Set theories with classes

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ASL 2019 North American Meeting

May 22, 2019

Classes in naive set theory

The absence of a formal distinction between sets and classes in naive set theory led to supposed paradoxes.

Russell's Paradox

Let X be the set of all sets that are not members of themselves. Assuming $X \in X$ implies $X \notin X$ and assuming $X \notin X$ implies $X \in X$.

Burali-Forti's Paradox

The set of all ordinals is itself an ordinal larger than all ordinals and therefore cannot be an ordinal.

Classes in first-order set theory

The objects of first-order set theory are sets.

Definition: A class is a first-order definable (with parameters) collection of sets.

Classes play an important role in modern set theory.

- Inner models
- Ord-length products and iterations of forcing notions
- Elementary embeddings of the universe

But first-order set theory does not provide a framework for understanding the structure of classes.

In first-order set theory we cannot study:

- non-definable collections of sets,
- properties which quantify over classes.

Second-order set theory is a formal framework in which both sets and classes are objects of the set-theoretic universe. In this framework, we can study:

- non-definable classes
- general properties of classes

Kunen's Inconsistency

How do we formalize the statement of Kunen's Inconsistency that there is no non-trivial elementary embedding $j : V \to V$?

The following result is nearly trivial to prove.

Theorem: If $V \models \mathrm{ZF}$, then there is no definable non-trivial elementary embedding $j : V \to V$.

A non-trivial formulation must involve the existence and formal properties of non-definable collections.

Kunen's Inconsistency: A model of Kelley-Morse second-order set theory cannot have a non-trivial elementary embedding $j : V \to V$.

Properties of class forcing

Question: Does every class forcing notion satisfy the Forcing Theorem?

Fails in weak systems, holds in stronger systems.

Question: Does every class forcing notion have a Boolean completion?

Depends on how you define Boolean completion, with the right definition holds in strong systems.

Question: If two class forcing notions densely embed must they be forcing equivalent?

No.

Question: Does the Intermediate Model Theorem (all intermediate models between a universe and its forcing extension are forcing extensions) hold for class forcing?

Fails in weak systems, holds partially in stronger systems.

Truth predicates

Informal definition: A truth predicate is a collection of Gödel codes

$$T = \{\ulcorner\varphi(\bar{a})\urcorner \mid V \models \varphi(\bar{a})\},$$

where $\varphi(x)$ are first-order formulas.

Theorem: (Tarski) A truth predicate is never definable.

Models of sufficiently strong second-order set theories have truth predicates, as well as iterated truth predicates.

Second-order set theory

Second-order set theory has two sorts of objects: sets and classes.

Syntax: Two-sorted logic

- Separate variables for sets and classes
- Separate quantifiers for sets and classes
- Convention: upper-case letters for classes, lower-case letters for sets
- Notation:
  - Σ^0_n - first-order Σ_n-formula
  - Σ^1_n - n-alternations of class quantifiers followed by a first-order formula

Semantics: A model is a triple $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$.

- V consists of the sets.
- $\mathcal{C}$ consists of the classes.
- Every set is a class: $V \subseteq \mathcal{C}$.
- $C \subseteq V$ for every $C \in \mathcal{C}$.

Alternatively, we can formalize second-order set theory with classes as the only objects and define that a set is a class that is an element of some class.

Gödel-Bernays set theory GBC

Axioms

- Sets: ZFC
- Classes:
  - Extensionality
  - Replacement: If F is a function and a is a set, then $F \upharpoonright a$ is a set.
  - Global well-order: There is a class well-order of sets.
  - Comprehension scheme for first-order formulas:

    If $\varphi(x, A)$ is a first-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Models

- L together with its definable collections is a model of GBC.
- Suppose $V \models \mathrm{ZFC}$ has a definable global well-order. Then V together with its definable collections is a model of GBC.
- **Theorem**: (folklore) Every model of ZFC has a forcing extension with the same sets and a global well-order. Force to add a Cohen sub-class to Ord.

Strength

- GBC is equiconsistent with ZFC.
- GBC has the same first-order consequences as ZFC.

$\mathrm{GBC} + \Sigma^1_1$-Comprehension

Axioms

- GBC
- Comprehension for Σ^1_1-formulas:
  If $\varphi(x, A) := \exists X \psi(x, X, A)$ with ψ first-order, then $\{x \mid \varphi(x, A)\}$ is a class.

Note: Σ^1_1-Comprehension is equivalent to Π^1_1-Comprehension.

Question: How strong is $\mathrm{GBC} + \Sigma^1_1$-Comprehension?

Meta-ordinals

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: A meta-ordinal is a well-order $(\Gamma, \leq) \in \mathcal{C}$.

- Examples: Ord, $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \cdot \omega$.
- Notation: For $a \in \Gamma$, $\Gamma \restriction a$ is the restriction of the well-order to $\leq$-predecessors of a.

Theorem: $\mathrm{GBC} + \Sigma^1_1$-Comprehension proves that any two meta-ordinals are comparable.

- Suppose $(\Gamma, \leq)$ and $(\Delta, \leq)$ are meta-ordinals.
- Let $A = \{a \in \Gamma \mid \Gamma \restriction a \text{ is isomorphic to an initial segment of } (\Delta, \leq)\}$ (exists by Σ^1_1-Comprehension)
- If $A = \Gamma$, then $(\Gamma, \leq)$ embeds into $(\Delta, \leq)$.
- Otherwise, let a be $\leq$-least not in A. Then $\Gamma \restriction b$ is isomorphic to $(\Delta, \leq)$, where b is immediate $\leq$-predecessor of a.

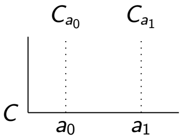
Question: Does GBC imply that any two meta-ordinals are comparable?

Problem: Meta-ordinals don't need to have unique representations! Unless...

Elementary Transfinite Recursion ETR

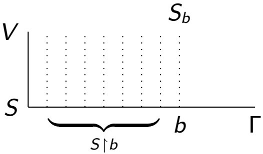
Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: Suppose $A \in \mathcal{C}$ is a class. A sequence of classes $\langle C_a \mid a \in A\rangle$ is a single class C such that $C_a = \{x \mid \langle a, x\rangle \in C\}$.



Definition: Suppose $\Gamma \in \mathcal{C}$ is a meta-ordinal. A solution along Γ to a first-order recursion rule $\varphi(x, b, F)$ is a sequence of classes S such that for every $b \in \Gamma$, $S_b = \varphi(x, b, S \restriction b)$.

Elementary Transfinite Recursion ETR: For every meta-ordinal Γ, every first-order recursion rule $\varphi(x, b, F)$ has a solution along Γ.



Theorem: $\mathrm{GBC}+\Sigma^1_1$-Comprehension implies ETR. Let $A = \{a \mid \varphi(x, b, F)$ has a solution along $\Gamma \restriction a\}$.

ETR_Γ: Elementary transfinite recursion for a fixed Γ.

- $\mathrm{ETR}_{\mathrm{Ord}\cdot\omega}$, $\mathrm{ETR}_{\mathrm{Ord}}$, ETR_ω

Theorem: (Williams) If $\Gamma \geq \omega^\omega$ is a (meta)-ordinal, then $\mathrm{GBC} + \mathrm{ETR}_{\Gamma\cdot\omega}$ implies $\mathrm{Con}(\mathrm{GBC} + \mathrm{ETR}_\Gamma)$.

Truth Predicates

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: A class $T \in \mathcal{C}$ is a truth predicate for $\langle V, \in\rangle$ if it satisfies Tarksi's truth conditions: For every $\ulcorner\varphi\urcorner \in V$ (φ possibly nonstandard),

- if φ is atomic, $V \models \varphi(\bar{a})$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$,
- $\ulcorner\neg\varphi(\bar{a})\urcorner \in T$ iff $\ulcorner\varphi(\bar{a})\urcorner \notin T$,
- $\ulcorner\varphi(\bar{a}) \wedge \psi(\bar{a})\urcorner \in T$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$ and $\ulcorner\psi(\bar{a})\urcorner \in T$,
- $\ulcorner\exists x\varphi(x, \bar{a})\urcorner \in T$ iff $\exists b\, \ulcorner\varphi(b, \bar{a})\urcorner \in T$.

Observation: If T is a truth predicate, then $V \models \varphi(\bar{a})$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$.

Corollary: GBC cannot imply that there is a truth predicate.

Observation: If T is a truth predicate and $\ulcorner\varphi\urcorner \in \mathrm{ZFC}^{\mathrm{V}}$ (φ possibly nonstandard), then $\varphi \in T$.

Theorem: If there is a truth predicate $T \in \mathcal{C}$, then V is the union of an elementary chain of its rank initial segments V_α:

$$V_{\alpha_0} \prec V_{\alpha_1} \prec \cdots \prec V_{\alpha_\xi} \prec \cdots \prec V,$$

such that V thinks that each $V_{\alpha_\xi} \models \mathrm{ZFC}$.

Let $\langle V_\alpha, \in, T \cap V_\alpha\rangle \prec_{\Sigma_1} \langle V, \in, T\rangle$.

Iterated truth predicates

Theorem: $\mathrm{GBC} + \mathrm{ETR}_\omega$ implies that for every class A, there is a truth predicate for $\langle V, \in, A\rangle$. A truth predicate is defined by a recursion of length ω.

Theorem: (Fujimoto) GBC together with the assertion that for every class A, there is a truth predicate for $\langle V, \in, A\rangle$ implies ETR_ω.

Theorem: $\mathrm{GBC} + \mathrm{ETR}_\omega$ (and therefore $\mathrm{GBC} + \Sigma^1_1$-Comprehension) implies Con(ZFC), Con(Con(ZFC)), etc.

Definition: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$ and $(\Gamma, \leq)$ is a meta-ordinal. A sequence of classes $\vec{T} = \langle T_a \mid a \in \Gamma\rangle$ is an iterated truth predicate of length Γ if for every $a \in \Gamma$, T_a is a truth predicate for $\langle V, \in, \vec{T} \restriction a\rangle$.

Theorem: (Fujimoto)

- Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$ and $\Gamma \geq \omega^\omega$ is a meta-ordinal. ETR_Γ is equivalent to the existence of an iterated truth predicate of length Γ.
- Over GBC, ETR is equivalent to the existence of an iterated truth predicate of length Γ for every meta-ordinal Γ.

A single recursion, the iterated truth recursion, suffices to give all other recursions.

The second-order constructible universe

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBC} + \mathrm{ETR}$.

Theorem: $\mathrm{GBC} + \mathrm{ETR}$ implies that any two meta-ordinals are comparable.

Problem: Meta-ordinals don't need to have unique representations! Unless...

Given a meta-ordinal Γ, we can build a meta-constructible universe L_Γ by a recursion of length Γ.

Definition: A meta-ordinal Γ is constructible if there is another meta-ordinal Δ such that L_Δ has a well-order of Ord isomorphic to Γ. ("$\Gamma \in L_{\mathrm{Ord}^+}$")

Theorem: (Tharp) Constructible meta-ordinals have unique representations.

Definition:

- A class $A \in \mathcal{C}$ is constructible if there is a constructible meta-ordinal Γ such that $A \in L_\Gamma$.
- The second-order constructible universe is $\mathscr{L} = \langle L, \in, \mathcal{L} \rangle$, where $\mathcal{L}$ consists of the constructible classes.

Theorem: If $\mathscr{V} \models \mathrm{GBC} + \Sigma^1_1$-Comprehension, then $\mathscr{L} \models \mathrm{GBC} + \Sigma^1_1$-Comprehension. It also satisfies a version of the Axiom of Choice for classes. (Coming up!)

The Class Forcing Theorem

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is a class partial order.

Definition:

- A class $\mathbb{P}$-name is a collection of pairs $\langle \sigma, p \rangle$ such that $p \in \mathbb{P}$ and $\sigma \in V^{\mathbb{P}}$.
- G is $\mathscr{V}$-generic for $\mathbb{P}$ if G meets every dense class $D \in \mathcal{C}$ of $\mathbb{P}$.
- The forcing extension $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G] \rangle$.

Observation: $\mathscr{V}[G]$ may not satisfy GBC. Force with $\mathrm{Coll}(\omega, \mathrm{Ord})$.

The Class Forcing Theorem: There is a solution to the recursion defining the forcing relation for atomic formulas.

Observation: Suppose the Class Forcing Theorem holds.

- The forcing relation for all first-order formulas with a fixed class parameter is a class.
- For every second-order formula $\varphi(x, Y)$ the relation $p \Vdash \varphi(\tau, \Gamma)$ is (second-order) definable.

Theorem: (Krapf, Njegomir, Holy, Lücke, Schlicht) GBC does not imply the Class Forcing Theorem.

Theorem: (G., Hamkins, Holy, Schlicht, Williams) Over GBC, $\mathrm{ETR}_{\mathrm{Ord}}$ is equivalent to the Class Forcing Theorem.

Class games and determinacy

Let Ord^ω be the topological space of ω-length sequences of ordinals with the product topology.

Fix a class $A \subseteq \mathrm{Ord}^\omega$.

The Game $\mathcal{G}_A$

- Player I (Alice) and Player II (Bob) alternately play ordinals for ω-many steps.

I	α_0		α_1		α_2		$\dots$		α_n		
II		β_0		β_1		β_2		$\dots$		β_n	

- Alice wins if $\vec{\alpha} = \langle \alpha_0, \beta_0, \alpha_1, \beta_1 \dots, \alpha_n, \beta_n \dots \rangle \in A$. Otherwise, Bob wins.
- The game is determined if one of the players has a winning strategy.

Note: A strategy for a player in the game $\mathcal{G}_A$ is a class.

Open Class Determinacy: $\mathcal{G}_A$ is determined for every open $A \subseteq \mathrm{Ord}^\omega$.

Clopen Class Determinacy: $\mathcal{G}_A$ is determined for every clopen $A \subseteq \mathrm{Ord}^\omega$.

Theorem: (G., Hamkins) $\mathrm{GBC}+\Sigma^1_1$-Comprehension implies Open Class Determinacy.

Theorem: (G., Hamkins) Over GBC, ETR is equivalent to Clopen Class Determinacy.

Beyond ETR

Theorem: (Sato) Over GBC, Open Class Determinacy is stronger than ETR.

Question: Is ETR preserved by tame forcing? Tame forcings preserve GBC.

Theorem: (G., Hamkins) Suppose $\mathscr{V} \models \mathrm{GBC} + \mathrm{ETR}_\Gamma$, $\mathbb{P}$ is a tame forcing notion and $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic. Then $\mathscr{V}[G] \models \mathrm{GBC} + \mathrm{ETR}_\Gamma$.

Theorem: (Hamkins) Tame countably strategically-closed forcing preserves ETR.

Question: Can forcing add meta-ordinals?

Theorem: (Hamkins, Woodin) Over GBC, Open Class Determinacy

- is preserved by tame forcing,
- implies that tame forcing does not add meta-ordinals.

The hierarchy so far

GBC + Σ^1_1-Comprehension

↓

GBC + Open Determinacy

↓

GBC + For every well-order Γ, there is an iterated truth predicate of length Γ

GBC + Clopen Class Determinacy

GBC + ETR

↓

GBC + There is an iterated truth predicate of length Ord

GBC + Class Forcing Theorem

GBC + $\mathrm{ETR}_{\mathrm{Ord}}$

↓

GBC + ETR_ω

↓

GBC + There exists a truth predicate

↓

GBC

Kelley-Morse set theory KM

Axioms

- GBC
- Full comprehension:
  If $\varphi(x, A)$ is a second-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Models

Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal. Then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}$.

Theorem: (Antos) The theory KM is preserved by tame forcing.

Theorem: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}$. Then its constructible universe $\mathscr{L} = \langle L, \in, \mathcal{L}\rangle \models \mathrm{KM}$. It also satisfies a choice principle for classes. (Next slide!)

A choice principle for classes

Choice Scheme (CC): Given a second-order formula $\varphi(x, X, A)$, if for every set x, there is a class X witnessing $\varphi(x, X, A)$, then there is a sequence of classes collecting witnesses for every x:

$$\forall x\, \exists X\, \varphi(x, X, A) \to \exists Y\, \forall x\, \varphi(x, Y_x, A).$$

Σ^1_n-Choice Scheme (Σ^1_n-CC): Choice Scheme for Σ^1_n-formulas.

Set Choice Scheme (Set-CC): Given a second-order formula $\varphi(x, X, A)$ and a set a:

$$\forall x \in a\, \exists X\, \varphi(x, X, A) \to \exists Y\, \forall x \in a\, \varphi(x, Y_x, A).$$

Proposition: Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal. Then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}+\mathrm{CC}$.

Theorem: (Marek, Mostowski, Ratajczyk) If $\mathscr{V} \models \mathrm{GBC} + \Sigma^1_n$-Comprehension, then its constructible universe $\mathscr{L} \models \mathrm{GBC} + \Sigma^1_n$-Comprehension$+\Sigma^1_n$-$\mathrm{CC}$.

- If $\mathscr{V} \models \mathrm{KM}$, then its constructible universe $\mathscr{L} \models \mathrm{KM}+\mathrm{CC}$.
- The theories KM and $\mathrm{KM}+\mathrm{CC}$ are equiconsistent.

Theorem: (G., Hamkins) There is a model of KM in which the Choice Scheme fails for for ω-many choices for a first-order formula.

Theorem: (Antos, Friedman, G.) The theory $\mathrm{KM}+\mathrm{CC}$ is preserved by tame forcing.

Fodor's Lemma for classes

The Class Fodor Principle: Every regressive class function $f : S \to \mathrm{Ord}$ from a stationary class S into Ord is constant on a stationary subclass $T \subseteq S$.

Theorem: $\mathrm{GBC} + \Sigma^0_1\text{-CC}$ implies the Class Fodor Principle.

- Suppose each $A_n = f^{-1}(\{\alpha\})$ is not stationary.
- By $\Sigma^1_0\text{-CC}$, there is a sequence of clubs $\langle C_n \mid n < \alpha\rangle$ such that A_α misses C_α.
- Let $C = \Delta_{\alpha\in\mathrm{Ord}} C_\alpha$. Then $C \cap S = \emptyset$.

Theorem: (G., Hamkins, Karagila) Every model of KM has an extension to a model of KM with the same sets in which the Class Fodor Principle fails.

- Force to add a class Cohen function $f : \mathrm{Ord} \to \omega$.
- Let $A_n = \{\alpha \mid f(\alpha) > n\}$.
- Let $\mathbb{Q} = \Pi_{n<\omega}\mathbb{Q}_n$ be the product forcing with $\mathbb{Q}_n$ shooting a club through A_n.
- Let $G \subseteq \mathbb{Q}$ be $\mathscr{V}$-generic and take only classes added by some finite stage n.

Question: How strong is the Class Fodor Principle? Does it imply $\mathrm{Con}(\mathrm{ZFC})$?

Theorem: (G., Hamkins, Karagila) The Class Fodor Principle is preserved by set forcing.

Question: Is the Class Fodor Principle preserved by tame forcing?

The Łoś Theorem for second-order ultrapowers

Suppose $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{KM}$.

- Suppose U is an ultrafilter on a cardinal κ.
- Define that functions $f : \kappa \to V$ and $g : \kappa \to V$ are equivalent when $\{\xi < \kappa \mid f(\xi) = g(\xi)\} \in U$.
- Let M be the collection of the equivalence classes $[f]_U$.
- Define that $[f]_U \,\mathrm{E}\, [g]_U$ when $\{\xi < \kappa \mid f(\xi) \in g(\xi)\} \in U$.
- Define that class sequences F and G are equivalent when $\{\xi < \kappa \mid F_\xi = G_\xi\} \in U$.
- Let $\mathcal{M}$ be the collection of the equivalence classes $[F]_U$.
- $[f]_U \,\mathrm{E}\, [F]_U$ when $\{\xi < \kappa \mid f(\xi) \in F(\xi)\} \in U$.
- $\langle M, \mathrm{E}, \mathcal{M}\rangle$ is the ultrapower of $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$ by U.
- Define $j : \langle V, \in, \mathcal{C}\rangle \to \langle M, \mathrm{E}, \mathcal{M}\rangle$ by $j(a) = [c_a]_U$ and $j(A) = [C_A]_U$.

The Łoś Theorem

Problem: For the class existential quantifier, we need

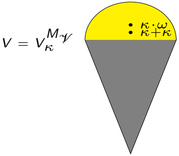
$$\mathscr{V} \models \forall \xi < \kappa\, \exists X\, \varphi(X, f(\xi)) \to \exists Y\, \forall \xi < \kappa\, \varphi(Y_\xi, f(\xi)).$$

Theorem: (G., Hamkins) Over KM, the Łoś Theorem for second-order ultrapowers is equivalent to Set-CC.

Back to first-order with $\mathrm{KM}+\mathrm{CC}$

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}+\mathrm{CC}$.

- View each extensional well-founded class relation $R \in \mathcal{C}$ as coding a transitive set.
- Define a membership relation E on the collection of all such relations R (modulo isomorphism).
  - $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \cdot \omega$.
  - $V \cup \{V\}$.
- Let $\langle M_{\mathscr{V}}, \mathrm{E}\rangle$, the companion model of $\mathscr{V}$, be the resulting first-order structure.
  - $M_{\mathscr{V}}$ has the largest cardinal $\kappa \cong \mathrm{Ord}^{\mathscr{V}}$.
  - $V_\kappa^{M_{\mathscr{V}}} \cong V$.
  - $\mathcal{P}(V_\kappa)^{M}_{\mathscr{V}} \cong \mathcal{C}$.
  - $\langle M_{\mathscr{V}}, \mathrm{E}\rangle \models \mathrm{ZFC}^-_{\mathrm{I}}$. (Next slide!)



The theory $\mathrm{ZFC}^-_{\mathrm{I}}$

Axioms

- ZFC without powerset (Collection scheme instead of Replacement scheme).
- There is a largest cardinal κ.
- κ is inaccessible. (κ is regular and for all $\alpha < \kappa$, 2^α exists and $2^\alpha < \kappa$.)

Models

Suppose that $V \models \mathrm{ZFC}$ and κ is inaccessible. Then $H_{\kappa^+} \models \mathrm{ZFC}^-_{\mathrm{I}}$.

Moving to second-order

Suppose $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with a largest cardinal κ.

- $V = V_\kappa^M$
- $\mathcal{C} = \{X \in M \mid X \subseteq V_\kappa^M\}$
- $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM{+}CC}$
- $M_{\mathscr{V}} \cong M$ is the companion model of $\mathscr{V}$.

Theorem: (Marek, Mostowski) The theory $\mathrm{KM{+}CC}$ is bi-interpretable with the theory $\mathrm{ZFC}^-_{\mathrm{I}}$.

The Class Reflection Principle

Reflection Principle: Every formula is reflected by a transitive set:

For every first-order formula $\varphi(x)$, there is a transitive set M such that for all $a \in M$, $\varphi(a)$ holds if and only if $M \models \varphi(a)$.

Theorem: (Lévy) ZFC proves the reflection principle.
Every first-order formula is reflected by some V_α.

Class Reflection Principle: Every formula is reflected by a sequence of classes:

For every second-order formula $\varphi(X)$, there is a sequence of classes $S = \langle S_\xi \mid \xi \in \mathrm{Ord}\rangle$ such that for all $\xi \in \mathrm{Ord}$, $\varphi(S_\xi)$ if and only if $\langle V, \in, S\rangle \models \varphi(S_\xi)$.

Theorem: Suppose $\mathscr{V} \models \mathrm{KM}+\mathrm{CC}$ and $M_{\mathscr{V}} \models \mathrm{ZFC}^-_{\mathrm{I}}$ is its companion model.
Then $\mathscr{V}$ satisfies the Class Reflection Principle if and only if $M_{\mathscr{V}}$ satisfies the Reflection Principle.

Another class choice principle

ω-Dependent Choice Scheme (ω-DC): Given a second-order formula $\varphi(X, Y, A)$, if for every class X, there is another class Y such that $\varphi(X, Y, A)$ holds, then we can make ω-many dependent choices according to φ:

$$\forall X \exists Y \varphi(X, Y, A) \rightarrow \exists Y \forall n \varphi(Y_n, Y_{n+1}, A).$$

Theorem: Over KM+CC, ω-DC is equivalent to the Class Reflection Principle.

Question: Does KM+CC imply ω-DC?

Conjecture: (Friedman, G.) KM+CC does not imply the ω-DC.

Suppose that α is an uncountable regular cardinal or $\alpha = \mathrm{Ord}$.

α-Dependent Choice Scheme (α-DC: Given a second-order formula $\varphi(X, Y, A)$, if for every class X, there is another class Y such that $\varphi(X, Y, A)$ holds, then we can make α-many dependent choices according to φ.

Proposition: If $\mathscr{V} \models \mathrm{KM}+\alpha\text{-DC}$, then its companion model $M_{\mathscr{V}}$ satisfies that every first-order formula reflects to a transitive set that is closed under $<\alpha$-sequences.

Theorem: If $\mathscr{V} \models \mathrm{KM}$, then its constructible universe $\mathscr{L} \models \mathrm{KM}+\mathrm{CC}+\mathrm{Ord}\text{-DC}$.

- The theories KM and KM+CC+Ord-DC are equiconsistent.

Proposition: Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal. Then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}+\mathrm{CC}+\mathrm{Ord}\text{-DC}$.

The Class Intermediate Model Theorem

Intermediate Model Theorem: (Solovay)

- If $V \models \mathrm{ZFC}$ and $W \models \mathrm{ZFC}$ is an intermediate model between V and its set-forcing extension $V[G]$, then W is a set-forcing extension of V.
- If $V \models \mathrm{ZF}$ and $V[a] \models \mathrm{ZF}$, with $a \subseteq V$, is an intermediate model between V and its set-forcing extension $V[G]$, then $V[a]$ is a set-forcing extension of V.

Definition: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{GBC}$. Then $\mathscr{W} = \langle W, \in, \mathcal{C}^* \rangle$ is a simple extension of $\mathscr{V}$ if $\mathcal{C}^*$ is generated by $\mathcal{C}$ together with a single new class.

- Forcing extension are simple extensions.

Definition: Suppose T is a second-order set theory.

- The Intermediate Model Theorem holds for T if whenever $\mathscr{V} \models T$ and $\mathscr{W} \models T$ is an intermediate model between $\mathscr{V}$ and its class-forcing extension $\mathscr{V}[G]$, then $\mathscr{W}$ is a class-forcing extension of $\mathscr{V}$.
- The simple Intermediate Model Theorem holds for T if whenever $\mathscr{V} \models T$ and $\mathscr{W} \models T$ is a simple extension of $\mathscr{V}$ between $\mathscr{V}$ and its class-forcing extension $\mathscr{V}[G]$, then $\mathscr{W}$ is a class-forcing extension of $\mathscr{V}$.

The Class Intermediate Model Theorem (continued)

Theorem:

- (Friedman) The simple Intermediate Model Theorem for GBC fails.
- (Hamkins, Reitz) The simple Intermediate Model Theorem for GBC fails even for Ord-cc forcing.

Theorem: (Antos, Friedman, G.) The simple Intermediate Model Theorem for $\mathrm{KM}+\mathrm{CC}$ holds.

Theorem: (Antos, Friedman, G.) Every model $\mathscr{V}\models\mathrm{KM}+\mathrm{CC}$ has a forcing extension $\mathscr{V}[G]$ with a non-simple intermediate model. Therefore the Intermediate Model Theorem for $\mathrm{KM}+\mathrm{CC}$ fails.

Question: Does the simple Intermediate Model Theorem for KM hold?

Boolean completions of class partial orders

Definition: Suppose $\mathbb{B}$ is a class Boolean algebra.

- $\mathbb{B}$ is set-complete if all its subsets have suprema.
- $\mathbb{B}$ is class-complete if all its subclasses have suprema.

Theorem: (Krapf, Njegomir, Holy, Lücke, Schlicht)

- Suppose $\mathscr{V} \models \mathrm{GBC}$. A class partial order $\mathbb{P}$ has a set-complete Boolean completion if and only if the Class Forcing Theorem holds for $\mathbb{P}$.
- If a Boolean algebra is class-complete, then it has the Ord-cc. Therefore a partial order has a class-complete Boolean completion if and only if it has the Ord-cc.

Definition: In a model of second-order set theory, a hyperclass is a definable collection of classes.

Definition: Suppose $\mathbb{B}$ is a hyperclass Boolean algebra.

- $\mathbb{B}$ is class-complete if all its subclasses have suprema.
- $\mathbb{B}$ is complete if all its sub-hyperclasses have suprema.

Boolean completions of class partial orders (continued)

Theorem: Suppose $\mathscr{V} \models \mathrm{GBC}$. Every class partial order $\mathbb{P}$ has a class-complete hyperclass Boolean completion. Standard regular cuts construction.

Theorem: (Antos, Friedman, G.) Suppose $\mathscr{V} \models \mathrm{KM}$. Then every class partial order $\mathbb{P}$ has a complete hyperclass Boolean completion.

Theorem: (Antos, Friedman, G.) Suppose $\mathscr{V} \models \mathrm{GBC}$. If some non-$\mathrm{Ord}$-cc class partial order $\mathbb{P}$ has a complete hyperclass Boolean completion, then $\mathscr{V} \models \mathrm{KM}$.

Suppose $\mathscr{V} \models \mathrm{KM}+\mathrm{CC}$ and $\mathbb{P}$ is class partial order. In the companion model $M_{\mathscr{V}}$ (with the largest cardinal κ):

- $\mathbb{P}$ is a set partial order.
- $\mathbb{P}$ has a class-complete Boolean completion $\mathbb{B}$.