A $\square(\kappa)$-LIKE PRINCIPLE CONSISTENT WITH WEAK COMPACTNESS

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Abstract. Sun proved that when κ is weakly compact, the 1-*club* subsets of κ provide a filter base for the weakly compact ideal, and hence can also be used to give a characterization of weakly compact sets which resembles the definition of stationarity: a set $S \subseteq \kappa$ is weakly compact (or equivalently Π^1_1-indescribable) if and only if $S \cap C \neq \varnothing$ for every 1-club $C \subseteq \kappa$. By replacing clubs with 1-clubs in the definition of $\square(\kappa)$, we obtain a $\square(\kappa)$-like principle we call $\square_1(\kappa)$ that is consistent with the weak compactness of κ but inconsistent with the Π^1_2-indescribability of κ. By generalizing the standard forcing to add a $\square(\kappa)$-sequence, we show that if κ is κ^+-weakly compact and GCH holds then there is a cofinality-preserving forcing extension in which κ remains κ^+-weakly compact and $\square_1(\kappa)$ holds. If κ is Π^1_2-indescribable and GCH holds then there is a cofinality-preserving forcing extension in which κ is κ^+-weakly compact, $\square_1(\kappa)$ holds and every weakly compact subset of κ has a weakly compact proper initial segment. As an application, we prove that, relative to a Π^1_2-indescribable cardinal, it is consistent that κ is κ^+-weakly compact, every weakly compact subset of κ has a weakly compact proper initial segment, and there exist two weakly compact subsets S^0 and S^1 of κ such that there is no $\beta < \kappa$ for which both $S^0 \cap \beta$ and $S^1 \cap \beta$ are weakly compact.

Contents

1. Introduction

In this paper, we introduce and investigate an incompactness principle we call $\square_1(\kappa)$, which is closely related to $\square(\kappa)$ but is consistent with weak compactness. Let us begin by recalling the basic facts about $\square(\kappa)$.

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The principle $\square(\kappa)$ asserts that there is a κ-length coherent sequence of clubs $\vec{C} = \langle C_\alpha : \alpha \in \lim(\kappa)\rangle$ that cannot be threaded. For an uncountable cardinal κ, a sequence $\vec{C} = \langle C_\alpha : \alpha \in \lim(\kappa)\rangle$ of clubs $C_\alpha \subseteq \alpha$ is called *coherent* if whenever β is a limit point of C_α we have $C_\beta = C_\alpha \cap \beta$. Given a coherent sequence $\vec{C}$, we say that C is a *thread* through $\vec{C}$ if C is a club subset of κ and $C \cap \alpha = C_\alpha$ for every limit point α of C. A coherent sequence $\vec{C}$ is called a $\square(\kappa)$-sequence if it cannot be threaded, and $\square(\kappa)$ holds if there is a $\square(\kappa)$-sequence. It is easy to see that $\square(\kappa)$ implies that κ is not weakly compact, and thus $\square(\kappa)$ can be viewed as asserting that κ exhibits a certain amount of incompactness. The principle $\square(\kappa)$ was isolated by Todorčević [Tod87], building on work of Jensen [Jen72], who showed that, if $V = L$, then $\square(\kappa)$ holds for every regular uncountable κ that is not weakly compact.

The natural $\leq\kappa$-strategically closed forcing to add a $\square(\kappa)$-sequence [LH14, Lemma 35] preserves the inaccessibility as well as the Mahloness of κ, but kills the weak compactness of κ and indeed adds a non-reflecting stationary set. However, if κ is weakly compact, there is a forcing [HLH17] which adds a $\square(\kappa)$-sequence and also preserves the fact that every stationary subset of κ reflects. Thus, relative to the existence of a weakly compact cardinal, $\square(\kappa)$ is consistent with $\mathrm{Refl}(\kappa)$, the principle that every stationary set reflects. However, $\square(\kappa)$ implies the failure of the simultaneous stationary reflection principle $\mathrm{Refl}(\kappa, 2)$ which states that if S and T are any two stationary subsets of κ, then there is some $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \omega$ such that $S \cap \alpha$ and $T \cap \alpha$ are both stationary in α. In fact, $\square(\kappa)$ implies that every stationary subset of κ can be partitioned into two stationary sets that do not simultaneously reflect [HLH17, Theorem 2.1].

If κ is a weakly compact cardinal, then the collection of non–Π^1_1-indescribable subsets of κ forms a natural normal ideal called the Π^1_1-indescribability ideal:

$$\Pi^1_1(\kappa) = \{X \subseteq \kappa : X \text{ is not } \Pi^1_1\text{-indescribable}\}.$$

A set $S \subseteq \kappa$ is Π^1_1-*indescribable* if for every $A \subseteq V_\kappa$ and every Π^1_1-sentence φ, whenever $(V_\kappa, \in, A) \models \varphi$ there is an $\alpha \in S$ such that $(V_\alpha, \in, A \cap V_\alpha) \models \varphi$. More generally, a Π^1_n-indescribable cardinal κ carries the analogously defined Π^1_n-indescribability ideal. It is natural to ask the question: which results concerning the nonstationary ideal can be generalized to the various ideals associated to large cardinals, such as the Π^1_n-indescribability ideals? The work of Sun [Sun93] and Hellsten [Hel03a] showed that when κ is Π^1_n-indescribable the collection of n-*club* subsets of κ (see the next section for definitions) is a filter-base for the filter $\Pi^1_n(\kappa)^*$ dual to the Π^1_n-indescribability ideal, yielding a characterization of Π^1_n-indescribable sets that resembles the definition of stationarity: when κ is Π^1_n-indescribable, a set $S \subseteq \kappa$ is Π^1_n-indescribable if and only if $S \cap C \neq \varnothing$ for every n-club $C \subseteq \kappa$. Several recent results have used this characterization ([Hel06], [Hel10], [Cod] and [CS]) to generalize theorems concerning the nonstationary ideal to the Π^1_1-indescribability ideal. For technical reasons discussed below in Section 8, there has been less success with the Π^1_n-indescribability ideals for $n > 1$. In this article we continue this line of research: by replacing "clubs" with "1-clubs" we obtain a $\square(\kappa)$-like principle $\square_1(\kappa)$ (see Definition 2.1) that is consistent with weak compactness but not with Π^1_2-indescribability.

We will see that the principle $\square_1(\kappa)$ holds trivially at weakly compact cardinals κ below which stationary reflection fails. (This is analogous to the fact that $\square(\kappa)$ holds trivially for every κ of cofinality ω_1.) Thus, the task at hand is not just to show that $\square_1(\kappa)$ is consistent with the weak compactness of κ, but to show that it is consistent with the weak compactness of κ even when stationary reflection holds at many cardinals below κ, so that nontrivial coherence of the sequence is obtained. Recall that when κ is κ^+-weakly compact, the set of weakly compact cardinals below κ is weakly compact and much more, so, in particular, the set of inaccessible $\alpha < \kappa$ at which stationary reflection holds is weakly compact.

Theorem 1.1. *If κ is κ^+-weakly compact and the* GCH *holds, then there is a cofinality-preserving forcing extension in which*

1. *κ remains κ^+-weakly compact and*
2. *$\square_1(\kappa)$ holds.*

We will also investigate the relationship between $\square_1(\kappa)$ and weakly compact reflection principles. The *weakly compact reflection principle* $\mathrm{Refl}_1(\kappa)$ states that κ is weakly compact and for every weakly compact $S \subseteq \kappa$ there is an $\alpha < \kappa$ such that $S \cap \alpha$ is weakly compact. It is straightforward to see that if κ is Π^1_2-indescribable, then $\mathrm{Refl}_1(\kappa)$ holds, and if $\mathrm{Refl}_1(\kappa)$ holds, then κ is ω-weakly compact (see [Cod, Section 2]). However, the following results show that neither of these implications can be reversed. The first author [Cod] showed that if $\mathrm{Refl}_1(\kappa)$ holds then there is a forcing which adds a non-reflecting weakly compact subset of κ and preserves the ω-weak compactness of κ, hence the ω-weak compactness of κ does not imply $\mathrm{Refl}_1(\kappa)$. The first author and Hiroshi Sakai [CS] showed that $\mathrm{Refl}_1(\kappa)$ can hold at the least ω-weakly compact cardinal, and hence $\mathrm{Refl}_1(\kappa)$ does not imply the Π^1_2-indescribability of κ. Just as $\square(\kappa)$ and $\mathrm{Refl}(\kappa)$ can hold simultaneously relative to a weakly compact cardinal, we will prove that $\square_1(\kappa)$ and $\mathrm{Refl}_1(\kappa)$ can hold simultaneously relative to a Π^1_2-indescribable cardinal.

Theorem 1.2. *Suppose that κ is Π^1_2-indescribable and the* GCH *holds. Then there is a cofinality-preserving forcing extension in which*

1. *$\square_1(\kappa)$ holds,*
2. *$\mathrm{Refl}_1(\kappa)$ holds and*
3. *κ is κ^+-weakly compact.*

In Section 2, using n-club subsets of κ, we formulate a generalization of $\square_1(\kappa)$ to higher degrees of indescribability. It is easily seen that $\square_n(\kappa)$ implies that κ is not Π^1_{n+1}-indescribable (see Proposition 2.8 below). However, for technical reasons outlined in Section 8, our methods do not seem to show that $\square_n(\kappa)$ can hold nontrivially (see Definition 2.9). when κ is Π^1_n-indescribable. Our methods do allow for a generalization of Hellsten's 1-club shooting forcing to n-club shooting, and we also show that, if S is a Π^1_n-indescribable set, a 1-club can be shot through S while preserving the Π^1_n-indescribability of all Π^1_n-indescribable subsets of S.

Finally, we consider the influence of $\square_n(\kappa)$ on *simultaneous* reflection of Π^1_n-indescribable sets. We let $\mathrm{Refl}_n(\kappa,\mu)$ denote the following simultaneous reflection principle: κ is Π^1_n-indescribable and whenever $\{S_\alpha : \alpha < \mu\}$ is a collection of Π^1_n-indescribable sets, there is a $\beta < \kappa$ such that $S_\alpha \cap \beta$ is Π^1_n-indescribable for all $\alpha < \mu$. In Section 7, we show that for $n \geq 1$, if $\square_n(\kappa)$ holds at a Π^1_n-indescribable cardinal, then the simultaneous reflection principle $\mathrm{Refl}_n(\kappa, 2)$ fails (see Theorem 7.1). As a consequence, we show that relative to a Π^1_2-indescribable cardinal, it is consistent that $\mathrm{Refl}_1(\kappa)$ holds and $\mathrm{Refl}_1(\kappa, 2)$ fails (see Corollary 7.4).

2. The principles $\square_n(\kappa)$

Suppose that κ is a cardinal. A set $S \subseteq \kappa$ is Π^1_n-*indescribable* if for every $A \subseteq V_\kappa$ and every Π^1_n-sentence φ, whenever $(V_\kappa, \in, A) \models \varphi$ there is an $\alpha \in S$ such that $(V_\alpha, \in, A \cap V_\alpha) \models \varphi$. The cardinal κ is said to be Π^1_n-*indescribable* if κ is a Π^1_n-indescribable subset of κ. The Π^1_0-indescribable cardinals are precisely the inaccessible cardinals, and, if κ is inaccessible, then $S \subseteq \kappa$ is Π^1_0-indescribable if and only if it is stationary. The Π^1_1-indescribable cardinals are precisely the weakly compact cardinals.

The Π^1_n-*indescribability ideal* on κ is

$$\Pi^1_n(\kappa) = \{X \subseteq \kappa : X \text{ is not } \Pi^1_n\text{-indescribable}\},$$

the corresponding collection of positive sets is

$$\Pi^1_n(\kappa)^+ = \{X \subseteq \kappa : X \text{ is } \Pi^1_n\text{-indescribable}\}$$

and the dual filter is

$$\Pi^1_n(\kappa)^* = \{\kappa \setminus X : X \in \Pi^1_n(\kappa)\}.$$

Clearly, if κ is not Π^1_n-indescribable, then $\Pi^1_n(\kappa) = P(\kappa)$. Lévy proved [Lév71] that if κ is Π^1_n-indescribable, then $\Pi^1_n(\kappa)$ is a nontrivial normal ideal on κ.

A set $C \subseteq \kappa$ is called 0-*club* if it is a club. A set $X \subseteq \kappa$ is said to be n-*closed* if it contains all of its Π^1_{n-1}-indescribable reflection points: whenever $\alpha < \kappa$ and $X \cap \alpha$ is Π^1_{n-1}-indescribable, then $\alpha \in X$ (note that such α must be Π^1_{n-1}-indescribable). If a set $C \subseteq \kappa$ is both n-closed and Π^1_{n-1}-indescribable, then C is said to be an n-*club* subset of κ. For example, $C \subseteq \kappa$ is 1-club if and only if it is stationary and contains all of its inaccessible stationary reflection points, and $C \subseteq \kappa$ is 2-club if and only if it is weakly compact and contains all of its weakly compact reflection points. Building on work of Sun [Sun93], Hellsten showed [Hel03b] that when κ is a Π^1_n-indescribable cardinal, a set $S \subseteq \kappa$ is Π^1_n-indescribable if and only if $S \cap C \neq \varnothing$ for every n-club $C \subseteq \kappa$. Thus, when κ is Π^1_n-indescribable, the collection of n-club subsets of κ generates the filter $\Pi^1_n(\kappa)^*$. In particular, this implies that n-club sets are themselves Π^1_n-indescribable.

For $n < \omega$ and $X \subseteq \kappa$, we define the n-*trace of* X to be

$$\mathrm{Tr}_n(X) = \{\alpha < \kappa : X \cap \alpha \in \Pi^1_n(\alpha)^+\}.$$

Notice that when $X = \kappa$, $\mathrm{Tr}_n(\kappa)$ is the set of Π^1_n-indescribable cardinals below κ, and in particular $\mathrm{Tr}_0(\kappa)$ is the set of inaccessible cardinals less than κ. For uniformity of notation, let us say that an ordinal α is Π^1_{-1}-*indescribable* if it is a limit ordinal, and if α is a limit ordinal, $S \subseteq \alpha$ is Π^1_{-1}-*indescribable* if it is unbounded in α. Thus, if $X \subseteq \kappa$, then $\mathrm{Tr}_{-1}(\kappa) = \{\alpha < \kappa : \sup(X \cap \alpha) = \alpha\}$.

Definition 2.1. Suppose $n < \omega$ and $\mathrm{Tr}_{n-1}(\kappa)$ is cofinal in κ. A sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ is called a *coherent sequence of* n-*clubs* if

(1) for all $\alpha \in \mathrm{Tr}_{n-1}(\kappa)$, C_α is an n-club subset of α and
(2) for all $\alpha < \beta$ in $\mathrm{Tr}_{n-1}(\kappa)$, $C_\beta \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$ implies $C_\alpha = C_\beta \cap \alpha$.

We say that a set $C \subseteq \kappa$ is a *thread* through a coherent sequence of n-clubs

$$\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$$

if C is n-club and for all $\alpha \in \mathrm{Tr}_{n-1}(\kappa)$, $C \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$ implies $C_\alpha = C \cap \alpha$. A coherent sequence of n-clubs $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ is called a $\square_n(\kappa)$-*sequence* if there is no thread through $\vec{C}$. We say that $\square_n(\kappa)$ holds if there is a $\square_n(\kappa)$-sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$.

Remark 2.2. Note that $\square_0(\kappa)$ is simply $\square(\kappa)$. For $n = 1$, the principle $\square_1(\kappa)$ states that there is a coherent sequence of 1-clubs

$$\langle C_\alpha : \alpha < \kappa \text{ is inaccessible}\rangle$$

that cannot be threaded.

Generalizing the fact that $\square(\kappa)$ implies κ is not weakly compact, let us show that $\square_n(\kappa)$ implies κ is not Π^1_{n+1}-indescribable. To do this, we first recall the Hauser characterization of Π^1_n-indescribability.

We say that a transitive model $\langle M, \in\rangle$ is a *κ-model* if $|M| = \kappa$, $\kappa \in M$, $M^{<\kappa} \subseteq M$, and $M \models \mathrm{ZFC}^-$ (ZFC without the power set axiom). It is not difficult to see that if κ is inaccessible, then V_κ is an element of every κ-model M.

Definition 2.3 (Hauser)**.** Suppose κ is inaccessible. For $n \geq 0$, a κ-model N is *Π^1_n-correct at κ* if and only if

$$V_\kappa \models \varphi \iff (V_\kappa \models \varphi)^N$$

for all Π^1_n-formulas φ whose parameters are contained in $N \cap V_{\kappa+1}$.

Remark 2.4. Notice that every κ-model is Π^1_0-correct at κ.

Theorem 2.5 (Hauser)**.** *The following statements are equivalent for every inaccessible cardinal κ, every subset $S \subseteq \kappa$, and all $0 < n < \omega$.*

(1) S is Π^1_n-indescribable.
(2) For every κ-model M with $S \in M$, there is a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$ such that $\kappa \in j(S)$.
(3) For every $A \subseteq \kappa$ there is a κ-model M with $A, S \in M$ for which there is a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$ such that $\kappa \in j(S)$.
(4) For every $A \subseteq \kappa$ there is a κ-model M with $A, S \in M$ for which there is a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$ such that $\kappa \in j(S)$ and $j, M \in N$.

Lemma 2.6. *Suppose κ is a cardinal. If $S \in \Pi^1_n(\kappa)^+$ and $S_\alpha \in \Pi^1_n(\alpha)^+$ for each $\alpha \in S$, then $\bigcup_{\alpha\in S} S_\alpha \in \Pi^1_n(\kappa)^+$.*

Proof. Fix an n-club C in κ. The set $\mathrm{Tr}_{n-1}(C)$ is n-closed because if $\mathrm{Tr}_{n-1}(C)\cap\alpha \in \Pi^1_{n-1}(\alpha)^+$, then $C \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$ since, by n-closure of C, $\mathrm{Tr}_{n-1}(C) \subseteq C$. Also, $\mathrm{Tr}_{n-1}(C)$ meets every n-club D because the intersection $C \cap D$ is an n-club. Thus, $\mathrm{Tr}_{n-1}(C)$ is an n-club. It follows that there is an $\alpha \in S \cap \mathrm{Tr}_{n-1}(C)$. Since S_α is Π^1_n-indescribable in α and $C \cap \alpha$ is an n-club in α, we have $S_\alpha \cap C \cap \alpha \neq \varnothing$, and hence $\left(\bigcup_{\alpha\in S} S_\alpha\right) \cap C \neq \varnothing$. $\square$

A simple complexity calculation shows that for every $n < \omega$, there is a Π^1_{n+1}-formula $\chi_n(X)$ such that for every κ and every $S \subseteq \kappa$, $(V_\kappa, \in) \models \chi_n(S)$ if and only if S is Π^1_n-indescribable (see [Kan03, Corollary 6.9]). It therefore follows that there is a Π^1_n-formula $\psi_n(X)$ such that for every κ and every $C \subseteq \kappa$, $(V_\kappa, \in) \models \psi_n(C)$ if and only if C is an n-club subset of κ. Thus, in particular, a Π^1_n-correct model N is going to be correct about Π^1_{n-1}-indescribable sets as well as n-clubs.

Corollary 2.7. *Suppose κ is Π^1_n-indescribable. If $S \in \Pi^1_n(\kappa)^+$, then*

$$\mathrm{Tr}_{n-1}(S) = \{\alpha < \kappa : S \cap \alpha \in \Pi^1_{n-1}(\alpha)^+\}$$

is an n-club.

Proof. Suppose S is Π^1_n-indescribable. First, let us argue that $\mathrm{Tr}_{n-1}(S)$ is Π^1_n-indescribable. Let M be a κ-model with $S, \mathrm{Tr}_{n-1}(S) \in M$ and let $j : M \to N$ be an elementary embedding with critical point κ such that N is Π^1_{n-1}-correct and $\kappa \in j(S)$. The Π^1_{n-1}-correctness of N implies that $j(S) \cap \kappa = S$ is a Π^1_{n-1}-indescribable subset of κ in N. Thus, $\kappa \in j(\mathrm{Tr}_{n-1}(S))$. Hence $\mathrm{Tr}_{n-1}(S)$ is Π^1_n-indescribable.

It remains to show that $\mathrm{Tr}_{n-1}(S)$ is n-closed, which is equivalent to showing that if $\mathrm{Tr}_{n-1}(S) \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$, then $S \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$. More generally, observe that if $X \subseteq \alpha$ and $\mathrm{Tr}_m(X)$ is Π^1_m-indescribable, then $X = \bigcup_{\beta \in \mathrm{Tr}_m(X)} X \cap \beta$ must be Π^1_m-indescribable by Lemma 2.6. □

Proposition 2.8. *For every $n < \omega$, $\square_n(\kappa)$ implies that κ is not Π^1_{n+1}-indescribable.*

Proof. Suppose $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_n(\kappa)\rangle$ is a $\square_n(\kappa)$-sequence and κ is Π^1_{n+1}-indescribable. Let M be a κ-model with $\vec{C} \in M$. Since κ is Π^1_{n+1}-indescribable, we may let $j : M \to N$ be an elementary embedding with critical point κ and a Π^1_n-correct N as in Theorem 2.5 (2). By elementarity, it follows that $j(\vec{C}) = \langle \bar{C}_\alpha : \alpha \in \mathrm{Tr}^N_n(j(\kappa))\rangle$ is a $\square_n(j(\kappa))$-sequence in N. Since N is Π^1_n-correct, we know that $\kappa \in \mathrm{Tr}^N_n(j(\kappa))$ and $\bar{C}_\kappa$ must also be n-club in V. Since $j(\vec{C})$ is a $\square_n(j(\kappa))$-sequence in N, it follows that for every Π^1_n-indescribable $\alpha < \kappa$ if $\bar{C}_\kappa \cap \alpha \in \Pi^1_n(\alpha)^+$, then $\bar{C}_\kappa \cap \alpha = C_\alpha$, and hence $\bar{C}_\kappa$ is a thread through $\vec{C}$, a contradiction. □

Let us now describe the sense in which $\square_n(\kappa)$ can hold trivially when κ is Π^1_n-indescribable and certain reflection principles fail often below κ.

Definition 2.9. Suppose $n < \omega$ and $\mathrm{Tr}_{n-1}(\kappa)$ is cofinal in κ. We say that $\square_n(\kappa)$ *holds trivially* if there is a $\square_n(\kappa)$-sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ and a club $E \subseteq \kappa$ such that for all $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$, C_α is trivially an n-club subset of α in the sense that C_α is a Π^1_{n-1}-indescribable subset of α and has no Π^1_{n-1}-indescribable proper initial segment.

Notice that $\square(\kappa)$ holds trivially if $\mathrm{cf}(\kappa) = \omega_1$. In this case we can find a club $E \subseteq \kappa$ consisting of ordinals of countable cofinality, namely, let $\langle \alpha_\xi : \xi < \omega_1\rangle$ be an increasing continuous cofinal sequence in κ, and let E consist of α_ξ for ξ a limit ordinal. For all $\alpha \in E$, we can let C_α be a cofinal subset of α of order type ω. Then, for every limit ordinal $\beta \in \kappa \setminus E$, we can let $\alpha_\beta = \max(E \cap \beta)$ and set C_β to be the interval (α_β, β). It is easily verified that a sequence thus defined is a $\square(\kappa)$-sequence.

Recall that the principle $\mathrm{Refl}_n(\kappa)$ holds if and only if κ is Π^1_n-indescribable and for every Π^1_n-indescribable subset X of κ, there is an $\alpha < \kappa$ such that $X \cap \alpha$ is Π^1_n-indescribable (see [Cod] and [CS] for more details).

Proposition 2.10. *Suppose $1 \leq n < \omega$ and κ is Π^1_n-indescribable. Then $\square_n(\kappa)$ holds trivially if and only if there is a club $E \subseteq \kappa$ such that $\neg\mathrm{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$.*

Proof. If $\square_n(\kappa)$ holds trivially, then there is a $\square_n(\kappa)$-sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ and a club $E \subseteq \kappa$ such that for $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$, C_α is a Π^1_{n-1}-indescribable set with no Π^1_{n-1}-indescribable initial segment, in which case C_α is a witness to the fact that $\mathrm{Refl}_{n-1}(\alpha)$ fails.

Conversely, suppose that $E \subseteq \kappa$ is a club and $\neg\mathrm{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$. For each $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$, let C_α be a Π^1_{n-1}-indescribable subset of α which has no Π^1_{n-1}-indescribable proper initial segment. Then each C_α is trivially n-club in α. For all $\beta \in \mathrm{Tr}_{n-1}(\kappa) \setminus E$, let $\alpha_\beta = \max(E \cap \beta)$, and let C_β be the interval (α_β, β). Then $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ is easily seen to be a coherent sequence of n-clubs, since there are no points at which coherence needs to be checked for indices in E and coherence is easily checked for indices outside of E because of the uniformity of the definition. We must argue that $\vec{C}$ has no thread. Suppose there is a thread $C \subseteq \kappa$ through $\vec{C}$. Since κ is Π^1_n-indescribable and C is an n-club subset of κ it follows, by Corollary 2.7, that $\mathrm{Tr}_{n-1}(C)$ is an n-club in κ. Thus we can choose $\alpha, \beta \in \mathrm{Tr}_{n-1}(C) \cap E$ with $\alpha < \beta$. Since C is a thread we have $C_\alpha = C_\beta \cap \alpha = C \cap \alpha$, which contradicts the fact that C_β has no Π^1_{n-1}-indescribable proper initial segment. This shows that $\square_n(\kappa)$ holds trivially. □

Remark 2.11. It seems like it might be more optimal to change Definition 2.9 to instead say that $\square_n(\kappa)$ holds trivially if there is a $\square_n(\kappa)$-sequence $\vec{C}$ and an *n-club* $E \subseteq \kappa$ such that for all $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$, C_α is trivially an n-club subset of α. However, we were not able to prove the analogue of Proposition 2.10 corresponding to this alternative definition, namely that $\square_n(\kappa)$ holds trivially if and only if there is an *n-club* $E \subseteq \kappa$ such that $\neg\mathrm{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$.

Corollary 2.12. *In L, if κ is the least Π^1_n-indescribable cardinal, then $\square_n(\kappa)$ holds trivially.*

Proof. Generalizing a result of Jensen [Jen72], Bagaria, Magidor and Sakai proved [BMS15] that in L a cardinal κ is Π^1_n-indescribable if and only if $\mathrm{Refl}_{n-1}(\kappa)$ holds. Suppose $V = L$ and κ is the least Π^1_n-indescribable cardinal. Then $\mathrm{Refl}_{n-1}(\alpha)$ fails for all $\alpha < \kappa$. Hence by Proposition 2.10, $\square_n(\kappa)$ holds trivially. □

Another consequence of Proposition 2.10 is that we can force $\square_1(\kappa)$ to hold *trivially* at a Π^1_1-indescribable cardinal by killing certain stationary reflection principles below κ.

Recall that a partial order $\mathbb{P}$ is said to be *α-strategically closed*, for an ordinal α, if Player II has a winning strategy in the following two-player game $\mathcal{G}_\alpha(\mathbb{P})$ of perfect information. In a run of $\mathcal{G}_\alpha(\mathbb{P})$, the two players take turns playing elements of a decreasing sequence $\langle p_\beta : \beta < \alpha\rangle$ of conditions from $\mathbb{P}$. Player I plays at all odd ordinal stages, and Player II plays at all even ordinal stages (in particular, at limits). Player II goes first and must play $\mathbb{1}_{\mathbb{P}}$. Player I wins if there is a limit ordinal $\gamma < \alpha$ such that $\langle p_\beta : \beta < \gamma\rangle$ has no lower bound (i.e., if Player II is unable to play at stage γ). If the game continues successfully for α-many moves, then Player II wins. Clearly, for a cardinal α, if $\mathbb{P}$ is α-strategically closed, then $\mathbb{P}$ is $<\alpha$-distributive, and hence adds no new α-sequences of ground model sets.

We will use the following general proposition about indestructibility of weakly compact cardinals.

Definition 2.13. Suppose κ is an inaccessible cardinal. We say that a forcing iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa,\ \beta < \kappa \rangle$$

is *good* if it has Easton support and, for all $\alpha < \kappa$, if α is inaccessible, then $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset such that $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha \in \dot{V}_\kappa$, where $\dot{V}_\kappa$ is a $\mathbb{P}_\alpha$-name for $(V_\kappa)^{V^{\mathbb{P}_\alpha}}$ and, otherwise, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for trivial forcing.

If $\mathbb{P}_\kappa$ is a good iteration, then we can argue by induction on α that every $\mathbb{P}_\alpha \in V_\kappa$ because if $\mathbb{P}_\alpha \in V_\kappa$ and $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha \in \dot{V}_\kappa$, then $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha \in V_\kappa$. The following standard proposition about good iterations can be found, for example, in [Cum10].

Proposition 2.14. *Suppose κ is a Mahlo cardinal. Then a good iteration $\mathbb{P}_\kappa$ has size κ and is κ-c.c.*

Lemma 2.15. *Suppose κ is weakly compact and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa,\ \beta < \kappa \rangle$ is a good iteration which at non-trivial stages α has $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is α-strategically closed", and let G be $\mathbb{P}$-generic over V. Then κ remains weakly compact in $V[G]$.*

Proof. By Proposition 2.14, we can assume without loss that $\mathbb{P} \subseteq V_\kappa$. Since κ is weakly compact, there are κ-models M and N with $A \in M$ for which there is an elementary embedding $j : M \to N$ with critical point κ. A nice-name counting argument, using the κ-c.c. and the fact that the tails of the forcing iteration are eventually α-distributive for every $\alpha < \kappa$, shows that κ is inaccessible in $V[G]$.

Suppose $A \in P(\kappa)^{V[G]}$ and let $\dot{A} \in H(\kappa^+)^V$ be a $\mathbb{P}_\kappa$-name such that $\dot{A}_G = A$. Let M be a κ-model with $\dot{A} \in M$ for which there are a κ-model N and an elementary embedding $j : M \to N$ with critical point κ. Since $N^{<\kappa} \cap V \subseteq N$, we have $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{\kappa,j(\kappa)}$, where N believes that $\mathbb{1}_{\mathbb{P}_\kappa} \Vdash$ "$\dot{\mathbb{Q}}_\kappa$ is κ-strategically closed", and $\dot{\mathbb{P}}_{\kappa,j(\kappa)}$ is a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$-name for N's version of the tail of the iteration $j(\mathbb{P}_\kappa)$ of length $j(\kappa)$. By the generic closure criterion (Lemma 3.2), since $\mathbb{P}_\kappa$ has the κ-c.c., $N[G]$ is a κ-model in $V[G]$. The poset $(\dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{\kappa,j(\kappa)})_G$ is κ-strategically closed in $N[G]$, so, by diagonalizing, we can build an $N[G]$-generic filter $H * G' \in V[G]$ for $(\dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{\kappa,j(\kappa)})_G$. Since conditions in $\mathbb{P}_\kappa$ have supports of size less than the critical point of j we have $j\text{"}\,G \subseteq \hat{G} =_{\text{def}} G * H * G'$. Thus j lifts to $j : M[G] \to N[\hat{G}]$. Since $A = \dot{A}_G \in M[G]$, this shows that κ remains weakly compact in $V[G]$. □

Proposition 2.16. *If κ is Π^1_1-indescribable (weakly compact), then there is a forcing extension in which $\square_1(\kappa)$ holds trivially and κ remains Π^1_1-indescribable.*

Proof. For regular $\alpha > \omega$, let $\mathbb{S}_\alpha$ denote the usual forcing to add a nonreflecting stationary subset of $\alpha \cap \text{cof}(\omega)$ (see Example 6.5 in [Cum10]). Recall that conditions in $\mathbb{S}_\alpha$ are bounded subsets p of $\alpha \cap \text{cof}(\omega)$ such that for every $\beta \leq \sup(p)$ with $\text{cf}(\beta) > \omega$, the set $p \cap \beta$ is nonstationary in β. It is not difficult to see that the poset $\mathbb{S}_\alpha$ is α-strategically closed.

Now we let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an Easton-support iteration of length κ such that if $\alpha < \kappa$ is inaccessible, then $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathbb{S}_\alpha^{V^{\mathbb{P}_\alpha}}$, and otherwise $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for trivial forcing.

Suppose G is generic for $\mathbb{P}_\kappa$ over V. By Lemma 2.15, since $\mathbb{P}_\kappa$ has all the right properties, κ remains weakly compact in $V[G]$. Also, in $V[G]$, for each inaccessible $\alpha < \kappa$, by a routine genericity argument and the fact that the tail of the forcing iteration from stage $\alpha + 1$ to κ is α^+-strategically closed, the stage α generic H_α

obtained from G yields a nonreflecting stationary subset of α: $S_\alpha = \bigcup H_\alpha$. Thus in $V[G]$, $\text{Refl}_0(\alpha)$ fails for all inaccessible $\alpha < \kappa$, and hence $\square_1(\kappa)$ holds trivially by Proposition 2.10. □

In Section 5 we will show that $\square_1(\kappa)$ can hold non-trivially at a weakly compact cardinal.

3. Preserving Π^1_n-indescribability by forcing

In this section, we will provide some results to be used in indestructibility arguments for Π^1_n-indescribable cardinals in later sections.

The following two folklore lemmas (and their variants) are widely used in indestructibility arguments for large cardinals characterized by the existence of elementary embeddings.

Lemma 3.1 (Ground closure criterion)**.** *Suppose κ is a cardinal, M is a κ-model, $\mathbb{P} \in M$ is a forcing notion, and $G \in V$ is generic for $\mathbb{P}$ over M. Then $M[G]$ is a κ-model.*

Lemma 3.2 (Generic closure criterion)**.** *Suppose κ is a cardinal, M is a κ-model, $\mathbb{P} \in M$ is a forcing notion with the κ-c.c., and G is generic for $\mathbb{P}$ over V. Then $M[G]$ is a κ-model in $V[G]$.*

Lemma 3.3. *Suppose κ is inaccessible, $\mathbb{P}$ is a κ-strategically closed forcing and G is generic for $\mathbb{P}$ over V. Then $(V_\kappa, \in, A) \models \forall X \psi(X, A)$ implies $((V_\kappa, \in, A) \models \forall X \psi(X, A))^{V[G]}$ for all $A \in V^V_{\kappa+1}$ and all first order ψ.*

Proof. First, observe that since $\mathbb{P}$ is $<\kappa$-distributive, κ remains inaccessible in $V[G]$ and $V_\kappa = V_\kappa^{V[G]}$. Suppose towards a contradiction that $(V_\kappa, \in, A) \models \forall X \psi(X, A)$, but for some $B \subseteq V_\kappa$ in $V[G]$, $(V_\kappa, \in, A) \models \neg\psi(B, A)$. Let $\dot{B}$ be a $\mathbb{P}$-name for B. Since κ is inaccessible in $V[G]$, the set

$$C = \{\alpha < \kappa : (V_\alpha, \in, A \cap \alpha, B \cap \alpha) \models \neg\psi(B \cap \alpha, A \cap \alpha))\}$$

contains a club in $V[G]$. Let $\dot{C}$ be a $\mathbb{P}$-name for such a club. In V, we can use Player II's winning strategy in $\mathcal{G}_\kappa(\mathbb{P})$ together with the names $\dot{B}$ and $\dot{C}$ to build $\hat{B}$ and $\hat{C}$ such that $\hat{C}$ is club in κ and for each $\alpha \in \hat{C}$ we have

$$(V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \neg\psi(\hat{B} \cap V_\alpha, A \cap V_\alpha).$$

Since $(V_\kappa, \in, A) \models \forall X \psi(X, A)$, we have $(V_\kappa, \in, A, \hat{B}) \models \psi(\hat{B}, A)$, and since κ is inaccessible, the set

$$\{\alpha < \kappa : (V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \psi(\hat{B} \cap \alpha, A \cap \alpha)\}$$

contains a club. Thus, there is an $\alpha \in \hat{C}$ such that

$$(V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \psi(\hat{B} \cap V_\alpha, A \cap V_\alpha),$$

a contradiction. □

Corollary 3.4. *Suppose κ is inaccessible, $\mathbb{P}$ is a κ-strategically closed forcing notion and G is generic for $\mathbb{P}$ over V. If N is a Π^1_1-correct κ-model in V, then N remains a Π^1_1-correct κ-model in $V[G]$.*

Proof. Clearly N remains a κ-model because $\mathbb{P}$ is $<\kappa$-distributive. Let φ be a Π^1_1-statement, and suppose first that $(V_\kappa \models \varphi)^N$. By Π^1_1-correctness, $V_\kappa \models \varphi$, and so by Lemma 3.3, $(V_\kappa \models \varphi)^{V[G]}$. On the other hand, if $(V_\kappa \models \neg\varphi)^N$, then there is a $B \subseteq V_\kappa$ in N witnessing this failure. Since N, V, and $V[G]$ all have the same V_κ, B witnesses the failure of φ in both V and $V[G]$ as well, so $(V_\kappa \models \neg\varphi)^{V[G]}$ □

Proposition 3.5. *Suppose κ is inaccessible, $\mathbb{P}$ is κ-strategically closed, and G is generic for $\mathbb{P}$ over V. If $S \in P(\kappa)^V$ is Π^1_1-indescribable in $V[G]$, then S is Π^1_1-indescribable in V.*

Proof. Suppose towards a contradiction that there is $S \in P(\kappa)^V$ that is Π^1_1-indescribable in $V[G]$ but not Π^1_1-indescribable in V. In V, find a subset $A \subseteq V_\kappa$ and a Π^1_1 statement $\varphi = \forall X \psi(X, A)$ such that $(V_\kappa, \in, A) \models \varphi$ and for all $\alpha \in S$ we have $(V_\alpha, \in, A \cap V_\alpha) \models \neg\varphi$. Since $\mathbb{P}$ is $<\kappa$-distributive, V and $V[G]$ have the same V_κ, so it follows that in $V[G]$, by the Π^1_1-indescribability of S, it must be the case that $(V_\kappa, \in, A) \models \exists X \neg\psi(X, A)$. Working in $V[G]$, we fix $B \subseteq V_\kappa$ such that $(V_\kappa, \in, A) \models \neg\psi(B, A)$ and observe that the set

$$C = \{\alpha < \kappa : V_\alpha \models \neg\psi(B \cap V_\alpha, A \cap V_\alpha)\}$$

contains a club. Let $\dot{C}$ be a $\mathbb{P}$-name for such a club, and let $\dot{B}$ be a $\mathbb{P}$-name for B. In V, we can use Player II's winning strategy in $\mathcal{G}_\kappa(\mathbb{P})$ together with $\dot{B}$ and $\dot{C}$ to build $\hat{B}$ and $\hat{C}$ such that $\hat{C} \subseteq \kappa$ is club and $\forall \alpha \in \hat{C}$, $V_\alpha \models \neg\psi(\hat{B} \cap V_\alpha, A \cap V_\alpha)$. But this implies that $V_\kappa \models \neg\psi(\hat{B}, A)$, a contradiction. □

The converse of Proposition 3.5 is clearly false because the forcing $\mathrm{Add}(\kappa, 1)$ to add a Cohen subset to κ with bounded conditions can destroy the weak compactness of κ and it is $<\kappa$-closed and therefore κ-strategically closed. We will see in Section 6 (Remark 6.4) that Proposition 3.5 can fail for Π^1_2-indescribable sets.

A good iteration $\mathbb{P}_\kappa$ of length κ is said to be *progressively closed* if for every $\alpha < \kappa$, there is $\alpha \leq \beta_\alpha < \kappa$ such that every stage after β_α is forced to be α-strategically closed. In this case, it is not difficult to see that $\mathbb{P}_{\beta_\alpha}$ forces that the tail of the iteration is α-strategically closed. Next, we will show that good progressively closed κ-length iterations preserve Π^1_n-correctness.

Let $\mathbb{P}$ be a forcing notion and suppose σ is a $\mathbb{P}$-name. Recall that τ is a *nice name for a subset of* σ if

$$\tau = \bigcup(\{\pi\} \times A_\pi : \pi \in \mathrm{dom}(\sigma)\},$$

where each A_π is an antichain of $\mathbb{P}$. It is well known and easy to verify that for every $\mathbb{P}$-name μ, there is a nice name τ for a subset of σ such that $\mathbb{1}_\mathbb{P} \Vdash \mu \subseteq \sigma \rightarrow \mu = \tau$. We call such τ the *nice replacement for* μ.

Lemma 3.6. *Suppose σ is a $\mathbb{P}$-name and $n \geq 0$. Let X_σ be the set of nice names for subsets of σ, let p be a condition in $\mathbb{P}$ and let φ be any Π_n-assertion in the forcing language of the form*

$$(\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \ldots, x_n).$$

Then $p \Vdash \varphi$ if and only if

$$(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \ldots, \tau_n).$$

The analogous statement holds for Σ_n-assertions in the forcing language.

Proof. We will prove the lemma simultaneously for Π_n and Σ_n statements by induction on n. Clearly the lemma holds for $n = 0$. Assume inductively that the lemma holds for some n, and suppose φ is an assertion in the forcing language of complexity Π_{n+1}. Let

$$\varphi = (\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \ldots, x_{n+1}) = (\forall x_1 \subseteq \sigma)\bar{\varphi}(x_1),$$

where ψ is Δ_0 and $\bar{\varphi}(x)$ is Σ_n. For the forward direction, clearly $p \Vdash \varphi$ implies $(\forall \tau_1 \in X_\sigma)(p \Vdash \bar{\varphi}(\tau_1))$. By the inductive hypothesis applied to $p \Vdash \bar{\varphi}(\tau_1)$, we conclude that $(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \ldots, \tau_n)$. For the converse, suppose $(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \ldots, \tau_n)$ holds. Let us argue that $p \Vdash (\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \ldots, x_n)$. If not, there is some $q \leq p$ and some $\mathbb{P}$-name μ for a subset of σ such that $q \Vdash (\forall x_2 \subseteq \sigma) \cdots \neg\psi(\mu, x_2, \ldots, x_n)$. Let τ be a nice replacement for μ so that $q \Vdash (\forall x_2 \subseteq \sigma) \cdots \neg\psi(\tau, x_2, \ldots, x_n)$, or in other words, $q \Vdash \neg\bar{\varphi}(\tau)$. By assumption $(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau, \tau_2, \ldots, \tau_n)$, so applying the inductive hypothesis, we obtain $p \Vdash (\exists x_2 \subseteq \sigma) \cdots \psi(\tau, x_2, \ldots, x_n)$ and hence $p \Vdash \bar{\varphi}(\tau)$, a contradiction. The proof of the lemma for Σ_{n+1} statements is similar. $\square$

Theorem 3.7. *Suppose κ is a Mahlo cardinal, N is a Π^1_n-correct κ-model and $\mathbb{P} \in N$ is a progressively closed good Easton-support iteration of length κ. If $G \subseteq \mathbb{P}$ is generic over V, then $N[G]$ is a Π^1_n-correct κ-model in $V[G]$.*

Proof. By Proposition 2.14, $\mathbb{P}$ has the κ-c.c. and without loss of generality $\mathbb{P} \subseteq V_\kappa$. Thus, by the generic closure criterion Lemma 3.2, $N[G]$ remains a κ-model in $V[G]$. By the progressive closure of the iteration, $V_\kappa^{V[G]} = V_\kappa[G]$. Thus, $V_\kappa^{N[G]} = V_\kappa^{V[G]}$. Let $\sigma \in N$ be a $\mathbb{P}$-name such that $\sigma_G = V_\kappa^{N[G]} = V_\kappa^{V[G]}$ and $\mathrm{dom}(\sigma) \subseteq V_\kappa$.

Let us argue that $N[G]$ is Π^1_n-correct. Suppose $(V_\kappa^{N[G]}, \in, A) \models \varphi$ in $N[G]$, where

$$\varphi = \forall X_1 \exists X_2 \cdots \psi(X_1, \ldots, X_n, A)$$

is Π^1_n and all quantifiers appearing in ψ are first-order over $V_\kappa^{N[G]}$. Let $\dot{A}$ be a $\mathbb{P}$-name for A such that $\mathrm{dom}(\dot{A}) \subseteq V_\kappa$. Let $\bar{\psi}(x_1, \ldots, x_n, \dot{A})$ be a formula in the forcing language obtained from ψ by replacing all parameters with $\mathbb{P}$-names and all first-order quantifiers "Qx" with "$Qx \in \sigma$" for $Q = \forall, \exists$. Let $\bar{\varphi}(\sigma, \dot{A})$ denote the following formula in the forcing language:

$$(\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \bar{\psi}(x_1, \ldots, x_n, \dot{A}).$$

Since $(V_\kappa^{N[G]}, \in, A) \models \varphi$ holds in $N[G]$, it follows that $N[G] \models \bar{\varphi}(\sigma_G, \dot{A}_G)$. Thus, we may choose $p \in G$ with $(p \Vdash \bar{\varphi}(\sigma, \dot{A}))^N$. By Lemma 3.6,

$$(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \ldots, \tau_n, \dot{A}) \tag{3.1}$$

holds in N. The statement $p \Vdash \psi(\tau_1, \ldots, \tau_n, \dot{A})$ is first-order in the structure $(V_\kappa, \in, \tau_1, \ldots, \tau_n, \sigma, \dot{A}, \mathbb{P})$.[1] Furthermore, since "$\tau \in X_\sigma$" can be expressed by a first-order formula $\chi(\tau, \sigma)$ over $(V_\kappa, \in, \sigma, \tau, \mathbb{P})$, it follows that the statement in (3.1) is Π^1_n over $(V_\kappa, \in, \sigma, \dot{A})$. Since $N \models$ "(3.1) holds in $(V_\kappa, \in, \sigma, \dot{A})$" and N is Π^1_n-correct at κ, it follows that (3.1) holds in $(V_\kappa, \in, \sigma, \dot{A})$. Hence by Lemma 3.6, $p \Vdash \bar{\varphi}(\sigma, \dot{A})$ over V, and since $p \in G$, we conclude that $V[G] \models \bar{\varphi}(\sigma, \dot{A}_G)$, which implies $(V_\kappa^{V[G]}, \in, A) \models \varphi$ in $V[G]$.

[1]This can be proved by using the definition of the forcing relation and induction on complexity of formulas.

An analogous argument establishes the converse, verifying that if $(V_\kappa^{V[G]}, \in, A) \models \varphi$ for a Π^1_n-assertion φ and $A \in N[G]$, then the same assertion holds in $N[G]$. $\square$

A similar argument yields the following result.

Corollary 3.8. *Suppose κ is an inaccessible cardinal, N is a Π^1_n-correct κ-model and $\mathbb{P} \in N$ is a $<\kappa$-distributive forcing notion of size κ. If $G \subseteq \mathbb{P}$ is generic over V, then $N[G]$ remains a Π^1_n-correct κ-model in $V[G]$.*

Proof. Without loss of generality we can assume that $\mathbb{P} \subseteq V_\kappa$. Since $\mathbb{P}$ is $<\kappa$-distributive, N remains a κ-model in $V[G]$, and, since $G \in V[G]$, it follows that $N[G]$ is a κ-model in $V[G]$ by the ground closure criterion Lemma 3.1. The $<\kappa$-distributivity of $\mathbb{P}$ entails that $V_\kappa^{N[G]} = V_\kappa^{V[G]} = V_\kappa$. Since the statement "$\tau$ is a nice name for a subset of $\check{V}_\kappa$" is first-order over the structure $(V_\kappa, \in, \tau, \mathbb{P})$, the rest of the argument can be carried out as in the proof of Theorem 3.7. $\square$

The conclusion of Corollary 3.8 need not hold if the N-generic filter G is not fully V-generic (see Remark 5.7).

4. Shooting n-clubs

Hellsten [Hel10] showed that if $W \subseteq \kappa$ is any Π^1_1-indescribable (i.e., weakly compact) subset of κ, then there is a forcing extension in which W contains a 1-club and all weakly compact subsets of W remain weakly compact. We will define a generalization of Hellsten's forcing to shoot an n-club through a Π^1_n-indescribable subset of a cardinal κ while preserving the Π^1_n-indescribability of all its subsets, so that, in particular, κ remains Π^1_n-indescribable in the forcing extension.

Suppose γ is an inaccessible cardinal and $A \subseteq \gamma$ is cofinal. For $n \geq 1$, we define a poset $T^n(A)$ consisting of all bounded n-closed $c \subseteq A$ ordered by end extension: $c \leq d$ if and only if $d = c \cap \sup_{\alpha \in d}(\alpha + 1)$.

Lemma 4.1. *For $n \geq 1$, if γ is inaccessible and $A \subseteq \gamma$ is cofinal, then $T^n(A)$ is γ-strategically closed.*

Proof. We describe a winning strategy for player II in the game $\mathcal{G}_\kappa(T^n(A))$. Player II begins the game by playing $c_0 = \varnothing$. At an even successor stage $\alpha + 2$, player II chooses a condition $c_{\alpha+2} \in T^n(A)$ such that $c_{\alpha+2} \lneq c_{\alpha+1}$. At limit stages $\alpha < \gamma$, player II records an ordinal $\gamma_\alpha = \bigcup_{\beta<\alpha} c_\beta$, chooses an element $\eta_\alpha \in A \setminus (\gamma_\alpha + 1)$ and plays $c_\alpha = \left(\bigcup_{\beta<\alpha} c_\beta\right) \cup \{\eta_\alpha\}$. In order to argue that c_α is a condition in $T^n(A)$, we need to verify, letting $c = \bigcup_{\beta<\alpha} c_\beta$, that c is not a Π^1_{n-1}-indescribable subset of γ_α. We can assume that γ_α is Π^1_{n-1}-indescribable, as otherwise $c \cap \gamma_\alpha$ is clearly not Π^1_{n-1}-indescribable. But then, by construction, $\{\gamma_\xi : \xi < \alpha \text{ is a limit ordinal}\}$ is a club (and hence an $(n-1)$-club) in γ_α disjoint from c, which implies that c is not a Π^1_{n-1}-indescribable subset of γ_α. Thus, c_α is a valid play by Player II, and we have described a winning strategy in $\mathcal{G}_\kappa(T^n(A))$. $\square$

Remark 4.2. In fact, for $n \geq 2$, $T^n(A)$ satisfies the following strengthening of γ-strategic closure. For $X \subseteq \gamma$ and a poset $\mathbb{P}$, let $\mathcal{G}_{\gamma,X}(\mathbb{P})$ be the modification of $\mathcal{G}_\gamma(\mathbb{P})$ in which Player I plays at all stages indexed by an ordinal in X and Player II plays elsewhere, and it is still the case that Player I wins if and only if there is a limit ordinal $\beta < \gamma$ such that $\langle p_\alpha : \alpha < \beta\rangle$ has no lower bound in $\mathbb{P}$. So,

$\mathcal{G}_\gamma(\mathbb{P})$ is precisely the game $\mathcal{G}_{\gamma,X}(\mathbb{P})$, where X is the set of odd ordinals less than γ. A routine modification of the proof of the preceding lemma shows that Player II has a winning strategy in the game $\mathcal{G}_{\gamma,X}(T^n(A))$ if, for all $\beta < \gamma$, $X \cap \beta$ is not Π^1_{n-1}-indescribable. In particular, this is the case if X is the set of all $\alpha < \gamma$ such that α is not Π^1_{n-2}-indescribable.

Theorem 4.3. *Suppose that $n \geq 1$ and $S \subseteq \kappa$ is Π^1_n-indescribable. Then there is a forcing extension in which S contains a 1-club and all Π^1_n-indescribable subsets of S from V remain Π^1_n-indescribable.*

Proof. Let $\mathbb{P}_{\kappa+1} = \langle(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta) : \alpha \leq \kappa+1,\ \beta \leq \kappa\rangle$ be an Easton-support iteration such that

- if $\gamma \leq \kappa$ is inaccessible and $S \cap \gamma$ is cofinal in γ, then $\dot{\mathbb{Q}}_\gamma = (T^1(S \cap \gamma))^{V^{\mathbb{P}_\gamma}}$;
- otherwise, $\dot{\mathbb{Q}}_\gamma$ is a $\mathbb{P}_\gamma$-name for trivial forcing.

Since κ is Π^1_n-indescribable, Proposition 2.14 implies that $\mathbb{P}_\kappa$ has size κ and the κ-c.c.. Forcing with $\mathbb{P}_{\kappa+1}$ therefore preserves the inaccessibility of κ because $\mathbb{P}_\kappa$ has the κ-c.c. and is progressively closed and $\dot{\mathbb{Q}}_\kappa$ is forced to be $<\kappa$-distributive.

Suppose $G * H \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ is generic over V. Clearly, $C(\kappa) =_{\text{def}} \bigcup H$ is a 1-closed subset of S; to show that $C(\kappa)$ is a 1-club subset of κ, it remains to show that $C(\kappa)$ is a stationary subset of κ in $V[G * H]$.

Suppose $T \subseteq S$ is Π^1_n-indescribable in V. We will simultaneously show that in $V[G * H]$, $C(\kappa)$ intersects every club subset of κ and T remains Π^1_n-indescribable (in particular, κ remains Π^1_n-indescribable). Fix $A, C \in P(\kappa)^{V[G*H]}$ such that C is a club subset of κ in $V[G * H]$. Let $\dot{A}, \dot{C}, \dot{C}(\kappa) \in H(\kappa^+)$ be $\mathbb{P}_{\kappa+1}$-names such $\dot{A}_{G*H} = A$, $\dot{C}_{G*H} = C$ and $\dot{C}(\kappa)_{G*H} = C(\kappa)$. In V, let M be a κ-model with $\dot{A}, \dot{C}, \dot{C}(\kappa), \mathbb{P}_{\kappa+1}, T, S \in M$. Since T is Π^1_n-indescribable in V, it follows by Theorem 2.5 that there is a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $\kappa \in j(T)$.

Since $N^{<\kappa} \cap V \subseteq N$ and $j(S) \cap \kappa = S$, it follows that $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{T}^1(S) * \dot{\mathbb{P}}_{\kappa,j(\kappa)}$, where $\dot{\mathbb{P}}_{\kappa,j(\kappa)}$ is a $\mathbb{P}_{\kappa+1}$-name for the tail of the iteration $j(\mathbb{P}_\kappa)$. Since $\mathbb{P}_\kappa$ has the κ-c.c., by the generic closure criterion (Lemma 3.2), $N[G]$ is a κ-model in $V[G]$. Since $T^1(S)$ is κ-strategically closed, $N[G]$ remains a κ-model in $V[G * H]$, and hence by the ground closure criterion (Lemma 3.1), $N[G * H]$ is a κ-model in $V[G * H]$. Since $\mathbb{P}_{\kappa,j(\kappa)} = (\dot{\mathbb{P}}_{\kappa,j(\kappa)})_{G*H}$ is κ-strategically closed in $N[G * H]$ and $N[G * H]$ is a κ-model in $V[G * H]$, it follows that there is a filter $G' \in V[G * H]$ which is generic for $\mathbb{P}_{\kappa,j(\kappa)}$ over $N[G * H]$ and the embedding j lifts to $j : M[G] \to N[\hat{G}]$, where $\hat{G} \cong G * H * G'$.

Notice that $p = C(\kappa) \cup \{\kappa\} = \bigcup H \cup \{\kappa\} \in N[\hat{G}]$. Since $\kappa \in j(T) \subseteq j(S)$, we see that $N[\hat{G}] \models$ "p is a closed subset of $j(S)$". Thus, $p \in j(T^1(S))$. Since $j(T^1(S))$ is $j(\kappa)$-strategically closed in $N[\hat{G}]$ and $N[\hat{G}]$ is a κ-model in $V[G * H]$ by the ground closure criterion, there is a filter $\hat{H} \in V[G * H]$ generic for $j(T^1(S))$ over $N[\hat{G}]$ with $p \in \hat{H}$. Since p is below every condition in $j\,"H$, we have $j\,"H \subseteq \hat{H}$, and thus j lifts to $j : M[G * H] \to N[\hat{G} * \hat{H}]$, where $\kappa \in j(C(\kappa))$. By Theorem 3.7 and Corollary 3.8, $N[G * H]$ is a Π^1_{n-1}-correct κ-model in $V[G * H]$. Since $\mathbb{P}_{\kappa,j(\kappa)}$ and $j(T^1(S))$ are $(\kappa+1)$-strategically closed in $N[G * H]$, it follows that $N[G * H]$ and $N[\hat{G} * \hat{H}]$ have the same subsets of V_κ, so, in particular, $N[\hat{G} * \hat{H}]$ is a Π^1_{n-1}-correct κ-model in $V[G * H]$. Thus, by Theorem 2.5, we have verified that T remains Π^1_n-indescribable in $V[G * H]$.

It remains to show that $C(\kappa) \cap C \neq \varnothing$. Recall that C is a club subset of κ in $V[G * H]$, so $j(C)$ is a club subset of $j(\kappa)$ in $N[\hat{G} * \hat{H}]$. Since $j(C) \cap \kappa = C$, it follows that $\kappa \in j(C)$, and hence $\kappa \in j(C(\kappa) \cap C)$. By elementarity, $C(\kappa) \cap C \neq \varnothing$, so $C(\kappa)$ is a stationary and hence 1-club subset of κ in $V[G * H]$. □

Remark 4.4. In the proof of Theorem 4.3, for any $m \leq n$ we can force with $T^m(S \cap \gamma)$ at every relevant $\gamma \leq \kappa$ instead of $T^1(S \cap \gamma)$. This iteration will still preserve the Π^1_n-indescribability of every subset of S that is Π^1_n-indescribable in V, and it will shoot an m-club through S. If $m > 1$, then this forcing will have slightly better closure properties then $T^1(S \cap \gamma)$ (see Remark 4.2), which could be useful for certain applications, though we have not found any such applications as of yet.

5. $\square_1(\kappa)$ can hold nontrivially at a weakly compact cardinal

In this section, we will show that the principle $\square_1(\kappa)$ can hold at a weakly compact cardinal that has many weakly compact cardinals below it. First, we define a forcing to add a generic coherent sequence of 1-clubs to a Mahlo cardinal κ.

Definition 5.1. Suppose κ is a Mahlo cardinal. We define a forcing $\mathbb{Q}(\kappa)$ such that q is a condition in $\mathbb{Q}(\kappa)$ if and only if

- q is a sequence with $\mathrm{dom}(q) = \mathrm{inacc}(\kappa) \cap (\gamma^q + 1)$ for some $\gamma^q < \kappa$,
- $q(\alpha) = C^q_\alpha$ is a 1-club subset of α for each $\alpha \in \mathrm{dom}(q)$ and
- for all $\alpha, \beta \in \mathrm{dom}(q)$, if $C^q_\beta \cap \alpha \in \Pi^1_0(\alpha)^+$, then $C^q_\alpha = C^q_\beta \cap \alpha$.[2]

The ordering on $\mathbb{Q}(\kappa)$ is defined by letting $p \leq q$ if and only if p is an end extension of q.

Proposition 5.2. *Suppose κ is a Mahlo cardinal. The poset $\mathbb{Q}(\kappa)$ is κ-strategically closed.*

Proof. We describe a winning strategy for Player II in the game $\mathcal{G}_\kappa(\mathbb{Q}(\kappa))$. We will recursively arrange so that, if $\delta < \kappa$ and $\langle q_\alpha : \alpha < \delta \rangle$ is a partial play of the game with Player II playing according to her winning strategy, then, for all limit ordinals $\beta < \delta$, we have $\{\gamma^{q_\alpha} : \alpha < \beta,\ \alpha \text{ even}\}$ is a club in its supremum and, if γ^{q_β} is inaccessible, is a subset of $C^{q_\beta}_{\gamma^{q_\beta}}$. We will also arrange that, for all even successor ordinals $\alpha < \beta < \delta$, γ^{q_α} and γ^{q_β} are inaccessible cardinals, $C^{q_\beta}_{\gamma^{q_\beta}} \cap \gamma^{q_\alpha} = C^{q_\alpha}_{\gamma^{q_\alpha}}$ and $\{\gamma^{q_\alpha} : \alpha < \beta,\ \alpha \text{ even}\} \subseteq C^{q_\beta}_{\gamma^{q_\beta}}$.

We first deal with successor ordinals. Suppose that $\delta < \kappa$ is an even ordinal and $\langle q_\alpha : \alpha \leq \delta + 1 \rangle$ has been played. Suppose first that γ^{q_δ} is an inaccessible cardinal (in particular, by our recursion hypotheses, this must be the case if δ is a successor ordinal). In this case, let $\gamma^{q_{\delta+2}}$ be the least inaccessible cardinal above $\gamma^{q_{\delta+1}}$ and let $q_{\delta+2}$ be the condition extending $q_{\delta+1}$ by setting

$$C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}} = C^{q_\delta}_{\gamma^{q_\delta}} \cup \{\gamma^{q_\delta}\} \cup [\gamma^{q_{\delta+1}}, \gamma^{q_{\delta+2}}).$$

The fact that $C^{q_\delta}_{\gamma^{q_\delta}} \cup \{\gamma^{q_\delta}\} \subseteq C^{q_\delta+2}_{\gamma^{q_{\delta+2}}}$ ensures that the recursion hypothesis is maintained. The set $C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}}$ is stationary in $\gamma^{q_{\delta+2}}$ because it contains a tail, and it has

[2]Equivalently, for all $\alpha, \beta \in \mathrm{dom}(q)$, if α is inaccessible and $C^q_\beta \cap \alpha$ is stationary, then $C^q_\alpha = C^q_\beta \cap \alpha$.

all its inaccessible stationary reflection points because those are $\leq\gamma^{q_\delta}$. The coherence property holds because we have omitted the interval $(\gamma^{q_\delta},\gamma^{q_{\delta+1}})$ from $C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}}$ ensuring that for no α in that interval is $C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}}\cap\alpha$ stationary. It follows that $q_{\delta+2}$ is a condition and a valid play for Player II.

If γ^{q_δ} is not inaccessible, then δ is a limit ordinal (by our recursion hypothesis). In this case, again let $\gamma^{q_{\delta+2}}$ be the least inaccessible cardinal above $\gamma^{q_{\delta+1}}$, and define $q_{\delta+2}$ by setting

$$C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}} = \bigcup_{\substack{\alpha<\delta \\ \alpha \text{ even}}} C^{q_\delta}_{\gamma^{q_{\alpha+2}}} \cup \{\gamma^{q_\delta}\} \cup [\gamma^{q_{\delta+1}},\gamma^{q_{\delta+2}}).$$

A similar argument as above verifies that $q_{\delta+2}$ is a valid play in the game and maintains our recursion hypotheses.

Finally, suppose that $\delta<\kappa$ is a limit ordinal and $\langle q_\alpha : \alpha<\delta\rangle$ has been played. Let $\gamma^{q_\delta}=\sup\{\gamma^{q_\alpha} : \alpha<\delta\}$. If γ^{q_δ} is not inaccessible, then we can simply set $q_\delta=\bigcup_{\alpha<\delta}q_\alpha$. If γ^{q_δ} is inaccessible, then we must additionally define $C^{q_\delta}_{\gamma^{q_\delta}}$. We do this by setting

$$C^{q_\delta}_{\gamma^{q_\delta}} = \bigcup_{\substack{\alpha<\delta \\ \alpha \text{ even}}} C^{q_\delta}_{\gamma^{q_{\alpha+2}}}.$$

It is easy to verify that this is as desired. The fact that $C^{q_\delta}_{\gamma^{q_\delta}}$ is stationary in γ^{q_δ} follows from the fact that $\{\gamma^{q_\alpha} : \alpha<\delta,\ \alpha \text{ even}\}\subseteq C^{q_\delta}_{\gamma^{q_\delta}}$, so it in fact contains a club in γ^{q_δ}. $\square$

It follows from Proposition 5.2 that $\mathbb{Q}(\kappa)$ is $<\kappa$-distributive. In particular, if $G\subseteq\mathbb{Q}(\kappa)$ is a generic filter, then $\bigcup G$ is a coherent sequence of 1-clubs of length κ, since V_κ remains unchanged.

Next, we define a forcing which will be used to generically thread a coherent sequence of 1-clubs.

Definition 5.3. Suppose that $\vec{C}(\kappa)=\langle C_\alpha(\kappa) : \alpha\in\operatorname{inacc}(\kappa)\rangle$ is a coherent sequence of 1-clubs. The poset $\mathbb{T}(\vec{C}(\kappa))$ consists of all conditions t such that

- t is a 1-closed bounded subset of κ and
- for every $\alpha<\kappa$, if $t\cap\alpha\in\Pi^1_0(\alpha)^+$, then $C_\alpha(\kappa)=t\cap\alpha$.[3]

The ordering on $\mathbb{T}(\vec{C}(\kappa))$ is defined by letting $t\leq s$ if and only if t end-extends s.

Lemma 5.4. *Suppose κ is a regular cardinal and $\vec{C}(\kappa)$ is a coherent sequence of 1-clubs. Then the poset $\mathbb{T}(\vec{C}(\kappa))$ is κ-strategically closed.*[4]

Proof. We describe a winning strategy for player II in $\mathcal{G}_\kappa(\mathbb{T}(\vec{C}(\kappa)))$. Player II's strategy at successor ordinal stages can be arbitrary provided that Player II chooses conditions properly extending Player I's previous play.

So let δ be a limit stage and let $\langle t_\alpha : \alpha<\delta\rangle$ be the sequence of conditions played at previous stages of the game. Player II then plays $t_\delta=\left(\bigcup_{\alpha<\delta}t_\alpha\right)\cup\{\kappa_\delta+1\}$, where $\kappa_\delta=\sup\left(\bigcup_{\alpha<\delta}t_\alpha\right)$. We will also assume recursively that Player II has played according to this strategy successfully at previous limit stages of the game, so that, if $\lambda<\delta$ is a limit ordinal, then $\kappa_\lambda\notin t_\delta$. It remains to show that $t_\delta\in\mathbb{T}(\vec{C}(\kappa))$.

[3]Equivalently, for every inaccessible cardinal $\alpha<\kappa$, if $t\cap\alpha$ is stationary in α then $C_\alpha(\kappa)=t\cap\alpha$.

[4]Note that the forcing to thread a $\square(\kappa)$-sequence is never κ-strategically closed.

To argue that t_δ is a 1-closed subset of κ, it suffices to see that $t_\delta \cap \kappa_\delta$ is not stationary in κ_δ. By our recursive assumption, $\{\kappa_\lambda : \lambda < \delta\}$ is a club subset of κ_δ disjoint from t_δ, and hence $t_\delta \cap \kappa_\delta$ is not stationary in κ_δ. The coherence condition follows easily. □

Lemma 5.5. *Suppose κ is a regular cardinal and $\vec{C}(\kappa) = \langle C_\alpha(\kappa) : \alpha \in \operatorname{inacc}(\kappa)\rangle$ is a coherent sequence of 1-clubs. If $G \subseteq \mathbb{T}(\vec{C}(\kappa))$ is generic over V, then $C_\kappa = \bigcup G$ threads $\vec{C}(\kappa)$ in $V[G]$.*

Proof. By the $<\kappa$-distributivity of $\mathbb{T}(\vec{C}(\kappa))$ and the definition of its conditions, C_κ meets the coherence requirements and contains all its inaccessible stationary reflection points. So it remains to check that C_κ is stationary.

Fix a club $C \subseteq \kappa$ in $V[G]$ and let $\dot{C}$ be a $\mathbb{T}(\vec{C}(\kappa))$-name for C. Assume towards a contradiction that $C \cap C_\kappa = \varnothing$. Fix $t_0 \in \mathbb{T}(\vec{C}(\kappa))$ forcing that $\dot{C}$ is a club and $\dot{C} \cap \dot{C}_\kappa = \varnothing$, where $\dot{C}_\kappa$ is the canonical $\mathbb{T}(\vec{C}(\kappa))$-name for C_κ, and let β_0 be the supremum of t_0. Recursively define a decreasing sequence $\langle t_n : n < \omega\rangle$ of conditions from $\mathbb{T}(\vec{C}(\kappa))$ as follows, letting β_n denote $\sup(t_n)$. Given $n < \omega$, if t_n is defined, find an ordinal α_n with $\beta_n < \alpha_n < \kappa$ and a condition $t_{n+1} \leq t_n$ such that $t_{n+1} \Vdash \check{\alpha}_n \in \dot{C}$. Let $\alpha = \bigcup_{n<\omega} \alpha_n = \bigcup_{n<\omega} \beta_n$, and let $t = \bigcup_{n<\omega} t_n \cup \{\alpha\}$. Clearly t is a condition in $\mathbb{T}(\vec{C}(\kappa))$ and $t \Vdash \alpha \in \dot{C} \cap \dot{C}_\kappa$, which is the desired contradiction. □

Theorem 5.6. *Suppose κ is weakly compact and the* GCH *holds. There is a cofinality-preserving forcing extension in which*

(1) *for all $\gamma \leq \kappa$, every set $W \in P(\gamma)^V$ which is weakly compact in V remains weakly compact and*
(2) *$\square_1(\kappa)$ holds.*

Proof. Define an Easton-support iteration $\langle(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta) : \alpha \leq \kappa+1,\ \beta \leq \kappa\rangle$ as follows.

- If $\gamma < \kappa$ is Mahlo, let $\dot{\mathbb{Q}}_\gamma = (\mathbb{Q}(\gamma) * \dot{\mathbb{T}}(\vec{C}(\gamma)))^{V^{\mathbb{P}_\gamma}}$, where $\vec{C}(\gamma)$ is the generic coherent sequence of 1-clubs of length γ added by $\mathbb{Q}(\gamma)$.
- If $\gamma = \kappa$, let $\dot{\mathbb{Q}}_\kappa = (\mathbb{Q}(\gamma))^{V^{\mathbb{P}_\kappa}}$.
- Otherwise, let $\dot{\mathbb{Q}}_\gamma$ be a $\mathbb{P}_\gamma$-name for trivial forcing.

Let $G * H \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ be generic over V. $V[G*H]$ is our desired model. Standard arguments using progressive closure of the iteration $\mathbb{P}_\kappa$ together with the GCH show that cofinalities are preserved in $V[G*H]$.

The argument for the preservation of weakly compact subsets of $\gamma < \kappa$ is similar to and easier than the argument for the preservation of weakly compact subsets of κ, and we leave it to the reader.

Recall that $\vec{C}(\kappa) = \bigcup H$ is a coherent sequence of 1-clubs of length κ. Fix $W \in P(\kappa)^V$ which is weakly compact in V. It remains to argue that in $V[G*H]$, W is weakly compact and $\vec{C}(\kappa)$ has no thread.

Fix a set $C \in P(\kappa)^{V[G*H]}$ which is a 1-club subset of κ in $V[G*H]$. We will simultaneously show that C is not a thread through $\vec{C}(\kappa)$ and that W remains weakly compact in $V[G*H]$. Fix $A \in P(\kappa)^{V[G*H]}$ and let $\dot{C}, \dot{A}, \tau \in H(\kappa^+)^V$ be $\mathbb{P}_{\kappa+1}$-names with $\dot{C}_{G*H} = C$, $\dot{A}_{G*H} = A$ and $\tau_{G*H} = \vec{C}(\kappa)$. Let M be a κ-model with $W, \dot{C}, \dot{A}, \tau, \mathbb{P}_{\kappa+1} \in M$. Since W is weakly compact in V, there is a κ-model N and an elementary embedding $j : M \to N$ such that $\operatorname{crit}(j) = \kappa$ and $\kappa \in j(W)$.

Since $N^{<\kappa} \cap V \subseteq N$, we have, in N,

$$j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * (\dot{\mathbb{Q}}(\kappa) * \dot{\mathbb{T}}(\vec{C}(\kappa))) * \dot{\mathbb{P}}_{\kappa,j(\kappa)},$$

where $\dot{\mathbb{P}}_{\kappa,j(\kappa)}$ is a $\mathbb{P}_{\kappa+1} * \dot{\mathbb{T}}(\vec{C}(\kappa))$-name for the iteration from $\kappa+1$ to $j(\kappa)$. By Lemma 5.4, $\mathbb{T}(\vec{C}(\kappa))$ is κ-strategically closed in $N[G*H]$, and hence, using standard arguments, we can build a filter $h \in V[G*H]$ for $\mathbb{T}(\vec{C}(\kappa))$ which is generic over $N[G*H]$. Let $C_\kappa = \bigcup h$ and notice that $C_\kappa \neq C$ because $C \in N[G*H]$ and C_κ is generic over $N[G*H]$. Similarly, we can build a filter $G' \in V[G*H]$ which is generic for $\mathbb{P}_{\kappa,j(\kappa)} = (\dot{\mathbb{P}}_{\kappa,j(\kappa)})_{G*H*h}$ over $N[G*H*h]$. Since $j\,``\,G \subseteq G*H*h*G'$, the embedding can be extended to $j: M[G] \to N[\hat{G}]$, where $\hat{G} = G*H*h*G'$.

Let $\mathbb{Q}(\kappa) = (\dot{\mathbb{Q}}(\kappa))_G$. Working in $N[\hat{G}]$, since

$$\vec{C}(\kappa) = \bigcup H = \langle C_\alpha(\kappa) : \alpha \in \operatorname{inacc}(\kappa)\rangle$$

is a coherent sequence of 1-clubs and C_κ is a thread through $\vec{C}(\kappa)$ by Lemma 5.5, it follows that the function

$$q = \langle C_\alpha(\kappa) : \alpha \in \operatorname{inacc}(\kappa)\rangle \cup \{(\kappa, C_\kappa)\}$$

is a condition in $j(\mathbb{Q}(\kappa))$ below every element of $j\,``\,H$. We may build a filter $\hat{H} \in V[G*H]$ which is generic for $j(\mathbb{Q}(\kappa))$ over $N[\hat{G}]$ with $q \in \hat{H}$. Since $j\,``\,H \subseteq \hat{H}$, it follows that j extends to $j: M[G*H] \to N[\hat{G}*\hat{H}]$. Now $A \in M[G*H]$ and $\kappa \in j(W)$, so W is weakly compact in $V[G*H]$.

It remains to show that C is not a thread through $\vec{C}(\kappa)$. For the sake of contradiction, assume C is a thread through $\vec{C}(\kappa)$. By elementarity we see that in $N[\hat{G}*\hat{H}]$,

$$j(\vec{C}(\kappa)) = \langle \bar{C}_\alpha(j(\kappa)) : \alpha \in \operatorname{inacc}(j(\kappa))\rangle$$

is a coherent sequence of 1-clubs. Since $q = \vec{C}(\kappa) ^\frown \langle C_\kappa\rangle \in \hat{H}$ we have $\bar{C}_\kappa(j(\kappa)) = C_\kappa$. Now since C is a thread for $\vec{C}(\kappa)$ in in $M[G*H]$, by elementarity, $j(C)$ is a thread for $j(\vec{C}(\kappa)) = \langle \bar{C}_\alpha(j(\kappa)) : \alpha \in \operatorname{inacc}(j(\kappa))\rangle$. Since κ is inaccessible in $N[\hat{G}*\hat{H}]$ and $\kappa \in \operatorname{Tr}_0(j(C))$, it follows that $C_\kappa = \bar{C}_\kappa(j(\kappa)) = j(C) \cap \kappa = C$, a contradiction. $\square$

Remark 5.7. Observe that in the proof of Theorem 5.6, if we assume that κ is Π^1_2-indescribable and that the target N of the embedding $j: M \to N$ we start with is Π^1_1-correct, then the κ-model $N[G*H]$ from the proof of Theorem 5.6 is Π^1_1-correct by Theorem 3.7 and Corollary 3.8. However, the κ-model $N[G*H*h]$ cannot be Π^1_1-correct because otherwise we would have shown that, in the extension $V[G*H]$, κ is Π^1_2-indescribable, contradicting Proposition 2.8. Thus, a forcing extension of a Π^1_1-correct κ-model, even by a κ-strategically closed forcing notion, need not be Π^1_1-correct if the generic filter is not fully V-generic.

For the next theorem, let us recall what it means for a cardinal κ to be α-weakly compact, where $\alpha \leq \kappa^+$. Suppose κ is a weakly compact cardinal. It is not difficult to see that if sets $X, Y \in P(\kappa)$ are equivalent modulo the ideal $\Pi^1_1(\kappa)$, then their traces $\operatorname{Tr}_1(X)$ and $\operatorname{Tr}_1(Y)$ are equivalent as well. Thus, the trace operation $\operatorname{Tr}_1 : P(\kappa) \to P(\kappa)$ leads to a well defined operation $\operatorname{Tr}_1 : P(\kappa)/\Pi^1_1(\kappa) \to P(\kappa)/\Pi^1_1(\kappa)$ on the collection $P(\kappa)/\Pi^1_1(\kappa)$ of equivalence classes of subsets of κ modulo the ideal $\Pi^1_1(\kappa)$. By taking diagonal intersections at limit ordinals, we can iterate the trace operation on the equivalence classes κ^+-many times. To be more precise, fix a sequence $\langle e_\beta \mid \kappa \leq \beta < \kappa^+,\ \beta \text{ limit}\rangle$, where $e_\beta : \kappa \to \beta$ is a bijection for all relevant β. To start, let $\mathrm{Tr}_1^1 = \mathrm{Tr}_1$. Given $\alpha < \kappa^+$, if $\mathrm{Tr}_1^\alpha : P(\kappa)/\Pi_1^1(\kappa) \to P(\kappa)/\Pi_1^1(\kappa)$ has been defined, let $\mathrm{Tr}_1^{\alpha+1} = \mathrm{Tr}_1 \circ \mathrm{Tr}_1^\alpha$. If $\beta < \kappa$ is a limit ordinal and Tr_1^α has been defined for all $\alpha < \beta$, then define Tr_1^β by letting $\mathrm{Tr}_1^\beta([S]) = [\bigcap_{\alpha<\beta} S_\alpha]$, where S_α is a representative element of $\mathrm{Tr}_1^\alpha([S])$ for all $\alpha < \beta$. Finally, if β is a limit ordinal and $\kappa \leq \beta < \kappa^+$, then let $\mathrm{Tr}_1^\beta([S]) = [\triangle_{\eta<\kappa} S_{e_\beta(\eta)}]$.

It is straightforward to verify that each of these functions is well-defined and does not depend on our choice of e_β for limit β. For $\alpha < \kappa^+$, the cardinal κ is then said to be *α-weakly compact* if $\mathrm{Tr}_1^\alpha([\kappa]) \neq [\varnothing]$, and κ is *κ^+-weakly compact* if it is α-weakly compact for all $\alpha < \kappa^+$. For more details, the reader is referred to [Cod].

If we start with a κ^+-weakly compact cardinal κ in Theorem 5.6, then it will remain κ^+-weakly compact in the extension $V[G * H]$. Because weakly compact subsets of all cardinals $\gamma \leq \kappa$ are preserved to $V[G * H]$, it is easy to show by induction on $\alpha \leq \kappa^+$ that if a set X is in the equivalence class $\mathrm{Tr}^\alpha([\kappa])$ as computed in V, then the equivalence class $\mathrm{Tr}^\alpha([\kappa])$ as computed in $V[G * H]$ contains some $Y \supseteq X$. Thus, we get the following.

Theorem 1.1. *If κ is κ^+-weakly compact and* GCH *holds then there is a cofinality preserving forcing extension in which*

(1) *κ remains κ^+-weakly compact and*
(2) *$\square_1(\kappa)$ holds.*

Next we will show that, if κ is Mahlo, one can characterize precisely when $\square_1(\kappa)$ holds after forcing with $\mathbb{Q}(\kappa)$. Notice that, if there is a stationary subset of κ that does not reflect at an inaccessible cardinal (i.e., if $\mathrm{Refl}_0(\kappa)$) fails, then $\square_1(\kappa)$ must fail, since any such non-reflecting stationary set of κ would then be a thread through any coherent sequence of 1-clubs of length κ. We will see in Theorem 5.9 that $\mathrm{Refl}_0(\kappa)$ holding in the extension by $\mathbb{Q}(\kappa)$ is in fact sufficient for $\square_1(\kappa)$ to hold. First, we need the following general proposition. Recall that $\mathrm{Refl}_n(\kappa)$ holds if and only if κ is Π^1_n-indescribable and, for every Π^1_n-indescribable subset S of κ, there is an $\alpha < \kappa$ such that $S \cap \alpha$ is Π^1_n-indescribable.

Proposition 5.8. *Fix $n < \omega$. If κ is a cardinal,* $\mathrm{Refl}_n(\kappa)$ *holds and $S \in \Pi^1_n(\kappa)^+$, then the set*

$$T = \{\alpha < \kappa : (S \cap \alpha \in \Pi^1_n(\alpha)^+) \wedge (\mathrm{Refl}_n(\alpha) \textit{ fails})\}$$

is Π^1_n-indescribable.

Proof. We proceed by induction on κ. Suppose the proposition holds for all cardinals $\alpha < \kappa$, $\mathrm{Refl}_n(\kappa)$ holds and $S \in \Pi^1_n(\kappa)^+$. It suffices to show that $T \cap C \neq \varnothing$ for every n-club subset C of κ. Fix an n-club set C and note that $S \cap C$ is Π^1_n-indescribable. Thus, by $\mathrm{Refl}_n(\kappa)$, there is some $\alpha_0 < \kappa$ such that $S \cap C \cap \alpha_0 \in \Pi^1_n(\alpha_0)^+$. It follows that $\alpha_0 \in \mathrm{Tr}_n(S)$, but also $\alpha_0 \in C$ because C contains all of its Π^1_{n-1}-reflection points. If $\alpha_0 \in T$, we have shown that $T \cap C \neq \varnothing$. So suppose that $\alpha_0 \notin T$, so $\mathrm{Refl}_n(\alpha_0)$ holds. We can now appeal to the inductive hypothesis at α_0, applied to the Π^1_n-indescribable set $S \cap \alpha_0$ and the n-club $C \cap \alpha_0$, to find a cardinal $\alpha_1 \in T \cap C$. □

Theorem 5.9. *Suppose κ is Mahlo and $p \in \mathbb{Q}(\kappa)$. The following are equivalent:*

(1) $p \Vdash_{\mathbb{Q}(\kappa)} \mathrm{Refl}_0(\kappa)$
(2) $p \Vdash_{\mathbb{Q}(\kappa)} \square_1(\kappa)$

Proof. The implication (2) $\Rightarrow$ (1) follows immediately from the observation that a stationary subset of κ that does not reflect at any inaccessible cardinal is a thread through any putative $\square_1(\kappa)$-sequence.

We now show (1) $\Rightarrow$ (2). Suppose for the sake of contradiction that $p \Vdash_{\mathbb{Q}(\kappa)} \mathrm{Refl}_0(\kappa)$ and there is $p_1 \leq_{\mathbb{Q}(\kappa)} p$ such that $p_1 \Vdash_{\mathbb{Q}(\kappa)} \neg\square_1(\kappa)$. In particular, p_1 forces that $\bigcup \dot{G}$ is not a $\square_1(\kappa)$-sequence, so there is a $\mathbb{Q}(\kappa)$-name $\dot{C}$ that is forced by p_1 to be a thread through $\bigcup \dot{G}$.

Let G be $\mathbb{Q}(\kappa)$-generic over V with $p_1 \in G$, and move to $V[G]$. Let $C = \dot{C}_G$. Since C is stationary in κ and $\mathrm{Refl}_0(\kappa)$ holds, Proposition 5.8 implies that there are stationarily many inaccessible $\lambda < \kappa$ such that C reflects at λ and $\mathrm{Refl}_0(\lambda)$ fails.

Next, observe that every sequence of elements of G of size less than κ has a lower bound in G. Suppose that $\beta < \kappa$, and fix in $V[G]$ a sequence $\vec{p} = \langle p_\xi : \xi < \beta\rangle$ of elements of G. The sequence $\vec{p}$ must be in V by the $<\kappa$-distributivity of $\mathbb{Q}(\kappa)$, and so there is a condition $p \in G$ forcing that $\vec{p}$ is contained in G. But then p is a lower bound for $\vec{p}$. Observe also that, for all $\gamma < \kappa$, the initial segment $C^{(\gamma)} = C \cap \gamma$ of C is in V.

Now, in $V[G]$, we build a strictly decreasing sequence of conditions $\langle q_\alpha : \alpha < \kappa\rangle$ from G such that

(1) $q_0 = p_1$,
(2) $\{\gamma^{q_\alpha} : \alpha < \kappa\}$, the set of suprema of the domains of the conditions, is a club and
(3) for all $\alpha < \kappa$, $q_{\alpha+1} \Vdash_{\mathbb{Q}(\kappa)} \dot{C} \cap \gamma^{q_\alpha} = \check{C}^{(\alpha)}$.

We can ensure that (2) holds as follows. At a limit stage $\lambda < \kappa$, given that we have already constructed $\langle q_\alpha : \alpha < \lambda\rangle$, we know that there is some $q \in G$ below our sequence. So we let $\gamma_\lambda = \bigcup_{\alpha<\lambda} \gamma_\alpha$ and take $q_\lambda = q \restriction \gamma_\lambda + 1$.

Thus, we can find an inaccessible cardinal λ such that $\lambda = \gamma^{q_\lambda}$, C reflects at λ, and $\mathrm{Refl}_0(\lambda)$ fails. Since $\mathrm{Refl}_0(\lambda)$ fails (in $V[G]$ and hence also in V, since forcing with $\mathbb{Q}(\kappa)$ did not add any bounded subsets to κ), we can fix in V a stationary $C_\lambda \subseteq \lambda$ that is different from $C^{(\lambda)} = C \cap \lambda$ and that does not reflect at any inaccessible cardinal below λ. Now form a condition $q^*_\lambda \in \mathbb{Q}$ with $\gamma^{q^*_\lambda} = \gamma^{q_\lambda} = \lambda$ by letting $q^*_\lambda \restriction \lambda = \bigcup_{\alpha<\lambda} q_\alpha$ and $C^{q^*_\lambda}_\lambda = C_\lambda$. This is easily seen to be a valid condition, because everything needed to construct it is in V and since C_λ does not reflect at any inaccessible cardinal. Since $q^*_\lambda \leq_{\mathbb{Q}(\kappa)} q_\alpha$ for all $\alpha < \lambda$, we have

$$q^*_\lambda \Vdash_{\mathbb{Q}(\kappa)} \dot{C} \cap \lambda = \check{C}^{(\lambda)}.$$

In particular, since $C^{(\gamma)} = C \cap \lambda$ is stationary in λ, and since q^*_λ extends p_1 and thus forces that $\dot{C}$ is a thread through $\bigcup \dot{G}$, it must be the case that q^*_λ forces that the λ-th entry in $\bigcup \dot{G}$ is $C^{(\lambda)}$. However, q^*_λ forces the λ-th entry in $\bigcup \dot{G}$ to be C_λ, which is different from $C \cap \lambda$. This gives the desired contradiction. $\square$

Remark 5.10. Since the weak compactness of κ implies $\mathrm{Refl}_0(\kappa)$, by Theorem 5.9 it follows that in the proof of Theorem 5.6, in order to show that $\square_1(\kappa)$ holds in $V[G * H]$ it suffices to show that κ remains weakly compact.

6. Consistency of $\square_1(\kappa)$ with $\mathrm{Refl}_1(\kappa)$

In this section, we will show that the principle $\square_1(\kappa)$ is consistent with $\mathrm{Refl}_1(\kappa)$. First, we will need a lemma showing that we can force the existence of a fast function while preserving Π^1_2-indescribability.

The fast function forcing $\mathbb{F}_\kappa$, introduced by Woodin, consists of conditions that are partial functions $p : \kappa \to \kappa$ such that for every $\gamma \in \mathrm{dom}(p)$, the following conditions hold:

- γ is inaccessible,
- $p\," \gamma \subseteq \gamma$, and
- $|p \upharpoonright \gamma| < \gamma$.

The union $f : \kappa \to \kappa$ of a generic filter for $\mathbb{F}_\kappa$ is called a *fast function*. Let $\mathbb{F}_{[\gamma,\kappa)}$ denote the subset of $\mathbb{F}_\kappa$ consisting of conditions p with $\mathrm{dom}(p) \subseteq [\gamma, \kappa)$ and observe that $\mathbb{F}_{[\gamma,\kappa)}$ is $\leq\gamma$-closed. It is not difficult to see that for any condition $p \in \mathbb{F}_\kappa$ and $\gamma \in \mathrm{dom}(p)$, the forcing $\mathbb{F}_\kappa$ factors below p as

$$\mathbb{F}_\gamma \upharpoonright p \cong \mathbb{F}_\kappa \upharpoonright (p \upharpoonright \gamma) \times \mathbb{F}_{[\gamma,\kappa)} \upharpoonright (p \upharpoonright [\gamma, \kappa)).$$

Lemma 6.1. *Suppose κ is Π^1_n-indescribable. In a generic extension $V[f]$ by fast function forcing, κ remains Π^1_n-indescribable and the fast function f has the following property. For every $A \in H(\kappa^+)$ and $\alpha < \kappa^+$, there are a κ-model M with $f, A \in M$, a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j(f)(\kappa) = \alpha$ and $j, M \in N$.*

Proof. The cardinal κ remains inaccessible in $V[f]$ because for unboundedly many inaccessible $\alpha < \kappa$, there is a condition $p \in G$ with $\alpha \in \mathrm{dom}\, p$, so $\mathbb{F}_\kappa$ below p factors with a first factor of size α and a second factor that is $\leq\alpha$-closed.

Fix $A \in H(\kappa^+)^{V[f]}$ and $\alpha < \kappa^+$ (note that V and $V[f]$ have the same κ^+). Let $\dot{A}$ be an $\mathbb{F}_\kappa$-name for A and let $B \subseteq \kappa$ code α. By Theorem 2.5 (4), there are a κ-model M with $\mathbb{F}_\kappa, \dot{A}, B \in M$, a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j, M \in N$. We will lift j to $M[G]$. Let $p = \langle \kappa, \alpha \rangle$ be a condition in $j(\mathbb{F}_\kappa)$. Below p, $j(\mathbb{F}_\kappa)$ factors as $j(\mathbb{F}_\kappa) \upharpoonright p \cong \mathbb{F}_\kappa \times \mathbb{F}_{[\kappa,j(\kappa))} \upharpoonright p$, where the second factor is $\leq\kappa$-closed in N. In V, we can build an N-generic function f' for $\mathbb{F}_{[\kappa,j(\kappa))}$ containing p, and so $f \times f'$ is N-generic for $j(\mathbb{F}_\kappa)$. Thus, we can lift j to $j : M[f] \to N[f][f']$, and clearly $M[f]$ and j are in $N[f][f']$.

It remains to verify that $M[f]$ is a κ-model and $N[f][f']$ is a Π^1_{n-1}-correct κ-model. The argument to show that $M[f]$ is a κ-model in $V[f]$ will be more involved than usual because, as $\mathbb{F}_\kappa$ is not κ-c.c., we cannot apply the generic closure criterion. Fixing $\beta < \kappa$, we will show that $M[f]^\beta \subseteq M[f]$ in $V[f]$. By density, there is an inaccessible cardinal $\alpha > \beta$ and a condition $p = \langle \{\gamma, \delta\} \rangle \in G$ such that $\gamma < \alpha < \delta$. Below p, $\mathbb{F}_\kappa$ factors as $\mathbb{F}_\gamma \times \mathbb{F}_{(\delta,\kappa)}$ and f factors as $f_\gamma \times f_{(\delta,\kappa)}$. Since $\mathbb{F}_\gamma$ clearly has the α-c.c., by the generic closure criterion, $M[f_\gamma]^\beta \subseteq M[f_\gamma]$ in $V[f_\gamma]$. Also, since $\mathbb{F}_{(\delta,\kappa)}$ is $\leq\alpha$-closed, $M[f_\gamma]^\beta \subseteq M[f_\gamma]$ in $V[f]$. Finally, by the ground closure criterion, $M[f_\gamma][f_{(\delta,\kappa)}]^\beta \subseteq M[f_\gamma][f_{(\delta,\kappa)}]$ in $V[f]$. The same argument shows that $N[f]$ is a κ-model in $V[f]$, and therefore, $N[f][f']$ is a κ-model as well. To show that $N[f]$ is Π^1_{n-1}-correct, we argue essentially as in the proof of Theorem 3.7. The arguments in that proof go through noting only that $V_\kappa^{V[f]} = V_\kappa^{N[f]} = V_\kappa[f]$ and $\mathbb{F}_\kappa \subseteq V_\kappa$. Finally, the model $N[f][f']$ must also be Π^1_{n-1}-correct because the tail forcing $\mathbb{F}_{[\kappa,j(\kappa))}$ does not add any subsets to $V_\kappa[f]$ by closure. □

It is not difficult to see that once we have a fast function, we also get a weak Laver function [Ham02].

Lemma 6.2. *Suppose κ is Π^1_n-indescribable. In the generic extension $V[f]$ by fast function forcing, there is a function $\ell : \kappa \to V_\kappa$ satisfying the following property. For all $A, B \in H(\kappa^+)^{V[f]}$, there are a κ-model M with $\ell, A, B \in M$, a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\ell)(\kappa) = B$ and $j, M \in N$.*

Proof. Fix any bijection $b : \kappa \to V_\kappa$ in V. In $V[f]$, define $\ell : \kappa \to V_\kappa$ by letting $\ell(\gamma) = b(f(\gamma))_{f\restriction\gamma}$ provided that $f \restriction \gamma$ is $\mathbb{F}_\gamma$-generic over V and $b(f(\gamma))$ is an $\mathbb{F}_\gamma$-name. By adapting the proof of Lemma 6.1, we verify that ℓ has the desired properties as follows. Working in $V[f]$, fix $A, B \in H(\kappa^+)^{V[f]}$ and let $\dot{\ell}, \dot{A}, \dot{B}$ be nice $\mathbb{F}_\kappa$-names for ℓ, A and B respectively. By Theorem 2.5 (4), there are a κ-model M with $\dot{\ell}, \dot{A}, \dot{B}, \mathbb{F}_\kappa, b \in M$, a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j, M \in N$. Since $\mathbb{F}_\kappa$ is κ^+-c.c., we can assume without loss of generality that $\dot{B} \in j(V_\kappa)$. By elementarity $j(b) : j(\kappa) \to j(V_\kappa)$ is a bijection, and thus there is some ordinal $\alpha < j(\kappa)$ such that $j(b)(\alpha) = \dot{B}$. As in the proof of Lemma 6.1, we may lift j to $j : M[f] \to N[f][f']$ such that $j(f)(\kappa) = \alpha$. Now we have $j(\ell)(\kappa) = j(b)(j(f)(\kappa))_{j(f)\restriction\kappa} = j(b)(\alpha)_f = \dot{B}_f = B$. Now one may prove that $M[f]$ is a κ-model and $N[f][f']$ is a Π^1_{n-1}-correct κ-model exactly as in the proof of Lemma 6.1. $\square$

Theorem 1.2. *Suppose κ is Π^1_2-indescribable and* GCH *holds. Then there is a cofinality-preserving forcing extension $V[G]$ in which*

(1) *$\square_1(\kappa)$ holds,*
(2) *$\mathrm{Refl}_1(\kappa)$ holds and*
(3) *κ is κ^+-weakly compact.*

Proof. By passing to an extension with a fast function, we can assume without loss of generality that there is a function $\ell : \kappa \to V_\kappa$ such that for any $A, B \in H(\kappa^+)$ there are a κ-model M with $\ell, A, B \in M$, a Π^1_1-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\ell)(\kappa) = B$.

Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa,\ \beta < \kappa \rangle$ be the Easton-support iteration defined as follows.

- If $\alpha < \kappa$ is inaccessible and $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for an α-strategically closed, α^+-c.c. forcing notion, then $\dot{\mathbb{Q}}_\alpha = \ell(\alpha)$.
- Otherwise, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for trivial forcing.

Let G be generic for $\mathbb{P}_\kappa$ over V. In $V[G]$, we define a 2-step iteration

$$\mathbb{Q}_\kappa = \mathbb{Q}_{\kappa,0} * (\dot{\mathbb{Q}}_{\kappa,1} \times \dot{\mathbb{Q}}_{\kappa,2})$$

as follows.

- $\mathbb{Q}_{\kappa,0}$ is the forcing to add a $\square_1(\kappa)$-sequence from Definition 5.1.
- $\dot{\mathbb{Q}}_{\kappa,2}$ is a $\mathbb{Q}_{\kappa,0}$-name for the forcing $\mathbb{T}(\vec{C}(\kappa))$ to thread the generic $\square_1(\kappa)$-sequence.
- $\dot{\mathbb{Q}}_{\kappa,1}$ is a $\mathbb{Q}_{\kappa,0}$-name for an iteration $\langle \mathbb{R}_\eta, \dot{\mathbb{S}}_\xi : \eta \leq \kappa^+,\ \xi < \kappa^+ \rangle$ with supports of size $<\kappa$ defined as follows. For each $\eta < \kappa^+$, a $\mathbb{Q}_{\kappa,0} * \dot{\mathbb{R}}_\eta$-name $\dot{S}_\eta$ is chosen for a stationary subset of κ such that

  $$\Vdash_{\mathbb{Q}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})} \text{“there is a 1-club in } \kappa \text{ disjoint from } \dot{S}_\eta\text{”},$$

  and then $\dot{\mathbb{S}}_\eta$ is a $\mathbb{Q}_{\kappa,0} * \dot{\mathbb{R}}_\eta$-name for the forcing $T^1(\kappa \setminus \dot{S}_\eta)$ to shoot a 1-club through the complement of $\dot{S}_\eta$.

Notice that $\mathbb{P}_\kappa$ is κ-c.c. and preserves GCH and, in $V[G]$, the forcing $\mathbb{Q}_\kappa$ is κ-strategically closed and κ^+-c.c.. By standard chain condition arguments and bookkeeping, we can ensure that in $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * \dot{\mathbb{Q}}_{\kappa,1}}$, if $S \subseteq \kappa$ is stationary and

$$\Vdash_{\mathbb{Q}_{\kappa,2}} \text{“there is a 1-club in } \kappa \text{ disjoint from } \check{S}\text{”},$$

then there is already a 1-club in κ disjoint from S.

Let $H = h_0 * (h_1 \times h_2)$ be generic for $\mathbb{Q}_\kappa$ over $V[G]$. Our desired model will be $V[G * h_0 * h_1]$. We must show that in $V[G * h_0 * h_1]$, κ is κ^+-weakly compact, $\mathrm{Refl}_1(\kappa)$ holds and $\square_1(\kappa)$ holds.

In order to show that κ is κ^+-weakly compact in $V[G * h_0 * h_1]$, we will first prove the following.

Claim 6.3. *κ is Π^1_2-indescribable in $V[G * H]$.*

Proof. Fix $A \in P(\kappa)^{V[G*H]}$. We must find a κ-model M with $A \in M$, a Π^1_1-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ.

Let $\dot{A} \in V$ be a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$-name for A. Since $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{Q}}_{\kappa,1} \times \dot{\mathbb{Q}}_{\kappa,2})$ has the κ^+-c.c., we can fix $\eta < \kappa^+$ such that $\dot{A}$ is a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$-name. Moreover, we can assume that $\dot{A}$, $\mathbb{P}_\kappa$ and $\dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$ are in $H(\kappa^+)$. For $\eta < \kappa^+$, let $h_1 \upharpoonright \eta$ be the generic for $\mathbb{R}_\eta$ induced by h_1.

By Proposition 6.2, there are a κ-model M with $\ell, \mathbb{P}_\kappa, \dot{A}, \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2}) \in M$, a Π^1_1-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$ and $j, M \in N$. Without loss of generality we may additionally assume that $M \models |\eta| = \kappa$ since a bijection witnessing this can easily be placed into such a κ-model.

Notice that $j(\mathbb{P}_\kappa)$ is an Easton-support iteration in N of length $j(\kappa)$ and

$$j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * (\dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})) * \dot{\mathbb{P}}_{\kappa,j(\kappa)}$$

by our choice of $j(l)(\kappa)$. By Theorem 3.7, $N[G]$ is Π^1_1-correct in $V[G]$, and by Corollary 3.8, $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$ is Π^1_1-correct in $V[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$. Hence $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$ is Π^1_1-correct in $V[G * h_0 * (h_1 \times h_2)]$ by Corollary 3.4.

Since $(\dot{\mathbb{P}}_{\kappa,j(\kappa)})_{G * h_0 * (h_1 \upharpoonright \eta \times h_2)} = \mathbb{P}_{\kappa,j(\kappa)}$ is κ-strategically closed in $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$ and since $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$ is a κ-model in $V[G * H]$, we can build a filter $G'_{\kappa,j(\kappa)}$ which is generic for $\mathbb{P}_{\kappa,j(\kappa}$ over $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$. Since $j \upharpoonright G$ is the identity function, it follows that $j\,``\,G \subseteq \hat{G} =_{\mathrm{def}} G * h_0 * (h_1 \upharpoonright \eta \times h_2) * G_{\kappa,j(\kappa)}$, and thus j lifts to $j : M[G] \to N[\hat{G}]$.

Let $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{inacc}(\kappa) \rangle$ be the generic $\square_1(\kappa)$-sequence added by h_0, and let T be the thread added by $h_2 \subseteq \mathbb{Q}_{\kappa,2} = \mathbb{T}(\vec{C}(\kappa))$. By Lemma 5.5, T is a 1-club in $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$. Let $p_0 = \vec{C} \cup \{(\kappa, T)\}$. Then $p_0 \in N[\hat{G}]$ and $p_0 \in j(\mathbb{Q}_{\kappa,0})$. Moreover, $p_0 \leq_{j(\mathbb{Q}_{\kappa,0})} j(q)$ for all $q \in h_0$. By the strategic closure of $j(\mathbb{Q}_{\kappa,0})$ and the fact that $N[\hat{G}]$ is a κ-model in $V[G * H]$, we can build a filter $p_0 \in \hat{h}_0 \subseteq j(\mathbb{Q}_{\kappa,0})$ which is generic over $N[\hat{G}]$. Thus, j extends to $j : M[G * h_0] \to N[\hat{G} * \hat{h}_0]$.

Similarly, in $N[\hat{G} * \hat{h}_0]$, the set $p_2 = T \cup \{\kappa\}$ is a condition in $j(\mathbb{Q}_{\kappa,2})$ and $p_2 \leq_{j(\mathbb{Q}_{\kappa,2})} j(q)$ for all $q \in h_2$. Again, since $j(\mathbb{Q}_{\kappa,2})$ is κ-strategically closed in $N[\hat{G} * \hat{h}_0]$, which is a κ-model in $V[G * H]$, we can build a filter $p_2 \in \hat{h}_2 \subseteq j(\mathbb{Q}_{\kappa,2})$ which is generic for $j(\mathbb{Q}_{\kappa,2})$ over $N[\hat{G} * \hat{h}_0]$, and lift j to

$$j : M[G * h_0 * h_2] \to N[\hat{G} * \hat{h}_0 * \hat{h}_2].$$

Now we lift the embedding through $h_1 \restriction \eta$. Let $\mathbb{R}_\eta = (\dot{\mathbb{R}}_\eta)_{G*h_0}$. By elementarity, $j(\mathbb{R}_\eta)$ is an iteration of length $j(\eta)$ with supports of size less than $j(\kappa)$. For each $\xi < \eta$, $\dot{\mathbb{S}}_{j(\xi)} = j(\dot{\mathbb{S}}_\xi)$ is, in $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$, a $j(\mathbb{R}_\xi) = \mathbb{R}_{j(\xi)}$-name for the forcing to shoot a 1-club disjoint from $j(\dot{S}_\xi)$. For all $\xi < \eta$, let

$$D_\xi = \bigcup\{(p(\xi))_{h_1 \restriction \xi} : p \in h_1 \text{ and } \xi \in \mathrm{dom}(p)\}$$

and note that D_ξ is a 1-club subset of κ in $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ because the forcing after $\mathbb{R}_{\xi+1}$ is κ-strategically closed and therefore cannot affect Π^1_1-truths by Lemma 3.3. Since $h_1 \restriction \eta, j \in N[\hat{G} * \hat{h}_0 * \hat{h}_2]$, we can define a function $p^* \in N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ such that $\mathrm{dom}(p^*) = j\text{ " }\eta$ by letting $p^*(j(\xi))$ be a $j(\mathbb{R}_\xi)$-name for $D_\xi \cup \{\kappa\}$ for all $\xi < \eta$. In order to verify that $p^* \in j(\mathbb{R}_\eta)$, we must show that for all $\xi < \eta$, $p^* \restriction j(\xi) \Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) = \varnothing$.

Suppose this is not the case, and let $\xi < \eta$ be the minimal counterexample. It follows that $p^* \restriction j(\xi) \in j(\mathbb{R}_\xi)$ and, for all $p \in h_1 \restriction \xi$ we have $p^* \restriction j(\xi) \leq j(p)$. By assumption,

$$p^* \restriction j(\xi) \not\Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) = \varnothing$$

and thus we may let $p^{**} \leq_{j(\mathbb{R}_\xi)} p^* \restriction j(\xi)$ be such that

$$p^{**} \Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) \neq \varnothing.$$

Since $j(\mathbb{R}_\xi)$ is sufficiently strategically closed, we can build a filter $\hat{h}_1 \subseteq j(\mathbb{R}_\xi)$ in $V[G * H]$ which is generic over $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ with $p^{**} \in \hat{h}_1$ and lift to

$$j : M[G * h_0 * h_2 * (h_1 \restriction \xi)] \to N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1].$$

It follows that in $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$ we have $(D_\xi \cup \{\kappa\}) \cap j(S_\xi) \neq \varnothing$, where $S_\xi = (\dot{S}_\xi)_{h_0 \restriction \xi}$. Since $j(S_\xi) \cap \kappa = S_\xi$, we know that $D_\xi \cap j(S_\xi) = \varnothing$, so it must be the case that $\kappa \in j(S_\xi)$. However, in $M[G * h_0 * (h_1 \restriction \xi)]$, we have

$$\Vdash_{\mathbb{Q}_{\kappa,2}} \text{"there is a 1-club in } \kappa \text{ disjoint from } \check{S}_\xi\text{"}.$$

Therefore, we can fix such a 1-club E in $M[G * h_0 * h_2 * (h_1 \restriction \xi)]$. Note that E is actually stationary because $M[G * h_0 * h_2 * (h_1 \restriction \xi)]$ is Π^1_1-correct by Theorem 3.7 and Corollary 3.8. But then $\kappa \in j(E)$ since in $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$, $j(E)$ is 1-club in $j(\kappa)$ and $j(E) \cap \kappa = E$ is stationary in κ. Thus $\kappa \in j(E) \cap j(S_\xi) = \varnothing$, a contradiction.

Thus, $p^* \in j(\mathbb{R}_\eta)$ and we can build a filter $p^* \in \hat{h}_1$ in $V[G * H]$ which is generic over $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$. This implies that the embedding lifts to

$$j : M[G * h_0 * h_2 * (h_1 \restriction \eta)] \to N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1].$$

As we argued above, $N[G * h_0 * h_2 * (h_1 \restriction \eta)]$ is Π^1_1-correct in $V[G * H]$, and since the forcing

$$\mathbb{P}_{(\kappa, j(\kappa))} * j(\dot{\mathbb{Q}}_{\kappa,0}) * (j(\dot{\mathbb{Q}}_{\kappa,2}) \times j(\dot{\mathbb{R}}_\eta))$$

is $\leq\kappa$-distributive, it follows that $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$ is Π^1_1-correct in $V[G * H]$. Since $A = \dot{A}_{G * h_0 * (h_1 \restriction \eta \times h_2)} \in M[G * h_0 * h_2 * (h_1 \restriction \eta)]$, this shows that κ is Π^1_2-indescribable in $V[G * H]$. $\square$

Now let us argue that κ is κ^+-weakly compact in $V[G * h_0 * h_1]$. Fix $\zeta < \kappa^+$. We must argue that $\mathrm{Tr}^\zeta_1([\kappa])^{V[G * h_0 * h_1]} \neq [\varnothing]$. Since κ is Π^1_2-indescribable in $V[G * h_0 * (h_1 \times h_2)]$ by Claim 6.3, and since $\mathbb{Q}_{\kappa,2}$ is κ-strategically closed, it follows that $\mathrm{Tr}_1^\zeta([\kappa])^{V[G*h_0*(h_1\times h_2)]} = [S]$, where $S \in V[G*h_0*h_1]$ is Π^1_2-indescribable in $V[G*h_0*(h_1\times h_2)]$. It follows that S is weakly compact in $V[G*h_0*h_1]$ by Proposition 3.5, and clearly $\mathrm{Tr}_1^\zeta([\kappa])^{V[G*h_0*h_1]} = [S]$. Thus, κ is κ^+-weakly compact in $V[G*h_0*h_1]$.

We next argue that $\mathrm{Refl}_1(\kappa)$ holds in $V[G*h_0*h_1]$. Fix a weakly compact set $S\subseteq\kappa$ in $V[G*h_0*h_1]$. Since S intersects every 1-club in κ, our construction of $\mathbb{Q}_{\kappa,1}$ implies that there is $p\in\mathbb{Q}_{\kappa,2}$ such that

$$p \Vdash_{\mathbb{Q}_{\kappa,2}} \text{"there is no 1-club in } \kappa \text{ disjoint from } \check{S}\text{"}.$$

Let $g_2\subseteq\mathbb{Q}_{\kappa,2}$ be generic over $V[G*h_0*h_1]$ with $p\in g_2$. By the proof of Claim 6.3, κ is Π^1_2-indescribable in $V[G*h_0*h_1*g_2]$. Therefore, in $V[G*h_0*h_1*g_2]$, $\mathrm{Refl}_1(\kappa)$ holds and S is a weakly compact subset of κ, and thus there is some $\alpha<\kappa$ such that $S\cap\alpha$ is a weakly compact subset of α. But $V[G*h_0*h_1*g_2]$ and $V[G*h_0*h_1]$ have the same V_κ, so $S\cap\alpha$ is a weakly compact subset of α in $V[G*h_0*h_1]$. Thus, $\mathrm{Refl}_1(\kappa)$ holds in $V[G*h_0*h_1]$.

Finally, we argue that $\square_1(\kappa)$ holds in $V[G*h_0*h_1]$. The sequence

$$\bigcup h_0 = \vec{C} = \langle C_\alpha : \alpha\in\mathrm{inacc}(\kappa)\rangle$$

is a $\square_1(\kappa)$-sequence in $V[G*h_0]$ by Theorem 5.9 because we can show that $\mathrm{Refl}_0(\kappa)$ holds by essentially the same argument as for $\mathrm{Refl}_1(\kappa)$ above. Suppose that $\vec{C}$ is no longer a $\square_1(\kappa)$-sequence in $V[G*h_0*h_1]$. This implies that there is a condition $p\in h_1$ such that in $V[G*h_0]$,

$$p\Vdash_{\mathbb{Q}_{\kappa,1}} \text{"there is a 1-club } \dot{E}\subseteq\check{\kappa} \text{ that threads } \check{\vec{C}}\text{"}.$$

Let g_1 be generic for $\mathbb{Q}_{\kappa,1}$ over $V[G*h_0*h_1]$ with $p\in g_1$. In $V[G*h_0*(h_1\times g_1)]$, let $E=\dot{E}_{h_1}$ and $E^*=\dot{E}_{g_1}$. By mutual genericity, we may fix $\alpha\in E\setminus E^*$. A proof almost identical to that of Claim 6.3 shows that κ is Π^1_2-indescribable in $V[G*h_0*(h_1\times g_1\times h_2)]$ and hence weakly compact in $V[G*h_0*(h_1\times g_1)]$. Now, in $V[G*h_0*(h_1\times g_1)]$, fix any $j:M\to N$ with critical point κ and $E,E^*\in M$. Since both are 1-clubs, $\kappa\in j(E)\cap j(E^*)$, and so by elementarity there is an inaccessible $\beta\in\kappa\setminus(\alpha+1)$ such that $E\cap\beta$ and $E^*\cap\beta$ are both stationary in β. But then, as they both thread $\vec{C}$, it must be the case that $E\cap\beta=C_\beta=\hat{E}\cap\beta$. This contradicts the fact that $\alpha\in E\setminus E^*$ and finishes the proof of the theorem. □

Remark 6.4. Observe that κ cannot be Π^1_2-indescribable in $V[G*h_0*h_1]$ because $\square_1(\kappa)$ holds there. Thus, the set S, where $\mathrm{Tr}_1^\zeta([\kappa])^{V[G*h_0*h_1]}=[S]$, cannot be Π^1_2-indescribable in $V[G*h_0*h_1]$, which shows that Proposition 3.5 can fail for Π^1_2-indescribable sets.

7. An application to simultaneous reflection

In this section we will show that the simultaneous reflection principle $\mathrm{Refl}_n(\kappa,2)$ is incompatible with $\square_n(\kappa)$.

Theorem 7.1. *Suppose that $1\le n<\omega$, κ is Π^1_n-indescribable and $\square_n(\kappa)$ holds. Then there are two Π^1_n-indescribable subsets $S_0,S_1\subseteq\kappa$ that do not reflect simultaneously, i.e., there is no $\beta<\kappa$ such that $S_0\cap\beta$ and $S_1\cap\beta$ are both Π^1_n-indescribable subsets of β.*

Proof. Suppose for the sake of contradiction that every pair of Π^1_n-indescribable subsets of κ reflects simultaneously. Already, $\mathrm{Refl}_n(\kappa)$ implies that κ is ω-Π^1_n-indescribable (see [Cod]), so the set $E = \{\alpha < \kappa : \mathrm{Refl}_{n-1}(\alpha) \text{ holds}\}$ is a Π^1_n-indescribable subset of κ because the set of Π^1_n-indescribable cardinals below κ is Π^1_n-indescribable and $(n-1)$-reflection holds at each of them.

Let $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ be a $\square_n(\kappa)$-sequence. For all $\alpha \in \mathrm{Tr}_{n-1}(\kappa)$, let

$$S^0_\alpha = \{\beta \in \mathrm{Tr}_{n-1}(\kappa) \setminus (\alpha+1) : C_\beta \cap \alpha \in \Pi^1_{n-1}(\alpha)^+\} \text{ and}$$
$$S^1_\alpha = \mathrm{Tr}_{n-1}(\kappa) \setminus ((\alpha+1) \cup S^0_\alpha).$$

Let $A = \{\alpha \in \mathrm{Tr}_{n-1}(\kappa) : S^0_\alpha \in \Pi^1_n(\kappa)^+\}$.

Claim 7.2. *A is Π^1_n-indescribable in κ.*

Proof. Fix an n-club $C \subseteq \kappa$. Since $\mathrm{Tr}_{n-1}(C)$ is an n-club in κ, it follows that $E \cap \mathrm{Tr}_{n-1}(C)$ is Π^1_n-indescribable in κ. For each $\beta \in E \cap \mathrm{Tr}_{n-1}(C)$, $\mathrm{Refl}_{n-1}(\beta)$ holds and $C_\beta \cap C$ is a Π^1_{n-1}-indescribable subset of β. Thus, for $\beta \in E \cap \mathrm{Tr}_{n-1}(C)$, we may let α_β be the least Π^1_{n-1}-indescribable cardinal such that $C_\beta \cap C \cap \alpha_\beta$ is Π^1_{n-1}-indescribable in α_β. Notice that $\alpha_\beta \in C$ for all $\beta \in E \cap \mathrm{Tr}_{n-1}(C)$ because C is an n-club. Since the map $\beta \mapsto \alpha_\beta$ is regressive on $E \cap \mathrm{Tr}_{n-1}(C)$, it follows by the normality of $\Pi^1_n(\kappa)$ that there is a fixed $\alpha \in C$ and a Π^1_n-indescribable set $T \subseteq E \cap \mathrm{Tr}_{n-1}(C)$ such that $\alpha_\beta = \alpha$ for all $\beta \in T$. This implies that $T \subseteq S^0_\alpha$, and thus $\alpha \in A \cap C$. □

Claim 7.3. *There is $\alpha \in A$ such that S^1_α is a Π^1_n-indescribable subset of κ.*

Proof. Suppose not, and let $\alpha_0 < \alpha_1$ be elements of A. Since E is Π^1_n-indescribable in κ and $S^1_{\alpha_0}$ and $S^1_{\alpha_1}$ are both in the Π^1_n-indescribability ideal on κ, we can find $\beta \in E \setminus ((\alpha_1+1) \cup S^1_{\alpha_0} \cup S^1_{\alpha_1})$. It follows that $\beta \in S^0_{\alpha_0} \cap S^0_{\alpha_1}$, so, by the coherence properties of the $\square_n(\kappa)$-sequence, we have $C_\beta \cap \alpha_0 = C_{\alpha_0}$ and $C_\beta \cap \alpha_1 = C_{\alpha_1}$, and hence $C_{\alpha_1} \cap \alpha_0 = C_{\alpha_0}$. But then by Lemma 2.6 and Claim 7.2, we see that $\bigcup_{\alpha \in A} C_\alpha$ is a Π^1_n-indescribable subset of κ. Thus, $\bigcup_{\alpha \in A} C_\alpha$ is a thread through $\vec{C}$, which is a contradiction. □

We can therefore fix $\alpha \in \mathrm{Tr}_{n-1}(\kappa)$ such that both S^0_α and S^1_α are Π^1_n-indescribable subsets of κ. Let $S_0 = S^0_\alpha$ and $S_1 = S^1_\alpha$. We claim that S_0 and S_1 cannot reflect simultaneously. Otherwise, there is γ such that $S_0 \cap \gamma$ and $S_1 \cap \gamma$ are both Π^1_n-indescribable subsets of γ. Consider the n-club C_γ. Since γ is Π^1_n-indescribable, $\mathrm{Tr}_{n-1}(C_\gamma)$ is also an n-club in γ. We can therefore find $\beta_0 < \beta_1$ in $\mathrm{Tr}_{n-1}(C_\gamma)$ such that $\beta_0 \in S_0$ and $\beta_1 \in S_1$. But note that $C_{\beta_0} = C_\gamma \cap \beta_0$ and $C_{\beta_1} = C_\gamma \cap \beta_1$, so $C_{\beta_0} = C_{\beta_1} \cap \beta_0$, contradicting the fact that $C_{\beta_0} \cap \alpha$ is Π^1_{n-1}-indescribable in α whereas $C_{\beta_1} \cap \alpha$ is not Π^1_{n-1}-indescribable in α. □

As a direct consequence of Theorem 1.2 and Theorem 7.1 we obtain the following.

Corollary 7.4. *Suppose κ is Π^1_2-indescribable. Then there is a forcing extension in which $\mathrm{Refl}_1(\kappa)$ and $\neg\mathrm{Refl}_1(\kappa, 2)$ both hold.*

8. Questions

The theorems proved in this article about the principle $\square_1(\kappa)$ do not easily generalize to $\square_n(\kappa)$ because several key technical results about Π^1_1-indescribability which we used crucially in the proofs no longer hold for higher orders of indescribability. For example, given an embedding $j : M \to N$, where N is Π^1_n-correct, we cannot necessarily use a generic G for a poset $\mathbb{P} \in N$ from the ground model to lift j because $N[G]$ may no longer be Π^1_n-correct. An illustration of this is given in Remark 5.7. Also, while κ-strategically closed forcing cannot make a subset of κ Π^1_1-indescribable if it was not so already in the ground model by Proposition 3.5, a set can become Π^1_2-indescribable after κ-strategically closed forcing by Remark 6.4.

Question 8.1. Relative to large cardinals, for $n > 1$, is it consistent that κ is Π^1_n-indescribable and $\square_n(\kappa)$ holds nontrivially?

Question 8.2. Relative to large cardinals, for $n > 1$, is it consistent that $\mathrm{Refl}_n(\kappa)$ and $\square_n(\kappa)$ both hold?

Question 8.3. Relative to large cardinals, is it consistent that $\mathrm{Refl}_1(\kappa, 2) + \neg\mathrm{Refl}_1(\kappa, 3)$?

Question 8.4. Can we force any indestructibility of $\mathrm{Refl}_1(\kappa)$?

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