Elementary embeddings and smaller large cardinals

Victoria Gitman

vgitman@nylogic.org
http://victoriagitman.github.io

Oxford set theory seminar

April 28, 2020

Elementary embeddings and larger large cardinals

A common theme in the definitions of larger large cardinals is the existence of elementary embeddings from the universe V into some inner model M.

- A cardinal κ is measurable if there exists an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$.
- A cardinal κ is strong if for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $V_\lambda \subseteq M$.
- A cardinal κ is supercompact if for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $M^\lambda \subseteq M$.

The closer M is to V the stronger the large cardinal.

Elementary embeddings and ultrafilters

Suppose κ is a cardinal and $U \subseteq P(\kappa)$ is an ultrafilter.

- U is α-complete, for a cardinal α, if whenever $\beta < \alpha$ and $\{A_\xi \mid \xi < \beta\}$ is a sequence of sets such that $A_\xi \in U$, then $\bigcap_{\xi<\beta} A_\xi \in U$.
- U is normal if whenever $\{A_\xi \mid \xi < \kappa\}$ is a sequence of sets such that $A_\xi \in U$, then the diagonal intersection $\Delta_{\xi<\kappa} A_\xi \in U$. $\Delta_{\xi<\kappa} A_\xi = \{\alpha < \kappa \mid \alpha \in \bigcap_{\xi<\alpha} A_\xi\}$

Theorem: The ultrapower of V by U is well-founded if and only if U is an ω_1-complete.

Observations:

- If U is normal and all the tails sets $\kappa \setminus \alpha \in U$ for $\alpha < \kappa$, then U is κ-complete.
- If U is ω_1-complete, then we get an elementary embedding $j_U : V \to M$, where M is the Mostowski collapse of the ultrapower.
- If U is κ-complete, then $j_U : V \to M$ has critical point κ.

Proposition: Suppose $j : V \to M$ is an elementary embedding with $\text{crit}(j) = \kappa$. Then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is a normal ultrafilter.

We call U the ultrafilter generated by κ via j.

Iterated ultrapowers

Suppose κ is a cardinal and $U \subseteq P(\kappa)$ is an ultrafilter.

The ultrapower construction with U can be iterated as follows.

Let $V = M_0$ and $j_{01} : M_0 \to M_1$ be the ultrapower of V by U.

- Let $j_{12} : M_1 \to M_2$ be the ultrapower of M_1 by $j_{01}(U)$, which is an ultrafilter on $j_{01}(\kappa)$ in M_1.
- Let $j_{12} \circ j_{01} = j_{0,2} : M_0 \to M_2$.

Inductively, given $j_{\xi\gamma} : M_\xi \to M_\gamma$ for $\xi < \gamma < \delta$, define:

- if $\delta = \alpha + 1$, let $j_{\alpha,\delta} : M_\alpha \to M_\delta$ be the ultrapower of M_α by $j_{0\alpha}(U)$.
- if δ is a limit, let M_δ be the direct limit the system of iterated ultrapower embeddings constructed so far.

Theorem: (Gaifman) If U is ω_1-complete, then the iterated ultrapowers M_ξ for $\xi \in \mathrm{Ord}$ are well-founded.

- If M_ξ is well-founded, then $M_{\xi+1}$ is well-founded, since $j_{0\xi}(U)$ is ω_1-complete in M_ξ.
- It suffices to see that the countable limit stages M_ξ for $\xi < \omega_1$ are well-founded.

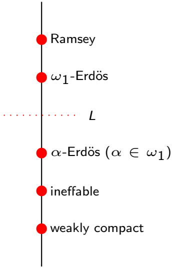
Smaller large cardinals

Definition: A cardinal κ is weakly compact if every coloring $f : [\kappa]^2 \to 2$ of pairs of elements of κ in 2 colors has a homogeneous set of size κ.

Theorem: The following are equivalent:

- κ is weakly compact.
- (Erdös, Tarski) κ is inaccessible and the tree property holds at κ.
- (Kiesler, Tarski) Every $<\kappa$-satisfiable theory of size κ in $L_{\kappa,\kappa}$ is satisfiable.

Definition: A cardinal κ is ineffable if for every sequence $\{A_\xi \mid \xi < \kappa\}$ with $A_\xi \subseteq \xi$, there is a $A \subseteq \kappa$ and a stationary set S such that for all $\xi \in S$, $A \cap \xi = A_\xi$.



Theorem: (Kunen, Jensen) A cardinal κ is ineffable if and only if every coloring $f : [\kappa]^2 \to 2$ of pairs of elements of κ in 2 colors has a stationary homogeneous set.

Definition:

- A cardinal κ is α-Erdös if every coloring $f : [\kappa]^{<\omega} \to 2$ of finite tuples of elements of κ in 2 colors has a homogeneous set of order-type α.
- A cardinal κ is Ramsey if κ is κ-Erdös.

Weak κ-models

Smaller large cardinals κ usually imply existence of elementary embeddings of models of (weak) set theory of size κ.

Suppose κ is a cardinal.

Definition:

- A weak κ-model is a transitive model $M \models \mathrm{ZFC}^-$ of size κ with $\kappa \in M$.

  ZFC^- is the theory ZFC without the powerset axiom with the collection scheme instead of the replacement scheme.
- A κ-model M is a weak κ-model such that $M^{<\kappa} \subseteq M$.

  This is the maximum possible closure for a model of size κ.
- A weak κ-model is simple if κ is the largest cardinal of M.

Natural simple weak κ-models arise as elementary substructures of H_{κ^+}. $H_\theta = \{x \mid |\mathrm{TCl}(x)| < \theta\}$

Observations:

- If $M \prec H_{\kappa^+}$ has size κ and $\kappa \subseteq M$, then M is a simple weak κ-model.
- If κ is inaccessible, then there are simple κ-models $M \prec H_{\kappa^+}$.

Small ultrafilters and elementary embeddings

Suppose M is a weak κ-model.

Let $P^M(\kappa) = \{A \subseteq \kappa \mid A \in M\}$. $P^M(\kappa)$ typically won't be an element of M.

Definition: A set $U \subseteq P^M(\kappa)$ is an M-ultrafilter if it contains the tail sets $\kappa \setminus \alpha$ and the structure

$$\langle M, \in, U\rangle \models \text{“}U \text{ is a normal ultrafilter on } \kappa.\text{”}$$

- U is an ultrafilter measuring $P^M(\kappa)$.
- U is closed under diagonal intersections $\Delta_{\xi<\kappa} A_\xi$ for sequences $\{A_\xi \mid \xi < \kappa\} \in M$.
- Typically, $U \notin M$.
- Typically, separation and collection will fail badly in the structure $\langle M, \in, U\rangle$.

  We will see why later on.

Definition: Suppose U is an M-ultrafilter.

- U is α-complete, for a cardinal α, if whenever $\beta < \alpha$ and $\{A_\xi \mid \xi < \beta\}$ is a sequence of sets such that $A_\xi \in U$, then $\bigcap_{\xi<\beta} A_\xi \neq \emptyset$.
- U is good if the ultrapower of M by U is well-founded.

Small elementary embeddings

Suppose M is a weak κ-model and U is an M-ultrafilter.

Observations:

- If U is ω_1-complete, then U is good.

  We will see shortly that the converse fails.

- If M is a κ-model, then U is ω_1-complete.

Proposition:

- If U is a good M-ultrafilter, then the Mostowski collapse of the ultrapower yields an elementary embedding $j_U : M \to N$ with $\text{crit}(j_U) = \kappa$.
- Suppose $j : M \to N$ is an elementary embedding with $\text{crit}(j) = \kappa$.

  Then $U = \{A \in M \mid A \subseteq \kappa \text{ and } \kappa \in j(A)\}$ is a good M-ultrafilter.

  We call U the M-ultrafilter generated by κ via j.

Iterating small ultrapowers

Suppose M is a weak κ-model, U is an M-ultrafilter, and $j_U : M \to N$ is the ultrapower embedding.

To iterate the ultrapower construction, we need to define "$j_U(U)$".

Definition: An M-ultrafilter U is weakly amenable if for every $A \in M$ with $|A|^M \leq \kappa$, $U \cap A \in M$.

- If M is simple, then U is fully amenable.
- $j_U(U) = \{A \subseteq j(\kappa) \mid A = [f] \text{ and } \{\xi < \kappa \mid f(\xi) \in U\} \in U\}$.

Weakly amenable M-ultrafilters U are "partially internal to M".

Weakly amenable M-ultrafilters

Suppose M is a weak κ-model and U is an M-ultrafilter.

Proposition: U is weakly amenable if and only if $\langle M, \in, U\rangle$ satisfies Σ_0-separation.

Definition: An elementary embedding $j : M \to N$ with $\text{crit}(j) = \kappa$ is κ-powerset preserving if $P^M(\kappa) = P^N(\kappa)$.

Proposition:

- If U is good and weakly amenable, then the ultrapower $j_U : M \to N$ is κ-powerset preserving.
  - If M is simple, then $M = H^N_{\kappa^+}$.
- If $j : M \to N$ is κ-powerset preserving, then U, the M-ultrafilter generated by κ via j, is weakly amenable.

In an ultrapower $j_U : M \to N$ by a weakly amenable M-ultrafilter, κ-powerset preservation creates reflection between M and its ultrapower N.

Elementary embedding characterizations of weakly compact cardinals

Theorem: The following are equivalent for an inaccessible cardinal κ.

- κ is weakly compact.
- For every $A \subseteq \kappa$, there is a weak κ-model M, with $A \in M$, for which there is a good M-ultrafilter.
- For every $A \subseteq \kappa$, there is a κ-model M, with $A \in M$, for which there is an M-ultrafilter.
- For every $A \subseteq \kappa$, there is a κ-model $M \prec H_{\kappa^+}$, with $A \in M$, for which there is an M-ultrafilter.
- For every weak κ-model M, there is a good M-ultrafilter.

Question: Can we get weakly amenable M-ultrafilters U?

We will see that the more "internal" the M-ultrafilter U is to M, the stronger the associated large cardinal.

α-iterable cardinals

Suppose M is a weak κ-model.

Definition: An M-ultrafilter U is α-iterable if it is weakly amenable and has α-many well-founded iterated ultrapowers. U is iterable if it is α-iterable for every $\alpha \in \mathrm{Ord}$.

Proposition: (Gaifman) If an M-ultrafilter U is ω_1-iterable, then U is iterable.

Theorem: (Kunen) If an M-ultrafilter U is ω_1-complete, then U is iterable.

Definition: (G., Welch) A cardinal κ is α-iterable, for $1 \leq \alpha \leq \omega_1$, if for every $A \subseteq \kappa$ there is a weak κ-model M, with $A \in M$, for which there is an α-iterable M-ultrafilter.

Theorem:

- (G.) A 1-iterable cardinal κ is a limit of ineffable cardinals.
- (G., Schindler) Suppose λ is additively indecomposable. A $\lambda + 1$-iterable cardinal has a λ-Erdös cardinal below it. A λ-Erdös cardinal is a limit of λ-iterable cardinals.
- (G., Welch) An α-iterable cardinal is a limit of β-iterable cardinals for all $\beta < \alpha$.
- (G., Welch) If $\alpha < \omega_1$, then an α-iterable cardinal is downward absolute to L.

Elementary embedding characterization of Ramsey cardinals

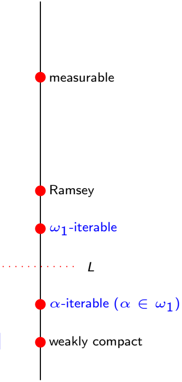
Theorem: (Mitchell) A cardinal κ is Ramsey if and only if for every $A \subseteq \kappa$ there is a weak κ-model M, with $A \in M$, for which there is a weakly amenable ω_1-complete M-ultrafilter.

Theorem: (Sharpe, Welch) A Ramsey cardinal is a limit of ω_1-iterable cardinals.

Question: Can we strengthen the Ramsey embedding characterization by replacing weak κ-model with κ-model or κ-model elementary in H_{κ^+}, etc.?

Definition:

- A cardinal κ is strongly Ramsey if for every $A \subseteq \kappa$ there is a κ-model M, with $A \in M$, for which there is a weakly amenable M-ultrafilter.
- A cardinal κ is super Ramsey if for every $A \subseteq \kappa$ there is a κ-model $M \prec H_{\kappa^+}$, with $A \in M$, for which there is a weakly amenable M-ultrafilter.



Strongly and super Ramsey cardinals

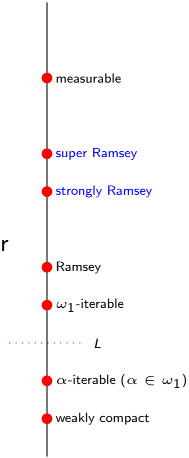
Theorem: (G.)

- A measurable cardinal is a limit of super Ramsey cardinals.
- A super Ramsey cardinal is a limit of strongly Ramsey cardinals.
- A strongly Ramsey cardinal is a limit of Ramsey cardinals.
- It is inconsistent for every κ-model to have a weakly amenable M-ultrafilter.

We can weaken strongly Ramsey cardinals to assert that for every $A \subseteq \kappa$ there is a weak κ-model M, with $A \in M$, such that $M^\omega \subseteq M$ for which there is a weakly amenable M-ultrafilter. Such a cardinal is already a limit of Ramsey cardinals.

Question: Can we stratify by closure on the weak κ-model M?

Question: Can we have elementary embeddings on models elementary in some large H_θ?



α-Ramsey cardinals

Definition:

- An imperfect weak κ-model is an $\in$-model $M \models \mathrm{ZFC}^-$ such that $\kappa + 1 \subseteq M$.
- An imperfect κ-model is an imperfect weak κ-model M such that $M^{<\kappa} \subseteq M$.

Definition: (Holy, Schlicht) A cardinal κ is α-Ramsey for a regular α, with $\omega_1 \leq \alpha \leq \kappa$, if for every $A \subseteq \kappa$ and arbitrarily large regular θ, there is an imperfect weak κ-model $M \prec H_\theta$, with $A \in M$, such that $M^{<\alpha} \subseteq M$ for which there is a weakly amenable M-ultrafilter.

Proposition: (Holy, Schlicht) The following are equivalent.

- κ is α-Ramsey.
- For every A and arbitrarily large regular θ there is an imperfect weak κ-model $M \prec H_\theta$, with $A \in M$, such that $M^{<\alpha} \subseteq M$ for which there is a weakly amenable M-ultrafilter.
- For arbitrarily large regular θ there is an imperfect weak κ-model $M \prec H_\theta$ such that $M^{<\alpha} \subseteq M$ for which there is a weakly amenable M-ultrafilter.

Theorem: (Holy, Schlicht)

- A measurable cardinal is a limit of κ-Ramsey cardinals κ.
- A κ-Ramsey cardinal κ is a limit of super Ramsey cardinals.
- (G.) A strongly Ramsey cardinal is a limit of cardinals α which are α-Ramsey.
- An ω_1-Ramsey cardinal is a limit of Ramsey cardinals.

Games with κ-models and small ultrafilters

Definition: (Holy, Schlicht) Fix regular α and θ such that $\omega_1 \leq \alpha \leq \kappa$ and $\theta > \kappa$. The game $\text{Ramsey}\,G_\alpha^\theta(\kappa)$ is played by the challenger and the judge.

At every stage $\gamma < \alpha$:

- the challenger plays an imperfect κ-model $M_\gamma \prec H_\theta$ extending his previous moves,
- the judge responds with an M_γ-ultrafilter U_γ extending her previous moves,
- $\{\langle M_{\bar\gamma}, \in, U_{\bar\gamma}\rangle \mid \bar\gamma < \gamma\} \in M_\gamma$.

The judge wins if she can play for α-many moves and otherwise the challenger wins.

Observations: Suppose the judge wins a run of the game $\text{Ramsey}\,G_\alpha^\theta(\kappa)$.

- $M = \bigcup_{\gamma<\alpha} M_\gamma$ is closed under $<\alpha$-sequences.
- $U = \bigcup_{\gamma<\alpha} U_\gamma$ is a weakly amenable M-ultrafilter.

Definition: The game $\text{Ramsey}\,G_\alpha^{*\theta}(\kappa)$ is played like $\text{Ramsey}\,G_\alpha^\theta(\kappa)$, but now the judge plays structures $\langle N_\gamma, \in, U_\gamma\rangle$ such that N_γ is a κ-model with $P^{M_\gamma}(\kappa) \subseteq N_\gamma$ and U_γ is an N_γ-ultrafilter.

Question: Why games?

Theorem: (G.) Suppose κ is weakly compact. The property that given a κ-model M, an M-ultrafilter U, and a κ-model $\bar M$ extending M, we can always find a $\bar M$-ultrafilter $\bar U$ extending U is inconsistent.

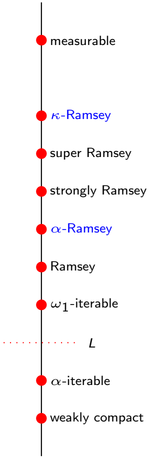
Games and α-Ramsey cardinals

Theorem: (Holy, Schlicht) The existence of a winning strategy for either player in the games $\text{Ramsey}G^{\theta}_{\alpha}(\kappa)$ or $\text{Ramsey}G^{*\theta}_{\alpha}(\kappa)$ is independent of θ.

Theorem: (Holy, Schlicht) The following are equivalent.

- κ is α-Ramsey.
- The challenger doesn't have a winning strategy in the game $\text{Ramsey}G^{\theta}_{\alpha}(\kappa)$ for some/all θ.
- The challenger doesn't have a winning strategy in the game $\text{Ramsey}G^{*\theta}_{\alpha}(\kappa)$ for some/all θ.
- For every $A \in H_{(2^\kappa)^+}$, there is an imperfect weak κ-model $M \prec H_{(2^\kappa)^+}$, with $A \in M$, such that $M^{<\alpha} \subseteq M$ for which there is a weakly amenable M-ultrafilter.

Theorem: (Holy, Schlicht) Every β-Ramsey cardinal is a limit of α-Ramsey cardinals for $\alpha < \beta$.



The structure $\langle M, \in, U\rangle$

Suppose κ is inaccessible, M is a simple weak κ-model, with $V_\kappa \in M$ and U is a weakly amenable M-ultrafilter.

Proposition: The structure $\langle M, \in, U\rangle$ has a Δ_1-definable global well-order.

Proof:

- The (possibly ill-founded) ultrapower N of M by U has a well-order $<$ of $M = H^N_{\kappa^+}$.
- $<$ is represented by the equivalence class $[f]$.
- $a < b$ if $\{\xi < \kappa \mid a\, f(\xi)\, b\} \in U$. $\square$

Proposition: The structure $\langle M, \in, U\rangle$ has a Δ_1-definable truth predicate for $\langle M, \in\rangle$.

Proof:

- Let $(\kappa^+)^N = \mathrm{Ord}^M$ be represented by $[f]$ in the ultrapower N of M by U.
- $\langle M, \in\rangle \models \varphi(a)$ if $\{\xi < \kappa \mid H_{f(\xi)} \models \varphi(a)\} \in U$. $\square$

Proposition: The structure $\langle M, \in, U\rangle$ has for every $n < \omega$, a Σ_n-definable truth predicate for Σ_n-formulas in the language with U.

- To check the truth of a Δ_0-formula $\varphi(a)$, we need $U \cap \mathrm{TCl}(a)$.
- The truth predicate $\mathrm{Tr}_{\Delta_0}(\varphi(x), a)$ is defined as usual, but with parameter $U \cap \mathrm{TCl}(a)$.
- The remaining truth predicates are defined by induction on complexity. $\square$

The structure $\langle M, \in, U\rangle$ with some set theory

Suppose κ is inaccessible, M is a simple weak κ-model, with $V_\kappa \in M$, and U is an M-ultrafilter.

Let ZFC_n^- denote the theory ZFC with the separation and collections schemes restricted to Σ_n-assertions.

Theorem: (G., Schlicht) If $\langle M, \in, U\rangle \models \mathrm{ZFC}_{n+1}^-$ for $n \geq 1$, then for every $A \in M$, there is a κ-model $\bar{M} \in M$, with $A \in \bar{M}$, such that $\langle \bar{M}, \in, U\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle$ and $\bar{M} \prec M$.

Proof:

- Use Σ_{n+1}-collection to show that every set X can be extended to a set $\bar{X}$ closed under existential witnesses for Σ_n-formulas in the language with U with parameters from X.
- Use the well-order $<$ and Σ_{n+1}-collection to build unique sequences of length α for $\alpha < \kappa$ of a chain of models $\bar{M}_\xi$ such that:
  - The odd stages $\xi + 1$ are models closed under existential witnesses for Σ_n-formulas in the language with U with parameters from M_ξ.
  - The even stages $\xi + 1$ are models elementary in M.
- M is correct about κ-models because $V_\kappa \in M$. $\square$

Proposition: If for every $A \in M$, there is a κ-model $\bar{M} \in M$, with $A \in \bar{M}$, such that $\langle \bar{M}, \in, U\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle$, then $\langle M, \in, U\rangle \models \mathrm{ZFC}_n^-$.

A foray into second-order set theory

Definition

- Let ZFC^-_U denote the theory ZFC^- in the language with a unary predicate U.
- Let KM_U denote the theory Kelley-Morse in the language with a unary predicate U on classes.

Theorem: (Marek?) The following theories are equiconsistent.

(1) ZFC^-_U, U is an M-ultrafilter, there is a largest cardinal κ and it is inaccessible.

(2) $\mathrm{KM}_U + U$ is a normal ultrafilter on Ord.

Baby measurable cardinals

Definition: (Bovykin, McKenzie, G., Schlicht)

- A cardinal κ is weakly n-baby measurable if for every $A \subseteq \kappa$, there is a weak κ-model M, with $A \in M$, for which there is a good M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}_n^-$.
- A cardinal κ is n-baby measurable if we replace weak κ-model by κ-model in the definition of weakly n-baby measurable cardinal.
- A cardinal κ is very weakly baby measurable if for every $A \subseteq \kappa$, there is a weak κ-model M, with $A \in M$, for which there is a M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$.
- A cardinal κ is weakly baby measurable if for every $A \subseteq \kappa$, there is a weak κ-model M, with $A \in M$, for which there is a good M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$.
- A cardinal κ is baby measurable if we replace weak κ-model by κ-model in the definition of weakly baby measurable cardinal.

The n-baby measurable cardinals were introduced by Bovykin and McKenzie.

Theorem: (Bovykin, McKenzie) The following theories are equiconsistent.

(1) ZFC together with the scheme consisting of assertions for every $n < \omega$

"There exist an n-baby measurable cardinal κ such that $V_\kappa \prec_{\Sigma_n} V$."

(2) NFUM - A natural strengthening of New Foundations with Urelements

Baby measurable cardinals in the hierarchy

Proposition: A weakly $n+2$-baby measurable cardinal is an n-baby measurable limit of n-baby measurable cardinals.

Proof: Suppose $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_{n+2}$.

- There is a κ-model $\bar{M} \in M$ such that $\langle \bar{M}, \in, U\rangle \prec_{\Sigma_{n+1}} \langle M, \in, U\rangle$.
- $\langle \bar{M}, \in, U\rangle \models \mathrm{ZFC}^-_n$. $\square$

Theorem: (G., Schlicht) A weakly 0-baby measurable cardinal below which the GCH holds is a limit of 1-iterable cardinals.

Proof: Fix a simple weak κ-model M, with $V_\kappa \in M$, for which there is a good M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_0$.

- Use the GCH to show that $2^\kappa = \kappa^+$ in the ultrapower N of M by U, and therefore the well-order $<$ has order-type Ord^M.
- Use the well-order $<$ and Σ_1-collection to show that there are sequences $\{M_i \mid i < n\}$ of weak κ-models such that $U \cap M_i \in M_{i+1}$ for $n < \omega$.
- Use Σ_1-collection to collect the sequences into a set X.
- Build an ill-founded tree inside X of such sequences from X witnessing that U is weakly amenable for some $\bar{M} \in M$. $\square$

Baby measurable cardinals in the hierarchy (continued)

Theorem: (G., Schlicht) A weakly 1-baby measurable cardinal is a limit of cardinal α that are α-Ramsey.

Proof: Fix a simple weak κ-model M, with $V_\kappa \in M$, for which there is a good M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}_1^-$.

- Let N be the ultrapower of M by U.
- Suppose $[f] = \sigma \in N$ is a winning strategy for the challenger in Ramsey$G_\kappa^{\kappa^+}(\kappa)$.
- In M, use U and $[f]$ to construct a winning run of the game for the judge. $\square$.

Theorem: (G., Schlicht) A weakly 1-baby measurable cardinal below which the GCH holds is strongly Ramsey.

Proof: Fix a simple weak κ-model M, with $V_\kappa \in M$, for which there is a good M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}_1^-$.

- Use the well-order $<$ and Σ_1-collection to show that there are sequences $\{M_i \mid \xi < \alpha\}$ of κ-models such that $U \cap M_\xi \in M_{\xi+1}$ for $\alpha < \kappa$.
- Use Σ_1-collection to collect the sequences into a set X.
- Use Σ_1-separation to pick out the sequences from X. $\square$

Theorem: (G., Schlicht) A weakly 2-baby measurable cardinal is strongly Ramsey.

Baby measurable cardinals in the hierarchy

Theorem: (G., Schlicht) A very weakly baby measurable cardinal is n-baby measurable for every $n < \omega$.

Proof:

- Fix a weak κ-model M for which there is an M-ultrafilter U such that $\langle M, \in, U\rangle \models \text{ZFC}^-$.
- For every $A \in M$, there is a κ-model $\bar{M} \in M$ such that $\langle \bar{M}, \in, U\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle$.
- $U \cap \bar{M}$ is a good $\bar{M}$-ultrafilter. $\square$

Theorem: (G., Schlicht) A weakly baby measurable cardinal is a limit of very weakly baby measurable cardinals.

Theorem: (G., Schlicht) A baby measurable cardinal is a limit of weakly baby measurable cardinals.

Proposition: A measurable cardinal is a limit of baby measurable cardinals.

Games with structures $\langle M, \in, U\rangle$

Definition: (G., Schlicht) Suppose α and θ are regular such that $\omega_1 \leq \alpha \leq \kappa$ and $\theta > \kappa$. The game weak$G_\alpha^\theta(\kappa)$ is played by the challenger and the judge.
At every stage $\gamma < \alpha$:

- the challenger plays an imperfect κ-model $M_\gamma \prec H_\theta$ extending his previous moves.
- the judge responds with a structure $\langle N_\gamma, \in, U_\gamma\rangle$, where N_γ is a κ-model with $P^{M_\gamma}(\kappa) \subseteq N_\gamma$ and U_γ is an N_γ-ultrafilter, extending her previous moves.

Let $M = \bigcup_{\gamma<\alpha} M_\gamma$ and $U = \bigcup_{\gamma<\alpha} U_\gamma$.

The judge wins if she can play for α-many moves such that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$ and otherwise the challenger wins.

Note that $H^M_{\kappa^+} = \bigcup_{\gamma<\alpha} N_\gamma$.

Definition: (G., Schlicht) The game $G_\alpha^\theta(\kappa)$ is played like weak$G_\alpha^\theta(\kappa)$, but now the judge has to extend her moves elementarily: if $\bar\gamma < \gamma$, then $\langle N_{\bar\gamma}, \in, U\rangle \prec \langle N_\gamma, \in, U\rangle$.

Note that $H^M_{\kappa^+} = \bigcup_{\gamma<\alpha} N_\gamma$ and $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$.

Definition: (G., Schlicht) The game strong$G_\alpha^\theta(\kappa)$ is played like weak$G_\alpha^\theta(\kappa)$, but now the judge has to respond with structures $\langle N_\gamma, \in, U_\gamma\rangle$, where $N_\gamma \prec H_\theta$ is an imperfect κ-model and U_γ is an N_γ-ultrafilter.

Game baby measurable cardinals

Definition: (G., Schlicht)

- A cardinal κ is weakly α-game baby measurable for a regular α, with $\omega_1 \leq \alpha \leq \kappa$, if for every $A \subseteq \kappa$ and arbitrarily large θ there is an imperfect weak κ-model $M \prec H_\theta$, with $A \in M$, such that $M^{<\alpha} \subseteq M$ for which there is an M-ultrafilter U such that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$.
- A cardinal κ is α-game baby measurable if we replace the assumption that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$ with the assumption that for every $B \subseteq \kappa$, with $B \in M$, there is an imperfect κ-model $\bar{M} \in M$, with $B \in \bar{M}$, such that $\langle \bar{M}, \in, U\rangle \prec \langle H^M_{\kappa^+}, \in, U\rangle$.
- A cardinal κ is strongly α-game baby measurable if we further strengthen to say that for every $B \in M$, there is an imperfect κ-model $\bar{M} \in M$, with $B \in \bar{M}$, such that $\langle \bar{M}, \in, U\rangle \prec \langle M, \in, U\rangle$.

Games and game baby measurable cardinals

Theorem: (G., Schlicht) The existence of a winning strategy for either player in the game weak$G_\alpha^\theta(\kappa)$ or the game $G_\alpha^\theta(\kappa)$ is independent of θ.

Theorem: (G., Schlicht) A cardinal κ is weakly α-game baby measurable if and only the challenger doesn't have a winning strategy in the game weak$G_\alpha^\theta(\kappa)$ for some/all cardinals θ. A cardinal κ is α-game baby measurable if and only if the challenger doesn't have a winning strategy in the game $G_\alpha^\theta(\kappa)$ for some/all cardinals θ.

Theorem: (G., Schlicht) Every weakly β-game baby measurable cardinal is a limit of cardinals $\delta > \alpha$ that are α-game baby measurable for every $\alpha < \beta$. An analogous result holds for α-game measurable cardinals.

Proposition: A weakly ω_1-game baby measurable cardinal is a limit of weakly baby measurable cardinals.

Theorem: (G., Schlicht) A baby measurable cardinal is a limit of cardinals α that are weakly $<\alpha$-game baby measurable. A weakly κ-game baby measurable cardinal is a limit of baby measurable cardinals.

Theorem: (G., Schlicht) A ω_1-game baby measurable cardinal is a limit of cardinals α that are weakly α-game baby measurable.

Theorem: (G., Schlicht) A measurable cardinal is a limit of cardinals α that are strongly α-game baby measurable.

Strongly game baby measurable cardinals

Theorem: (G., Schlicht) A cardinal κ is strongly α-game baby measurable if and only if the challenger doesn't have a winning strategy in the game strong $G^\theta_\alpha(\kappa)$ for any θ.

Open Question: Is the existence of winning strategies for either player in the game strong G^θ_α independent of θ?

Open Question: Is a strongly β-game baby measurable cardinal a limit of strongly α-game baby measurable cardinals for $\alpha < \beta$?

Open Question: Are strongly α-game baby measurable cardinals stronger than α-game baby measurable cardinals?

The hierarchy