Elementary embeddings and smaller large cardinals

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A common theme in the definitions of larger large cardinals is the existence of elementary embeddings from the universe $V$ into some inner model $M$.

- A cardinal $\kappa$ is **measurable** if there exists an elementary embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$.
- A cardinal $\kappa$ is **strong** if for every $\lambda > \kappa$, there is an elementary embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$ and $V_\lambda \subseteq M$.
- A cardinal $\kappa$ is **supercompact** if for every $\lambda > \kappa$, there is an elementary embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$ and $M^\lambda \subseteq M$.

The closer $M$ is to $V$ the **stronger** the large cardinal.
Elementary embeddings and ultrafilters

Suppose $\kappa$ is a cardinal and $U \subseteq \mathcal{P}(\kappa)$ is an ultrafilter.

- $U$ is $\alpha$-complete, for a cardinal $\alpha$, if whenever $\beta < \alpha$ and $\{A_\xi \mid \xi < \beta\}$ is a sequence of sets such that $A_\xi \in U$, then $\bigcap_{\xi < \beta} A_\xi \in U$.
- $U$ is normal if whenever $\{A_\xi \mid \xi < \kappa\}$ is a sequence of sets such that $A_\xi \in U$, then the diagonal intersection $\Delta_{\xi<\kappa} A_\xi \in U$. $\Delta_{\xi<\kappa} A_\xi = \{\alpha < \kappa \mid \alpha \in \bigcap_{\xi<\alpha} A_\xi\}$

**Theorem:** The ultrapower of $V$ by $U$ is well-founded if and only if $U$ is an $\omega_1$-complete.

**Observations:**

- If $U$ is normal and all the tails sets $\kappa \setminus \alpha \in U$ for $\alpha < \kappa$, then $U$ is $\kappa$-complete.
- If $U$ is $\omega_1$-complete, then we get an elementary embedding $j_U : V \to M$, where $M$ is the Mostowski collapse of the ultrapower.
- If $U$ is $\kappa$-complete, then $j_U : V \to M$ has critical point $\kappa$.

**Proposition:** Suppose $j : V \to M$ is an elementary embedding with $\text{crit}(j) = \kappa$. Then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is a normal ultrafilter.

We call $U$ the ultrafilter generated by $\kappa$ via $j$. 
Iterated ultrapowers

Suppose \( \kappa \) is a cardinal and \( U \subseteq P(\kappa) \) is an ultrafilter.

The ultrapower construction with \( U \) can be iterated as follows.

Let \( V = M_0 \) and \( j_{01} : M_0 \to M_1 \) be the ultrapower of \( V \) by \( U \).

- Let \( j_{12} : M_1 \to M_2 \) be the ultrapower of \( M_1 \) by \( j_{01}(U) \), which is an ultrafilter on \( j_{01}(\kappa) \) in \( M_1 \).
- Let \( j_{12} \circ j_{01} = j_{0,2} : M_0 \to M_2 \).

Inductively, given \( j_{\xi\gamma} : M_\xi \to M_\gamma \) for \( \xi < \gamma < \delta \), define:

- if \( \delta = \alpha + 1 \), let \( j_{\alpha,\delta} : M_\alpha \to M_\delta \) be the ultrapower of \( M_\alpha \) by \( j_{0\alpha}(U) \).
- if \( \delta \) is a limit, let \( M_\delta \) be the direct limit the system of iterated ultrapower embeddings constructed so far.

**Theorem:** (Gaifman) If \( U \) is \( \omega_1 \)-complete, then the iterated ultrapowers \( M_\xi \) for \( \xi \in \text{Ord} \) are well-founded.

- If \( M_\xi \) is well-founded, then \( M_{\xi+1} \) is well-founded, since \( j_{0\xi}(U) \) is \( \omega_1 \)-complete in \( M_\xi \).
- It suffices to see that the countable limit stages \( M_\xi \) for \( \xi < \omega_1 \) are well-founded.
Smaller large cardinals

**Definition:** A cardinal $\kappa$ is **weakly compact** if every coloring $f : [\kappa]^2 \to 2$ of pairs of elements of $\kappa$ in 2 colors has a homogeneous set of size $\kappa$.

**Theorem:** The following are equivalent:
- $\kappa$ is weakly compact.
- (Erdős, Tarski) $\kappa$ is inaccessible and the tree property holds at $\kappa$.
- (Kiesler, Tarski) Every $<\kappa$-satisfiable theory of size $\kappa$ in $L_{\kappa,\kappa}$ is satisfiable.

**Definition:** A cardinal $\kappa$ is **ineffable** if for every sequence $\{A_\xi \mid \xi < \kappa\}$ with $A_\xi \subseteq \xi$, there is a $A \subseteq \kappa$ and a stationary set $S$ such that for all $\xi \in S$, $A \cap \xi = A_\xi$.

**Theorem:** (Kunen, Jensen) A cardinal $\kappa$ is ineffable if and only if every coloring $f : [\kappa]^2 \to 2$ of pairs of elements of $\kappa$ in 2 colors has a stationary homogeneous set.

**Definition:**
- A cardinal $\kappa$ is $\alpha$-Erdős if every coloring $f : [\kappa]^{<\omega} \to 2$ of finite tuples of elements of $\kappa$ in 2 colors has a homogeneous set of order-type $\alpha$.
- A cardinal $\kappa$ is Ramsey if $\kappa$ is $\kappa$-Erdős.
Weak $\kappa$-models

Smaller large cardinals $\kappa$ usually imply existence of elementary embeddings of models of (weak) set theory of size $\kappa$.

Suppose $\kappa$ is a cardinal.

Definition:

- A weak $\kappa$-model is a transitive model $M \models ZFC^-$ of size $\kappa$ with $\kappa \in M$.
  
  $ZFC^-$ is the theory $ZFC$ without the powerset axiom with the collection scheme instead of the replacement scheme.

- A $\kappa$-model $M$ is a weak $\kappa$-model such that $M^{<\kappa} \subseteq M$.
  
  This is the maximum possible closure for a model of size $\kappa$.

- A weak $\kappa$-model is simple if $\kappa$ is the largest cardinal of $M$.

Natural simple weak $\kappa$-models arise as elementary substructures of $H_{\kappa^+}$. $H_\theta = \{x \mid |\text{TC}(x)| < \theta\}$

Observations:

- If $M \prec H_{\kappa^+}$ has size $\kappa$ and $\kappa \subseteq M$, then $M$ is a simple weak $\kappa$-model.

- If $\kappa$ is inaccessible, then there are simple $\kappa$-models $M \prec H_{\kappa^+}$. 
Small ultrafilters and elementary embeddings

Suppose $M$ is a weak $\kappa$-model.

Let $P^M(\kappa) = \{ A \subseteq \kappa \mid A \in M \}$. $P^M(\kappa)$ typically won’t be an element of $M$.

**Definition:** A set $U \subseteq P^M(\kappa)$ is an $M$-ultrafilter if it contains the tail sets $\kappa \setminus \alpha$ and the structure

$$\langle M, \in, U \rangle \models \text{"}U \text{ is a normal ultrafilter on } \kappa.\"$$

- $U$ is an ultrafilter measuring $P^M(\kappa)$.
- $U$ is closed under diagonal intersections $\Delta_{\xi<\kappa} A_\xi$ for sequences $\{ A_\xi \mid \xi < \kappa \} \in M$.
- Typically, $U \notin M$.
- Typically, separation and collection will fail badly in the structure $\langle M, \in, U \rangle$.

We will see why later on.

**Definition:** Suppose $U$ is an $M$-ultrafilter.

- $U$ is $\alpha$-complete, for a cardinal $\alpha$, if whenever $\beta < \alpha$ and $\{ A_\xi \mid \xi < \beta \}$ is a sequence of sets such that $A_\xi \in U$, then $\bigcap_{\xi<\beta} A_\xi \neq \emptyset$.
- $U$ is good if the ultrapower of $M$ by $U$ is well-founded.
Small elementary embeddings

Suppose $M$ is a weak $\kappa$-model and $U$ is an $M$-ultrafilter.

Observations:

- If $U$ is $\omega_1$-complete, then $U$ is good.
  
  We will see shortly that the converse fails.

- If $M$ is a $\kappa$-model, then $U$ is $\omega_1$-complete.

Proposition:

- If $U$ is a good $M$-ultrafilter, then the Mostowski collapse of the ultrapower yields an elementary embedding $j_U : M \to N$ with $\text{crit}(j_U) = \kappa$.

- Suppose $j : M \to N$ is an elementary embedding with $\text{crit}(j) = \kappa$.
  
  Then $U = \{ A \in M \mid A \subseteq \kappa \text{ and } \kappa \in j(A) \}$ is a good $M$-ultrafilter.

  We call $U$ the $M$-ultrafilter generated by $\kappa$ via $j$. 

Iterating small ultrapowers

Suppose $M$ is a weak $\kappa$-model, $U$ is an $M$-ultrafilter, and $j_U : M \to N$ is the ultrapower embedding.

To iterate the ultrapower construction, we need to define “$j_U(U)$”.

**Definition:** An $M$-ultrafilter $U$ is **weakly amenable** if for every $A \in M$ with $|A|^M \leq \kappa$, $U \cap A \in M$.

- If $M$ is simple, then $U$ is fully amenable.
- $j_U(U) = \{ A \subseteq j(\kappa) \mid A = [f] \text{ and } \{ \xi < \kappa \mid f(\xi) \in U \} \in U \}$.

Weakly amenable $M$-ultrafilters $U$ are “partially internal to $M$”.
Weakly amenable $M$-ultrafilters

Suppose $M$ is a weak $\kappa$-model and $U$ is an $M$-ultrafilter.

**Proposition:** $U$ is weakly amenable if and only if $\langle M, \in, U \rangle$ satisfies $\Sigma_0$-separation.

**Definition:** An elementary embedding $j : M \to N$ with $\text{crit}(j) = \kappa$ is $\kappa$-powerset preserving if $P^M(\kappa) = P^N(\kappa)$.

**Proposition:**
- If $U$ is good and weakly amenable, then the ultrapower $j_U : M \to N$ is $\kappa$-powerset preserving.
  - If $M$ is simple, then $M = H^N_{\kappa^+}$.
- If $j : M \to N$ is $\kappa$-powerset preserving, then $U$, the $M$-ultrafilter generated by $\kappa$ via $j$, is weakly amenable.

In an ultrapower $j_U : M \to N$ by a weakly amenable $M$-ultrafilter, $\kappa$-powerset preservation creates reflection between $M$ and its ultrapower $N$. 
Elementary embedding characterizations of weakly compact cardinals

**Theorem:** The following are equivalent for an inaccessible cardinal $\kappa$.
- $\kappa$ is weakly compact.
- For every $A \subseteq \kappa$, there is a weak $\kappa$-model $M$, with $A \in M$, for which there is a good $M$-ultrafilter.
- For every $A \subseteq \kappa$, there is a $\kappa$-model $M$, with $A \in M$, for which there is an $M$-ultrafilter.
- For every $A \subseteq \kappa$, there is a $\kappa$-model $M \prec H_{\kappa^+}$, with $A \in M$, for which there is an $M$-ultrafilter.
- For every weak $\kappa$-model $M$, there is a good $M$-ultrafilter.

**Question:** Can we get weakly amenable $M$-ultrafilters $U$?

We will see that the more “internal” the $M$-ultrafilter $U$ is to $M$, the stronger the associated large cardinal.
$\alpha$-iterable cardinals

Suppose $M$ is a weak $\kappa$-model.

**Definition:** An $M$-ultrafilter $U$ is $\alpha$-iterable if it is weakly amenable and has $\alpha$-many well-founded iterated ultrapowers. $U$ is iterable if it is $\alpha$-iterable for every $\alpha \in \text{Ord}$.

**Proposition:** (Gaifman) If an $M$-ultrafilter $U$ is $\omega_1$-iterable, then $U$ is iterable.

**Theorem:** (Kunen) If an $M$-ultrafilter $U$ is $\omega_1$-complete, then $U$ is iterable.

**Definition:** (G., Welch) A cardinal $\kappa$ is $\alpha$-iterable, for $1 \leq \alpha \leq \omega_1$, if for every $A \subseteq \kappa$ there is a weak $\kappa$-model $M$, with $A \in M$, for which there is an $\alpha$-iterable $M$-ultrafilter.

**Theorem:**
- (G.) A 1-iterable cardinal $\kappa$ is a limit of ineffable cardinals.
- (G., Schindler) Suppose $\lambda$ is additively indecomposable. A $\lambda + 1$-iterable cardinal has a $\lambda$-Erdős cardinal below it. A $\lambda$-Erdős cardinal is a limit of $\lambda$-iterable cardinals.
- (G., Welch) An $\alpha$-iterable cardinal is a limit of $\beta$-iterable cardinals for all $\beta < \alpha$.
- (G., Welch) If $\alpha < \omega_1$, then an $\alpha$-iterable cardinal is downward absolute to $L$. 
Elementary embedding characterization of Ramsey cardinals

**Theorem**: (Mitchell) A cardinal $\kappa$ is **Ramsey** if and only if for every $A \subseteq \kappa$ there is a weak $\kappa$-model $M$, with $A \in M$, for which there is a weakly amenable $\omega_1$-complete $M$-ultrafilter.

**Theorem**: (Sharpe, Welch) A **Ramsey** cardinal is a limit of $\omega_1$-iterable cardinals.

**Question**: Can we strengthen the Ramsey embedding characterization by replacing weak $\kappa$-model with $\kappa$-model or $\kappa$-model elementary in $H_{\kappa^+}$, etc.?

**Definition**:

- A cardinal $\kappa$ is **strongly Ramsey** if for every $A \subseteq \kappa$ there is a $\kappa$-model $M$, with $A \in M$, for which there is a weakly amenable $M$-ultrafilter.

- A cardinal $\kappa$ is **super Ramsey** if for every $A \subseteq \kappa$ there is a $\kappa$-model $M \prec H_{\kappa^+}$, with $A \in M$, for which there is a weakly amenable $M$-ultrafilter.
Strongly and super Ramsey cardinals

**Theorem:** (G.)

- A measurable cardinal is a limit of super Ramsey cardinals.
- A super Ramsey cardinal is a limit of strongly Ramsey cardinals.
- A strongly Ramsey cardinal is a limit of Ramsey cardinals.
- It is inconsistent for every $\kappa$-model to have a weakly amenable $M$-ultrafilter.

We can weaken strongly Ramsey cardinals to assert that for every $A \subseteq \kappa$ there is a weak $\kappa$-model $M$, with $A \in M$, such that $M^\omega \subseteq M$ for which there is a weakly amenable $M$-ultrafilter. Such a cardinal is already a limit of Ramsey cardinals.

**Question:** Can we stratify by closure on the weak $\kappa$-model $M$?

**Question:** Can we have elementary embeddings on models elementary in some large $H_\theta$?
**α-Ramsey cardinals**

**Definition:**
- An imperfect weak $\kappa$-model is an $\in$-model $M \models \text{ZFC}^-$ such that $\kappa + 1 \subseteq M$.
- An imperfect $\kappa$-model is an imperfect weak $\kappa$-model $M$ such that $M^{<\kappa} \subseteq M$.

**Definition:** (Holy, Schlicht) A cardinal $\kappa$ is $\alpha$-Ramsey for a regular $\alpha$, with $\omega_1 \leq \alpha \leq \kappa$, if for every $A \subseteq \kappa$ and arbitrarily large regular $\theta$, there is an imperfect weak $\kappa$-model $M < H_\theta$, with $A \in M$, such that $M^{<\alpha} \subseteq M$ for which there is a weakly amenable $M$-ultrafilter.

**Proposition:** (Holy, Schlicht) The following are equivalent.
- $\kappa$ is $\alpha$-Ramsey.
- For every $A$ and arbitrarily large regular $\theta$ there is an imperfect weak $\kappa$-model $M < H_\theta$, with $A \in M$, such that $M^{<\alpha} \subseteq M$ for which there is a weakly amenable $M$-ultrafilter.
- For arbitrarily large regular $\theta$ there is an imperfect weak $\kappa$-model $M < H_\theta$ such that $M^{<\alpha} \subseteq M$ for which there is a weakly amenable $M$-ultrafilter.

**Theorem:** (Holy, Schlicht)
- A measurable cardinal is a limit of $\kappa$-Ramsey cardinals $\kappa$.
- A $\kappa$-Ramsey cardinal $\kappa$ is a limit of super Ramsey cardinals.
- (G.) A strongly Ramsey cardinal is a limit of cardinals $\alpha$ which are $\alpha$-Ramsey.
- An $\omega_1$-Ramsey cardinal is a limit of Ramsey cardinals.
Games with $\kappa$-models and small ultrafilters

**Definition:** (Holy, Schlicht) Fix regular $\alpha$ and $\theta$ such that $\omega_1 \leq \alpha \leq \kappa$ and $\theta > \kappa$. The game $\text{Ramsey} G^\theta_\alpha(\kappa)$ is played by the challenger and the judge.

At every stage $\gamma < \alpha$:
- the challenger plays an imperfect $\kappa$-model $M_\gamma \prec H_\theta$ extending his previous moves,
- the judge responds with an $M_\gamma$-ultrafilter $U_\gamma$ extending her previous moves,
- $\{\langle M_\tilde{\gamma}, \in, U_\tilde{\gamma} \rangle \mid \tilde{\gamma} < \gamma \} \in M_\gamma$.

The judge wins if she can play for $\alpha$-many moves and otherwise the challenger wins.

**Observations:** Suppose the judge wins a run of the game $\text{Ramsey} G^\theta_\alpha(\kappa)$.
- $M = \bigcup_{\gamma < \alpha} M_\gamma$ is closed under $<\alpha$-sequences.
- $U = \bigcup_{\gamma < \alpha} U_\gamma$ is a weakly amenable $M$-ultrafilter.

**Definition:** The game $\text{Ramsey} G^* G^\theta_\alpha(\kappa)$ is played like $\text{Ramsey} G^\theta_\alpha(\kappa)$, but now the judge plays structures $\langle N_\gamma, \in, U_\gamma \rangle$ such that $N_\gamma$ is a $\kappa$-model with $P^{M_\gamma}(\kappa) \subseteq N_\gamma$ and $U_\gamma$ is an $N_\gamma$-ultrafilter.

**Question:** Why games?

**Theorem:** (G.) Suppose $\kappa$ is weakly compact. The property that given a $\kappa$-model $M$, an $M$-ultrafilter $U$, and a $\kappa$-model $\bar{M}$ extending $M$, we can always find a $\bar{M}$-ultrafilter $\bar{U}$ extending $U$ is inconsistent.
Games and \( \alpha \)-Ramsey cardinals

**Theorem:** (Holy, Schlicht) The existence of a winning strategy for either player in the games Ramsey\( G^\theta_\alpha(\kappa) \) or Ramsey\( G^{*\theta}_\alpha(\kappa) \) is independent of \( \theta \).

**Theorem:** (Holy, Schlicht) The following are equivalent.

- \( \kappa \) is \( \alpha \)-Ramsey.
- The challenger doesn’t have a winning strategy in the game Ramsey\( G^\theta_\alpha(\kappa) \) for some/all \( \theta \).
- The challenger doesn’t have a winning strategy in the game Ramsey\( G^{*\theta}_\alpha(\kappa) \) for some/all \( \theta \).
- For every \( A \in H_{(2^\kappa)^+} \), there is an imperfect weak \( \kappa \)-model \( M \prec H_{(2^\kappa)^+} \), with \( A \in M \), such that \( M^{<\alpha} \subseteq M \) for which there is a weakly amenable \( M \)-ultrafilter.

**Theorem:** (Holy, Schlicht) Every \( \beta \)-Ramsey cardinal is a limit of \( \alpha \)-Ramsey cardinals for \( \alpha < \beta \).
The structure $\langle M, \in, U \rangle$

Suppose $\kappa$ is inaccessible, $M$ is a simple weak $\kappa$-model, with $V_\kappa \in M$ and $U$ is a weakly amenable $M$-ultrafilter.

**Proposition:** The structure $\langle M, \in, U \rangle$ has a $\Delta_1$-definable global well-order.

**Proof:**

- The (possibly ill-founded) ultrapower $N$ of $M$ by $U$ has a well-order $<$ of $M = H_{\kappa^+}$.
- $<$ is represented by the equivalence class $[f]$.
- $a < b$ if $\{ \xi < \kappa \mid a f(\xi) b \} \in U$. □

**Proposition:** The structure $\langle M, \in, U \rangle$ has a $\Delta_1$-definable truth predicate for $\langle M, \in \rangle$.

**Proof:**

- Let $(\kappa^+)^N = \text{Ord}^M$ be represented by $[f]$ in the ultrapower $N$ of $M$ by $U$.
- $\langle M, \in \rangle \models \varphi(a)$ if $\{ \xi < \kappa \mid H_f(\xi) \models \varphi(a) \} \in U$. □

**Proposition:** The structure $\langle M, \in, U \rangle$ has for every $n < \omega$, a $\Sigma_n$-definable truth predicate for $\Sigma_n$-formulas in the language with $U$.

- To check the truth of a $\Delta_0$-formula $\varphi(a)$, we need $U \cap \text{TCl}(a)$.
- The truth predicate $\text{Tr}_{\Delta_0}(\varphi(x), a)$ is defined as usual, but with parameter $U \cap \text{TCl}(a)$.
- The remaining truth predicates are defined by induction on complexity. □
The structure $\langle M, \in, U \rangle$ with some set theory

Suppose $\kappa$ is inaccessible, $M$ is a simple weak $\kappa$-model, with $V_\kappa \in M$, and $U$ is an $M$-ultrafilter.

Let $\text{ZFC}_n^-$ denote the theory $\text{ZFC}$ with the separation and collections schemes restricted to $\Sigma_n$-assertions.

**Theorem:** (G., Schlicht) If $\langle M, \in, U \rangle \models \text{ZFC}_{n+1}^-$ for $n \geq 1$, then for every $A \in M$, there is a $\kappa$-model $\bar{M} \in M$, with $A \in \bar{M}$, such that $\langle \bar{M}, \in, U \rangle \prec \Sigma_n \langle M, \in, U \rangle$ and $\bar{M} \prec M$.

**Proof:**

- Use $\Sigma_{n+1}$-collection to show that every set $X$ can be extended to a set $\bar{X}$ closed under existential witnesses for $\Sigma_n$-formulas in the language with $U$ with parameters from $X$.
- Use the well-order $<$ and $\Sigma_{n+1}$-collection to build unique sequences of length $\alpha$ for $\alpha < \kappa$ of a chain of models $\bar{M}_\xi$ such that:
  - The odd stages $\xi + 1$ are models closed under existential witnesses for $\Sigma_n$-formulas in the language with $U$ with parameters from $M_\xi$.
  - The even stages $\xi + 1$ are models elementary in $M$.
- $M$ is correct about $\kappa$-models because $V_\kappa \in M$. □

**Proposition:** If for every $A \in M$, there is a $\kappa$-model $\bar{M} \in M$, with $A \in \bar{M}$, such that $\langle \bar{M}, \in, U \rangle \prec \Sigma_n \langle M, \in, U \rangle$, then $\langle M, \in, U \rangle \models \text{ZFC}_n^-$. 
A foray into second-order set theory

Definition
- Let $\text{ZFC}_U^-$ denote the theory $\text{ZFC}^-$ in the language with a unary predicate $U$.
- Let $\text{KM}_U$ denote the theory Kelley-Morse in the language with a unary predicate $U$ on classes.

Theorem: (Marek?) The following theories are equiconsistent.
1. $\text{ZFC}_U^-$, $U$ is an $M$-ultrafilter, there is a largest cardinal $\kappa$ and it is inaccessible.
2. $\text{KM}_U + U$ is a normal ultrafilter on $\text{Ord}$.
Baby measurable cardinals

**Definition:** (Bovykin, McKenzie, G., Schlicht)

- A cardinal $\kappa$ is **weakly $n$-baby measurable** if for every $A \subseteq \kappa$, there is a weak $\kappa$-model $M$, with $A \in M$, for which there is a good $M$-ultrafilter $U$ such that $\langle M, \in, U \rangle \models \text{ZFC}_n^-$. 

- A cardinal $\kappa$ is **$n$-baby measurable** if we replace weak $\kappa$-model by $\kappa$-model in the definition of weakly $n$-baby measurable cardinal.

- A cardinal $\kappa$ is **very weakly baby measurable** if for every $A \subseteq \kappa$, there is a weak $\kappa$-model $M$, with $A \in M$, for which there is a $M$-ultrafilter $U$ such that $\langle M, \in, U \rangle \models \text{ZFC}^-$. 

- A cardinal $\kappa$ is **weakly baby measurable** if for every $A \subseteq \kappa$, there is a weak $\kappa$-model $M$, with $A \in M$, for which there is a good $M$-ultrafilter $U$ such that $\langle M, \in, U \rangle \models \text{ZFC}^-$. 

- A cardinal $\kappa$ is **baby measurable** if we replace weak $\kappa$-model by $\kappa$-model in the definition of weakly baby measurable cardinal.

The $n$-baby measurable cardinals were introduced by Bovykin and McKenzie.

**Theorem:** (Bovykin, McKenzie) The following theories are equiconsistent.

1. ZFC together with the scheme consisting of assertions for every $n < \omega$

   “There exist an $n$-baby measurable cardinal $\kappa$ such that $V_\kappa \prec \Sigma_n^1 V$.”

2. **NFUM** - A natural strengthening of New Foundations with Urelements
Baby measurable cardinals in the hierarchy

**Proposition:** A weakly $n + 2$-baby measurable cardinal is an $n$-baby measurable limit of $n$-baby measurable cardinals.

**Proof:** Suppose $\langle M, \in, U \rangle \models ZFC_{n+2}^-$.  
- There is a $\kappa$-model $\bar{M} \in M$ such that $\langle \bar{M}, \in, U \rangle \prec_{\Sigma_{n+1}} \langle M, \in, U \rangle$.
- $\langle \bar{M}, \in, U \rangle \models ZFC_n^-$. □

**Theorem:** (G., Schlicht) A weakly 0-baby measurable cardinal below which the GCH holds is a limit of 1-iterable cardinals.

**Proof:** Fix a simple weak $\kappa$-model $M$, with $V_\kappa \in M$, for which there is a good $M$-ultrafilter $U$ such that $\langle M, \in, U \rangle \models ZFC_0^-$.  
- Use the GCH to show that $2^\kappa = \kappa^+$ in the ultrapower $N$ of $M$ by $U$, and therefore the well-order $<$ has order-type $\text{Ord}^M$.
- Use the well-order $<$ and $\Sigma_1$-collection to show that there are sequences $\{M_i \mid i < n\}$ of weak $\kappa$-models such that $U \cap M_i \in M_{i+1}$ for $n < \omega$.
- Use $\Sigma_1$-collection to collect the sequences into a set $X$.
- Build an ill-founded tree inside $X$ of such sequences from $X$ witnessing that $U$ is weakly amenable for some $\bar{M} \in M$. □
Baby measurable cardinals in the hierarchy (continued)

**Theorem**: (G., Schlicht) A weakly 1-baby measurable cardinal is a limit of cardinal $\alpha$ that are $\alpha$-Ramsey.

**Proof**: Fix a simple weak $\kappa$-model $M$, with $V_\kappa \in M$, for which there is a good $M$-ultrafilter $U$ such that $\langle M, \in, U \rangle \models \text{ZFC}_1^-$. 

- Let $N$ be the ultrapower of $M$ by $U$.
- Suppose $[f] = \sigma \in N$ is a winning strategy for the challenger in Ramsey $G^\kappa_\kappa (\kappa)$.
- In $M$, use $U$ and $[f]$ to construct a winning run of the game for the judge. □.

**Theorem**: (G., Schlicht) A weakly 1-baby measurable cardinal below which the GCH holds is strongly Ramsey.

**Proof**: Fix a simple weak $\kappa$-model $M$, with $V_\kappa \in M$, for which there is a good $M$-ultrafilter $U$ such that $\langle M, \in, U \rangle \models \text{ZFC}_1^-$. 

- Use the well-order $<$ and $\Sigma_1$-collection to show that there are sequences $\{M_i \mid \xi < \alpha\}$ of $\kappa$-models such that $U \cap M_\xi \in M_{\xi+1}$ for $\alpha < \kappa$.
- Use $\Sigma_1$-collection to collect the sequences into a set $X$.
- Use $\Sigma_1$-separation to pick out the sequences from $X$. □

**Theorem**: (G., Schlicht) A weakly 2-baby measurable cardinal is strongly Ramsey.
Baby measurable cardinals in the hierarchy

**Theorem:** (G., Schlicht) A very weakly baby measurable cardinal is $n$-baby measurable for every $n < \omega$.

**Proof:**
- Fix a weak $\kappa$-model $M$ for which there is an $M$-ultrafilter $U$ such that $\langle M, \in, U \rangle \models \text{ZFC}^-$.
- For every $A \in M$, there is a $\kappa$-model $\bar{M} \in M$ such that $\langle \bar{M}, \in, U \rangle \prec_{\Sigma_n} \langle M, \in, U \rangle$.
- $U \cap \bar{M}$ is a good $\bar{M}$-ultrafilter. □

**Theorem:** (G., Schlicht) A weakly baby measurable cardinal is a limit of very weakly baby measurable cardinals.

**Theorem:** (G., Schlicht) A baby measurable cardinal is a limit of weakly baby measurable cardinals.

**Proposition:** A measurable cardinal is a limit of baby measurable cardinals.
Games with structures $\langle M, \in, U \rangle$

**Definition:** (G., Schlicht) Suppose $\alpha$ and $\theta$ are regular such that $\omega_1 \leq \alpha \leq \kappa$ and $\theta > \kappa$. The game $\text{weak}G^\theta_\alpha(\kappa)$ is played by the challenger and the judge. At every stage $\gamma < \alpha$:

- the **challenger** plays an imperfect $\kappa$-model $M_\gamma \prec H_\theta$ extending his previous moves.
- the **judge** responds with a structure $\langle N_\gamma, \in, U_\gamma \rangle$, where $N_\gamma$ is a $\kappa$-model with $P^{M_\gamma}(\kappa) \subseteq N_\gamma$ and $U_\gamma$ is an $N_\gamma$-ultrafilter, extending her previous moves.

Let $M = \bigcup_{\gamma < \alpha} M_\gamma$ and $U = \bigcup_{\gamma < \alpha} U_\gamma$.

The judge wins if she can play for $\alpha$-many moves such that $\langle H^M_{\kappa^+}, \in, U \rangle \models \text{ZFC}^-$ and otherwise the challenger wins.

Note that $H^M_{\kappa^+} = \bigcup_{\gamma < \alpha} N_\gamma$.

**Definition:** (G., Schlicht) The game $G^\theta_\alpha(\kappa)$ is played like $\text{weak}G^\theta_\alpha(\kappa)$, but now the judge has to extend her moves elementarily: if $\bar{\gamma} < \gamma$, then $\langle N_{\bar{\gamma}}, \in, U \rangle \prec \langle N_\gamma, \in, U \rangle$.

Note that $H^M_{\kappa^+} = \bigcup_{\gamma < \alpha} N_\gamma$ and $\langle H^M_{\kappa^+}, \in, U \rangle \models \text{ZFC}^-$.

**Definition:** (G., Schlicht) The game $\text{strong}G^\theta_\alpha(\kappa)$ is played like $\text{weak}G^\theta_\alpha(\kappa)$, but now the judge has to respond with structures $\langle N_\gamma, \in, U_\gamma \rangle$, where $N_\gamma \prec H_\theta$ is an imperfect $\kappa$-model and $U_\gamma$ is an $N_\gamma$-ultrafilter.
Game baby measurable cardinals

**Definition**: (G., Schlicht)

- A cardinal $\kappa$ is **weakly $\alpha$-game baby measurable** for a regular $\alpha$, with $\omega_1 \leq \alpha \leq \kappa$, if for every $A \subseteq \kappa$ and arbitrarily large $\theta$ there is an imperfect weak $\kappa$-model $M \prec H_\theta$, with $A \in M$, such that $M^{<\alpha} \subseteq M$ for which there is an $M$-ultrafilter $U$ such that $\langle H^M_{\kappa^+}, \in, U \rangle \models ZFC^-$.  

- A cardinal $\kappa$ is **$\alpha$-game baby measurable** if we replace the assumption that $\langle H^M_{\kappa^+}, \in, U \rangle \models ZFC^- $ with the assumption that for every $B \subseteq \kappa$, with $B \in M$, there is an imperfect $\kappa$-model $\tilde{M} \in M$, with $B \in \tilde{M}$, such that $\langle \tilde{M}, \in, U \rangle \prec \langle H^M_{\kappa^+}, \in, U \rangle$.  

- A cardinal $\kappa$ is **strongly $\alpha$-game baby measurable** if we further strengthen to say that for every $B \in M$, there is an imperfect $\kappa$-model $\tilde{M} \in M$, with $B \in \tilde{M}$, such that $\langle \tilde{M}, \in, U \rangle \prec \langle M, \in, U \rangle$.  

Games and game baby measurable cardinals

**Theorem:** (G., Schlicht) The existence of a winning strategy for either player in the game $\text{weak } G^\theta_\alpha(\kappa)$ or the game $G^\theta_\alpha(\kappa)$ is independent of $\theta$.

**Theorem:** (G., Schlicht) A cardinal $\kappa$ is weakly $\alpha$-game baby measurable if and only if the challenger doesn’t have a winning strategy in the game $\text{weak } G^\theta_\alpha(\kappa)$ for some/all cardinals $\theta$. A cardinal $\kappa$ is $\alpha$-game baby measurable if and only if the challenger doesn’t have a winning strategy in the game $G^\theta_\alpha(\kappa)$ for some/all cardinals $\theta$.

**Theorem:** (G., Schlicht) Every weakly $\beta$-game baby measurable cardinal is a limit of cardinals $\delta > \alpha$ that are $\alpha$-game baby measurable for every $\alpha < \beta$. An analogous result holds for $\alpha$-game measurable cardinals.

**Proposition:** A weakly $\omega_1$-game baby measurable cardinal is a limit of weakly baby measurable cardinals.

**Theorem:** (G., Schlicht) A baby measurable cardinal is a limit of cardinals $\alpha$ that are weakly $\alpha$-game baby measurable. A weakly $\kappa$-game baby measurable cardinal is a limit of baby measurable cardinals.

**Theorem:** (G., Schlicht) A $\omega_1$-game baby measurable cardinal is a limit of cardinals $\alpha$ that are weakly $\alpha$-game baby measurable.

**Theorem:** (G., Schlicht) A measurable cardinal is a limit of cardinals $\alpha$ that are strongly $\alpha$-game baby measurable.
Strongly game baby measurable cardinals

**Theorem:** (G., Schlicht) A cardinal $\kappa$ is strongly $\alpha$-game baby measurable if and only if the challenger doesn’t have a winning strategy in the game $\text{strong } G^\theta_\alpha(\kappa)$ for any $\theta$.

**Open Question:** Is the existence of winning strategies for either player in the game $\text{strong } G^\theta_\alpha$ independent of $\theta$?

**Open Question:** Is a strongly $\beta$-game baby measurable cardinal a limit of strongly $\alpha$-game baby measurable cardinals for $\alpha < \beta$?

**Open Question:** Are strongly $\alpha$-game baby measurable cardinals stronger than $\alpha$-game baby measurable cardinals?
The hierarchy

- measurable
- $\alpha$-game baby measurable
- weakly $\kappa$-game baby measurable
- baby measurable
- weakly $\alpha$-game baby measurable
- weakly baby measurable
- $\kappa$-Ramsey
- super Ramsey
- strongly Ramsey
- $\alpha$-Ramsey
- Ramsey
- $\omega_1$-iterable

... $L$

- $\alpha$-iterable ($\alpha \in \omega_1$)
- weakly compact