The hierarchy of Ramsey-like cardinals

Victoria Gitman

vgitman@nylogic.org
http://victoriagitman.github.io

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Models of set theory

A universe V of set theory satisfies the Zermelo-Fraenkel axioms ZFC.

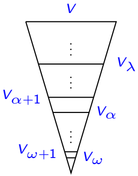
It is the union of the von Neamann hierarchy of the V_α's.

$V_0 = \emptyset$

$V_{\alpha+1} = \mathscr{P}(V_\alpha)$.

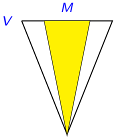
$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for a limit ordinal λ.

$V = \bigcup_{\alpha \in Ord} V_\alpha$.



Definition: A class M is an inner model of V if M is:

- transitive: if $a \in M$ and $b \in a$, then $b \in M$,
- $\mathrm{Ord} \subseteq M$,
- $M \models \mathrm{ZFC}$.



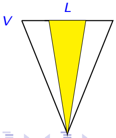
Gödel's constructible universe L is the smallest inner model.

$L_0 = \emptyset$

$L_{\alpha+1} =$ "All definable subsets of L_α"

$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ.

$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$.



A hierarchy of set-theoretic assertions

There are many set theoretic assertions independent of ZFC.

Some of these assertions are stronger than others.

A theory T is stronger than a theory S if from a model of T we can construct a model of S, but not the other way around.

Examples:

$\mathrm{ZFC} + \mathrm{CH}$ and $\mathrm{ZFC} + \neg\mathrm{CH}$ are equally strong.

- Given a model of $\mathrm{ZFC} + \mathrm{CH}$, we can use forcing to build a model of $\mathrm{ZFC} + \neg\mathrm{CH}$ and visa-versa.

ZFC and $\mathrm{ZFC} + V = L$ are equally strong.

- The axiom $V = L$: V is equal to its constructible universe.
- The constructible universe $L \models V = L$.

$\mathrm{ZFC} + \mathrm{Con}(\mathrm{ZFC})$ is stronger than ZFC.

- The axiom $\mathrm{Con}(\mathrm{ZFC})$: Gödel's arithmetized assertion that ZFC is consistent.
- The strength is a consequence of Gödel's Second Incompleteness Theorem.

$\mathrm{ZFC} + \mathrm{Con}(\mathrm{ZFC} + \mathrm{Con}(\mathrm{ZFC}))$ is stronger than $\mathrm{ZFC} + \mathrm{Con}(\mathrm{ZFC})$.

Large cardinal axioms

A large cardinal axiom LC asserts the existence of some very large infinite object.

- $\mathrm{ZFC} + \mathrm{LC}$ is stronger than ZFC.
- Large cardinal axioms form a hierarchy of increasingly strong set theoretic assertions.
- The strength of every known set theoretic assertion can be measured against the large cardinal hierarchy.

Motivating themes:

- ω is not unique: generalize combinatorial relationships between finite numbers and ω to uncountable cardinals.
- Reflection: properties of large V_α should reflect down.
- Elementary embeddings: the universe should elementarily embed into large inner models.

Smaller large cardinals are defined via combinatorial properties.

Larger large cardinals are defined via existence of elementary embeddings.

Inaccessible cardinals

Definition: (Sierpiński, Tarski) An uncountable cardinal κ is inaccessible if:

- κ is regular: there is no cofinal function $f : \alpha \to \kappa$ for any $\alpha < \kappa$,
- κ is a strong limit: $|\mathscr{P}(\alpha)| < \kappa$ for every $\alpha < \kappa$.

Observations:

- ω is inaccessible to the finite numbers.
- If κ is inaccessible, then $V_\kappa \models \mathrm{ZFC}$.
- $\mathrm{ZFC} +$ Inaccessible is stronger than ZFC.

Weakly compact cardinals

Definition: (Erdös, Tarski) An uncountable cardinal κ is weakly compact if every coloring $f : [\kappa]^2 \to 2$ has a homogeneous set of size κ.

- $[\kappa]^2 = \{\langle\alpha, \beta\rangle \mid \alpha < \beta < \kappa\}$.
- Equivalently we can use α-many colors for any $\alpha < \kappa$.

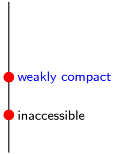
Definition: Suppose κ is a cardinal.

- A κ-tree is a tree of height κ with levels of size less than κ.
- The infinitary logic $L_{\kappa,\kappa}$ allows conjunctions, (disjunctions) and quantifier blocks of size less than κ.

Theorem: (Erdös, Tarski) An inaccessible cardinal κ is weakly compact iff Konig's Lemma holds for κ-trees (every κ-tree has a cofinal branch).

Theorem: (Kiesler, Tarski) An inaccessible cardinal κ is weakly compact iff every $<\kappa$-satisfiable theory of size κ of $L_{\kappa,\kappa}$ is satisfiable.

Theorem: A weakly compact cardinal is a limit of inaccessible cardinals.

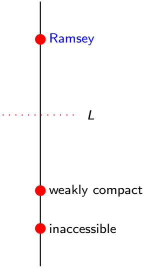


Ramsey cardinals

Definition: (Erdös, Hajnal) A cardinal κ is Ramsey if every coloring $f : [\kappa]^{<\omega} \to 2$ has a homogeneous set of size κ.

- $[\kappa]^{<\omega} = \bigcup_{n<\omega} [\kappa]^n$.
- A set $H \subseteq \kappa$ is homogeneous for a coloring $f : [\kappa]^{<\omega} \to 2$ if for every $n < \omega$, f is constant on $[H]^n$.
- Equivalently we can use α-many colors for any $\alpha < \kappa$.
- ω is not Ramsey.

Theorem: If κ is a Ramsey cardinal, then every model M in a language of size less than κ with $\kappa \subseteq M$ has a set of indiscernibles of size κ.



Theorem:

- Ramsey cardinals are weakly compact limits of weakly compact cardinals.
- (Rowbottom) Ramsey cardinals cannot exist in L.

Next up: large large cardinals...

Ultrafilters and ultrapowers

Suppose κ is a cardinal and $U \subseteq \mathscr{P}(\kappa)$ is an ultrafilter:

- If $A, B \in U$, then $A \cap B \in U$ (closed under intersections).
- If $A \in U$ and $B \supseteq A$, then $B \in U$ (closed under supersets),
- Either $A \in U$ or $\kappa \setminus A \in U$ (ultra).

Definition: Suppose $f : \kappa \to V$ and $g : \kappa \to V$.

- $f \sim g$ whenever $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U$,
- $f \in^* g$ whenever $\{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in U$.

Fact: $\sim$ is an equivalence relation with equivalence classes $[f]_U$ which respects $\in^*$.

Definition: Let the ultrapower $\text{Ult}\, V/U$ be the structure with universe $\{[f]_U \mid f : \kappa \to V\}$ and membership relation $\in^*$.

Łoś' Theorem:

- $\text{Ult}\, V/U \models \varphi([f]_U)$ iff $\{\alpha < \kappa \mid \varphi(f(\alpha))\} \in U$.
- $i : V \to \text{Ult}\, V/U$ defined by $i(a) = [c_a]_U$ is an elementary embedding ($c_a(x)$ is the constant function with value a).

The model $\text{Ult}\, V/U$ is usually ill-founded. Unless...

Well-founded ultrapowers

Definition: An ultrafilter $U \subseteq P(\kappa)$ is α-complete if it is closed under intersections of size less than α. ω_1-complete ultrafilters are called countably complete.

Theorem: An ultrafilter U is countably complete iff the ultrapower $\mathrm{Ult}\, V/U$ is well-founded.

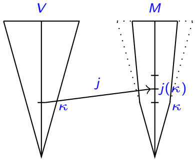
- (Mostowski Collapse) A class model with a well-founded set-like membership relation is isomorphic to an inner model.
- If there is a non-principal countably complete ultrafilter, then there is a non-trivial elementary embedding $j : V \to M$ for some inner model M.

Well-founded ultrapowers (continued)

Definition: The critical point crit(j) of an elementary embedding $j : V \to M$ is the least ordinal moved by j.

Theorem: Suppose $j : V \to M$ is an elementary embedding with $\text{crit}(j) = \kappa$. Then:

- $j \restriction V_\kappa = \text{id}$.
- $M^\kappa \subseteq M$ (every sequence of elements of M of length κ is in M).
- $V_{\kappa+1} \subseteq M$.
- $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is a non-principal κ-complete ultrafilter.
  - U is the ultrafilter generated by κ via j.
  - U is normal.



Theorem: If there is a non-principal κ-complete ultrafilter $U \subseteq P(\kappa)$, then there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$.

Normal ultrafilters

Definition:

- Suppose $\langle A_\alpha \mid \alpha < \kappa \rangle$ is a sequence of subsets of κ. The diagonal intersection

$$\Delta_{\alpha<\kappa} A_\alpha = \{\xi < \kappa \mid \xi \in \bigcap_{\alpha \in \xi} A_\alpha\}.$$

- An ultrafilter $U \subseteq P(\kappa)$ is normal if:
  - $\kappa \setminus \alpha \in U$ for all $\alpha < \kappa$,
  - U is closed under diagonal intersections.

Facts:

- A normal ultrafilter $U \subseteq P(\kappa)$ is κ-complete.
- The ultrafilter $U \subseteq P(\kappa)$ generated from an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ is normal.
- $\kappa = [\text{id}]_U$ $(\text{id}(x) = x)$.

Iterated ultrapowers

Suppose $U \subseteq P(\kappa)$ is a countably complete ultrafilter.

The ultrapower construction can be iterated along the ordinals producing a strictly decreasing chain

$$V = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_\xi \supset \cdots$$

of inner models with elementary embeddings $j_{\xi\eta} : M_\xi \to M_\eta$ between them.

- $j_{0,1} = j : V \to M_1$ is the ultrapower embedding by U.
- $j_{1,2} : M_1 \to M_2$ is the ultrapower embedding by $j_{0,1}(U) \in M_1$.
- $j_{\xi\xi+1} : M_\xi \to M_{\xi+1}$ is the ultrapower map by $U_\xi = j_{0\xi}(U)$.
- At limit stages λ, take the direct limit M_λ of the directed system of embeddings $\langle j_{\xi,\eta} : M_\xi \to M_\eta \mid \xi, \eta < \lambda \rangle$.

Theorem: (Gaifman) Countable completeness of the ultrafilter U implies that all direct limits are well-founded.

Very large cardinals

Definition: (Banach, Kuratowski, Tarski, Ulam) A cardinal κ is measurable if there exists a non-principal κ-complete ultrafilter on κ.

Fact: If κ is measurable, then there exists an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$, $V_{\kappa+1} \subseteq M$, $M^\kappa \subseteq M$.

Definition: (Mitchell) A cardinal κ is strong if for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $V_\lambda \subseteq M$.

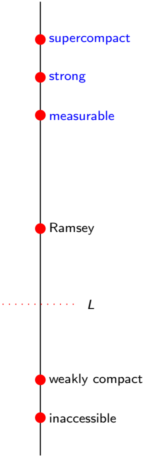
- A strongness embedding is a direct limit of a system of ultrapower embeddings.

Definition: (Reinhardt, Solovay) A cardinal κ is supercompact if for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $M^\lambda \subseteq M$.

- A supercompactness embedding is an ultrapower embedding.

Theorem:

- Measurable cardinals are Ramsey limits of Ramsey cardinals.
- Strong cardinals are measurable limits of measurable cardinals.
- Supercompact cardinals are strong limits of strong cardinals.



Small embeddings

Question: Can we characterize smaller large cardinals by the existence of elementary embeddings?

Answer: Yes, small embeddings.

Weak κ-models

Definition: ZFC^- is the theory ZFC without the power set axiom (with collection instead of replacement).

Definition Suppose λ is a cardinal. H_λ is the collection of all sets of whose transitive closure has size less than λ.

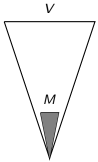
Fact: $H_{\kappa^+} \models \mathrm{ZFC}^-$.

Definition: Suppose κ is a cardinal.

- A weak κ-model is a transitive set $M \models \mathrm{ZFC}^-$ of size κ with $\kappa \in M$.
- A κ-model is a weak κ-model M such that $M^{<\kappa} \subseteq M$.

Facts:

- Every $M \prec H_{\kappa^+}$ of size κ with $\kappa \subseteq M$ is a weak κ-model.
- If κ is inaccessible, there are κ-models $M \prec H_{\kappa^+}$.
- If κ is inaccessible, then every κ-model contains V_κ.



M-ultrafilters

Suppose M is a weak κ-model.

Let $\mathscr{P}^M(\kappa) = \mathscr{P}(\kappa) \cap M$.

Definition: $U \subseteq \mathscr{P}^M(\kappa)$ is an M-ultrafilter if the structure

$$(M, \in, U) \models \text{“}U \text{ is a normal ultrafilter on } \kappa.\text{”}$$

- U is an ultrafilter on $\mathscr{P}^M(\kappa)$.
- U is closed under diagonal intersections of sequences from M.

Fact: We can form the ultrapower of M by an M-ultrafilter:

- Use functions $f : \kappa \to M$ with $f \in M$.
- Łoś' Theorem holds.
- An M-ultrafilter need not be countably complete (unless M is a κ-model).

M-ultrafilters and well-founded ultrapowers

Suppose M is a weak κ-model.

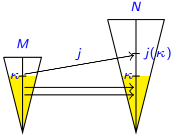
Definition: An M-ultrafilter is good if the ultrapower is well-founded.

Theorem: If an M-ultrafilter is good, then there is an elementary embedding $j : M \to N$ with $\text{crit}(j) = \kappa$.

Theorem: If there is an elementary embedding $j : M \to N$ with $\text{crit}(j) = \kappa$, then

$$U = \{A \in \mathscr{P}^M(\kappa) \mid \kappa \in j(A)\}$$

is a good M-ultrafilter.



Definition: An M-ultrafilter is α-complete if the intersection of any less than α-many sets in U is non-empty.

Fact: A countably complete M-ultrafilter is κ-complete.

Theorem: A countably complete M-ultrafilter is good.

Weakly amenable M-ultrafilters

Suppose M is a weak κ-model.

Definition: An M-ultrafilter U is weakly amenable if for every $A \in M$ with $|A|^M = \kappa$, $A \cap U \in M$.

Weakly amenable M-ultrafilters can be iterated.

- $j : M \to N$ is the ultrapower embedding by a weakly amenable M-ultrafilter U.
- Given $f : \kappa \to M$ in M, we can ask whether $\{\xi < \kappa \mid f(\xi) \in U\} \in U$.
- $j(U) = \{[f]_U \mid \{\xi < \kappa \mid f(\xi) \in U\} \in U\}$.

Definition: An elementary embedding $j : M \to N$ with $\text{crit}(j) = \kappa$ is κ-powerset preserving if $\mathscr{P}^M(\kappa) = \mathscr{P}^N(\kappa)$.

κ-powerset preservation creates reflection between M and N.

Theorem:

- If U is a good weakly amenable M-ultrafilter, then the ultrapower embedding $j : M \to N$ is κ-powerset preserving.
- If an elementary embedding $j : M \to N$ with $\text{crit}(j) = \kappa$ is κ-powerset preserving, then the M-ultrafilter U generated by κ via j is weakly amenable.

Iterable M-ultrafilters

Suppose M is a weak κ-model.

Definition: A weakly amenable M-ultrafilter U is α-good if U has α-many well-founded iterated ultrapowers.

Theorem: (Gaifman) An ω_1-good M-ultrafilter is α-good for all α.

Theorem: (Kunen) A weakly amenable countably complete M-ultrafilter is ω_1-good.

Back to weakly compact cardinals

Theorem: The following are equivalent for an inaccessible cardinal κ.

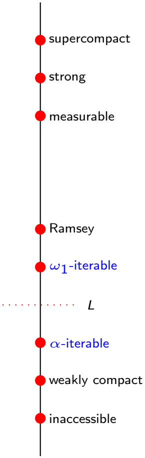
- κ is weakly compact.
- Every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a good M-ultrafilter.
- Every $A \subseteq \kappa$ is an element of a κ-model M for which there is a good M-ultrafilter.
- Every $A \subseteq \kappa$ is an element of a κ-model $M \prec H_{\kappa^+}$ for which there is a good M-ultrafilter.
- Every weak κ-model M has a good M-ultrafilter.

α-iterable cardinals

Definition: (G.) Supose $1 \leq \alpha \leq \omega_1$. A cardinal κ is α-iterable if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is an α-good M-ultrafilter.

Theorem:

- A 1-iterable cardinal κ is a weakly compact limit of weakly compact cardinals.
- (G., Welch) α-iterable cardinals for $\alpha < \omega_1$ can exist in L.
- ω_1-iterable cardinals cannot exist in L.
- (G., Welch) The α-iterable cardinals form a hierarchy of strength: An α-iterable cardinal is a limit of β-iterable cardinals for every $\beta < \alpha$.
- (Sharpe, Welch) A Ramsey cardinal is a limit of ω_1-iterable cardinals.



Back to Ramsey cardinals

Theorem: (Mitchell) The following are equivalent for a cardinal κ.

- κ is Ramsey.
- Every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a weakly amenable countably complete M-ultrafilter.
  - Ramsey ultrapower embeddings are κ-powerset preserving.
  - Ramsey ultrapower embeddings can be iterated along the ordinals.

Question: Can we replace weak κ-model with κ-model or κ-model elementary in H_{κ^+}?

Do Ramsey ultrapower embeddings exist for every weak κ-model?

Strongly and super Ramsey cardinals

Definition: (G.) A cardinal κ is:

- strongly Ramsey if every $A \subseteq \kappa$ is an element of a κ-model M for which there is a weakly amenable M-ultrafilter.
- super Ramsey if every $A \subseteq \kappa$ is an element of a κ-model $M \prec H_{\kappa^+}$ for which there is a weakly amenable M-ultrafilter.

Theorem: (G.)

- A strongly Ramsey cardinal is a Ramsey limit of Ramsey cardinals.
- A super Ramsey cardinal is a strongly Ramsey limit of strongly Ramsey cardinals.
- A measurable cardinal is a super Ramsey limit of super Ramsey cardinals.

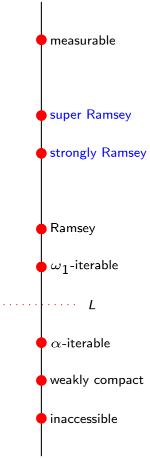
Theorem: (G.) A cardinal κ such that every weak κ-model M has a weakly amenable M-ultrafilter is inconsistent.

Theorem: (G.) A cardinal κ such that every $A \subseteq \kappa$ is an element of a weak κ-model M with $M^\omega \subseteq M$ for which there is a weakly amenable M-ultrafilter is a limit of Ramsey cardinals, but weaker than strongly Ramsey cardinal.

Questions: What large cardinal notions do we get if we:

- stratify by closure on the weak κ-model M?
- require that $M \prec H_\theta$ for some large θ? (We have to drop the requirement that M is transitive.)

The hierarchy so far



α-Ramsey cardinals

Definition: A imperfect weak κ-model is a set $M \models \mathrm{ZFC}^-$ of size κ with $\kappa + 1 \subseteq M$. An imperfect κ-model M is an imperfect weak κ-model such that $M^{<\kappa} \subseteq M$.

Definition: (Holy, Schlicht) Suppose $\omega < \alpha \leq \kappa$ is a regular cardinal. A cardinal κ is α-Ramsey if for arbitrarily large regular θ, every $A \subseteq \kappa$ an element of an imperfect weak κ-model $M \prec H_\theta$ with $M^{<\alpha} \subseteq M$ for which there is a weakly amenable M-ultrafilter.

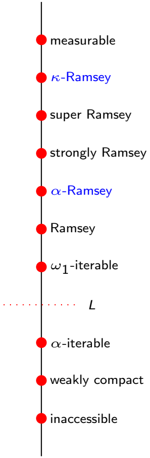
Theorem: (Holy, Schlicht) The following are equivalent.

- κ is α-Ramsey.
- For every regular θ, every $A \subseteq \kappa$ is an element...
- For arbitrarily large regular θ, every A is an element ...
- For arbitrarily large θ, there is an imperfect weak κ-model $M \prec H_\theta$ ...

Theorem:

- An ω_1-Ramsey cardinal is a Ramsey limit of Ramsey cardinals.
- (G.) A strongly Ramsey cardinal is a limit of α-Ramsey cardinals for every $\alpha < \kappa$.
- (Holy, Schlicht) A κ-Ramsey cardinal is a super Ramsey limit of super Ramsey cardinals.
- A measurable cardinal is a κ-Ramsey limit of κ-Ramsey cardinals.

The hierarchy so far



Game Ramsey cardinals

Definition: (Holy, Schlicht) Suppose κ is weakly compact, $\omega \leq \alpha \leq \kappa$ is an ordinal, and $\theta > \kappa$ is regular. Game $G^\theta_\alpha(\kappa)$ is played by the challenger and the judge.
At every stage $\gamma < \alpha$:

- the challenger plays an imperfect κ-model $M_\gamma \prec H_\theta$,
- the judge plays an M_γ-ultrafilter U_γ,
- $\langle M_{\bar{\gamma}} \mid \bar{\gamma} < \gamma\rangle$, $\langle U_{\bar{\gamma}} \mid \bar{\gamma} < \gamma\rangle \in M_\gamma$,
- $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\gamma$.

The judge wins if she can play for α-many moves and otherwise the challenger wins.

Facts: Suppose the judge wins.

- $U = \bigcup_{\gamma<\alpha} U_\gamma$ is a weakly amenable $M = \bigcup_{\gamma<\alpha} M_\gamma$-ultrafilter.
- If $\text{cf}(\alpha) > \omega$, then U is good.
- If α is regular, then $M^{<\alpha} \subseteq M$.

Game Ramsey cardinals (continued)

Theorem: (Holy, Schlicht) Suppose $\theta, \rho > \kappa$ are regular cardinals.

- The challenger has a winning strategy in $G_\alpha^\theta(\kappa)$ iff he has one in $G_\alpha^\rho(\kappa)$.
- The judge has a winning strategy in $G_\alpha^\theta(\kappa)$ iff she has one in $G_\alpha^\rho(\kappa)$.

Definition: (Holy, Schlicht) κ has the α-filter property if the challenger has no winning strategy in the game $G_\alpha^\theta(\kappa)$ for any regular $\theta > \kappa$.

Theorem: (Holy, Schlicht) A cardinal κ is α-Ramsey iff κ has the α-filter property.

Baby measurable cardinals

Definition: (Bovykin, McKenzie) A cardinal κ is baby measurable if every $A \subseteq \kappa$ is an element of a κ-model M for which there is a weakly amenable well-order $<$ of M and a weakly amenable M-ultrafilter U such that $\langle M, \in, <, U\rangle \models \mathrm{ZFC}^-$.

Theorem: (Bovykin, McKenzie) Suppose M is a κ-model, $<$ is a weakly amenable well-order of M, U is a weakly amenable M-ultrafilter, and $\langle M, \in, <, U\rangle \models \mathrm{ZFC}^-$. Then inside $\langle M, \in, <, U\rangle$, we construct the ultrapower of M by U.

Theorem:

- Measurable cardinals are baby measurable limit of baby measurable cardinals.
- (G., Schlicht) Baby measurable cardinals are limits of κ-Ramsey cardinals.

The Ramsey-like cardinals hierarchy