A set theoretic approach to Scott's Problem

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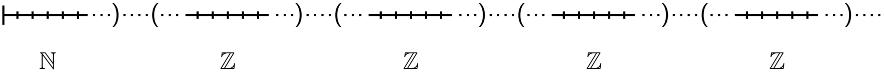
Models of Peano Arithmetic

- The first-order language of arithmetic is $\mathcal{L}_A = \langle +, \cdot, <, 0, 1 \rangle$.
- Peano Arithmetic (PA) is the usual axiomatization of number theory.
  - Commutativity and associativity of addition and multiplication, distributive law, ordering is discrete with least element 0, 0 is the additive identity, etc.
  - Induction scheme: for every $\mathcal{L}_A$-formula $\varphi(x, \vec{y})$,

$$\forall \vec{y}[(\varphi(0, \vec{y}) \wedge (\forall x\, \varphi(x, \vec{y}) \rightarrow \varphi(x+1, \vec{y}))) \rightarrow \forall x \varphi(x, \vec{y})].$$

- The natural numbers $\langle \mathbb{N}, +, \cdot, <, 0, 1 \rangle$ is the standard model of PA.
- The order-type of a nonstandard model of PA is $\mathbb{N} + \mathbb{A} \cdot \mathbb{Z}$ for a dense linear order $\mathbb{A}$ without endpoints.



- **Question**: If $M \models \mathrm{PA}$ is countable then $\mathbb{A} = \mathbb{Q}$. What can $\mathbb{A}$ be if M is uncountable?
- **Tennenbaum's Theorem**: (1959) There are no recursive nonstandard models of PA.

The standard system of a nonstandard model of PA

Suppose $M \models \mathrm{PA}$ is nonstandard.

Definition: (Friedman, 1973) The standard system of M, denoted $\mathrm{SSy}(M)$, is the collection of subsets of $\mathbb{N}$ that arise as intersections of definable subsets of M with $\mathbb{N}$.

$\mathrm{SSy}(M) = \{A \subseteq \mathbb{N} \mid A = \bar{A} \cap \mathbb{N} \text{ for some } \bar{A} \in \mathrm{Def(M)}\}$.

Definition:

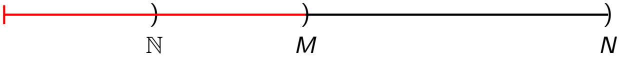
- An element $a \in M$ codes a set $A \subseteq \mathbb{N}$ if

$$A = \{n \in \mathbb{N} \mid M \models \text{“the } n\text{-th digit in the binary expansion of } a \text{ is } 1\text{”}\}.$$

- A set $A \subseteq \mathbb{N}$ is coded in M if there is $a \in M$ coding A.

Proposition: $\mathrm{SSy}(M)$ is the collection of all sets coded in M.

Corollary: If $M \models \mathrm{PA}$ is a submodel of $N \models \mathrm{PA}$ and M is an initial segment of N, then $\mathrm{SSy}(M) = \mathrm{SSy}(N)$.



The standard system of a nonstandard model of PA (continued)

Standard systems play an important role in the study of nonstandard models.

Here are some examples:

Proposition: For every $n \in \mathbb{N}$, the Σ_n-theory of a model is in its standard system.

Theorem: (Jensen and Ehrenfeucht, Wilmers?, 1970s) Two countable recursively saturated models M and N of PA are isomorphic if and only if they have the same theory and the same standard system.

A model is recursively saturated if it realizes all finitely realizable recursive types.

Friedman's Embedding Theorem: (1973) A countable model $M \models PA$ Σ_n-elementarily embeds into another model $N \models PA$ if and only if N satisfies the Σ_{n+1}-theory of M and $\mathrm{SSy}(M) \subseteq \mathrm{SSy}(N)$.

Properties of standard systems

Suppose $M \models \mathrm{PA}$ is nonstandard.

1. $\mathrm{SSy}(M)$ is a Boolean algebra.

2. $\mathrm{SSy}(M)$ is closed under relative recursion:
If $A \in \mathrm{SSy}(M)$ and $B \leq_T A$, then $B \in \mathrm{SSy}(M)$.

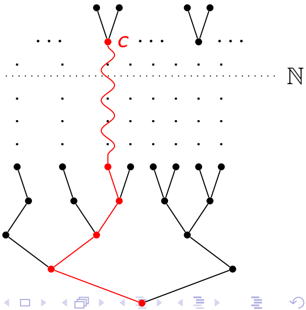
Proof: Suppose $A = \bar{A} \cap \mathbb{N}$ with $\bar{A} \in \mathrm{Def}(M)$.

- Let m be the Turing machine computing B from A.
- Define $\bar{B}$ to be the output of m with oracle $\bar{A}$.
- $B = \bar{B} \cap \mathbb{N}$. $\square$

3. $\mathrm{SSy}(M)$ has the tree property:
If $T \in \mathrm{SSy}(M)$ is an infinite binary tree, then $\mathrm{SSy}(M)$ has some infinite branch of T.

Proof: Suppose $T = \bar{T} \cap \mathbb{N}$ with $\bar{T} \in \mathrm{Def}(M)$.

- $\bar{T}$ is a binary tree up to some nonstandard level. (Otherwise, we could define $\mathbb{N}$.)
- Let $c \in \bar{T}$ be a nonstandard node.
- Let $\bar{B}$ be the predecessors of c in $\bar{T}$.
- $B = \bar{B} \cap \mathbb{N}$ is an infinite branch of T. $\square$



Scott sets and Scott's Problem

Definition: (Scott, 1962) A Scott set is a nonempty Boolean algebra of subsets of $\mathbb{N}$ that is closed under relative recursion and satisfies the tree property.

Aside: Scott sets are the ω-models of the second-order arithmetic theory WKL_0.

Suppose $\mathscr{X}$ is a Scott set.

- Every recursive set is in $\mathscr{X}$.
- Every consistent theory $S \in \mathscr{X}$ has a consistent completion $\bar{S} \in \mathscr{X}$.

  (Use the tree property.)

We just argued that every standard system is a Scott set.

Scott's Problem: Is every Scott set the standard system of some model of PA?

Scott's Problem for countable Scott sets

Theorem: (Scott, 1962) Suppose $\mathscr{X}$ is a countable Scott set and $S \in \mathscr{X}$ is a consistent theory extending PA. Then there is a model $M \models S$ such that $\mathrm{SSy}(M) = \mathscr{X}$.

Proof: We carry out the Henkin construction "inside $\mathscr{X}$".
Enumerate $\mathscr{X} = \{A_n \mid n < \omega\}$.

Stage 0:

Let $\mathcal{L}_0$ be $\mathcal{L}_A$ together with

- a new constant a_0,
- Henkin constants $\{c_i^0 \mid i < \omega\}$.

Let T_0' be S together with

- sentences $\{\varphi_i \mid i \in \mathbb{N}\}$ expressing that a_0 codes A_0:
  if $i \in A_0$, φ_i says "The i-th digit in the binary expansion of a_0 is 1",
  if $i \notin A_0$, φ_i says "The i-th digit in the binary expansion of a_0 is 0",
- Henkin sentences for formulas in $\mathcal{L}_A$.

$T_0' \in \mathscr{X}$ because it is recursive in S, A_0.
Let $T_0 \in \mathscr{X}$ be some consistent completion of T_0'.

Scott's Problem for countable Scott sets (continued)

Stage n:

Let $\mathcal{L}_n$ be $\mathcal{L}_{n-1}$ together with

- a new constant a_n,
- Henkin constants $\{c_i^n \mid i < \omega\}$.

Let T_n' be T_{n-1} together with

- sentences $\{\varphi_i \mid i \in \mathbb{N}\}$ expressing that a_n codes A_n.
- Henkin sentences for formulas in $\mathcal{L}_{n-1}$.

$T_n' \in \mathscr{X}$ because it is recursive in T_{n-1}, A_n.
Let $T_n \in \mathscr{X}$ be some consistent completion of T_n'.

The union:

$T := \bigcup_{n<\omega} T_n$ is a complete consistent Henkin theory.
The Henkin model $M \models T$ has $\mathrm{SSy}(M) = \mathscr{X}$:

- a_n codes A_n.
- If c_i^n codes A, then $A \in \mathscr{X}$ because A is recursive in the complete theory T_n. $\square$

Scott's Problem for Scott sets of size ω_1

Ehrenfeucht's Lemma: (1970s) Suppose M is a countable model of PA and $\mathscr{X}$ is a Scott set such that $\mathrm{SSy}(M) \subseteq \mathscr{X}$. For every $A \in \mathscr{X}$, there is an elementary extension N of M such that $A \in \mathrm{SSy}(N) \subseteq \mathscr{X}$.

Proof:

- Choose a countable Scott set $\mathscr{Y} \subseteq \mathscr{X}$ with $\mathrm{SSy}(M) \cup \{A\} \subseteq \mathscr{Y}$.
- The Σ_1-theory of M, $\mathrm{Th}_{\Sigma_1}(M) \in \mathrm{SSy}(M) \subseteq \mathscr{Y}$.
- By Scott's Theorem, there is a countable $K \models \mathrm{PA} + \mathrm{Th}_{\Sigma_1}(M)$ such that $\mathrm{SSy}(K) = \mathscr{Y}$.
- By Friedman's Embedding Theorem, we can assume $M \prec_{\Delta_0} K$.
- Let N be the closure under initial segment of M in K.

- Since $M \prec_{\Delta_0} N$ and M is cofinal in N, it follows that $M \prec N$.
- $\mathrm{SSy}(N) = \mathrm{SSy}(K) = \mathscr{Y}$ since N is an initial segment of K. $\square$

Scott's Problem for Scott sets of size ω_1 (continued)

Theorem: (Knight, Nadel, 1982) Every Scott set of size ω_1 is the standard system of some model of PA.

Proof: Let $\mathscr{X}$ be a Scott set of size ω_1.

- Enumerate $\mathscr{X} = \{A_\xi \mid \xi < \omega_1\}$.
- Choose a countable Scott set $\mathscr{X}_0 \subseteq \mathscr{X}$.
- By Scott's Theorem, there is $M_0 \models \mathrm{PA}$ with $\mathrm{SSy}(M_0) = \mathscr{X}_0$.
- By Ehrenfeucht's Lemma, given $M_\xi \models \mathrm{PA}$, let $M_\xi \prec M_{\xi+1}$ with $A_\xi \in \mathrm{SSy}(M_{\xi+1}) \subseteq \mathscr{X}$.
- For limit ordinals λ, let $M_\lambda = \bigcup_{\xi<\lambda} M_\xi$.
- In ω_1-steps, we obtain a continuous elementary chain:

$$M_0 \prec M_1 \prec \cdots \prec M_\xi \prec \cdots \prec M$$

- The model $M = \bigcup_{\xi<\omega_1} M_\xi$ has $\mathrm{SSy}(M) = \mathscr{X}$.

Corollary: Assuming CH, every Scott set is the standard system of some model of PA. Thus, it is consistent that Scott's Problem has a positive answer.

Open Questions

Question: What is the solution to Scott's Problem if CH fails?

Question: Does Ehrenfeucht's Lemma hold for uncountable models?

Incremental strategy:

- Assume $2^\omega = \omega_2$ and some additional set theoretic hypothesis.
- Make additional requirements on a Scott set $\mathscr{X}$ so that Ehrenfeucht's Lemma holds for models $M \models \mathrm{PA}$ of size ω_1 with $\mathrm{SSy}(M) \subseteq \mathscr{X}$.

Definition: A family $\mathscr{X} \subseteq \mathcal{P}(\mathbb{N})$ is called arithmetically closed (AC) if it is a nonempty Boolean algebra that is closed under the Turing jump operation.

Aside: Arithmetically closed families are the ω-models of the second-order arithmetic theory ACA_0.

Question: Is every AC family of size ω_2 the standard system of some model of PA?

An ultrapower construction

Set-up

- $M \models \mathrm{PA}$
- $\mathscr{X}$ is a Scott set with $\mathrm{SSy}(M) \subseteq \mathscr{X}$.
- U is an ultrafilter on $\mathscr{X}$.

Ultrapower construction (Kirby, Paris, 1977)

- Let $\Pi_{\mathbb{N}}^{\mathrm{Cod}} M = \{f : \mathbb{N} \to M \mid f = F \upharpoonright \mathbb{N}$ for some $F \in \mathrm{Def}(M)\}$.
- Define an equivalence relation $\sim_U$ on $\Pi_{\mathbb{N}}^{\mathrm{Cod}} M$ by

$$f \sim_U g \leftrightarrow \{n \in \mathbb{N} \mid f(n) = g(n)\} \in U.$$

- Let $\Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U$ be the collection of equivalence classes.
- $\Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U$ gets the usual ultrapower $\mathcal{L}_A$-structure, e.g.,

$$[f]_U + [g]_U = [h]_U \leftrightarrow \{n \in \mathbb{N} \mid M \models f(n) + g(n) = h(n)\} \in U.$$

- M embeds into $\Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U$ via the map sending a to $[c_a]_U$, where c_a is the constant function with value a.
- Łoś Theorem holds:

$$\Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U \models \varphi([f]_U) \leftrightarrow \{n \in \mathbb{N} \mid M \models \varphi(f(n))\} \in U.$$

- $|\Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U| = |M|$.

A forcing notion from a Scott set

Suppose A, B are infinite subsets of $\mathbb{N}$. We say that A is almost contained in B, $A \subseteq^* B$, if there is $n \in \mathbb{N}$ such that $A \setminus n \subseteq B$.

Definition: Suppose $\mathscr{X}$ is a Scott set. Let $\mathscr{X}/\mathrm{fin}$ be the partial order whose elements are infinite sets in $\mathscr{X}$ ordered by $\subseteq^*$.

Example: $\mathcal{P}(\omega)/\mathrm{fin}$ has been extensively studied.

Decisive sets

Definition: Suppose $\mathscr{X}$ is a Scott set and $\vec{B} = \langle B_n \mid n \in \mathbb{N} \rangle$ is coded in $\mathscr{X}$. A set $C \in \mathscr{X}$ decides $\vec{B}$ if whenever U is an ultrafilter on $\mathscr{X}$ with $C \in U$, then $\{n \mid B_n \in U\} \in \mathscr{X}$.

Observation: Suppose $\mathscr{X}$ is an AC family and $\vec{B}$ is a sequence coded in $\mathscr{X}$.

- If $C \in \mathscr{X}$ has $C \subseteq^* B_n$ or $C \subseteq^* \mathbb{N} \setminus B_n$ for all $n \in \mathbb{N}$, then C decides $\vec{B}$.
- There is $C \in \mathscr{X}$ such that $C \subseteq^* B_n$ or $C \subseteq^* \mathbb{N} \setminus B_n$ for all $n \in \mathbb{N}$.

Corollary: If $\mathscr{X}$ is an AC family, then for every sequence $\vec{B}$ coded in $\mathscr{X}$, there is $C \in \mathscr{X}$ deciding $\vec{B}$.

Lemma (G., 2007) TFAE for a Scott set $\mathscr{X}$.

1. $\mathscr{X}$ is arithmetically closed.
2. For every sequence $\vec{B}$ coded in $\mathscr{X}$, there is $C \in \mathscr{X}/\text{fin}$ deciding $\vec{B}$.
3. For every sequence $\vec{B}$ coded in $\mathscr{X}$, the set $\mathscr{D}_{\vec{B}}$ of C deciding $\vec{B}$ is dense in $\mathscr{X}/\text{fin}$.

A forcing notion from a Scott set (continued)

Lemma: (G., 2007) Suppose $M \models \mathrm{PA}$, $\mathscr{X}$ is an AC family with $\mathrm{SSy}(M) \subseteq \mathscr{X}$, and U is an ultrafilter on $\mathscr{X}$.

- For every $[f]_U \in \Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U$, there is a sequence $\vec{B}^f$ coded in $\mathscr{X}$ such that if there is $C \in U$ deciding $\vec{B}^f$, then the set coded by $[f]_U$ is in $\mathscr{X}$.
- For every $A \in \mathscr{X}$, there is $B^A \in \mathscr{X}/\mathrm{fin}$ such that if $B^A \in U$, then $A \in \mathrm{SSy}(\Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U)$.

Theorem: (G., 2007) Suppose

- $M \models \mathrm{PA}$,
- $\mathscr{X}$ is an AC family with $\mathrm{SSy}(M) \subseteq \mathscr{X}$,
- G is a filter on $\mathscr{X}/\mathrm{fin}$ meeting all dense sets $\mathcal{D}_{\vec{B}^f}$ for $f \in \Pi_{\mathbb{N}}^{\mathrm{Cod}} M$.

Then the ultrafilter U on $\mathscr{X}$ determined by G has the following properties:

- $\mathrm{SSy}(\Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U) \subseteq \mathscr{X}$.
- For every $A \in \mathscr{X}$, there is a $B_A \in \mathscr{X}/\mathrm{fin}$ such that if $B_A \in G$, then $A \in \Pi_{\mathbb{N}}^{\mathrm{Cod}} M/U$.

Forcing axioms

Question: How we do obtain such a partially generic filter G?

A forcing axiom asserts for a class $\mathcal{C}$ of partial orders and a cardinal κ that for every partial order $\mathbb{P} \in \mathcal{C}$ and every collection $\mathcal{D}$ of κ-many dense sets of $\mathbb{P}$ there is a filter on $\mathbb{P}$ meeting every set in $\mathcal{D}$.

Here are two examples:

- Martin's Axiom (MA): for every ccc partial order $\mathbb{P}$ and every collection $\mathcal{D}$ of $\kappa < 2^\omega$ many dense sets of $\mathbb{P}$, there a filter on $\mathbb{P}$ meeting every set in $\mathcal{D}$.
- Proper Forcing Axiom (PFA): for every proper partial order $\mathbb{P}$ and every collection $\mathcal{D}$ of ω_1-many dense sets of $\mathbb{P}$, there is a filter on $\mathbb{P}$ meeting every set in $\mathcal{D}$.

Forcing axioms and Ehrenfeucht's Lemma

Theorem:

- Assuming MA, Ehrenfeucht's Lemma holds for every AC family $\mathscr{X}$ such that $\mathscr{X}/\mathrm{fin}$ is ccc and model $M \models \mathrm{PA}$ of size $\kappa < 2^\omega$ with $\mathrm{SSy}(M) \subseteq \mathscr{X}$.
- Assuming PFA, Ehrenfeucht's Lemma holds for every AC family $\mathscr{X}$ such that $\mathscr{X}/\mathrm{fin}$ is proper and model $M \models \mathrm{PA}$ of size ω_1 with $\mathrm{SSy}(M) \subseteq \mathscr{X}$.

Theorem: (G., 2007) Assuming PFA, every AC family $\mathscr{X}$ of size ω_2 such that $\mathscr{X}/\mathrm{fin}$ is proper is the standard system of some model of PA.

Definition: Suppose $\mathscr{X}$ is an AC family of size ω_2. We say that $\mathscr{X}/\mathrm{fin}$ is piecewise proper if $\mathscr{X}$ is a chain

$$\mathscr{X}_0 \subseteq \mathscr{X}_1 \subseteq \cdots \subseteq \mathscr{X}_\xi \subseteq \cdots \subseteq \mathscr{X}$$

of AC families $\mathscr{X}_\xi$ of size ω_1 for $\xi < \omega_2$ such that each $\mathscr{X}_\xi/\mathrm{fin}$ is proper.

Theorem (G., 2007) Assuming PFA, every AC family $\mathscr{X}$ of size ω_2 such that $\mathscr{X}/\mathrm{fin}$ is piecewise proper is the standard system of some model of PA.

When is $\mathscr{X}/\mathrm{fin}$ ccc?

Lemma: If $\mathscr{X}$ is an uncountable Scott set, then $\mathscr{X}/\mathrm{fin}$ has uncountable antichains. Therefore every Scott set $\mathscr{X}$ such that $\mathscr{X}/\mathrm{fin}$ has the ccc is countable.

MA can't help in this approach to establish new instances of Ehrenfeucht's Lemma.

Question: What about PFA?

When is $\mathscr{X}/\mathrm{fin}$ proper?

Proper partial orders generalize ccc partial orders.

Theorem: (Todorčević, Veličković, 1992) Assuming PFA, $2^\omega = \omega_2$.

Observation: Suppose $\mathscr{X}$ is an AC family such that for every countable AC family $\mathscr{Y} \subseteq \mathscr{X}$ and countable collection $\mathcal{D}$ of dense sets of $\mathscr{Y}$, there is $A \in \mathscr{X}$ such that for every $D \in \mathcal{D}$, there is $B \in D$ with $A \subseteq^* B$. Then $\mathscr{X}/\mathrm{fin}$ is proper.

Theorem: (G., 2007) It is consistent that

- There are continuum many AC families $\mathscr{X}$ of size ω_1 such that $\mathscr{X}/\mathrm{fin}$ is proper.
- There are continuum many AC families $\mathscr{X}$ of size ω_2 such that $\mathscr{X}/\mathrm{fin}$ is piecewise proper.

Proof: An AC family $\mathscr{X}$ of size ω_1 such that $\mathscr{X}/\mathrm{fin}$ is proper can be added by a finite support iteration $\mathbb{P} = \{\mathbb{P}_\xi\}_{\xi<\omega_1}$ of length ω_1.

- Stage 0:
  - Let $\mathscr{X}_0$ be any countable AC family.
  - Force with $\mathscr{X}_0/\mathrm{fin}$.
- Stage 1:
  - Let $g_0 \subseteq \mathscr{X}_0/\mathrm{fin} = \mathbb{P}_0$ be generic.
  - In $V[g_0]$, choose an $A_0 \subseteq^* A$ for all $A \in g_0$.
  - Let $\mathscr{X}_1$ be the arithmetic closure of $\mathscr{X}_0$ and A_0.
  - Force with $\mathscr{X}_1/\mathrm{fin}$.

When is $\mathscr{X}/\mathrm{fin}$ proper (continued)

Proof: (continued)

- Stage $\xi+1$:
  - Let $G_\xi * g_\xi \subseteq \mathbb{P}_{\xi+1}$ be generic.
  - $\mathscr{X}_0 \subseteq \mathscr{X}_1 \subseteq \cdots \subseteq \mathscr{X}_\gamma \subseteq \cdots \subseteq \mathscr{X}_\xi$ are already constructed.
  - In $V[G_\xi][g_\xi]$, choose an $A_\xi \subseteq^* A$ for all $A \in g_\xi$.
  - Let $\mathscr{X}_{\xi+1}$ be the arithmetic closure of $\mathscr{X}_\xi$ and A_ξ.
  - Force with $\mathscr{X}_\gamma/\mathrm{fin}$ for some chosen* $\gamma \leq \xi+1$.
- Stage λ: (limit)
  - Let $G_\lambda \subseteq \mathbb{P}_\lambda$ be generic.
  - $\mathscr{X}_\gamma$ for $\gamma < \lambda$ are already constructed.
  - Let $\mathscr{X}_\lambda = \bigcup_{\gamma<\lambda} \mathscr{X}_\gamma$.
  - Force with $\mathscr{X}_\gamma/\mathrm{fin}$ for some chosen* $\gamma \leq \lambda$.
- * We force over each $\mathscr{X}_\gamma/\mathrm{fin}$ cofinally often because we keep adding new dense sets.
- Let $G \subseteq \mathbb{P}$ be generic.
- In $V[G]$, let $\mathscr{X} = \bigcup_{\xi<\omega_1} \mathscr{X}_\xi$.
- $\mathscr{X}/\mathrm{fin}$ is proper in $V[G]$. □

Theorem: (Enayat, 2008) There is an AC family $\mathscr{X}$ of size ω_1 such that $\mathscr{X}/\mathrm{fin}$ is not proper.

Question: In a model of PFA, are there AC families $\mathscr{X}$ of size ω_2 such that $\mathscr{X}/\mathrm{fin}$ is proper or piecewise proper?

Promising construction

A promising construction: Assuming $2^\omega = \omega_2$, build an AC family $\mathscr{X}$ in ω_2-steps such that $\mathscr{X}/\text{fin}$ is proper.

- Fix some enumeration $\{\langle \mathscr{Y}_\xi, \mathcal{D}_\xi\rangle \mid \xi < \omega_2\}$ of $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$.
- Stage 0: Let $\mathscr{X}_0$ be any countable AC family.
  - Consider $\langle \mathscr{Y}_0, \mathcal{D}_0\rangle$.
  - Suppose $\mathscr{Y}_0$ codes an AC family $\mathscr{Y} \subseteq \mathscr{X}_0$ and $\mathcal{D}_0$ codes a collection $\mathcal{D}$ of dense subsets of $\mathscr{Y}$.
  - Cleverly choose some partial generic filter $G_0 \subseteq \mathscr{Y}/\text{fin}$ meeting all sets in $\mathcal{D}$.
  - Let $A_0 \subseteq^* A$ for all $A \in G_0$.
  - Let $\mathscr{X}_1$ be the arithmetic closure of $\mathscr{X}_0$ and A_0.
- Stage ξ: $\mathscr{X}_0 \subseteq \mathscr{X}_1 \subseteq \cdots \subseteq \mathscr{X}_\gamma \subseteq \cdots \subseteq \mathscr{X}_\xi$ are already constructed.
  - Consider $\langle \mathscr{Y}_\gamma, \mathcal{D}_\gamma\rangle$ for some chosen* $\gamma \leq \xi$.
  - Suppose $\mathscr{Y}_\gamma$ codes an AC family $\mathscr{Y} \subseteq \mathscr{X}_\xi$ and $\mathcal{D}_\gamma$ codes a collection $\mathcal{D}$ of dense subsets of $\mathscr{Y}$.
  - Cleverly choose some partial generic filter $G_\xi \subseteq \mathscr{Y}/\text{fin}$ meeting all sets in $\mathcal{D}$.
  - Let $A_\xi \subseteq^* A$ for all $A \in G_\xi$.
  - Let $\mathscr{X}_{\xi+1}$ be the arithmetic closure of $\mathscr{X}_\xi$ and A_ξ.
- * Consider each $\langle \mathscr{Y}_\gamma, \mathcal{D}_\gamma\rangle$ cofinally often.
- $\mathscr{X} = \bigcup_{\xi<\omega_2} \mathscr{X}_\xi$ is proper.

Why do we have to be clever in choosing the partial generic filters G_ξ?

Otherwise, $\mathscr{X} = \mathcal{P}(\mathbb{N})$.

Promising construction (continued)

How can we be clever?

- At the start of the construction fix some $C \notin \mathscr{X}_0$.
- Choose G_ξ so that the arithmetic closure of $\mathscr{X}_\xi$ and A_ξ does not contain C.

Question: Suppose $\mathscr{X}$ is a countable AC family and $C \notin \mathscr{X}$. Is there a (partial) generic $G \subseteq \mathscr{X}/\mathrm{fin}$ and $A \in V[G]$ such that

- $A \subseteq^* B$ for all $B \in G$,
- the arithmetic closure of $\mathscr{X}$ and A avoids C?

Lower semicontinuous submeasures

Definition: A submeasure is a function $\mu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$,
- $\mu(X) \leq \mu(X \cup Y) \leq \mu(X) + \mu(Y)$ for all $X, Y \subseteq \mathbb{N}$.

Definition: A submeasure is lower semicontinuous (LS) if for all $A \subseteq \mathbb{N}$,

$$\mu(A) = \lim_{n \to \infty} \mu(A \cap n).$$

Definition: A code for a LS submeasure is a function $\mu : \mathcal{P}_{<\infty}(\mathbb{N}) \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$,
- $\mu(x) \leq \mu(x \cup y) \leq \mu(x) + \mu(y)$ for all $x, y \in \mathcal{P}_{<\infty}(\mathbb{N})$.

Examples:

- Fix $n \in \mathbb{N}$. Let $\mu(X) = n$ (the constant submeasure n).
- Fix $A \subseteq \mathbb{N}$. Let $\chi_A(X) = |A \cap X|$ (counting submeasure).
- Let $\mu(x) = n$, where n is the largest element of $x \in \mathcal{P}_{<\infty}(\mathbb{N})$.
- Fix $A \subseteq \mathbb{N}$. Let $\mu(X) = \Sigma_{i=0}^{k} \frac{1}{i!}$, where $k = |A \cap X|$.
- Let $x = \{a_1, a_2, \ldots, a_n\}$ be an increasing enumeration. Let $\mu(x) = \Sigma_{i=1}^{n} b_i$, where $b_i = a_i + 1$ if i is even and $b_i = a_i - 1$ if i is odd.

Lower semicontinuous submeasures (continued)

LS submeasures form a lattice.

Definition: Suppose μ and ν are LS submeasures.

- $\mu \leq \nu$ if $\mu(X) \leq \nu(X)$ for all $X \subseteq \mathbb{N}$.
- $\mu \vee \nu(X) = \max(\mu(X), \nu(X))$ (sumpremum).
- $\mu \wedge \nu(X) = \min\{\mu(Y) + \nu(Z) \mid Y \cup Z = X\}$ (infimum).

Say that a LS submeasure μ is good if $\mu(\mathbb{N}) = \infty$ and $\mu \leq \chi_{\mathbb{N}}$.

Definition: Suppose μ, ν are good LS submeasures. We say that $\mu \leq^* \nu$ if there is $n \in \mathbb{N}$ such that $\mu \leq \nu \vee n$ ($\mu(X) \leq \max(\nu(X), n)$).

Definition: Suppose $\mathscr{X}$ is a Scott set. Let $\mathbb{P}_{\mathscr{X}}$ be the partial order whose elements are good LS submeasures μ in $\mathscr{X}$ ordered by $\leq^*$.

Decisive measures

Definition: Suppose $\mathscr{X}$ is a Scott set and $\vec{B} = \langle B_n \mid n \in \mathbb{N}\rangle$ is coded in $\mathscr{X}$. A submeasure $\mu \in \mathbb{P}_{\mathscr{X}}$ decides $\vec{B}$ if whenever U is an ultrafilter on $\mathscr{X}$ of μ-infinite sets, then $\{n \mid B_n \in U\} \in \mathscr{X}$.

Observation: Suppose $\mathscr{X}$ is an AC family and $\vec{B}$ is a sequence coded in $\mathscr{X}$.

- If $\mu \in \mathbb{P}_{\mathscr{X}}$ has $\mu(B_n) < \infty$ or $\mu(\mathbb{N} \setminus B_n) < \infty$ for all $n \in \mathbb{N}$, then μ decides $\vec{B}$.
- There is $\mu \in \mathbb{P}_{\mathscr{X}}$ such that $\mu(B_n) < \infty$ or $\mu(\mathbb{N} \setminus B_n) < \infty$ for all $n \in \mathbb{N}$.

Corollary: If $\mathscr{X}$ is an AC family, then for every sequence $\vec{B}$ coded in $\mathscr{X}$, there is $\mu \in \mathbb{P}_{\mathscr{X}}$ deciding $\vec{B}$.

Lemma (Dorais, G.) TFAE for a Scott set $\mathscr{X}$.

1. $\mathscr{X}$ is arithmetically closed.
2. For every sequence $\vec{B}$ coded in $\mathscr{X}$, there is $\mu \in \mathbb{P}_{\mathscr{X}}$ deciding $\vec{B}$.
3. For every sequence $\vec{B}$ coded in $\mathscr{X}$, the set $\mathscr{D}_{\vec{B}}$ of $\mu \in \mathbb{P}_{\mathscr{X}}$ deciding $\vec{B}$ is dense in $\mathbb{P}_{\mathscr{X}}$.

Another forcing notion from a Scott set

Lemma (Dorais, G.) TFAE for a Scott set $\mathscr{X}$.

1. $\mathscr{X}$ is arithmetically closed.
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3. For every sequence $\vec{B}$ coded in $\mathscr{X}$, the set $\mathscr{D}_{\vec{B}}$ of $\mu \in \mathbb{P}_{\mathscr{X}}$ deciding $\vec{B}$ is dense in $\mathbb{P}_{\mathscr{X}}$.

Lemma: (Dorais, G.) Suppose $M \models \mathrm{PA}$, $\mathscr{X}$ is an AC family with $\mathrm{SSy}(M) \subseteq \mathscr{X}$, and U is an ultrafilter on $\mathscr{X}$.

- For every $[f]_U \in \Pi^{\mathrm{Cod}}_{\mathbb{N}} M/U$, there is a sequence $\vec{B}^f$ coded in $\mathscr{X}$ such that if there is $\mu \in \mathbb{P}_{\mathscr{X}}$ deciding $\vec{B}^f$ and U consists of μ-infinite sets, then the set coded by $[f]_U$ is in $\mathscr{X}$.
- For every $A \in \mathscr{X}$, there is $\mu \in \mathbb{P}_{\mathscr{X}}$ such that if U consists of μ-infinite sets, then $A \in \mathrm{SSy}(\Pi^{\mathrm{Cod}}_{\mathbb{N}} M/U)$.

Another forcing notion from a Scott set (continued)

Suppose $\mathscr{X}$ is an AC family. If G is a filter on $\mathbb{P}_{\mathscr{X}}$, then there is an ultrafilter U on $\mathscr{X}$ whose elements are μ-infinite for every $\mu \in G$. We will say that such U is G-good.

Theorem: (Dorais, G., 2013) Suppose

- $M \models \mathrm{PA}$,
- $\mathscr{X}$ is an AC family with $\mathrm{SSy}(M) \subseteq \mathscr{X}$,
- G is a filter on $\mathbb{P}_{\mathscr{X}}$ meeting all dense sets $\mathcal{D}_{\vec{B}f}$ for $f \in \Pi^{\mathrm{Cod}}_{\mathbb{N}} M$.

Then any G-good ultrafilter U has the following properties:

- $\mathrm{SSy}(\Pi^{\mathrm{Cod}}_{\mathbb{N}} M/U) \subseteq \mathscr{X}$.
- For every $A \in \mathscr{X}$, there is a $\mu \in \mathbb{P}_{\mathscr{X}}$ such that if $\mu \in G$, then $A \in \Pi^{\mathrm{Cod}}_{\mathbb{N}} M/U$.

Theorem: (Dorais, 2013) Suppose $\mathscr{X}$ is a countable AC family, $\mathcal{D}$ is a countable family of dense subsets of $\mathbb{P}_{\mathscr{X}}$ and $C \notin \mathscr{X}$. Then there is a filter G on $\mathbb{P}_{\mathscr{X}}$ meeting every set in $\mathcal{D}$ and a good LS submeasure $\mu \leq^* A$ for all $A \in G$ such that the arithmetic closure of $\mathscr{X}$ and μ does not contain C.

Theorem: (Dorais, G., 2013) Assuming CH, there are continuum many AC families $\mathscr{X}$ of size ω_1 such that $\mathbb{P}_{\mathscr{X}}$ is proper.

Thank you!

Observation: Suppose $\mu_0 \,{}^*\!\geq \mu_1 \,{}^*\!\geq \cdots {}^*\!\geq \mu_n \,{}^*\!\geq \cdots$ is a decreasing sequence of good LS submeasures. Then there is a good LS submeasure μ such that $\mu \leq^* \mu_n$ for all $n \in \mathbb{N}$.

Definition: A LS submeasure μ decides a sequence $\langle B_n \mid n \in \mathbb{N}\rangle$ if $\mu(B_n) < \infty$ or $\mu(\mathbb{N} \setminus B_n) < \infty$ for all $n \in \mathbb{N}$.

Observation: For every sequence $\vec{B} = \langle B_n \mid n \in \mathbb{N}\rangle$, there is a good LS submeasure μ deciding $\vec{B}$.