Incomparable ω_1-like models of set theory

Victoria Gitman

vgitman@nylogic.org
http://boolesrings.org/victoriagitman

March 31, 2014

This is joint work with Gunter Fuchs and Joel David Hamkins (CUNY).

The ω_1-like models of Peano Arithmetic and set theory

"ω_1-like models bridge the gap between countable and uncountable structures."
–Roman Kossak

Definition: A model $\mathcal{M} = \langle M, +, \cdot, <, 0, 1\rangle \models \mathrm{PA}$ is ω_1-like if:

- M has size ω_1,
- every proper initial segment is countable:
  for every $b \in M$, the initial segment $b^{\mathcal{M}} = \{a \in M \mid a < b\}$ is countable.

Definition: A model $\mathcal{M} = \langle M, \in\rangle \models \mathrm{ZFC}$ is ω_1-like if:

- M has size ω_1,
- every proper initial segment of the ordinals is countable:
  for every $\beta \in \mathrm{ORD}^{\mathcal{M}}$, the initial segment $\beta^{\mathcal{M}} = \{\alpha \in M \mid \alpha \in^{\mathcal{M}} \beta\}$ is countable
  (it follows that every rank initial segment $V_\alpha^{\mathcal{M}}$ is countable).

The ω_1-like models are locally countable,
yet they inherit the set-theoretic structure of ω_1.

This allows us to build exotic ω_1-like models with surprising properties.

Given any cardinal κ, we similarly define the notion of a κ-like model.

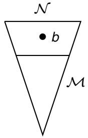
Existence of ω_1-like models of PA

Definition: Suppose $\mathcal{M} \prec \mathcal{N} \models \mathrm{PA}$. Then $\mathcal{N}$ is an end-extension of $\mathcal{M}$,

$$\mathcal{M} \prec_e \mathcal{N},$$

if for every $b \in N \setminus M$, we have $b > a$ for all $a \in M$.

Tagline: "New elements are added only on top."



Theorem: (MacDowell, Specker, 1961) Every model of PA has a proper end-extension.

Existence of ω_1-like models of PA (continued)

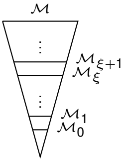
Corollary: There is an ω_1-like model of PA.

Proof: We construct an elementary chain of models of PA of length ω_1

$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_\omega \prec \cdots \mathcal{M}_\xi \prec \mathcal{M}_{\xi+1} \prec \cdots$$

using transfinite recursion:

- $\mathcal{M}_0$ is any countable model of PA,
- $\mathcal{M}_{\xi+1}$ is a proper countable end-extension of $\mathcal{M}_\xi$,
- $\mathcal{M}_\lambda = \bigcup_{\xi<\lambda} \mathcal{M}_\xi$ for limit ordinals λ.



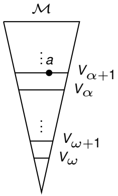
Clearly $\mathcal{M} = \bigcup_{\xi<\omega_1} \mathcal{M}_\xi$ is ω_1-like. □

Corollary: For every cardinal κ, there is a κ-like model of PA.

Models of set theory

A model $\mathcal{M} = \langle M, \in \rangle \models \mathrm{ZFC}$ is the union of its von Neumann hierarchy.
Inside $\mathcal{M}$, we define:

- $V_0^{\mathcal{M}} = \emptyset$,
- $V_{\alpha+1}^{\mathcal{M}} = P(V_\alpha^{\mathcal{M}})$ (powerset of $V_\alpha^{\mathcal{M}}$),
- $V_\lambda^{\mathcal{M}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathcal{M}}$ for limit ordinals λ.
- The rank of $a \in M$ is the least α such that $a \in V_{\alpha+1}^{\mathcal{M}} \setminus V_\alpha^{\mathcal{M}}$.



Note: The set membership relation $\in^{\mathcal{M}}$ is (externally) not necessarily well-founded.
So the order $\langle \mathrm{ORD}^{\mathcal{M}}, \in^{\mathcal{M}} \rangle$ is not necessarily well-founded.

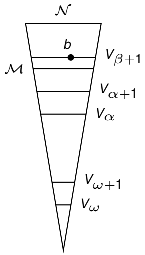
End-extensions of models of set theory

Definition: Suppose $\mathcal{M} \prec \mathcal{N} \models \mathrm{ZFC}$. Then $\mathcal{N}$ is an end-extension of $\mathcal{M}$,

$$\mathcal{M} \prec_e \mathcal{N},$$

if for every $b \in N \setminus M$, the rank β of b is greater than all $\alpha \in \mathrm{ORD}^{\mathcal{M}}$.

Tagline: “New elements are added only on top.”



End-extensions of models of set theory (continued)

Sneak Preview: It is not true that every model of ZFC has a proper end-extension!

Theorem: (Keisler, Morley, 1968) Every countable model of ZFC has a proper end-extension.

- Original proof is an omitting types argument.
- Alternative proof uses an ultrapower construction.
- The ultrapower argument is flexible: build desired properties into end-extensions.

Skolem ultrapowers

Suppose $\mathcal{M} = \langle M, \in \rangle \models \mathrm{ZFC}$.

$A \subseteq M$ is a class of $\mathcal{M}$ if there is a formula $\varphi(x, y)$ and $s \in M$ such that

$$A = \{a \in M \mid \mathcal{M} \models \varphi(a, s)\}.$$

Skolem ultrapower:

- Class functions $F : \mathrm{ORD}^{\mathcal{M}} \to M$
- Ultrafilter U on O - the collection of classes of ordinals

Question: Do Skolem ultrapowers satisfy the Łoś Theorem?

Skolem ultrapowers: Łoś Theorem

Inductive assumption:

$$\prod_O \mathcal{M}/U \models \varphi([G]_U, [F]_U) \Leftrightarrow \{\alpha \in \mathrm{ORD}^{\mathcal{M}} \mid \mathcal{M} \models \varphi(G(\alpha), F(\alpha))\} \in U.$$

Existential quantifier: $\exists x\, \varphi(x, [F]_U)$

Suppose $\{\alpha \in \mathrm{ORD}^{\mathcal{M}} \mid \mathcal{M} \models \exists x\, \varphi(x, F(\alpha))\} \in U$.

Is there a class function $G : \mathrm{ORD}^{\mathcal{M}} \to M$ such that

$$\mathcal{M} \models \exists x\, \varphi(x, F(\alpha)) \Rightarrow \mathcal{M} \models \varphi(G(\alpha), F(\alpha))?$$

We must choose one witness for every $\varphi(x, F(\alpha))$.

This needs a definable global choice function:

$$C : M \setminus \emptyset \to M \text{ such that } C(x) \in x.$$

Remark: There are models of set theory without a definable global choice function.

Moral: To take Skolem ultrapowers, we may need to "add classes" to $\mathcal{M}$.

Gödel-Bernays set theory - GBC

To allow non-definable classes, we need a foundation for set theory with both set and class objects.

Formalization:

- Two-sorted logic: separate variables and quantifiers for sets and classes.
- Typical model: $\mathcal{M} = \langle M, \in, S\rangle$, where M is the sets and S is the classes.

  We can also formalize in first-order logic: take classes as elements of the model, define sets as any class that is an element of some class.

GBC **axioms**:

Sets: Extensionality, Regularity, Pairing, Infinity, Union, Powerset.

Classes:

- Class comprehension limited to formulas with set quantifiers: if $\varphi(x, y, Y)$ is a formula with set quantifiers, then $\forall a \in M \forall A \in S$
  $$\{x \mid \varphi(x, a, A)\} \text{ is a class.}$$
- Replacement: if F is a class function and a is a set, then $F \restriction a$ is a set.
- There is a global choice function.

If we allow comprehension for any formula in the two-sorted logic, we get the Kelley-Morse set theory - KM.

Models of GBC

- Suppose $\langle M, \in\rangle \models \mathrm{ZFC}$ and D is its collection of (definable) classes. If D has a global choice function, then $\langle M, \in, D\rangle \models \mathrm{GBC}$.

Theorem: (Solovay) Every countable model $\langle M, \in\rangle \models \mathrm{ZFC}$ can be extended to a countable model $\langle M, \in, S\rangle \models \mathrm{GBC}$.

- The definable classes can be expanded to include a global choice function.
- We force to add a global well-order, using a variant of forcing which adds classes without adding sets.

  The class partial order $\mathbb{P}$ consists of all set well-orders ordered by extension.
- GBC is conservative over ZFC: any property of sets provable in GBC is already provable in ZFC.

  Kelley-Morse set theory is not conservative over ZFC: Con(ZFC) is provable in KM.

Theorem: If $\langle M, \in, S\rangle \models \mathrm{GBC}$, then a Skolem ultrapower of $\langle M, \in\rangle$ satisfies the Łoś Theorem.

Moral: Forcing to add classes without adding sets can create rich collections of classes for Skolem ultrapowers of countable models.

Skolem ultrapower end-extensions

Suppose $\langle M, \in\rangle \models \mathrm{ZFC}$ is countable.
Extend to a countable $\mathcal{M} = \langle M, \in, S\rangle \models \mathrm{GBC}$.
Let U be some ultrafilter on O - the collection of all classes of ordinals.

Observation:

- If $[F]_U$ has rank $\beta \in \mathrm{ORD}^{\mathcal{M}}$ in $\prod_O \mathcal{M}/U$, then we can assume $F : \mathrm{ORD}^{\mathcal{M}} \to V_{\beta+1}^{\mathcal{M}}$.
- If $\mathcal{M} \prec_e \prod_O \mathcal{M}/U$, then every class $F : \mathrm{ORD}^{\mathcal{M}} \to V_\alpha^{\mathcal{M}}$ is constant on a set in U.

Goal: Construct U so that every class $F : \mathrm{ORD}^{\mathcal{M}} \to V_\alpha^{\mathcal{M}}$ is constant on a set in U.

Skolem ultrapower end-extensions (continued)

Theorem: There is an ultrafilter U on O such that $\mathcal{M} \prec_e \prod_O \mathcal{M}/U$.

Proof: U is generated by a descending ω-sequence of proper classes in O:

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

- Enumerate all class $F : \mathrm{ORD}^{\mathcal{M}} \to V_\alpha^{\mathcal{M}}$ as $\langle F_n \mid n < \omega \rangle$.
- F_0 is constant on some proper class A_0.
- F_{n+1} is constant on some proper class $A_{n+1} \subseteq A_n$.

This generates an ultrafilter because for every $A \in O$, we considered F such that

- $F(\alpha) = 1$ for all $\alpha \in A$,
- $F(\alpha) = 0$ for all $\alpha \notin A$. $\square$

Existence of ω_1-like models of set theory

Theorem: (Keisler, Morley, 1968) If ZFC is consistent, then there is an ω_1-like model of ZFC.

Proof: We construct an elementary chain of models of ZFC of length ω_1:

$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_\omega \prec \cdots \mathcal{M}_\xi \prec \mathcal{M}_{\xi+1} \prec$$

- $\mathcal{M}_0$ is any countable model of ZFC,
- if $\mathcal{M}_{\xi+1}$ is a countable end-extension of $\mathcal{M}_\xi$,
- $\mathcal{M}_\lambda = \bigcup_{\xi<\lambda} \mathcal{M}_\xi$ for limit ordinals λ λ.

Clearly $\mathcal{M} = \bigcup_{\xi<\omega_1} \mathcal{M}_\xi$ is ω_1-like. $\square$

The existence of a well-founded ω_1-like model is equiconsistent with an inaccessible cardinal.

Theorem: (Enayat, 2001) It is consistent relative to a weakly compact cardinal that there are no ω_2-like models of ZFC.

Notable ω_1-like models

Theorem: (Kaufmann, 1983) There is an ω_1-like model of ZFC without a proper end-extension.

Definition: A model of set theory is Leibnizian if every element has a unique type.

Theorem: (Enayat, 2003) There is an ω_1-like Leibnizian model of ZFC.

Definition: A model of cardinality κ is Jónsson if it has no elementary substructures of cardinality κ.

Theorem: (Knight, 1976) There is an ω_1-like Jónsson model of ZFC (PA).

Embeddings of models of PA

Definition: Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of PA. A map $j : M \to N$ is an embedding of $\mathcal{M}$ into $\mathcal{N}$ if

- $j(0) = 0$ and $j(1) = 1$,
- $j(a + b) = j(a) + j(b)$ and $j(a \cdot b) = j(a) \cdot j(b)$,
- $a < b \to j(a) < j(b)$.

Definition: The standard system of a model $\mathcal{M} \models \mathrm{PA}$ is the collection:

$$\mathrm{SSy}(\mathcal{M}) = \{A \cap \mathbb{N} \mid A \text{ is definable (with parameters) over } \mathcal{M}\}.$$

Theorem: Every embedding $j : M \to N$ of models $\mathcal{M}, \mathcal{N}$ of PA is Δ_0-elementary.

Proof: By the MRDP Theorem. $\square$

MRDP (Matiyasevich, Robinson, Davis, Putnam) Theorem: Over PA, every Σ_1-formula is equivalent to a formula with a single existential quantifier (a set of integers is Diophantine iff it is computably enumerable).

Embeddings of models of PA (continued)

Theorem: (Friedman, 1973) There is an embedding $j : M \to N$ between countable models $\mathcal{M}$ and $\mathcal{N}$ of PA iff

- $\mathrm{SSy}(\mathcal{M}) \subseteq \mathrm{SSy}(\mathcal{N})$,
- $\mathcal{N}$ satisfies the Σ_1-theory of $\mathcal{M}$.

Proof:
($\Rightarrow$) Every embedding is Δ_0-elementary.
($\Leftarrow$) Back and forth argument. $\square$

Theorem: (Kossak, 1985) There are ω_1-like models $\mathcal{M}$ and $\mathcal{N}$ of PA such that

- $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$,
- $\mathcal{M}$ and $\mathcal{N}$ have the same theory,

but there is no embedding between $\mathcal{M}$ and $\mathcal{N}$.

Embeddings of models of set theory

Definition: Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of ZFC. A map $j : M \to N$ is an embedding if $a \in b \to j(a) \in j(b)$.

Theorem: (Hamkins, 2012) For any two countable models $\mathcal{M}$ and $\mathcal{N}$ of ZFC either $\mathcal{M}$ embeds into $\mathcal{N}$ or conversely.

- $\mathcal{M}$ embeds into $\mathcal{N}$ if and only if $\mathrm{ORD}^{\mathcal{M}}$ embeds into $\mathrm{ORD}^{\mathcal{N}}$.
- The embedding may not be Δ_0-elementary:

  Observation: There cannot be a Δ_0-elementary embedding between a well-founded $\mathcal{M} \models \mathrm{ZFC}$ and its constructible universe $L^{\mathcal{M}}$.

Theorem: (Fuchs, G., Hamkins, 2013) Assuming $\diamondsuit$, if ZFC is consistent, then there is a collection $\mathcal{C}$ of the maximum possible size 2^{ω_1} of ω_1-like models of ZFC such that there is no embedding between any pair of models in $\mathcal{C}$.

Building incomparable ω_1-like models

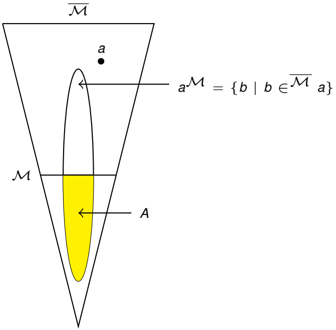
Strategy:

- Simultaneously build elementary chains of countable models of ZFC:
$$\mathcal{M}_0 \prec_e \mathcal{M}_1 \prec_e \cdots \prec_e \mathcal{M}_\xi \prec_e \mathcal{M}_{\xi+1} \prec_e \cdots$$
and
$$\mathcal{N}_0 \prec_e \mathcal{N}_1 \prec_e \cdots \prec_e \mathcal{N}_\xi \prec_e \mathcal{N}_{\xi+1} \prec_e \cdots.$$
of length ω_1.
- Let $\mathcal{M} = \bigcup_{\xi<\omega_1} \mathcal{M}_\xi$ and $\mathcal{N} = \bigcup_{\xi<\omega_1} \mathcal{N}_\xi$ be the resulting ω_1-like models.
- At each stage $\xi+1$, "guess" that some $j : M_\xi \to N_\xi$ extends to $j : M \to N$.
- Choose $\mathcal{M}_{\xi+1}$ and $\mathcal{N}_{\xi+1}$ so that j cannot be extended to $j : M_{\xi+1} \to N_{\xi+1}$.

Coded sets

Definition: Suppose $\mathcal{M} \prec_e \overline{\mathcal{M}} \models \mathrm{ZFC}$.

- A set $A \subseteq M$ is coded in $\overline{\mathcal{M}}$ if there is $a \in \overline{M}$ such that $A =$ "$a \cap M$":
$$A = \{b \in M \mid b \in^{\overline{\mathcal{M}}} a\}.$$
- We say that a codes A.

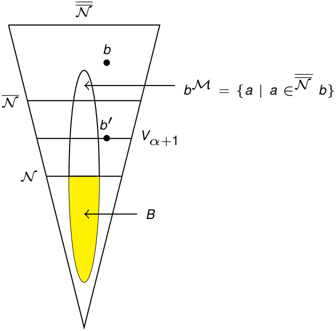


Coded sets (continued)

Observation: Suppose $\mathcal{N} \prec_e \overline{\mathcal{N}} \prec_e \overline{\overline{\mathcal{N}}}$. If $B \subseteq N$ is coded in $\overline{\overline{\mathcal{N}}}$, then B is already coded in $\overline{\mathcal{N}}$.

Proof:

- Fix $\alpha \in \mathrm{ORD}^{\overline{\mathcal{N}}} \setminus \mathrm{ORD}^{\mathcal{N}}$.
- Fix $b \in \overline{\overline{\mathcal{N}}}$ coding B.
- $b' = (b \cap V_\alpha) \in V_{\alpha+1}$ codes B. $\square$



Coded sets (continued)

Tagline: "If an end-extension omits to code a certain set, this cannot be fixed in a further end-extension."

Observation: Suppose

- $\mathcal{M} \prec_e \overline{\mathcal{M}} \models \mathrm{ZFC}$ and $\mathcal{N} \prec_e \overline{\mathcal{N}} \models \mathrm{ZFC}$,
- $j : M \to N$ is an embedding.

If j extends to $j : \overline{M} \to \overline{N}$ and $A \subseteq M$ is coded in $\overline{\mathcal{M}}$, then there is B coded in $\overline{\mathcal{N}}$ such that $B \cap j\text{"}M = j\text{"}A$.

Proof: If a codes A, then $j(a)$ codes B. $\square$

Killing-off embeddings

Key Lemma: (Fuchs, G., Hamkins) Suppose $\mathcal{M} \models \mathrm{ZFC}$ is countable. The collection $\mathcal{C}$ of subsets of M coded in some end-extension of $\mathcal{M}$ has size 2^ω.

Observation: Suppose

- $\mathcal{M}$ and $\mathcal{N}$ are countable models of ZFC,
- $j : M \to N$ is an embedding,
- $\mathcal{N} \prec_e \overline{\mathcal{N}}$, which is countable.

Then there is a countable end-extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ such that j cannot be extended to $j : \overline{\mathcal{M}} \to \overline{\mathcal{N}}$.

Proof:

- For $b \in \overline{\mathcal{N}}$, let $X_b = \{a \in \mathcal{M} \mid j(a) \in^{\overline{\mathcal{N}}} b\}$.
- There are countably many X_b.
- Let $\overline{\mathcal{M}}$ be an end-extension of $\mathcal{M}$ coding some $A \subseteq M$ such that
$$A \neq X_b \text{ for any } b \in \overline{\mathcal{N}}.$$
- j cannot be extended to $j : \overline{\mathcal{M}} \to \overline{\mathcal{N}}$ because there is no set B coded in $\overline{\mathcal{N}}$ such that $B \cap j\text{"}M = j\text{"}A$. $\square$

The guessing principle $\diamondsuit$

Definition: The principle $\diamondsuit$ states that there is a $\diamondsuit$-sequence

$$\langle A_\alpha \mid \alpha < \omega_1 \rangle \text{ with } A_\alpha \subseteq \alpha$$

such that for every $X \subseteq \omega_1$, the set

$$\{\alpha \in \omega_1 \mid X \cap \alpha = A_\alpha\}$$

is stationary in ω_1.

A subset of ω_1 is stationary if it has a non-empty intersection with every closed unbounded subset of ω_1.

Tagline: "Every subset of ω_1 is predicted cofinally often on the $\diamondsuit$-sequence."

The $\diamondsuit$-principle

- holds in the constructible universe L,
- can be forced over any model of set theory,
- implies CH.

Predicting embeddings with $\diamondsuit$

Observation: Suppose

- $\mathcal{M} = \bigcup_{\xi<\omega_1} \mathcal{M}_\xi$ where
$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_\xi \prec \mathcal{M}_{\xi+1} \prec \cdots$$
is an elementary chain of models of ZFC with unions taken at limit stages,
- $\mathcal{N} = \bigcup_{\xi<\omega_1} \mathcal{N}_\xi$ where
$$\mathcal{N}_0 \prec \mathcal{N}_1 \prec \cdots \prec \mathcal{N}_\xi \prec \mathcal{N}_{\xi+1} \prec \cdots$$
is an elementary chain of models of ZFC with unions taken at limit stages,
- $M = N = \omega_1$ (as sets without structure),
- $j : M \to N$ is an embedding.

Then there is a limit ordinal λ such that A_λ "codes" $j \restriction M_\lambda : M_\lambda \to N_\lambda$.

Fix a bijection $\varphi : \omega_1 \times \omega_1 \to \omega_1$.
On a club of ordinals λ:

- $M_\lambda = \lambda$ and $N_\lambda = \lambda$
- $j \restriction \lambda : \lambda \to \lambda$
- $\varphi : \lambda \times \lambda \to \lambda$

There is a limit λ such that $A_\lambda = \varphi$ " $j \restriction \lambda : \lambda \to \lambda$.

Incomparable ω_1-like models

"I was not predicting the future, I was trying to prevent it."
–Ray Bradbury

We simultaneously build elementary chains of countable models of ZFC of length ω_1:

$$\mathcal{M}_0 \prec_e \mathcal{M}_1 \prec_e \cdots \prec_e \mathcal{M}_\xi \prec_e \mathcal{M}_{\xi+1} \prec_e \cdots$$

and

$$\mathcal{N}_0 \prec_e \mathcal{N}_1 \prec_e \cdots \prec_e \mathcal{N}_\xi \prec_e \mathcal{N}_{\xi+1} \prec_e \cdots.$$

- Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be any countable models of ZFC.
- At stage limit ordinal λ:
  - $\mathcal{M}_\lambda = \bigcup_{\xi<\lambda} \mathcal{M}_\xi$ and $\mathcal{N}_\lambda = \bigcup_{\xi<\lambda} \mathcal{N}_\xi$,
  - $M_\lambda = N_\lambda = \lambda$ (as sets without structure).
- At stage successor ordinal $\xi + 1$:

  Let $\mathcal{N}_{\xi+1}$ be any proper countable end-extension of $\mathcal{N}_\xi$.
  - If A_ξ codes $j : M_\xi \to N_\xi$, choose $\mathcal{M}_{\xi+1}$ to kill-off j,
  - else let $\mathcal{M}_{\xi+1}$ be any proper end-extension of $\mathcal{M}_\xi$.

It follows that:

- $\mathcal{M} = \bigcup_{\xi<\omega_1} \mathcal{M}_\xi$ and $\mathcal{N} = \bigcup_{\xi<\omega_1} \mathcal{N}_\xi$ are ω_1-like models.
- There is no embedding from $\mathcal{M}$ to $\mathcal{N}$.

The Key Lemma: splitting functions

Definition: Suppose $\langle M, \in, S\rangle \models \mathrm{GBC}$. A class $F : \mathrm{ORD}^{\mathcal{M}} \to M$ such that $F(\alpha) \subseteq \alpha$ is splitting if for every proper $A \subseteq \mathrm{ORD}^{\mathcal{M}}$ in S, there is $\beta \in \mathrm{ORD}^{\mathcal{M}}$ such that A splits into

$$A^+_\beta = \{\alpha \in A \mid \beta \in^{\mathcal{M}} F(\alpha)\}$$

and

$$A^-_\beta = \{\alpha \in A \mid \beta \notin^{\mathcal{M}} F(\alpha)\}$$

both of which are proper.

Tagline: "When constructing a descending sequence generating an ultrafilter U for a Skolem ultrafilter, no initial segment of the sequence can decide the subset of $\mathrm{ORD}^{\mathcal{M}}$ coded by $[F]_U$ in the ultrapower."

The Key Lemma: splitting functions (continued)

Theorem: (Fuchs, G., Hamkins) A model $\langle \mathcal{M}, \in, S\rangle \models \mathrm{GBC}$ has a splitting function if and only if it has an $\mathrm{ORD}^{\mathcal{M}}$-tree without a cofinal branch.

Corollary: A model $\langle \mathcal{M}, \in, S\rangle \models \mathrm{GBC}$ may fail to have a splitting function.

Proof: If κ is weakly compact, then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{GBC}$, but every tree of height κ in $V_{\kappa+1}$ has a cofinal branch in $V_{\kappa+1}$.

Corollary: A splitting function can be added by forcing to any model of GBC without adding sets.

Proof: Use a class version of the forcing to add an ω_1-Souslin tree.

The Key Lemma: proof

Key Lemma: (Fuchs, G., Hamkins) Suppose $\mathcal{M} \models \mathrm{ZFC}$ is countable. The collection $\mathcal{C}$ of subsets of M coded in some end-extension of $\mathcal{M}$ has size 2^ω.

Proof:

- Suppose $\langle M, \in\rangle \models \mathrm{ZFC}$.
- Extend it to a model $\langle M, \in, S\rangle \models \mathrm{GBC}$ such that S has a splitting function F.
- Let O be the collection of all classes of ordinals.
- There is a family $\langle U_s \mid s : \mathbb{N} \to 2\rangle$ of ultrafilters on O such that
  - $\mathcal{M} \prec_e \prod_O \mathcal{M}/U_s$,
  - if $s \neq t$, then the subset of M coded by $[F]_{U_s}$ is not equal that coded by $[F]_{U_t}$.
- U_s is generated by the descending sequence $A_0^s \supseteq A_1^s \supseteq \cdots \supseteq A_n^s \supseteq \cdots$.
- Stage $2n$: ensure ultrapower is an end-extension.
- Stage $2n+1$:
  - choose β such that $(A_{2n}^s)_\beta^+$ and $(A_{2n}^s)_\beta^-$ are proper,
  - if $s(n) = 1$, let $A_{2n+1}^s = (A_{2n}^s)_\beta^+$,
  - if $s(n) = 0$, let $A_{2n+1}^s = (A_{2n}^s)_\beta^-$. $\square$

Thank you!