Indestructibility for Ramsey cardinals

Victoria Gitman

City University of New York

vgitman@nylogic.org
http://boolesrings.org/victoriagitman

April 2, 2012

This is joint work with Thomas Johnstone (CUNY)

Indestructibility

A large cardinal can be easily destroyed by forcing, e.g., force to collapse it to ω_1.

A large cardinal κ in a universe V is indestructible by a forcing notion $\mathbb{P}$ if in every forcing extension $V[G]$ by $G \subseteq \mathbb{P}$, κ retains its large cardinal property.
there is a V-generic filter $G \subseteq \mathbb{P}$ such that κ retains the large cardinal property in the forcing extension $V[G]$.

Standard indestructibility strategy:

Measurable cardinals and most stronger large cardinals κ are characterized by the existence of elementary embeddings $j : V \to M$ from the universe V into a transitive class M with critical point κ (and additional properties specific to the large cardinal).

Steps to show that κ retains the large cardinal property in $V[G]$:

- In $V[G]$, lift j to an elementary embedding $j : V[G] \to M[H]$ by finding a right M-generic filter H for the forcing notion $j(\mathbb{P}) \in M$.
- Verify that the lift j satisfies the additional properties specific to the large cardinal.

(to be continued...)

Question: Does the strategy apply to smaller large cardinals?

Elementary embeddings and smaller large cardinals

Many smaller large cardinals κ are characterized by the existence of elementary embeddings for "mini-universes" of set-theory.

The "mini-universes" are weak κ-models and κ-models of set theory.

Definitions:

- A weak κ-model of set theory is a transitive set $M \models \mathrm{ZFC}^-$ of size κ with $\kappa \in M$.
- A κ-model M of set theory is a weak κ-model such that $M^{<\kappa} \subseteq M$.
- ZFC^- is the theory ZFC without the powerset axiom and with the collection scheme instead of the replacement scheme.
- Natural examples are $M \prec H_{\kappa^+}$ of size κ with $\kappa \in M$.

Weakly compact cardinals

Definition: A cardinal κ is weakly compact if for every $f : [\kappa]^2 \to 2$, there is $H \subseteq \kappa$ of size κ such that f is constant on H.

Theorem: If $2^{<\kappa} = \kappa$, then κ is weakly compact if and only if any of the following hold:

- Every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an elementary embedding $j : M \to N$ with N transitive and $\text{cp}(j) = \kappa$.
- Every $A \subseteq \kappa$ is contained in a κ-model M for which there exists an elementary embedding $j : M \to N$ with N transitive and $\text{cp}(j) = \kappa$.
- For every κ-model M, there exists an elementary embedding $j : M \to N$ with N transitive and $\text{cp}(j) = \kappa$.

Elementary embeddings and ultrafilters

Fact: The existence of a (definable) elementary embedding $j : V \to M$ from the universe V into a transitive class M with $\text{cp}(j) = \kappa$ is equivalent to the existence of a κ-complete ultrafilter on κ.

Proof:

- The Mostowski collapse of the ultrapower of V is well-founded if and only if the ultrafilter is countably complete.
- If $j : V \to M$ is a (definable) elementary embedding with $\text{cp}(j) = \kappa$, then

$$U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$$

is a κ-complete ultrafilter on κ. □

Small elementary embeddings and M-ultrafilters

Definition: Suppose a transitive $M \models \mathrm{ZFC}^-$ and κ is a cardinal in M. A set $U \subseteq P(\kappa)^M$ is an *M-ultrafilter* if $\langle M, U\rangle \models$ "U is a normal ultrafilter on κ".

- U is κ-complete for sequences in M.
- The ultrapower of M by U is built from functions $f \in M$.
- The ultrapower of M need not be well-founded.

Fact: The existence of an elementary embedding $j : M \to N$ from a weak κ-model M into a transitive N with $\mathrm{cp}(j) = \kappa$ is equivalent to the existence of an M-ultrafilter on κ with a well-founded ultrapower.

Proof:

- If $j : M \to N$ is an elementary embedding with N transitive and $\mathrm{cp}(j) = \kappa$, then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is an M-ultrafilter.
- The ultrapower of M by U embeds into N, and is hence well-founded. □

Fact: If $j : M \to N$ is the ultrapower map by an M-ultrafilter on κ and M is a κ-model, then N is a κ-model as well. ("Ultrapowers preserve closure.")

M-ultrafilters with well-founded ultrapowers

Definition: An M-ultrafilter is countably complete if every countable sequence of its elements has a non-empty intersection.

Fact: The ultrapower of M by a countably complete M-ultrafilter is well-founded.

Fact: While countable completeness is sufficient for well-foundedness, it is not necessary.

Iterated ultrapowers

The ultrapower construction with a countably complete ultrafilter can be iterated ORD-many times:

- At successor stages, use the image of the ultrafilter from the previous stage. ($j_0 : V \to M_1$ is the ultrapower by U_0, $j_1 : M_1 \to M_2$ is the ultrapower by $U_1 = j_0(U_0)$,...)
- At limit stages, use direct limits.

Theorem: (Gaifman, 1974) The iterated ultrapowers are well-founded.

Question: Can the ultrapower construction by an M-ultrafilter be iterated?

- How to do we construct the successor stage models? If $j : M \to N$ is the ultrapower map by an M-ultrafilter U on κ, then $j(U)$ does not make sense!
- If the ultrapower is well-founded, are all the iterated ultrapowers well-founded?

Weakly amenable M-ultrafilters

Suppose $j : M \to N$ is the ultrapower map by an M-ultrafilter U on κ.

Idea: Define a predicate W on the ultrapower N corresponding to U using Łoś:

$$W = \{[f]_U \mid \{\alpha < \kappa \mid f(\alpha) \in U\} \in U\}$$

Obstacle: $\{\alpha < \kappa \mid f(\alpha) \in U\}$ might not be an element of M.

Definition: An M-ultrafilter U on κ is weakly amenable if $U \cap B$ is an element of M for every B of size κ in M.

- If $f : \kappa \to M$ is an element of M, then $\{\alpha < \kappa \mid f(\alpha) \in U\} \in M$.
- If $j : M \to N$ is the ultrapower map by a weakly amenable M-ultrafilter U, then W is a weakly amenable N-ultrafilter. ("Weak amenability propagates along the iteration.")

With a weakly amenable M-ultrafilter, the ultrapower construction can be iterated!

κ-powerset preserving embeddings

Definition: An elementary embedding $j : M \to N$ with $\text{cp}(j) = \kappa$ is κ-powerset preserving if $P(\kappa)^M = P(\kappa)^N$.

Fact: Weak amenability is equivalent to κ-powerset preservation.

- If $j : M \to N$ κ-powerset preserving, then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is weakly amenable.
- The ultrapower map $j : M \to N$ by a weakly amenable M-ultrafilter on κ is κ-powerset preserving.

Question: Does weak compactness imply the existence of κ-powerset preserving embeddings?

Definition: (G.) A cardinal κ is weakly Ramsey if every $A \subseteq \kappa$ is contained in in a weak κ-model M for which there exists a κ-powerset preserving elementary embedding.

Theorem: (G.) Weakly Ramsey cardinals are weakly compact and stationary limits of weakly compact cardinals.

Degrees of iterability

Suppose $j : M \to N$, with N transitive, is the ultrapower map by a weakly amenable M-ultrafilter U on κ.
Let $W = \{[f]_U \mid \{\alpha < \kappa \mid f(\alpha) \in U\} \in U\}$.

Question: Does the N-ultrafilter W produce a well-founded ultrapower?

Theorem: (Gaifman, 1974) If the first ω_1-many iterated ultrapowers are well-founded, then all the iterated ultrapowers are well-founded.

Theorem: (G., Welch) For every $\alpha < \omega_1$, it is consistent that there are weakly amenable M-ultrafilters producing exactly α-many well-founded iterated ultrapowers.

Theorem: (Kunen, 1970) If a weakly amenable M-ultrafilter U is countably complete, then all the iterated ultrapowers are well-founded.

Ramsey cardinals

Definition: A cardinal κ is Ramsey if for every $f : [\kappa]^{<\omega} \to 2$, there is $H \subseteq \kappa$ of size κ such that $f \upharpoonright [\kappa]^n$ is constant on H for every $n < \omega$. ("H is homogeneous for f.")

Theorem: (Mitchell, 1979) A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable countably complete (even κ-complete!) M-ultrafilter on κ.

Question: Is it equivalent to assume that the M-ultrafilters exist for κ-models (as with weakly compact cardinals)?

Definition: (G.) A cardinal κ is strongly Ramsey if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a weakly amenable M-ultrafilter on κ.

Theorem: (G.) A strongly Ramsey cardinal is Ramsey and a stationary limit of Ramsey cardinals.

Theorem: (G.) It is inconsistent to assume that for every κ-model M, there exists a weakly amenable M-ultrafilter on κ.

The indestructibility toolkit

Lifting Criterion: Suppose $j : M \to N$ is an elementary embedding of ZFC^- models having generic extensions $M[G]$ and $N[H]$ by forcing notions $\mathbb{P}$ and $j(\mathbb{P})$ respectively. The embedding j lifts to $j : M[G] \to N[H]$ with $j(G) = H$ if and only if $j \text{ " } G \subseteq H$.

Fact: The lift of an ultrapower embedding is again an ultrapower embedding.

Diagonalization Criterion: If $\mathbb{P}$ is a forcing notion in a transitive model $M \models \mathrm{ZFC}^-$ and for some cardinal κ the following criteria are satisfied:

- $M^{<\kappa} \subseteq M$,
- $\mathbb{P}$ is $<\kappa$-closed in M,
- M has at most κ many dense sets of $\mathbb{P}$,

then there is an M-generic filter for $\mathbb{P}$.

Closure Criterion: Suppose a transitive $M \models \mathrm{ZFC}^-$ and for some cardinal κ, $M^{<\kappa} \subseteq M$. If $G \subseteq \mathbb{P}$ is V-generic for a forcing notion $\mathbb{P} \in M$ having κ-cc in V, then $M[G]^{<\kappa} \subseteq M[G]$ in $V[G]$.

Indestructibility for weakly compact cardinals

Small forcing:

- $|\mathbb{P}| < \kappa$ for a large cardinal κ
- wlog $\mathbb{P} \in V_\kappa$

Theorem: (folklore) Weakly compact cardinals κ are indestructible by small forcing.
Proof:

- Fix $\mathbb{P} \in V_\kappa$ and a V-generic $G \subseteq \mathbb{P}$.
- Fix $A \subseteq \kappa$ in $V[G]$ and a nice name $\dot{A} \in H_{\kappa^+}$ such that $(\dot{A})_G = A$.
- Fix a weak κ-model M with $\dot{A} \in M$ and $j : M \to N$ with $\text{cp}(j) = \kappa$.
- Since $A \in M[G]$, it suffices to lift j.
- Lifting criterion: need an N-generic filter for $j(\mathbb{P}) = \mathbb{P}$ containing $j\,"\,G = G$.
- That's G!
- The embedding j lifts to $j : M[G] \to N[G]$. $\square$

Indestructibility for weakly compact cardinals

Canonical forcing of the GCH:

- ORD-length iteration $\mathbb{P}$
- Easton support: direct limits at inaccessibles, inverse limits elsewhere
- $\dot{\mathbb{Q}}_\alpha = \text{Add}(\alpha^+, 1)$ if α is a cardinal in $V^{\mathbb{P}_\alpha}$, trivial otherwise

Theorem: (folklore) Weakly compact cardinals κ are indestructible by the canonical forcing of the GCH.

Proof:

- It suffices to show that κ is indestructible by $\mathbb{P}_\kappa$: $\mathbb{P}_\kappa \subseteq V_\kappa$, $\mathbb{P}_\kappa$ has κ-cc.
- Fix a V-generic $G \subseteq \mathbb{P}_\kappa$.
- Fix $j : M \to N$ with critical point κ and M, N both κ-models (use an ultrapower!).
- It suffices to lift j to $M[G]$.
- Lifting criterion: need an N-generic for $j(\mathbb{P}) \cong \mathbb{P}_\kappa * \dot{\mathbb{P}}_{\text{tail}}$ containing j " $G = G$.
- Use G for $\mathbb{P}_\kappa$ and construct an $N[G]$-generic for $\mathbb{P}_{\text{tail}} = (\dot{\mathbb{P}}_{\text{tail}})_G$ in $V[G]$.
- Closure criterion: $N[G]^{<\kappa} \subseteq N[G]$ since $\mathbb{P}_\kappa$ has κ-cc
- Diagonalization criterion: $N[G]^{<\kappa} \subseteq N[G]$, $|N[G]| = \kappa$, $\mathbb{P}_{\text{tail}}$ is $\leq\kappa$-closed in $N[G]$.
- The embedding j lifts to $j : M[G] \to N[H]$ with $H = G * G_{\text{tail}}$. $\square$

Road map for Ramsey cardinals

- Come up with a diagonalization criterion that does not require closure: use the "opposite" of closure.

  Definition: (G., Johnstone) A weak κ-model M is special if it is the union of an elementary chain of transitive substructures $\langle m_i \mid i < \omega\rangle$ with $m_i \in M$ and $|m_i|^M = \kappa$.

  Fact: (G., Johnstone) If M is special and $j : M \to N$ is the ultrapower map by an M-ultrafilter on κ, then N is the union of an elementary chain of substructures $\langle x_i \mid i < \omega\rangle$ with $x_i \in N$ and $|x_i|^N = \kappa$.

  Theorem: (G., Johnstone) A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a special weak κ-model M for which there exists a weakly amenable countably complete M-ultrafilter on κ.

- Ensure that the lift is κ-powerset preserving.
  - This is almost free.
- Ensure that the lift is the ultrapower by a countably complete ultrafilter. This will work for:
  - small forcing
  - countably closed forcing

A new diagonalization criterion

Theorem: (G., Johnstone) If $\mathbb{P}$ is a forcing notion in a model M of ZFC^- and the following criteria are satisfied:

- $\mathbb{P}$ is $\leq\kappa$-closed in M,
- there is an increasing sequence $\langle X_i \mid i < \omega\rangle$ with $X_i \in M$, $|X_i|^M = \kappa$ and $M = \bigcup_{i<\omega} X_i$,

then there is an M-generic filter G for $\mathbb{P}$.

Proof:

- In M, construct a κ-length descending sequence of conditions meeting all dense sets of X_0. (Use $\leq\kappa$ closure of $\mathbb{P}$.)
- Choose p_0 below this sequence. (Use $\leq\kappa$ closure of $\mathbb{P}$.)
- Since $M = \bigcup_{i<\omega} X_i$, choose $i > 0$ such that $p_0 \in X_i$.
- In M, construct a κ-length descending sequence of conditions meeting all dense sets of X_i.
- Choose p_1 below this sequence, etc.
- Let G be the filter generated by $\langle p_n \mid n < \omega\rangle$. □

Indestructibility for Ramsey cardinals

Theorem: (G., Johnstone) Suppose

- M is a weak κ-model,
- $j : M \to N$ is the ultrapower map by a countably complete M-ultrafilter U on κ,
- $\mathbb{P} \in M$ is a countably closed forcing notion and $G \subseteq \mathbb{P}$ is V-generic.
- j lifts to an embedding $j : M[G] \to N[j(G)]$ in $V[G]$,

then the lift j is the ultrapower by a countably complete $M[G]$-ultrafilter in $V[G]$.

Proof:

- The lift $j : M[G] \to N[j(G)]$ is the ultrapower by an $M[G]$-ultrafilter W: $A \in W \Leftrightarrow \kappa \in j(A)$.
- Fix $\langle A_n \mid n < \omega \rangle \in V[G]$ with $A_n \subseteq \kappa$, $A_n \in M[G]$, $\kappa \in j(A_n)$.
- Fix $\mathbb{P}$-names $\dot{A}_n \in M$ such that $(\dot{A}_n)_G = A_n$.
- The sequence $\langle \dot{A}_n \mid n < \omega \rangle \in V$ by countable closure of $\mathbb{P}$.
- Fix a $\mathbb{P}$-name $\dot{S}$ such that
  - $1 \Vdash \dot{S}$ is an ω-sequence
  - for all $n \in \omega$, $1 \Vdash \dot{S}(\check{n}) = \dot{A}_n$
- Towards a contradiction, suppose $p \in G$ and $p \Vdash \bigcap \dot{S} = \emptyset$.

Indestructibility (continued)

- $p \in G \Rightarrow j(p) \in j(G)$
- In $j(G)$, choose $j(p) \geq p_0 \geq p_1 \geq \cdots \geq p_n \geq \cdots$ such that $p_n \Vdash \kappa \in j(\dot{A}_n)$ over N.
- Fix $f_n \in M$ such that $p_n = [f_n]_U$.
- The sequence $\langle f_n \mid n < \omega \rangle \in V$ by countable closure of $\mathbb{P}$.
- The following sets are in U:
  - $S_n = \{\xi < \kappa \mid f_n(\xi) \Vdash \check{\xi} \in \dot{A}_n \text{ over } M\}$ for $n < \omega$,
  - $T_n = \{\xi < \kappa \mid f_{n+1}(\xi) \leq f_n(\xi)\}$ for $n < \omega$,
  - $S = \{\xi < \kappa \mid f_0(\xi) \leq p\}$.
- Since U is countably complete, there is $\alpha < \kappa$ such that:
  - $f_n(\alpha) \Vdash \check{\alpha} \in \dot{A}_n$ over M for $n < \omega$,
  - $f_{n+1}(\alpha) \leq f_n(\alpha)$ for $n < \omega$,
  - $f_0(\alpha) \leq p$.
- Fix q below $p \geq f_0(\alpha) \geq f_1(\alpha) \geq \cdots \geq f_n(\alpha) \geq \cdots$.
  - $q \Vdash \check{\alpha} \in \dot{A}_n$ over M for $n < \omega$,
  - $q \Vdash \bigcap \dot{S} = \emptyset$
- Contradiction!

Indestructibility for Ramsey cardinals: summary

Theorem: (G., Johnstone, folklore) Ramsey cardinals κ are indestructible by:

- small forcing
- the canonical forcing of the GCH
- the forcing to add a fast function on κ
- the forcing to add a slim κ-Kurepa tree

Theorem: (G., Johnstone) If κ is Ramsey, then there is a forcing extension in which it becomes indestructible by $\mathrm{Add}(\kappa, \theta)$ for any cardinal θ.

Corollary: The GCH can be forced to fail at a Ramsey cardinal. (This is false for measurable cardinals)

Corollary: If Ramsey cardinals are consistent, then there is a model of ZFC in which κ is not Ramsey, but becomes Ramsey in a forcing extension.

Theorem: (G., Cody) Assuming GCH, if κ is Ramsey and *F* is a class function with domain regular cardinals such that:

- $F(\alpha) \leq F(\beta)$ for $\alpha < \beta$ and $\mathrm{cf}(F(\alpha)) > \alpha$, ("Easton's theorem")
- *F* has a closure point at κ, (necessary by inaccessibility)

then there is a cofinality preserving forcing extension in which κ remains Ramsey and $2^\delta = F(\delta)$ for every regular cardinal δ.