Working in set theory without powerset

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Axioms and applications

Intuitive axiomatization: remove powerset from ZFC.

Start with ZF.

ZF^-:

- Remove powerset.
- Replace the Replacement scheme with the Collection scheme.

ZFC^-:

- Replace AC with the Well-Ordering Principle: every set can be well-ordered.

Models

- H_{κ^+}: collection of all sets with transitive closure of size $\leq\kappa$ for a cardinal κ
- A forcing extension of a model of ZFC by pretame class forcing.
  - $\Pi_{\xi\in\mathrm{Ord}}\,\mathrm{Add}(\omega,1)$.
  - $\Pi_{\alpha\in\mathrm{Card}}\mathrm{Col}(\omega,\alpha)$.
- A first-order model bi-interpretable with a model of Kelley-Morse Set Theory with the Choice Scheme

The choice of axioms

Theorem: (Szczepaniak) There is a model of ZF^- in which AC holds (every family of sets has a choice function), but the Well-Ordering Principle fails.

Open Question: Are Zorn's Lemma and AC equivalent over ZF^-?

Consider the following theory.

$\mathrm{ZFC}-$:

- Remove powerset
- Replace AC with the Well-Ordering Principle.

Theorem: (Zarach) $\mathrm{ZFC}-$ does not imply the Collection scheme.

In ZFC, the proof that Replacement implies Collection replies on the existence of the von Neumann V_α-hierarchy.

Theorem: (G., Hamkins, Johnstone) It is consistent that there are models of $\mathrm{ZFC}-$ in which:

- ω_1 is singular,
- every set of reals is countable, but ω_1 exists,
- Łoś-Theorem fails for ultrapowers.

Although the above pathological behaviors are eliminated by replacing $\mathrm{ZFC}-$ with ZFC^-, we will see later that ZFC^- is still not as robust as desired.

Second-order set theory

In first-order set theory, classes are definable collections of sets (objects in the meta-theory).

Second-order set theory has two sorts of objects: sets and classes.

Syntax: Two-sorted logic

- Separate variables and quantifiers for sets and classes
- Convention: upper-case letters for classes, lower-case letters for sets
- Notation:
  - Σ^0_n - first-order Σ_n-formula
  - Σ^1_n - n-alternations of class quantifiers followed by a first-order formula

Semantics: A model is a triple $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle$.

- V consists of the sets.
- $\mathcal{C}$ consists of the classes.
- Every set is a class: $V \subseteq \mathcal{C}$.
- $C \subseteq V$ for every $C \in \mathcal{C}$.

Second-order axioms

- Sets: ZFC
- Classes:
  - Extensionality
  - Replacement: If F is a function and a is a set, then $F \upharpoonright a$ is a set.
  - Global well-order: There is a class bijection between Ord and V.

Gödel-Bernays set theory GBC:

Comprehension scheme for first-order formulas:

If $\varphi(x, A)$ is a first-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Kelley-Morse set theory KM:

Full comprehension:
If $\varphi(x, A)$ is a second-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Choice principles for classes (continued)

Choice scheme (CC): Given a second-order formula $\varphi(x, X, A)$, if for every set x, there is a class X witnessing $\varphi(x, X, A)$, then there is a class Y collecting witnesses for every x on its slices $Y_x = \{y \mid \langle y, x\rangle \in Y\}$ so that $\varphi(x, Y_x, A)$ holds. ("AC for classes")

Theorem: (G., Hamkins) It is consistent that there is a model of KM in which the Choice scheme fails ω-many choices for a first-order formula.

Theorem: (G., Hamkins) The Łoś-Theorem for second-order ultrapowers is equivalent to the Choice scheme for set-many choices.

Proposition: The Choice scheme implies that every formula is equivalent to a Σ^1_n-formula for some n.

Theorem: (G., Hamkins) KM fails to prove that every formula of the form $\forall x \varphi(x)$, where $\varphi(x)$ is Σ^1_1, is equivalent to a Σ^1_1-formula.

Suppose δ is a regular cardinal or $\delta = \mathrm{Ord}$.

Dependent Choice scheme DC_δ: Given a second-order formula $\varphi(X, Y, A)$, if for every class X, there is a class Y such that $\varphi(X, Y, A)$ holds, then there is a class Z such that for every $\xi < \delta$, $\varphi(Z \upharpoonright \xi, Z_\xi, A)$ holds ("DC for classes").

> *"We can make δ-many dependent choices over any definable relation on classes without terminal nodes."*

Models of $\mathrm{KM} + \mathrm{CC}$

Proposition: Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal. Then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM} + \mathrm{CC} + \mathrm{DC}_{\mathrm{Ord}}$.

Consider the following theory.

$\mathrm{ZFC}^-_{\mathrm{I}}$:

- ZFC^-
- There is the largest cardinal κ.
- κ is inaccessible: κ is regular and for all $\alpha < \kappa$, 2^α exists and $2^\alpha < \kappa$.
  - V_κ exists.
  - $V_\kappa \models \mathrm{ZFC}$.

Proposition: Suppose $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with the largest cardinal κ, then $\langle V_\kappa, \in, P(V_\kappa)\rangle \models \mathrm{KM} + \mathrm{CC}$.

Bi-interpretability of $\mathrm{KM} + \mathrm{CC}$ and $\mathrm{ZFC}^-_{\mathrm{I}}$

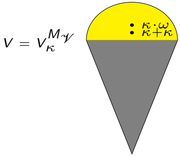
Theorem: (Marek) The theory $\mathrm{KM}{+}\mathrm{CC}$ is bi-interpretable with the theory $\mathrm{ZFC}^-_{\mathrm{I}}$.

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}{+}\mathrm{CC}$.

- View each extensional well-founded class relation $R \in \mathcal{C}$ as coding a transitive set.
  - $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \cdot \omega$
  - $V \cup \{V\}$
- Define a membership relation E on the collection of all such relations R (modulo isomorphism).
- Let $\langle M_{\mathscr{V}}, \mathsf{E}\rangle$, the companion model of $\mathscr{V}$, be the resulting first-order structure.
  - $M_{\mathscr{V}}$ has the largest cardinal $\kappa \cong \mathrm{Ord}^{\mathscr{V}}$.
  - $V_\kappa^{M_{\mathscr{V}}} \cong V$.
  - $\mathcal{P}(V_\kappa)^{M_{\mathscr{V}}} \cong \mathcal{C}$.
  - $\langle M_{\mathscr{V}}, \mathsf{E}\rangle \models \mathrm{ZFC}^-_{\mathrm{I}}$.

Suppose $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with the largest cardinal κ.

- $V = V_\kappa^M$
- $\mathcal{C} = \{X \in M \mid X \subseteq V_\kappa^M\}$
- $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}{+}\mathrm{CC}$
- $M_{\mathscr{V}} \cong M$ is the companion model of $\mathscr{V}$.



First-order Dependent Choice Scheme

Dependent Choice Scheme: DC_δ-scheme (δ regular)
Given a formula $\varphi(x, y, a)$, if for every b, there is a c such that $\varphi(b, c, a)$ holds, then there is a function f with domain δ such that for all $\xi < \delta$, $\varphi(f \upharpoonright \xi, f(\xi), a)$ holds.

$\mathrm{DC}_{<\mathrm{Ord}}$-scheme: the DC_δ-scheme for every regular δ.

Proposition: In ZFC, the $\mathrm{DC}_{<\mathrm{Ord}}$-scheme holds.

Proof: Fix a regular δ and a relation $\varphi(x, y, a)$ without terminal nodes.
Fix a V_γ, with $\mathrm{cof}(\gamma) \geq \delta$, such that V_γ reflects $\varphi(x, y, a)$ and $\forall x \exists y \varphi(x, y, a)$.

- $V_\gamma^{<\delta} \subseteq V_\gamma$.
- Use a well-ordering of V_γ together with closure to construct f. □

Models

- H_{κ^+}.
- Pretame forcing extensions of ZFC-models

  pretame forcing preserves $\mathrm{DC}_{<\mathrm{Ord}}$-scheme.
- A model $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle \models \mathrm{KM} + \mathrm{CC} + \mathrm{DC}_\delta$ if and only if its companion model $M_{\mathscr{V}} \models \mathrm{ZFC}_\mathrm{I}^- + \mathrm{DC}_\delta$-scheme.

Dependent Choice scheme without powerset

Theorem: (Friedman, G., Kanovei) There is a model $M \models \mathrm{ZFC}^-$ in which the DC_ω-scheme fails.

- Force with a tree iteration of Jensen's forcing along the tree $\omega_1^{<\omega}$.
- N is a symmetric submodel of the forcing extension.
- $M = H^N_{\omega_1}$ (ω is the largest cardinal in M)

Theorem: (G., Friedman) It is consistent that there is a model of $\mathrm{ZFC}^-_{\mathrm{I}}$ in which the DC_ω-scheme fails.

- Use a generalization of Jensen's forcing to an inaccessible κ.
- Force with a tree iteration of generalized Jensen's forcing along the tree $(\kappa^+)^{<\omega}$.

Corollary: It is consistent that there is a model of $\mathrm{KM} + \mathrm{CC}$ in which DC_ω fails.

Work in progress: (G.) It is consistent that there are models of $\mathrm{ZFC}^-_{\mathrm{I}}$ with the largest cardinal κ in which:

- DC_ω-scheme holds, but the DC_{ω_1}-scheme fails.
- DC_{ω_1}-scheme holds, but the DC_{ω_2}-scheme fails.
- DC_α-scheme holds for every regular $\alpha < \kappa$, but the DC_κ-scheme fails.

Dependent Choice scheme as a reflection principle

Proposition: (G., Hamkins, Johnstone) In ZFC^-, TFAE.

- DC_ω-scheme
- Every formula $\varphi(x, a)$ reflects to a transitive set model.

More generally, the following holds under a weak powerset existence assumption:

Proposition: (G.) In ZFC^-, TFAE for a regular δ such that $\gamma^{<\delta}$ exists for every γ.

- DC_δ-scheme
- Every formula $\varphi(x, a)$ reflects to a transitive model m such that $m^{<\delta} \subseteq m$.

Corollary: In $\mathrm{ZFC}^-_{\mathrm{I}}$, the DC_δ-scheme holds if and only if every formula reflects to a transitive set model closed under $<\delta$-sequences.

Other applications of the Dependent Choice scheme

Theorem: (Folklore) In ZFC^-, TFAE.

- $\mathrm{DC}_{<\mathrm{Ord}}$-scheme
- The partial order $\mathrm{Add}(\mathrm{Ord}, 1)$ is Ord-distributive.
- Global well-order can be forced without adding sets.

Proposition: In $\mathrm{ZFC}^- + \mathrm{DC}_\delta$-scheme, every class surjects onto δ.

Proposition: In $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Ord}}$-scheme, every class is big: surjects onto every ordinal.

ZFC^- and the Intermediate Model Theorem

Intermediate Model Theorem: (Solovay)

- If $V \models \mathrm{ZFC}$, $V[G]$ is a set forcing extension, and $W \models \mathrm{ZFC}$ such that $V \subseteq W \subseteq V[G]$, then $W = V[H]$ is a set forcing extension.
- If $V \models \mathrm{ZF}$, $V[G]$ is a set forcing extension, and $V[a] \models \mathrm{ZF}$ such that $a \subseteq V$ and $V[a] \subseteq V[G]$, then $V[a]$ is a set forcing extension.

Theorem: (Antos, G., Friedman) If $M \models \mathrm{ZFC}^-$, $M[G]$ is a set forcing extension, and $M[a] \models \mathrm{ZFC}^-$ such that $a \subseteq M$ and $M[a] \subseteq M[G]$, then $M[a] = M[H]$ is a set forcing extension.

Proof Sketch: Every poset $\mathbb{P} \in M$ densely embeds into a class complete Boolean algebra. $\square$.

Failure of the Intermediate Model Theorem

Theorem: (Antos, G., Friedman) If $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with the largest cardinal κ and $G \subseteq \mathrm{Add}(\kappa, 1)$ is M-generic, then there is a model $N \models \mathrm{ZFC}^-$ such that:

- $M \subseteq N \subseteq M[G]$,
- N is not a set forcing extension,
- if $M \models \mathrm{DC}_\kappa$-scheme, then $N \models \mathrm{DC}_\kappa$-scheme.

Proof Sketch:

- $G \subseteq \mathrm{Add}(\kappa, \kappa) \cong \mathrm{Add}(\kappa, 1)$
- $G_\xi = G \restriction \xi$ is the restriction of G to the first ξ-many coordinates of the product.
- $N = \bigcup_{\xi < \kappa} M[G_\xi]$
- Use an automorphism argument to show that N satisfies Collection. □

ZFC^- and ground model definability

Theorem: (Laver, Woodin) A model $V \models \mathrm{ZFC}$ is uniformly definable with parameters from V in all its set forcing forcing extensions.

Theorem: (G., Johnstone) If $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with the largest cardinal κ, then M is uniformly definable in its forcing extensions by any poset in V_κ.

Theorem: (G., Johnstone)

- Suppose $\kappa > \omega$ is regular and $W = V[G]$ is a forcing extension by $\mathrm{Add}(\omega, \kappa^+)$. Then $M = H^W_{\kappa^+}$ is not definable in its forcing extensions by $\mathrm{Add}(\omega, 1)$.
  - $P(\omega)$ is a proper class in M.
- Suppose κ is inaccessible and $W = V[G]$ is a forcing extension by $\mathrm{Add}(\kappa, \kappa^+)$. Then $M = H^W_{\kappa^+}$ is not definable in its forcing extensions by $\mathrm{Add}(\kappa, 1)$.
  - $M \models \mathrm{ZFC}^-_{\mathrm{I}}$.

Theorem: (Woodin) If there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$ (I_0), then $M = H_{\lambda^+}$ is not definable in its forcing extension by $\mathrm{Add}(\omega, 1)$.

- $P(\omega) \in M$.

Open Question: What is the consistency strength of having a model $M \models \mathrm{ZFC}^-$ which is not definable in a forcing extension by $\mathbb{P} \in M$ with $P(\mathbb{P}) \in M$?

ZFC^- and HOD

Theorem: The inner model HOD (hereditarily ordinal definable sets) is definable in any model $V \models \mathrm{ZFC}$.

Proof: A set a is in HOD if and only if there is α such that a is ordinal definable over V_α. □

Open Questions: Is HOD definable in

- models of ZFC^-?
- models of $\mathrm{ZFC}^-_{\mathrm{I}}$?
- models of $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Ord}}$-scheme?
- H_{κ^+}?

Strange models of ZFC^-

Set-up

$\mathbb{P} = \mathrm{Add}(\omega, \omega) \cong \mathrm{Add}(\omega, 1)$.

$G \subseteq \mathrm{Add}(\omega, \omega)$ is V-generic.

$G_n = G \upharpoonright n$ is the restriction of G to the first n-many coordinates of the product.

$N = \bigcup_{n<\omega} V[G_n]$.

Theorem: (Zarach)

- $N \models \mathrm{ZFC}^-$ automorphism argument
- V and N have the same cardinals and cofinalities forcing is ccc
- $N \models \mathrm{DC}_\omega$-scheme not obvious
- $P(\omega)$ is a proper class in N
- $P(\omega)$ is a small class in N: $P(\omega)$ does not surject onto $\gamma = (2^\omega)^+$
- $N \models \neg\mathrm{DC}_\gamma$-scheme if $2^\omega = \omega_1$, then $\gamma = \omega_2$

Does the DC_{ω_1}-scheme hold in N?

Strange models of ZFC^- (continued)

Does the DC_{ω_1}-scheme hold in N?

- $\varphi(x, y) :=$ x is a sequence of Cohen reals and y is Cohen generic over $V[x]$.
- In N, $\varphi(x, y)$ is a relation without terminal nodes.
- If $N \models \mathrm{DC}_{\omega_1}$-scheme, then there is a sequence of length ω_1 of dependent choices over φ.
- But...

Theorem: (Blass) A forcing extension by $\mathrm{Add}(\omega, 1)$ cannot have a sequence of Cohen reals $\langle r_\xi \mid \xi < \omega_1\rangle$ such that for every $\alpha < \omega_1$, r_α is Cohen generic over $V[\langle r_\xi \mid \xi < \alpha\rangle]$.

So $N \models \neg\mathrm{DC}_{\omega_1}$-scheme.

A modification of Blass's proof gives:

Theorem: Suppose κ is regular and $\kappa^{<\kappa} = \kappa$. A forcing extension by $\mathrm{Add}(\kappa, 1)$ cannot have a sequence of Cohen generic subsets $\langle A_\xi \mid \xi < \kappa^+\rangle$ of κ such that for every $\alpha < \kappa^+$, A_α is Cohen generic over $V[\langle A_\xi \mid \xi < \alpha\rangle]$.

Generalizing Zarach's construction

Set-up

$\mathbb{P} = \mathrm{Add}(\delta, \delta) \cong \mathrm{Add}(\delta, 1)$.

$G \subseteq \mathrm{Add}(\delta, \delta)$ is V-generic.

$G_\xi = G \upharpoonright \xi$ is the restriction of G to the first ξ-many coordinates of the product.

$N = \bigcup_{\xi<\delta} V[G_\xi]$.

Theorem: (G., Matthews)

- $N \models \mathrm{ZFC}^-$ automorphism argument
- V and N have almost the same cardinals and cofinalities
- $N \models \mathrm{DC}_\delta$-scheme (uses $N^{<\delta} \subseteq N$ in $V[G]$)
- $P(\delta)$ is a proper class in N
- $P(\delta)$ is a small class in N: $P(\delta)$ does not surject onto $\gamma = ((2^\delta)^+)^{V[G]}$
- $N \models \neg\mathrm{DC}_\gamma$-scheme
- If $\delta^{<\delta} = \delta$, then $N \models \neg\mathrm{DC}_{\delta^+}$- scheme.

Jensen's forcing and generalization to inaccessible κ

$\mathbb{J}$: (Jensen)

- subposet of Sacks forcing: perfect trees ordered by $\subseteq$
- constructed using $\diamondsuit$
- ccc
- adds a unique generic real

Theorem: (Lyubetsky, Kanovei, Abraham, G., Friedman) Products and iterations of $\mathbb{J}$ have "unique generics" properties.

Suppose κ is inaccessible.

$\mathbb{J}(\kappa)$ (G., Friedman)

- perfect κ-trees ordered by $\subseteq$
- constructed using $\diamondsuit_{\kappa^+}(\text{cof}(\kappa))$
- $<\kappa$-closed
- κ^+-cc
- adds a unique generic subset of κ

Theorem: (G., Friedman) Products and iterations of $\mathbb{J}(\kappa)$ have "unique generics" properties.

A different model of $\mathrm{ZFC}^- + \neg\mathrm{DC}_\omega$-scheme

Theorem: (G., Matthews) There is a model of $N \models \mathrm{ZFC}^-$ such that:

- $P(\omega)$ is a proper class.
- Every class is big.
- There are unboundedly many cardinals.
- DC_ω-scheme fails.

Proof Sketch: Force with the tree iteration $\mathbb{P}$ of Jensen's forcing along the tree $\mathrm{Ord}^{<\omega}$. Let $G \subseteq \mathbb{P}$ be V-generic.

- $\mathbb{P}$ has the ccc, and hence is pretame.
- $V[G] \models \mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Ord}}$-scheme.
- $N = \bigcup L[G_T]$, where T is a certain set subtree of $\mathrm{Ord}^{<\omega}$, $\mathbb{P}_T$ is the restriction of $\mathbb{P}$ to T, and G_T is the restriction of G to $\mathbb{P}_T$. $\square$