NOTES ON GITIK-SHELAH INDESTRUCTIBILITY RESULT

1. Preliminaries

These notes elaborate the arguments in [GS89] that demonstrate how to make a strong cardinal κ indestructible by $\leq \kappa$-weakly closed forcing with the Prikry condition.

Let $\langle \mathbb{P}, \leq, \leq^* \rangle$ be a set with two partial orders so that $p \leq^* q \longrightarrow p \leq q$ for every $p, q \in \mathbb{P}$. $\langle P, \leq \rangle$ is a used as a forcing notion.

Definition 1.1. We say that $\langle \mathbb{P}, \leq, \leq^* \rangle$ satisfies the **Prikry condition** if for every $p \in \mathbb{P}$ and every statement φ of the forcing language, there exists $q \leq^* p$ such that $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.
Let α be a cardinal. We say that $\mathbb{P}$ is $\leq \alpha$**-weakly closed** if $\leq^*$ is $\leq \alpha$-closed.

Example 1.2. Suppose κ is a measurable cardinal and let U be a normal measure on κ. Define $\mathbb{P}$ to be the set of pairs of the form $\langle s, A \rangle$ where s is a finite subset of κ, $A \in U$, and $min(A) > max(s)$. We say that $\langle s, A \rangle \leq \langle t, B \rangle$ whenever s is an end extension of t, $A \subseteq B$, $s \setminus t \subseteq B$. Finally, we say that $\langle s, A \rangle \leq^* \langle t, B \rangle$ whenever $s = t$ and $A \subseteq B$.
$\mathbb{P}$ is called the **Prikry forcing**. Prikry forcing has the Prikry condition and it is $< \kappa$-weakly closed.

Example 1.3. If $\mathbb{P}$ is $\leq \alpha$-closed, then $\langle \mathbb{P}, \leq, \leq \rangle$ is a $\leq \alpha$-weakly closed poset with the Prikry condition. Thus, the class of $\leq \alpha$-weakly closed posets with the Prikry condition extends the class of $\leq \alpha$-closed posets.

Even though the definition of the Prikry condition does not look first order, it can be reworded in a first order way:

Definition 1.4. Let $\mathbb{P}$ be a partial order. Let D be a dense open subset of $\mathbb{P}$. Call a partition of $D = D_0 \sqcup D_1$ **deciding** if

1. elements of D_0 are incompatible with elements of D_1,
2. if p is a condition such that all its strengthenings that are in D are necessarily in D_0, then $p \in D_0$,
3. if p is a condition such that all its strengthenings that are in D are necessarily in D_1, then $p \in D_1$.

Lemma 1.5. *Fix any sentence φ in the forcing language. Let D_1 consist of all conditions that force φ, and let D_0 consist of all conditions that force $\neg\varphi$. Then the partition $D = D_0 \sqcup D_1$ is deciding.*

Proof. D is clearly dense open. Suppose p is a condition such that every strengthening of p that is in D is also in D_0. Thus, every strengthening of p that decides φ forces φ. It follows that p forces φ as well and hence $p \in D_0$. The argument for D_1 is identical. $\square$

Lemma 1.6. *Suppose that $D = D_0 \sqcup D_1$ is a deciding partition. Then there is a sentence φ in the forcing language such that D_0 consists of all conditions that force φ, and D_1 consists of all conditions that force $\neg\varphi$.*

Proof. Let $\varphi := \check{D}_0 \cap \dot{G} \neq \emptyset$. Suppose $p \in D_0$, then $p \Vdash \varphi$. Suppose $p \Vdash \varphi$ and q is a strengthening of p in D. If $q \in D_1$, then any generic containing q will not meet D_0 but this is impossible. Thus, $q \in D_0$. It follows that $p \in D$. The argument for $\neg\varphi$ is identical. □

Theorem 1.7. *TFAE for a poset $\langle \mathbb{P}, \leq, \leq^* \rangle$:*

1. *$\mathbb{P}$ has the Prikry condition*
2. *For every deciding partition $D = D_0 \sqcup D_1$ of $\mathbb{P}$, and for all $p \in \mathbb{P}$, there exists $q \leq^* p$ such that $q \in D$.*

Proposition 1.8. *$\leq \alpha$-weakly closed forcing satisfying the Prikry condition does not add new subsets to α.*

Proof. Suppose $\mathbb{P}$ is $\leq \alpha$-weakly closed forcing satisfying the Prikry condition. Fix a name $\dot{f}$ for the characteristic function of a subset of α. Choose q_0 deciding $\dot{f}(0)$. Choose $q_1 \leq^* q_0$ deciding $\dot{f}(1)$. Continue in this manner for $\xi < \alpha$ using the weak closure of $\mathbb{P}$ to obtain the sequence $\langle q_\xi \mid \xi < \alpha \rangle$. Now choose any q *-below the sequence. This q clearly forces that $\dot{f} \in V$. □

Definition 1.9. A **Prikry iteration** $\mathbb{P}_\alpha$ of length α is a forcing iteration with Easton support taken at limits such that for $\beta < \alpha$, we have

$$\Vdash_{\mathbb{P}_\beta} \text{“}\langle \dot{\mathbb{Q}}_\beta, \leq, \leq^* \rangle \text{ has the Prikry condition”}.$$

If $\mathrm{dom}(p) = f$ and $\mathrm{dom}(q) = g$, then we have $p \leq q$ if:

1. $g \subseteq f$,
2. for all $\gamma \in g$, $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \dot{p_\gamma} \leq \dot{q_\gamma}$,
3. for all but finitely many $\gamma \in g$, $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \dot{p_\gamma} \leq^* \dot{q_\gamma}$.

We have $p \leq^* q$ if:
for all $\gamma \in g$, $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \dot{p_\gamma} \leq^* \dot{q_\gamma}$.

Theorem 1.10. *$\langle \mathbb{P}_\alpha, \leq, \leq^* \rangle$ has the Prikry condition.*

For details of proof, see Mote Gitik's article “Prikry type Forcings” in the Handbook of Set Theory (available on his website).

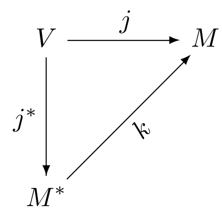
Theorem 1.11 (Laver function)**.** *If κ is a strong cardinal, then there is a partial function $l : \kappa \to V_\kappa$ satisfying:*

(1) *for every x and every $\lambda \geq |TC(x)|$, there is a λ-strongness embedding $j : V \to M$ with $j(l)(\kappa) = x$,*
(3) *for all $\gamma \in dom(l)$, we have $l \restriction \gamma \subseteq V_\gamma$,*
(2) *All $\gamma \in dom(l)$ are inaccessible cardinals.*

Remark 1.12. We can assume wlog that $j : V \to M$ above is an extender embedding.

Proof. Fix x and $\lambda \geq |\mathrm{TC}(x)|$. Choose a λ-strongness embedding $j : V \to M$ such that $j(l)(\kappa) = x$. Let $j^* : V \to M^*$ be the extender embedding derived from j

using finite subsets of $\beth_\lambda + 1$ as seeds. We get the following commutative diagram:



It is easy to see that $\mathrm{cp}(k) \geq \lambda + 1$. We will argue that $j^*(l)(\kappa) = x$. Observe that $k(j^*(l)(\kappa)) = k(j^*(l))(\kappa) = j(l)(\kappa) = x$. Let $y = j^*(l)(\kappa)$, then $k(y) = x$. Thus $\mathrm{rk}(y) \leq \mathrm{rk}(x) \leq \lambda$. But then $k(y) = y$. So $j^*(l)(\kappa) = x$. □

2. Main Result

We can assume without loss of generality that $V \models 2^\kappa = \kappa^+$ since if κ is strong, then there is a forcing extension in which this holds and κ remains strong [Ham07].

The preparatory forcing is a Prikry iteration $\mathbb{P}_\kappa$:

(1) If $\alpha \notin \mathrm{dom}(l)$, $\dot{\mathbb{Q}}_\alpha$ is the trivial forcing.
(2) Suppose $\alpha \in \mathrm{dom}(l)$ and $l(\alpha) = \langle \dot{\mathbb{Q}}, \lambda \rangle$ where $\dot{\mathbb{Q}} \in V_\lambda$ is a $\mathbb{P}_\alpha$-name.
   (a) If $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}$ is a $\leq \alpha$-weakly closed forcing has the Prikry condition", then $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}$.
   (b) Otherwise $\dot{\mathbb{Q}}_\alpha$ is the trivial forcing.

Let G be V-generic for $\mathbb{P}_\kappa$. We will show:

Theorem 2.1 (Main Theorem)**.** *The strongness of κ is indestructible in $V[G]$ by $\leq \kappa$-weakly closed forcing with the Prikry condition.*

In $V[G]$, fix a forcing $\mathbb{Q}$ that is $\leq \kappa$-weakly closed with the Prikry condition and let g be $V[G]$-generic for $\mathbb{Q}$. Fix a cardinal λ such that:

(1) $\lambda > \kappa$,
(2) $\mathbb{Q} \in V[G]_\lambda$,
(3) λ is a $\beth$-fixed point.
(4) $\mathrm{cof}(\lambda) > \omega$.

We will show that κ is λ-strong in $V[G][g]$.

Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}_\kappa$-name for $\mathbb{Q}$ such that

$$\dot{\mathbb{Q}} \in V_\lambda \text{ and } \Vdash_{\mathbb{P}_\kappa} \text{“}\dot{\mathbb{Q}} \text{ is } \leq \kappa\text{-weakly closed with Prikry condition”}.$$

Choose a λ-strongness embedding $j : V \to M$ such that $j(l)(\kappa) = \langle \dot{\mathbb{Q}}, \lambda \rangle$. We can assume that j is an extender embedding by the extender $\langle E_a \mid a \in [\lambda]^{<\omega} \rangle$.

We would like to argue that $j(\mathbb{P}_\kappa)$ factors as $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\mathrm{tail}}$. It will suffice to show that in M, the poset $\mathbb{P}_\kappa$ also forces that $\dot{q}$ has the Prikry condition. To see this, fix any M-generic H for $\mathbb{P}_\kappa$ and observe that H is V-generic as well. In $V[H]$, the poset $\dot{\mathbb{Q}}_H$ has the Prikry condition, meaning that all deciding partitions are *-dense. But, then M must agree since it has the same deciding partitions as V.

Note for the future that there is a function $F : \kappa \to V_\kappa$ such that $j(F)(\kappa) = \lambda$, namely the function defined by taking the second coordinates of the Laver function.

If we try to lift the embedding $j : V \to M$ to $V[G]$, we will run into the problem that we cannot construct a generic for $\mathbb{P}_{\mathrm{tail}}$ because it is not even countably closed. The strategy therefore will be to build a "pseudo-generic" G_{tail} for $\mathbb{P}_{\mathrm{tail}}$ that meets only the *-dense sets. The pseudo-generic $G*g*G_{\text{tail}}$ will suffice to allow us to build an extender in $V[G][g]$ with a well-founded direct limit. We will then show that the extender embedding is λ-strong. To build G_{tail}, we need to determine how much weak closure $\mathbb{P}_{\text{tail}}$ has. The next stage of forcing in $j(\mathbb{P}_\kappa)$ after κ cannot be earlier than the next inaccessible after λ since $j(l)(\kappa) \in M_{\lambda+1}$ and for $\alpha \in \text{dom}(j(l))$, we have $j(l) \restriction \alpha \subseteq M_\alpha$. Thus, $\mathbb{P}_{\text{tail}}$ is $\leq \lambda$-weakly closed.

The next two lemmas show how to construct G_{tail} meeting all the *-dense sets in $\mathbb{P}_{\text{tail}}$.

Lemma 2.2. *In V, there is a sequence $\langle \dot{D}_\alpha \mid \alpha < \kappa^+ \rangle$ such that*

(1) *in M, $\Vdash_{\mathbb{P}_{\kappa+1}}$ "$\dot{D}_\alpha$ is a *-dense open subset of $\dot{\mathbb{P}}_{tail}$",*
(2) *for every $\dot{D} \in M$, if $\Vdash_{\mathbb{P}_{\kappa+1}}$ "$\dot{D}$ is *-dense open in $\dot{\mathbb{P}}_{tail}$", there is $\alpha < \kappa^+$ such that $\Vdash_{\mathbb{P}_{\kappa+1}} \dot{D}_\alpha \subseteq \dot{D}$.*

Proof. Suppose in M, $\Vdash_{\mathbb{P}_{\kappa+1}}$ "$\dot{D}$ is *-dense open in $\dot{\mathbb{P}}_{\text{tail}}$", then $\dot{D} = j(h)(a)$. Let $\dot{C}$ be a name for a *-dense subset of $\mathbb{P}_{\text{tail}}$ that is the intersection of all *-dense open subsets whose names are given by $j(f)(b)$ for some $b \in [\lambda]^{<\omega}$. Note that $\dot{C}$ exists since $\mathbb{P}_{\text{tail}}$ is $\leq \lambda$-weakly closed. Also note that $\dot{C}$ is definable from κ, $j(h)$ and $j(\mathbb{P}_\kappa)$. Our $\dot{D}_\alpha$ are going to be precisely these $\dot{C}$. So we need to argue that there are at most κ^+ many such names $\dot{C}$. Let $X = \{j(f)(k) \mid f : \kappa \to V\}$, then $X \prec M$ and $X^\kappa \subseteq X$. Observe that $\mathbb{P}_{\kappa+1}$, $\dot{\mathbb{P}}_{\text{tail}}$, and each of the names $\dot{C}$ above is in X. Since $\mathbb{P}_{\text{tail}}$ has size $j(\kappa)$ in $M[G][g]$, it suffices to count the number of $\mathbb{P}_{\kappa+1}$-nice names for a subset of $j(\kappa)$ of the form $j(f)(\kappa)$. It is clear that a nice name for a subset of $j(\kappa)$ can be coded by a subset of $j(\kappa)$. Thus, we need to count the number of subsets of $j(\kappa)$ of the form $j(f)(\kappa)$. Suppose $j(f)(\kappa)$ is a subset of $j(\kappa)$, then we can assume wlog that $f : \kappa \to \mathcal{P}(\kappa)$. There are $(2^\kappa)^\kappa = 2^\kappa = \kappa^+$ many such f by our assumption. □

Thus, it suffices for G_{tail}, our pseudo-generic, to meet the sequence $\langle D_\alpha \mid \alpha < \kappa^+ \rangle$ above.

Lemma 2.3. *There is a sequence $\langle \dot{r}_\alpha \mid \alpha < \kappa^+ \rangle$ such that:*

(1) $\dot{r}_\alpha \in X$,
(2) $\Vdash_{\mathbb{P}_{\kappa+1}} \dot{r}_\alpha \in \dot{D}_\alpha$,
(3) $\Vdash_{\mathbb{P}_{\kappa+1}} \dot{r}_\alpha \leq_* \dot{r}_\beta$.

Proof. We will define the sequence by induction on α. Suppose $\dot{r}_\beta$ has been defined to satisfy the above properties for $\beta < \alpha$. Since $X^\kappa \subseteq X$ and each $\dot{r}_\alpha \in X$, the sequences $\langle \dot{r}_\beta \mid \beta < \alpha \rangle$ and $\langle \dot{D}_\beta \mid \beta < \alpha \rangle$ are in X. Since $\Vdash_{\mathbb{P}_{\kappa+1}}$ "$\dot{\mathbb{P}}_{\text{tail}}$ is at least $\leq \lambda$-weakly closed", there is a name $\dot{r}$ such that $\Vdash_{\mathbb{P}_{\kappa+1}} \dot{r} \leq^* \dot{r}_\beta$. In M, choose $\dot{r}_\alpha$ such that $\Vdash_{\mathbb{P}_{\kappa+1}}$ "$\dot{r}_\alpha \leq^* \dot{r}$ and $\dot{r}_\alpha \in \dot{D}_\alpha$". By elementarity, we can assume wlog that $\dot{r}_\alpha \in X$. □

Lemma 2.4. *If $D \subseteq j(\mathbb{P}_\kappa)$ is *-dense, then D meets $G * g * G_{tail}$.*

Proof. We will think of elements of D as triples $(p, \dot{q}, \dot{r})$ where $p \in \mathbb{P}_\kappa$, $\dot{q} \in \dot{\mathbb{Q}}$, and $\dot{r} \in \dot{\mathbb{P}}_{\text{tail}}$. Define $\dot{D}$ such that $\Vdash_{\mathbb{P}_{\kappa+1}}$ "$\dot{r} \in \dot{D}$ whenever $(p, \dot{q}) \in \dot{G} * \dot{g}$ and $(p, \dot{q}, \dot{r}) \in \check{D}$". I claim $\Vdash_{\mathbb{P}_{\kappa+1}} \dot{D}$ is *-dense. Fix $\dot{s}$ such that $\Vdash_{\mathbb{P}_{\kappa+1}} \dot{s} \in \dot{\mathbb{P}}_{\text{tail}}$ and define D' such that $(p, \dot{q}) \in D'$ whenever $(p, \dot{q}, \dot{r}) \in D$ and $(p, \dot{q}) \Vdash \dot{r} \leq^* \dot{s}$. It suffices to observe that D' is clearly dense. So $\dot{D}_{G*g}$ is *-dense in $M[G][g]$ and thus contains some r_α. It follows that there is $\dot r$ such that $\Vdash_{\mathbb{P}_{\kappa+1}} \dot r = \dot r_\alpha$ and $(p, \dot q, \dot r) \in D$ for some $(p, \dot q) \in G * g$. $\square$

Next we will define an extender on $[\kappa]^{<\omega_1}$ in $V[G][g]$. Notice that it suffices to define the extender on $[\kappa]^{<\omega_1}$ of $V[G]$ since $\mathbb{Q}$ is $\leq \kappa$-weakly closed and therefore does not add subsets to κ. Also, notice that usually an extender is defined on $[\kappa]^{<\omega}$. Our more general extender will guarantee the well-foundedness of the direct limit.

Fix $a \in [\lambda]^{<\omega_1}$ and let $\alpha = \mathrm{Ot}(a)$. Observe that since $\mathrm{cof}(\lambda) > \omega$, we have $a \in V_\lambda[G][g]$ and hence there is a $j(\mathbb{P}_\kappa)$-name $\dot a$ for a in V_λ such that

$$\Vdash_{j(\mathbb{P}_\kappa)} \dot a \subseteq \check\lambda \text{ and } \mathrm{Ot}(\dot a) = \check\alpha$$

We will define E^*_a on $[\kappa]^\alpha$ in $V[G][g]$ as follows:
Let $A \subseteq [\kappa]^\alpha$ in $V[G]$, then $A \in E^*_a$ if there is a $\mathbb{P}_\kappa$-name $\dot A$ for A such that there is $(p, \dot q, \dot r_\alpha) \Vdash \dot a \in j(\dot X)$ in $G * g * G_{\mathrm{tail}}$.

Remark 2.5. It is easy to see that equivalently we can require "for every $\mathbb{P}_\kappa$-name...." in the definition above. Suppose $\dot B$ is another name for A, then there is $p' \in G$ such that $p' \leq p$ and $p' \Vdash \dot A = \dot B$. It follows that $j(p') \Vdash j(\dot A) = j(\dot B)$. The condition $(p', \dot q, \dot r_\alpha)$ is below both $(p, \dot q, \dot r_\alpha)$ and $j(p')$. Therefore $(p', \dot q, \dot r_\alpha) \Vdash \dot a \in j(\dot B)$.

Lemma 2.6.

(1) *E^*_a is a κ-complete ultrafilter on $[\kappa]^{Ot(a)}$ in $V[G][g]$,*
(2) *if $a \in V$ is finite, then E^*_a extends E_a.*

Proof of (1). Suppose a is a countable subset of λ in $V[G][g]$ of order type α. We have

$$\Vdash_{j(\mathbb{P}_\kappa)} \dot a \in [j(\check\kappa)]^{\check\alpha}$$

Hence $[\kappa]^\alpha \in E^*_a$.

Suppose $A \subseteq [\kappa]^\alpha$ and $A \notin E^*_a$. Let $\dot A$ be a $\mathbb{P}_\kappa$-name for A such that $\Vdash_{\mathbb{P}_\kappa}$ "$\dot A \subseteq [\check\kappa]^{\check\alpha}$, then $\Vdash_{j(p_\kappa)} j(\dot A) \subseteq [j(\check\kappa)]^{\check\alpha}$. Thus,

$$\Vdash_{j(\mathbb{P}_\kappa)} \text{“}\dot a \in j(\dot A) \text{ or } \dot a \in j(\dot A)^C\text{”}.$$

The set deciding which situation occurs is *-dense and hence meets $G * g * G_{\mathrm{tail}}$ by Lemma 2.4.

Finally, suppose $\langle A_\alpha \mid \alpha < \beta\rangle$ is an element of $V[G]$ where $\beta < \kappa$ and each $A_\alpha \in E^*_a$. We will show that the intersection is an element of E^*_a as well. There are names $\dot A_\alpha$ for $\alpha < \beta$ such that $(p_\alpha, \dot q_\alpha, \dot r_{\gamma_\alpha}) \Vdash \dot a \in j(\dot A_\alpha)$. We can easily replace each r_{γ_α} by a fixed r_γ since r_γ just needs to get below all of them and there are only β many. By the remark above we can assume wlog that the sequence $\langle \dot A_\alpha \mid \alpha < \beta\rangle$ is an element of V. Let $\dot A$ be a name for the intersection of the sequence, then $\Vdash_{j(\mathbb{P}_\kappa)}$ "$\dot a \in j(\dot A)$ or $\dot a \in j(\dot A)^C$". Suppose there is $(p, \dot q, \dot r_\xi) \Vdash \dot a \in j(\dot A)^C$, then it also forces that there exists $\delta < \beta$ such that $\dot a \notin j(\dot A_\delta)$. Wlog we can assume that $\xi \leq \gamma$ and replace r_ξ by r_γ. Finally, if H is any generic for $\mathbb{P}_{\mathrm{tail}}$ containing r_γ, then $M[G][g][H]$ would have to satisfy that $\dot a_{G*g*H} \in j(\dot A_\alpha)_{G*g*H}$ and $\exists \delta < \beta$ such that $\dot a_{G*g*H}$ is not in the intersection of the $j(\dot A_\alpha)_{G*g*H}$. But this is clearly impossible. $\square$

Proof of (2). If $a \in V$ is finite and $A \in E_a$, then $a \in j(A)$. Thus $\Vdash_{j(\mathbb{P}_\kappa)} \check a \in j(\check A)$. $\square$

Let $j_a^* : V[G][g] \to M_a^*$ be the ultrapower embedding by E_a^*. For $a \subseteq b$, let $k_{ab} : M_a^* \to M_b^*$. Let $k_{a\infty} : M_a^* \to M^*$ and let $j^* : V[G][g] \to M^*$ be the direct limit embedding.

Lemma 2.7. *M^* is well-founded.*

Proof. Suppose M^* is not well-founded. Elements of M^* have the form $k_{a\infty}([f]_{E_a^*})$ where $f : [\kappa]^{\mathrm{Ot}(a)} \to V[G][g]$. Let

$$\cdots \ni k_{a_n\infty}([f_n]_{E_{a_n}^*}) \ni \cdots \ni k_{a_1\infty}([f_1]_{E_{a_1}^*}) \ni k_{a_0\infty}([f_0]_{E_{a_0}^*}).$$

Let $a = \cup_{n\in\omega} a_n$. Then $k_{a_n\infty}([f_n]_{E_{a_n}^*}) = k_{a\infty}(k_{a_n a}([f_n]_{E_{a_n}^*})) = k_{a\infty}([g_n]_{E_a^*})$. It follows that

$$\cdots \in [g_n]_{E_{a_n}^*} \in \cdots \in [g_1]_{E_{a_1}^*} \in [g_0]_{E_{a_0}^*}$$

which is impossible. So M^* is well-founded. □

We used the E_a^* with the infinite a to demonstrate well-foundedness. Observe that if we restrict only to E_a^* for finite a, the smaller direct limit embeds into the full direct limit and is therefore well-founded. By thus restricting, we are back to the usual extender definition.

We will show that $j^* : V[G][g] \to M^*$ is a λ-strongness embedding, that is $V[G][g]_\lambda \subseteq M^*$. The strategy will be to show that for $h : [\kappa]^n \to V_\kappa$ in V with a certain property $(*)$, we will have $j^*(h)(a) = j(h)(a)$. We will then show that every $t \in V_\lambda$ can be represented by $j(h)(a)$ where h has property $(*)$ and therefore $t \in M^*$.

Recall that $j(F)(\kappa) = \lambda$ for the initial embedding $j : V \to M$. Suppose $a \in [\lambda]^{<\omega}$, $\kappa \in a$, and $|a| = n$. Define $G : [\kappa]^n \to V$ by $G(\langle \alpha_0, \ldots, \alpha_n \rangle) = F(\alpha_i)$ where κ is the i^{th} element of a, then $j(G)(a) = j(F)(\kappa) = \lambda$. Let $r : \kappa \to V_\kappa$ be defined by $r(\alpha) = V_\alpha$.

Let us motivate property $(*)$. Suppose $t \in V_\lambda$, then $t = j(h)(a)$ for some $h : [\kappa]^n \to V_\kappa$. Thus, we have $j(h)(a) \in j(r \circ G)(a)$ and hence $h(\bar\alpha) \in r \circ G(\bar\alpha)$ on a set in E_a.

Suppose $f : [\kappa]^n \to V[G]_\kappa$ is in $V[G]$ and a is a finite subset of λ containing κ. We define that (f, a) has property $(*)$ if:

$$\{\bar\alpha \mid f(\bar\alpha) \in r \circ G(\bar\alpha)\} \in E_a^*.$$

Proposition 2.8.

1. *If $j^*(f)(a) = j^*(h)(b)$ where κ is an element of a and b, then (f, a) has property $(*)$ if and only if (h, b) has property $(*)$.*
2. *If (f, a) has property $(*)$ and $j^*(g)(b) \in j^*(f)(a)$ for some b containing κ, then (g, b) has property $(*)$.*
3. *$j^*(f)(a) = / \in j^*(h)(b)$ if and only if there if $(p, \dot q, \dot r_\alpha) \Vdash j(\dot f)(\check a) = / \in j(\dot h)(\check b)$ where $(p, \dot q) \in G * g$ and $\dot f$, $\dot h$ are some $\mathbb{P}_\kappa$-names for f and h.*

Lemma 2.9. *If (f, a) has property $(*)$, then there is a function $h : [\kappa]^m \to V_\kappa$ in V and a finite $b \subseteq \lambda$ such that $j^*(f)(a) = j^*(h)(b)$.*

Proof. By (3) above, we need a name $\dot f$ for f and a function h such that some $(p, \dot q, \dot r_\alpha) \Vdash j(\dot f)(\check a) = j(\check h)(\check a)$ in $G * g * G_{\text{tail}}$. Let $A = \{\bar\alpha \mid f(\bar\alpha) \in r(G(\bar\alpha))\} \in E_a^*$, then there are $\mathbb{P}_\kappa$-names $\dot A$ for A and $\dot f$ for f and a condition $(p, \dot q, \dot r_\alpha)$ in $G * g * G_{\text{tail}}$ such that

$$(p, \dot q, \dot r_\alpha) \Vdash j(\dot A) = \{\bar\alpha \mid j(\dot f)(\bar\alpha) \in j(\check r) \circ j(\check G)(\bar\alpha)\} \text{ and } \check a \in j(\dot A)$$

Thus
$$(p, \dot{q}, \dot{r}_\alpha) \Vdash j(\dot{f})(\check{a}) \in \underbrace{j(\check{r}) \circ j(\check{G})(\check{a})}_{V_\lambda}.$$
By replacing $\dot{f}$ with a different name if necessary we can assume wlog that $\Vdash_{j(\mathbb{P}_\kappa)} j(\dot{f})(\check{a}) \in V_\lambda$. Observe that there is a $\mathbb{P}_{\text{tail}}$-name τ such that for any generic $G*g*H$, $j(\dot{f})(\check{a})_{G*h*H} = \tau_H$. For every $t \in V_\lambda$, let $D_t = \{p \in \mathbb{P}_{\text{tail}} \mid p \Vdash \tau = \check{t} \text{ or } p \Vdash \tau \neq \check{t}\}$, then D_t is *-dense. The intersection D of D_t is *-dense since there are only λ-many t. If $p \in D$, then p decides which element of V_λ is represented by τ. Let $D_\alpha \subseteq D$, then $r_\alpha \Vdash_{\mathbb{P}_{\text{tail}}} \tau = \check{t}$. It follows that for any $(p, \dot{q}) \in G * g$, $(p, \dot{q}, \dot{r}_\alpha) \Vdash j(\dot{f})(\check{a}) = \check{t}$.

Since $t \in V_\lambda$, there is h in V such that $j(h)(b) = t$. Thus, $(p, \dot{q}, \dot{r}_\alpha) \Vdash j(\dot{f})(\check{a}) = j(\check{h})(\check{b})$. □

Let X be a subset of M consisting of all $j(f)(a)$ where (f, a) has property $(*)$ and let X^* be a subset of M^* consisting of all $j^*(f)(a)$ where (f, a) has property $(*)$.

Lemma 2.10. *X and X^* are transitive.*

Proof. By Proposition 2.8. □

Lemma 2.11. *If (h, a) has property $(*)$ with $h \in V$, then $j^*(h)(a) = j(h)(a)$.*

Proof. Let $\varphi : X \to X^*$ defined by $\varphi(j(h)(a)) = j^*(h)(a)$. Combining all the previous results shows that φ is an isomorphism. Since X and X^* are transitive, it follows that φ is the identity map. □

Corollary 2.12. *$V_\lambda \subseteq M^*$.*

Since $V_\lambda^{V[G][g]} = V_\lambda[G][g]$, to complete the argument that $j : V[G][g] \to M^*$ is a λ-strongness embedding, it remains to argue that $G * g \in M^*$.

Lemma 2.13. *$G * g \in M^*$.*

Proof. Observe that $\mathbb{P}_\kappa * \dot{\mathbb{Q}} \in M^*$ since it is an element of V_λ. Define $f : \kappa \to V_\kappa[G]$ by $f(\alpha) = G_\alpha * g_\alpha$ where $G_\alpha = G \cap \mathbb{P}_\alpha$ and $g_\alpha = G \cap \mathbb{Q}_\alpha$. We would like to show that $j^*(f)(\kappa) = G * g$. Since generics are maximal, it will suffice to show that $G * g \subseteq j^*(f)(\kappa)$. Let $(p\dot{q}) \in G * g$, then $p = j(h)(a)$ for some $h : [\kappa]^n \to V$ in V and $a \in [\lambda]^{<\omega}$. It follows that $j(h)(a) = j^*(h)(a) = (p, \dot{q})$. We can assume wlog that domain of f is $[\kappa]^{|a|}$. We need to show that $j^*(h)(a) \in j^*(f)(a)$. By proposition 2.8, we need a $\mathbb{P}_\kappa$-name $\dot{f}$ for f and a condition in $G * g * G_{\text{tail}}$ that forces
$$\underbrace{j(\check{h})(\check{a})}_{(p,\dot{q})} \in j(\dot{f})(\check{a}).$$
Choose a $\mathbb{P}_\kappa$-name $\dot{f}$ for f such that $\Vdash_{\mathbb{P}_\kappa} \dot{f}(\bar{\alpha}) = \dot{G} \cap \mathbb{P}_{\alpha_i} * \dot{G} \cap \dot{\mathbb{Q}}_{\alpha_i}$. Then
$$\Vdash_{j(\mathbb{P}_\kappa)} j(\dot{f})(\check{a}) = \dot{G} \cap \mathbb{P}_\kappa * \dot{G} \cap \dot{\mathbb{Q}}_\kappa.$$
where $\dot{G}$ is now the $j(\mathbb{P}_\kappa)$-name for the generic. Finally observe that
$$(p, \dot{q}, \dot{r}_\alpha) \Vdash \underbrace{(p, \dot{q})}_{j(\check{h})(\check{a})} \in \underbrace{\dot{G} \cap \mathbb{P}_\kappa * \dot{G} \cap \dot{\mathbb{Q}}_\kappa}_{j(\dot{f})(\check{a})}.$$
This completes the argument that $G * g \subseteq j^*(f)(\kappa)$. □

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