$ of ZFC either $\mathcal{M}$ embeds into $\mathcal{N}$ or conversely.

- $\mathcal{M}$ embeds into $\mathcal{N}$ if and only if $\text{ORD}^\mathcal{M}$ embeds into $\text{ORD}^\mathcal{N}$.
- The embedding may not be $\Delta_0$-elementary:

  **Observation:** There cannot be a $\Delta_0$-elementary embedding between a well-founded $\mathcal{M} \models \text{ZFC}$ and its constructible universe $L^\mathcal{M}$.

**Theorem:** (Fuchs, G., Hamkins, 2013) Assuming $\diamondsuit$, if ZFC is consistent, then there is a collection $\mathcal{C}$ of the maximum possible size $2^{\omega_1}$ of $\omega_1$-like models of ZFC such that there is no embedding between any pair of models in $\mathcal{C}$.
Building incomparable $\omega_1$-like models

Strategy:

- **Simultaneously** build elementary chains of countable models of ZFC:
  \[ M_0 \prec_e M_1 \prec_e \cdots \prec_e M_\xi \prec_e M_{\xi+1} \prec_e \cdots \]
  and
  \[ N_0 \prec_e N_1 \prec_e \cdots \prec_e N_\xi \prec_e N_{\xi+1} \prec_e \cdots \]
  of length $\omega_1$.

- Let $M = \bigcup_{\xi<\omega_1} M_\xi$ and $N = \bigcup_{\xi<\omega_1} N_\xi$ be the resulting $\omega_1$-like models.

- At each stage $\xi + 1$, “guess” that some $j : M_\xi \rightarrow N_\xi$ extends to $j : M \rightarrow N$.

- Choose $M_{\xi+1}$ and $N_{\xi+1}$ so that $j$ cannot be extended to $j : M_{\xi+1} \rightarrow N_{\xi+1}$.
Coded sets

**Definition:** Suppose $\mathcal{M} \prec_e \overline{\mathcal{M}} \models \text{ZFC}$.  
- A set $A \subseteq M$ is **coded** in $\overline{\mathcal{M}}$ if there is $a \in \overline{\mathcal{M}}$ such that $A = "a \cap M":$
  $$A = \{b \in M \mid b \in \overline{\mathcal{M}} \ a\}.$$  
- We say that $a$ codes $A$. 

![Diagram showing the concept of coding sets in a model](image-url)
Coded sets (continued)

Observation: Suppose $\mathcal{N} \prec_e \overline{\mathcal{N}} \prec_e \overline{\overline{\mathcal{N}}}$. If $B \subseteq N$ is coded in $\overline{\mathcal{N}}$, then $B$ is already coded in $\overline{\overline{\mathcal{N}}}$.

Proof:
- Fix $\alpha \in \text{ORD} \overline{\mathcal{N}} \setminus \text{ORD} \mathcal{N}$.
- Fix $b \in \overline{\mathcal{N}}$ coding $B$.
- $b' = (b \cap V_\alpha) \in V_{\alpha+1}$ codes $B$. □
Coded sets (continued)

**Tagline:** “If an end-extension omits to code a certain set, this cannot be fixed in a further end-extension.”

**Observation:** Suppose

- $\mathcal{M} \prec \mathcal{M}$ and $\mathcal{N} \prec \mathcal{N} \models \text{ZFC}$,
- $j : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding.

If $j$ extends to $j : \mathcal{M} \rightarrow \mathcal{N}$ and $A \subseteq \mathcal{M}$ is coded in $\mathcal{M}$, then there is $B$ coded in $\mathcal{N}$ such that $B \cap j " \mathcal{M} = j " A$.

**Proof:** If $a$ codes $A$, then $j(a)$ codes $B$. □
Killing-off embeddings

**Key Lemma**: (Fuchs, G., Hamkins) Suppose $\mathcal{M} \models \text{ZFC}$ is countable. The collection $\mathcal{C}$ of subsets of $M$ coded in some end-extension of $\mathcal{M}$ has size $2^\omega$.

**Observation**: Suppose

- $\mathcal{M}$ and $\mathcal{N}$ are countable models of ZFC,
- $j : M \to N$ is an embedding,
- $\mathcal{N} \prec_e \overline{\mathcal{N}}$, which is countable.

Then there is a countable end-extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ such that $j$ **cannot be extended** to $j : \overline{\mathcal{M}} \to \overline{\mathcal{N}}$.

**Proof**:

- For $b \in \overline{\mathcal{N}}$, let $X_b = \{ a \in \mathcal{M} \mid j(a) \in \overline{\mathcal{N}} \ b \}$.
- There are countably many $X_b$.
- Let $\overline{\mathcal{M}}$ be an end-extension of $\mathcal{M}$ coding some $A \subseteq M$ such that $A \neq X_b$ for any $b \in \overline{\mathcal{N}}$.
- $j$ **cannot be extended** to $j : \overline{\mathcal{M}} \to \overline{\mathcal{N}}$ because there is no set $B$ coded in $\overline{\mathcal{N}}$ such that $B \cap j " M = j " A$. □
The guessing principle ♦

**Definition:** The principle ♦ states that there is a ♦-sequence
\[ \langle A_\alpha \mid \alpha < \omega_1 \rangle \] 
with \( A_\alpha \subseteq \alpha \) such that for every \( X \subseteq \omega_1 \), the set
\[ \{ \alpha \in \omega_1 \mid X \cap \alpha = A_\alpha \} \]
is stationary in \( \omega_1 \).

A subset of \( \omega_1 \) is **stationary** if it has a non-empty intersection with every closed unbounded subset of \( \omega_1 \).

**Tagline:** “Every subset of \( \omega_1 \) is predicted cofinally often on the ♦-sequence.”

The ♦-principle
- holds in the constructible universe \( L \),
- can be forced over any model of set theory,
- implies CH.
Incomparable \(\omega_1\)-like models

Predicting embeddings with \(\Diamond\)

**Observation**: Suppose
- \(M = \bigcup_{\xi < \omega_1} M_\xi\) where
  \[ M_0 < M_1 < \cdots < M_\xi < M_{\xi+1} < \cdots \]
  is an elementary chain of models of ZFC with unions taken at limit stages,
- \(N = \bigcup_{\xi < \omega_1} N_\xi\) where
  \[ N_0 < N_1 < \cdots < N_\xi < N_{\xi+1} < \cdots \]
  is an elementary chain of models of ZFC with unions taken at limit stages,
- \(M = N = \omega_1\) (as sets without structure),
- \(j : M \to N\) is an embedding.

Then there is a limit ordinal \(\lambda\) such that \(A_\lambda\) “codes” \(j \upharpoonright \lambda : M_\lambda \to N_\lambda\).

Fix a bijection \(\varphi : \omega_1 \times \omega_1 \to \omega_1\).
On a club of ordinals \(\lambda\):
- \(M_\lambda = \lambda\) and \(N_\lambda = \lambda\)
- \(j \upharpoonright \lambda : \lambda \to \lambda\)
- \(\varphi : \lambda \times \lambda \to \lambda\)

There is a limit \(\lambda\) such that \(A_\lambda = \varphi \circ j \upharpoonright \lambda : \lambda \to \lambda\).
Incomparable $\omega_1$-like models

“I was not predicting the future, I was trying to prevent it.”
–Ray Bradbury

We simultaneously build elementary chains of countable models of ZFC of length $\omega_1$:

$\mathcal{M}_0 \prec_e \mathcal{M}_1 \prec_e \cdots \prec_e \mathcal{M}_\xi \prec_e \mathcal{M}_{\xi+1} \prec_e \cdots$

and

$\mathcal{N}_0 \prec_e \mathcal{N}_1 \prec_e \cdots \prec_e \mathcal{N}_\xi \prec_e \mathcal{N}_{\xi+1} \prec_e \cdots$.

- Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be any countable models of ZFC.
- At stage limit ordinal $\lambda$:
  - $\mathcal{M}_\lambda = \bigcup_{\xi < \lambda} \mathcal{M}_\xi$ and $\mathcal{N}_\lambda = \bigcup_{\xi < \lambda} \mathcal{N}_\xi$,
  - $M_\lambda = N_\lambda = \lambda$ (as sets without structure).
- At stage successor ordinal $\xi + 1$:
  - Let $\mathcal{N}_{\xi+1}$ be any proper countable end-extension of $\mathcal{N}_\xi$.
    - If $A_\xi$ codes $j : M_\xi \rightarrow N_\xi$, choose $\mathcal{M}_{\xi+1}$ to kill-off $j$,
    - else let $\mathcal{M}_{\xi+1}$ be any proper end-extension of $\mathcal{M}_\xi$.

It follows that:
- $\mathcal{M} = \bigcup_{\xi < \omega_1} \mathcal{M}_\xi$ and $\mathcal{N} = \bigcup_{\xi < \omega_1} \mathcal{N}_\xi$ are $\omega_1$-like models.
- There is no embedding from $\mathcal{M}$ to $\mathcal{N}$.
The Key Lemma: splitting functions

**Definition:** Suppose $\langle M, \in, S \rangle \models \text{GBC}$. A class $F : \text{ORD}^M \to M$ such that $F(\alpha) \subseteq \alpha$ is **splitting** if for every proper $A \subseteq \text{ORD}^M$ in $S$, there is $\beta \in \text{ORD}^M$ such that $A$ splits into

$$A^+_\beta = \{ \alpha \in A \mid \beta \in^M F(\alpha) \}$$

and

$$A^-_\beta = \{ \alpha \in A \mid \beta \notin^M F(\alpha) \}$$

both of which are proper.

**Tagline:** “When constructing a descending sequence generating an ultrafilter $U$ for a Skolem ultrafilter, no initial segment of the sequence can decide the subset of $\text{ORD}^M$ coded by $[F]_U$ in the ultrapower.”
The Key Lemma: splitting functions (continued)

**Theorem:** (Fuchs, G., Hamkins) A model $\langle M, \in, S \rangle \models \text{GBC}$ has a splitting function if and only if it has an $\text{ORD}^M$-tree without a cofinal branch.

**Corollary:** A model $\langle M, \in, S \rangle \models \text{GBC}$ may fail to have a splitting function.

**Proof:** If $\kappa$ is weakly compact, then $\langle V_\kappa, \in, V_{\kappa+1} \rangle \models \text{GBC}$, but every tree of height $\kappa$ in $V_{\kappa+1}$ has a cofinal branch in $V_{\kappa+1}$.

**Corollary:** A splitting function can be added by forcing to any model of $\text{GBC}$ without adding sets.

**Proof:** Use a class version of the forcing to add an $\omega_1$-Souslin tree.
The Key Lemma: proof

**Key Lemma:** (Fuchs, G., Hamkins) Suppose $\mathcal{M} \models \text{ZFC}$ is countable. The collection $C$ of subsets of $M$ coded in some end-extension of $\mathcal{M}$ has size $2^{\omega}$.

**Proof:**

- Suppose $\langle M, \in \rangle \models \text{ZFC}$.
- Extend it to a model $\langle M, \in, S \rangle \models \text{GBC}$ such that $S$ has a splitting function $F$.
- Let $O$ be the collection of all classes of ordinals.
- There is a family $\langle U_s \mid s : \mathbb{N} \to 2 \rangle$ of ultrafilters on $O$ such that
  - $\mathcal{M} \prec \prod_{O} \mathcal{M}/U_s$,
  - if $s \neq t$, then the subset of $M$ coded by $[F]_{U_s}$ is not equal that coded by $[F]_{U_t}$.
- $U_s$ is generated by the descending sequence $A^s_0 \supseteq A^s_1 \supseteq \cdots \supseteq A^s_n \supseteq \cdots$.
- **Stage 2n:** ensure ultrapower is an end-extension.
- **Stage 2n + 1:**
  - choose $\beta$ such that $(A^s_{2n})^+_{\beta}$ and $(A^s_{2n})^-_{\beta}$ are proper,
  - if $s(n) = 1$, let $A^s_{2n+1} = (A^s_{2n})^+_{\beta}$,
  - if $s(n) = 0$, let $A^s_{2n+1} = (A^s_{2n})^-_{\beta}$. □
Thank you!