Parameter-free comprehension in second-order arithmetic

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Second-order arithmetic

Second-order arithmetic has two types of objects: numbers and sets of numbers (reals).

Syntax: Two-sorted logic

- Separate variables and quantifiers for numbers and sets of numbers.
- Convention: lower-case letters for numbers, upper-case letters for sets.
- Notation:
  - Σ^0_n - first-order Σ_n-formula
  - Σ^1_n - n-alternations of set quantifiers followed by a first-order formula.

Semantics: A model is $\mathscr{M} = \langle M, +, \times, <, 0, 1, \in, \mathcal{S}\rangle$.

- M is the collection of numbers.
- $\mathcal{S}$ is the collection of sets of numbers: if $A \in \mathcal{S}$, then $A \subseteq M$.

Second-order axioms

- Numbers: PA
- Sets:
  - Extensionality
  - Induction axiom: $\forall X\ ((0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n\, n \in X)$

Weak axiom systems

Arithmetical comprehension ACA_0

Comprehension scheme for first-order formulas: for all n, Σ^0_n-CA_0
if $\varphi(n, A)$ is a first-order formula, then $\{n \mid \varphi(n, A)\}$ is a set.

- If $\langle M, +, \times, <, 0, 1\rangle \models \mathrm{PA}$ and $\mathcal{S}$ consists of definable subsets of M, then

$$\mathscr{M} = \langle M, +, \times, <, 0, 1, \in, \mathcal{S}\rangle \models \mathrm{ACA}_0.$$

- ACA_0 is conservative over PA.

Elementary Transfinite Recursion ATR_0

ACA_0

Transfinite recursion: every first-order recursion on sets along a well-order has a solution.

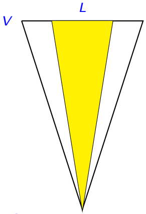
- A well-order is a linear order Γ whose every subset has a minimal element.
- A solution to a recursion is a code of a function $F : \mathrm{dom}(\Gamma) \to \mathcal{S}$.

  A code for F is $\bar{F} = \{\langle n, m\rangle \mid n \in \mathrm{dom}(\Gamma,) \, m \in F(n)\}$
- Iterate Turing jump.
- Build an internal constructible universe L.
- (Fujimoto) Equivalent (over ACA_0) to existence of iterated truth predicates along any well-order Γ.

Gödel's L in ATR_0

Gödel's constructible universe L

Suppose $V \models \mathrm{ZF}$.



- $L_0 = \emptyset$
- $L_{\alpha+1}$ is the set of all subsets of L_α definable over L_α.
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ.
- $L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$.

Suppose $\mathscr{M} = \langle M, +, \times, <, 0, 1, \in, \mathcal{S}\rangle \models \mathrm{ATR}_0$ and $\Gamma \in \mathcal{S}$ is a well-order.

- $\mathscr{M}$ can construct the L-hierarchy along Γ.
- There is a set coding a sequence of L_Δ for $\Delta \leq \Gamma$ obeying the definition of L.

A model of ATR_0 has its own constructible universe $L^{\mathscr{M}}$!

Definition: A well-order $\Gamma \in \mathcal{S}$ is constructible if there is a well-order Δ such that

$$L_\Delta \models \Gamma \text{ is countable.}$$

$L^{\mathscr{M}}_{\omega_1}$ is the union of L_Δ for constructible well-orders Δ.

The comprehension hierarchy

Increasing to the amount of comprehension to more complex second-order assertions produces a hierarchy of second-order set theories

Σ^1_n-comprehension $\Sigma^1_{\mathrm{n}}\text{-CA}_0$

If $\varphi(n, A)$ is a Σ^1_n-formula, then $\{n \mid \varphi(n, A)\}$ is a set.

- $\Sigma^1_1\text{-CA}_0$ is stronger than ATR_0.

culminating in:

Full second-order arithmetic Z_2

For all n, Σ^1_n-comprehension.

- If $\mathscr{M} \models \mathrm{Z}_2$, then $L^{\mathscr{M}}_{\omega_1} \models \mathrm{ZFC}^-$ ZFC without powerset

Definition: Z_2^{-p} is full comprehension without parameters.

Equiconsistency of Z_2 and Z_2^{-p}

Theorem: (H. Friedman) Z_2 and Z_2^{-p} are equiconsistent.

Proof idea: Suppose $\mathscr{M} = \langle M, +, \times, <, 0, 1, \in, \mathcal{S}\rangle \models \mathrm{Z}_2^{-p}$.

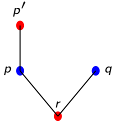
- A modified L-construction can be carried out without parameters.
- $L_{\omega_1}^{\mathscr{M}} \models \mathrm{ZFC}^-$.
- Let $\bar{\mathcal{S}} = \{A \in \mathcal{S} \mid A \in L_{\omega_1}^{\mathscr{M}}\}$ be the "constructible reals" of $\mathscr{M}$.
- $(M, +, \times, <, 0, 1, \in, \bar{\mathcal{S}}) \models \mathrm{Z}_2$.

Quick review of forcing

Suppose $V \models \mathrm{ZFC}$ and $\mathbb{P}$ is a forcing notion: partial order with largest element $\mathbb{1}$.

Dense sets and generic filters

$D \subseteq \mathbb{P}$ is dense if for every $p \in \mathbb{P}$, there is $q \in D$ with $q \leq p$.



$G \subseteq \mathbb{P}$ is a filter:

- $\mathbb{1} \in G$.
- (upward closure) If $p \in G$ and $p' \geq p$, then $p' \in G$.
- (compability) If $p, q \in G$, then $r \in G$ such that $r \leq p, q$.

A filter $G \subseteq \mathbb{P}$ is V-generic if it meets every dense set $D \in V$ of $\mathbb{P}$: $D \cap G \neq \emptyset$.

Theorem: V has no V-generic filters for $\mathbb{P}$.

The forcing extension $V[G]$ is constructed from V together with an external V-generic filter G.

Quick review of forcing (continued)

$\mathbb{P}$-**names**: names for elements of $V[G]$.

Defined recursively so that a $\mathbb{P}$-name σ consists of pairs $\langle \tau, p\rangle$: $p \in \mathbb{P}$ and τ is a $\mathbb{P}$-name.

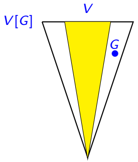
Special $\mathbb{P}$-names

- Given $a \in V$, $\check{a} = \{\langle \check{b}, \mathbb{1}\rangle \mid b \in a\}$.
- $\dot{G} = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$.

Forcing extension $V[G]$

Suppose $G \subseteq \mathbb{P}$ is V-generic and σ is a $\mathbb{P}$-name. The interpretation of σ by G: $\sigma_G = \{\tau_G \mid \langle \tau, p\rangle \in \sigma \text{ and } p \in G\}$.

Defined recursively.



The forcing extension $V[G] = \{\sigma_G \mid \sigma \text{ is a } \mathbb{P}\text{-name in } V\}$.

- $V \subseteq V[G]$: $\check{a}_G = a$.
- $G \in V[G]$: $\dot{G}_G = G$.
- $V[G] \models \mathrm{ZFC}$

Forcing relation $p \Vdash \varphi(\sigma)$: whenever G is V-generic and $p \in G$, then $V[G] \models \varphi(\sigma_G)$.

- For a fixed first-order formula $\varphi(x)$, the relation $p \Vdash \varphi(\sigma)$ is definable.
- If $q \leq p$ and $p \Vdash \varphi(\sigma)$, then $q \Vdash \varphi(\sigma)$.
- If $V[G] \models \varphi(\sigma_G)$, then there is $p \in G$ such that $p \Vdash \varphi(\sigma)$.

Cohen forcing

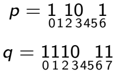
$\mathrm{Add}(\omega, 1)$ - adds a new real

Conditions: binary sequences $p : D \to 2$ with $D \subseteq \omega$ finite.

Order: $q \leq p$ if q extends p.

Suppose $G \subseteq \mathrm{Add}(\omega, 1)$ is V-generic.

- $r = \bigcup G$ is a new real
- $V[G]$ has continuum-many V-generic reals for $\mathrm{Add}(\omega, 1)$.



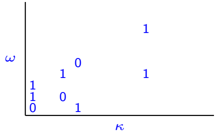
$\mathrm{Add}(\omega, \kappa)$ - adds (at least) κ-many reals

Conditions: functions $p : D \to 2$, where D is a finite subset of $\omega \times \kappa$.

Order: $q \leq p$ if q extends p.

Suppose $G \subseteq \mathrm{Add}(\omega, \kappa)$ is V-generic.

$\bigcup G$ gives κ-many new reals.



Sacks forcing

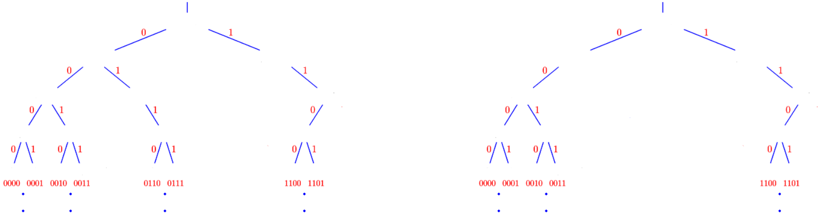
Sacks forcing $\mathbb{S}$ - adds a generic real

Conditions: Perfect trees $T \subseteq 2^{<\omega}$: every node has a splitting node above it.

Order: $S \leq T$ if S is a subtree of T.

Suppose $G \subseteq \mathbb{S}$ is V-generic.

- There is a real $b \in V[G]$ such that $T \in G$ iff b is a branch of T.
- The generic real b determines G.



Jensen's forcing

Jensen's forcing $\mathbb{J}$ - adds a unique generic real

- constructed using $\diamondsuit$ in L (construction is technical)
- $\mathbb{J} \subseteq \mathbb{S}$
- has ccc
- adds a unique generic Π^1_2-definable singleton real
- used by Jensen to show that it is consistent to have a non-constructible Π^1_2-definable singleton real.

  Every Σ^1_2-definable singleton real is in L by Shoenfield's Absoluteness.

Products and iterations of forcing notions

Products

Suppose $\mathbb{P}_\alpha$ for $\alpha < \beta$ are forcing notions.

A product $\mathbb{P} = \Pi_{\alpha<\beta}\mathbb{P}_\alpha$ is a natural forcing notion.

- Conditions: $\langle p_\alpha \mid \alpha < \beta\rangle$ with $p_\alpha \in \mathbb{P}_\alpha$.
- Common supports: finite, bounded, full.
- Example: $\mathrm{Add}(\omega,\kappa) = \Pi_{\alpha<\kappa}\mathrm{Add}(\omega,1)$ with finite support.
- Usage: adding several objects to a forcing extension.

Iterations

Suppose $\mathbb{P}$ is a forcing notion, $G \subseteq \mathbb{P}$ is V-generic, and $\mathbb{Q}$ is a forcing notion in $V[G]$.

V has a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for $\mathbb{Q}$. Every element of $V[G]$ has a $\mathbb{P}$-name in V.

In V, we define a forcing notion $\mathbb{P} * \dot{\mathbb{Q}}$ such that forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ is the same as forcing with $\mathbb{P}$ followed by forcing with $\mathbb{Q}$.

- Conditions: $(p, \dot{q})$ with $p \in \mathbb{P}$ and $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$.
- Order: $(p, \dot{q}) \leq (r, \dot{s})$ if $p \leq r$ and $p \Vdash \dot{q} \leq \dot{s}$.
- n-step iterations are defined similarly (infinite iterations can be defined as well).

Example: $\mathbb{S} * \dot{\mathbb{S}}$, where $\dot{\mathbb{S}}$ is the name for the Sacks forcing of the forcing extension.

Sacks forcing of $V[G]$ is different from Sacks forcing of V because $V[G]$ has new perfect trees.

Automorphisms of forcing notions

Suppose $\mathbb{P}$ is a notion of forcing and π is an automorphism of $\mathbb{P}$.

We apply π (recursively) to $\mathbb{P}$-names: $\langle\tau, p\rangle \in \sigma$ if and only if $\langle\pi(\tau), \pi(p)\rangle \in \pi(\sigma)$.

- $\pi(\check{a}) = \check{a}$

The forcing relation respects automorphisms: $p \Vdash \varphi(\sigma)$ if and only if $\pi(p) \Vdash \varphi(\pi(\sigma))$.

If $G \subseteq \mathbb{P}$ is V-generic, then $\pi\,"\,G$ is V-generic.

Examples

- For any $p, q \in \mathrm{Add}(\omega, 1)$, there is an automorphism π such that p and $\pi(q)$ are compatible.
- Every permutation of κ gives rise to a coordinate-switching automorphism of $\mathrm{Add}(\omega, \kappa)$.
- For any $p, q \in \mathrm{Add}(\omega, \kappa)$, there is an automorphism π such that p and $\pi(q)$ are compatible.
- Jensen's forcing $\mathbb{J}$ is rigid.

A very bad model of Z_2^{-p}

Theorem: (Lyubetsky and Kanovei) It is consistent that there is a model $\mathscr{M} = \langle M, +, \times, <, 0, 1, \in, \mathcal{S}\rangle \models \mathrm{Z}_2^{-p}$ such that $\mathcal{S}$ is not closed under complements.

Proof: Let $G \subseteq \mathrm{Add}(\omega, \omega)$ be V-generic.

- Let $\{a_n \mid n < \omega\}$ be the ω-many generic reals from G.
- Let $\mathcal{S} = P^V(\omega) \cup \{a_n \mid n < \omega\}$.
- $\mathcal{S}$ is not closed under complements.
- Let $\mathscr{M} = \langle \omega, +, \times, <, 0, 1, \in, \mathcal{S}\rangle$.
- Let $\dot{\mathscr{M}}$ be the canonical $\mathrm{Add}(\omega, \omega)$-name for $\mathscr{M}$.
- Fix a second-order formula $\varphi(x)$.
- Suppose, for $n < \omega$, $p \Vdash \dot{\mathscr{M}} \models \varphi(\check{n})$, but $q \Vdash \dot{\mathscr{M}} \models \neg\varphi(\check{n})$.
- Let π be a coordinate-switching automorphism such that there is $r \leq p, \pi(q)$.
- $\pi(q) \Vdash \dot{\mathscr{M}} \models \neg\varphi(\check{n})$ since $\pi(\dot{\mathscr{M}}) = \dot{\mathscr{M}}$.
- $r \Vdash \dot{\mathscr{M}} \models \varphi(\check{n})$ $(r \leq p)$ and $r \Vdash \dot{\mathscr{M}} \models \neg\varphi(\check{n})$ $(r \leq \pi(q))$. Impossible!
- If some $p \Vdash \dot{\mathscr{M}} \models \varphi(\check{n})$, then all $p \Vdash \dot{\mathscr{M}} \models \varphi(\check{n})$.
- By definability of the forcing relation, $\{n < \omega \mid \mathscr{M} \models \varphi(n)\} \in V$.
- $\mathscr{M} \models \mathrm{Z}_2^{-p}$. $\square$

A model of $\Sigma^1_2\text{-}\mathrm{CA}_0 + \mathrm{Z}_2^{-p}$

Theorem: (Lyubetsky and Kanovei) It is consistent that there is a model of $\Sigma^1_2\text{-}\mathrm{CA}_0 + \mathrm{Z}_2^{-p}$ in which $\Sigma^1_4\text{-}\mathrm{CA}_0$ fails.

The model is constructed in a forcing extension by a (non-linear) tree iteration of Sacks forcing.

Theorem: (G.) It is consistent that there is a model of $\Sigma^1_2\text{-}\mathrm{CA}_0 + \mathrm{Z}_2^{-p}$ in which $\Sigma^1_3\text{-}\mathrm{CA}_0$ fails.

The model is constructed in a forcing extension by a tree iteration of Jensen's forcing.

Work in progress: (Kanovei) It is consistent that there is a model of $\Sigma^1_n\text{-}\mathrm{CA}_0 + \mathrm{Z}_2^{-p}$ in which $\Sigma^1_{n+1}\text{-}\mathrm{CA}_0$ fails.

Proof idea: use a generalization of Jensen's forcing.

Finite iterations of Jensen's forcing

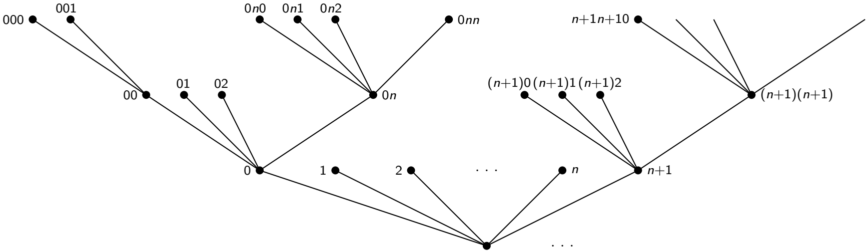
Theorem (Abraham) In L, for every $n<\omega$, there is an n-length iteration $\mathbb{J}_n=\mathbb{Q}_0\cdot\dot{\mathbb{Q}}_1\cdots\dot{\mathbb{Q}}_{n-1}$ such that:

- $\mathbb{Q}_0=\mathbb{J}$.
- If $G_i\subseteq\mathbb{J}_n\restriction i$ is L-generic, then in $L[G_i]$, $\mathbb{Q}=(\dot{\mathbb{Q}}_i)_{G_i}$ has all properties of Jensen's forcing.
- If $m>n$, then $\mathbb{J}_m\restriction n=\mathbb{J}_n$.
- $\mathbb{J}_n$ has the ccc.
- $\mathbb{J}_n$ adds a unique generic Π^1_2-definable n-length sequence of reals.

Let $\vec{\mathbb{J}}=\langle\mathbb{J}_n\mid n<\omega\rangle$.

Tree iteration of Jensen's forcing

The tree $\omega^{<\omega}$

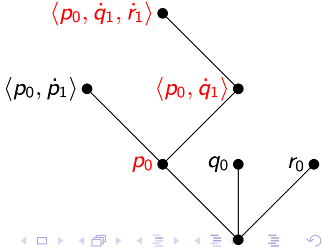


$\mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$: iteration along the tree $\omega^{<\omega}$

Conditions: $p : D_p \to \bigcup_{n<\omega} \mathbb{J}_n$ such that:

- D_p is a finite subtree of $\omega^{<\omega}$,
- for all $s \in D_p$, $p(s) \in \mathbb{J}_{\text{len}(s)}$,
- for $s \subseteq t$ in D_p, $p(s) = p(t) \upharpoonright \text{len}(s)$.

Order: $q \leq p$ if $D_q \supseteq D_p$ and for all $s \in D_p$, $q(s) \leq p(s)$.



Tree iteration of Jensen's forcing (continued)

The tree iteration $\mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$ adds a tree $\mathcal{T}^G$ (isomorphic to $\omega^{<\omega}$) such that each node on level n has an L-generic n-length sequence of reals for $\mathbb{J}_n$.

Theorem: (Friedman, G.) The tree iteration $\mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$ has the ccc.

Theorem: (Friedman, G.) Suppose $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$ is L-generic. In $L[G]$:

- The only L-generic n-length sequence of reals for $\mathbb{J}_n$ are those coming from the nodes of $\mathcal{T}^G$.
- The collection of all L-generic n-length sequences of reals for $\mathbb{J}_n$ (any n) is Π^1_2-definable.

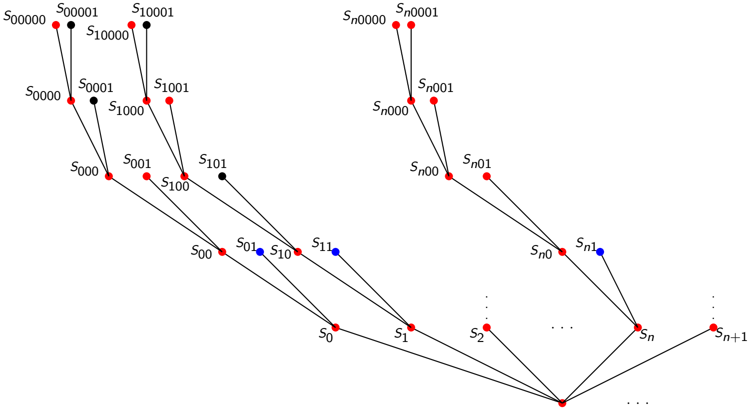
Proposition: Suppose

- $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$ is L-generic.
- T is a finite subtree of $\omega^{<\omega}$.
- $G_T = G \restriction T$.

Then the only L-generic n-length sequences of reals for $\mathbb{J}_n$ in $L[G_T]$ are those coming from the nodes of T.

Kanovei's tree

Suppose $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$ is L-generic. In $L[G]$, define the tree $S \subseteq \mathcal{T}^G$.



- $S_n \in S$ for every $n < \omega$
- $S_{n\vec{0}_m}$ for every $n, m < \omega$

  $\vec{0}_n$ is the sequence of $n \geq 1$-many zeroes
- $S_{n\vec{0}_{m+1}1} \in S$ whenever $S_{n1}(1)(m) = 1$

$S_{01}(1) = 100\ldots$

$S_{11}(1) = 010\ldots$

$S_{n1}(1) = 111\ldots$

A model of $\Sigma^1_2\text{-}\mathrm{CA}_0 + \mathrm{Z}_2^{-p}$ in which $\Sigma^1_3\text{-}\mathrm{CA}_0$ fails

Let

- $\mathscr{T} = \{T \subseteq S \mid T \text{ finite}\}$,
- $\mathcal{S} = \{A \in P^{L[G_T]}(\omega) \mid T \in \mathscr{T}\}$,
- $\mathscr{M} = (\omega, +, \times, <, 0, 1, \in, \mathcal{S})$,
- $\dot{\mathscr{M}}$ be a canonical $\mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$-name for $\mathscr{M}$.

Every permutation f of ω gives rise to an automorphism π_f of $\mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$ which permutes the subtrees $\mathcal{T}_n^G$ (sitting on node S_n) of $\mathcal{T}^G$, while preserving the rest of the tree structure.

- $\pi_f(\dot{\mathscr{M}}) = \dot{\mathscr{M}}$
- for any $p, q \in \mathbb{P}(\vec{\mathbb{J}}, \omega^{<\omega})$, there is an automorphism π_f such that p and $\pi_f(q)$ are compatible.

A model of $\Sigma^1_2\text{-}\mathrm{CA}_0 + \mathrm{Z}_2^{-p}$ in which $\Sigma^1_3\text{-}\mathrm{CA}_0$ fails (continued)

Theorem (G.): $\mathscr{M} = (M, +, \times, <, 0, 1, \in, \mathcal{S}) \models \Sigma^1_2\text{-}\mathrm{CA}_0 + \mathrm{Z}_2^{-p} + \neg\Sigma^1_3\text{-}\mathrm{CA}_0$.

Proof:

- $\mathscr{M} \models \Sigma^1_2\text{-}\mathrm{CA}_0$ by Shoenfield's Absoluteness.
- $\mathscr{M} \models \mathrm{Z}_2^{-p}$ because every parameter-free $A \in \mathcal{S}$ is in L by the automorphism argument.
- The collection $\{\vec{r} \mid \exists n\, \vec{r} \text{ is an } L\text{-generic } n\text{-length sequence for } \mathbb{J}_n\} = \{S_s \mid s \in T \text{ for some } T \in \mathscr{T}\}$ is Π^1_2-definable in $\mathscr{M}$ (uses the construction of the $\mathbb{J}_n$).
- $S_{01}(1) \notin \mathcal{S}$.
- $S_{01}(1)$ is Σ^1_3-definable in $\mathscr{M}$: $m \in S_{01}(1)$ if and only if there are two L-generic $m+2$-length sequences of reals for $\mathbb{J}_{m+2}$ whose first coordinate is S_0. $\square$