The Stable Core

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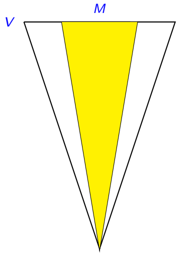
Classical inner models

An inner model of a ZFC-universe is a definable transitive sub-model $M \models \mathrm{ZF(C)}$ with $\mathrm{Ord} \subseteq M$.



A canonical inner model M, defined by $\varphi(x)$, should have the following properties:

- Fine structure.
- Regularity: GCH, $\square$, etc.
- Absoluteness: $M = \{x \mid \varphi^M(x)\}$.
- Forcing absolutness: $M = \{x \mid \varphi^{V[G]}(x)\}$.
- ...

Gödel's constructible universe L

- $L_0 = \emptyset$
- $L_{\alpha+1}$ consists of all definable subsets of L_α
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for a limit λ
- $L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$
- Most large cardinals are incompatible with L.
- In the presence of large cardinals, L is not "close to V".

Canonical inner models (continued)

The Dodd-Jensen core model K^{DJ}

- $L[\mathcal{M}]$, where $\mathcal{M}$ is the class of all mice.
- (oversimplified) A mouse N is a transitive model with fine-structure such that for some $\kappa \in N$, there is an iterable N-ultrafilter on κ.
- K^{DJ} is compatible with all known large cardinals up to a measurable cardinal.
- Measurable cardinals are incompatible with K^{DJ}.
- In the presence of larger large cardinals, K^{DJ} is not "close to V".

Kunen's canonical model for one measurable cardinal $L[U]$

- Suppose κ is measurable and U is a normal measure on κ.
- $L_0[U] = \emptyset$.
- $L_{\alpha+1}[U]$ consists of all definable subsets of $\langle L_\alpha[U], \in, U \cap L_\alpha[U]\rangle$.
- $L_\lambda[U] = \bigcup_{\alpha<\lambda} L_\alpha[U]$ for a limit λ.
- $L[U] = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha[U]$.
- $U \cap L[U]$ is the unique normal measure on κ and κ is the unique measurable in $L[U]$.
- If U and W are two normal measures on κ, then $L[U] = L[W]$.
- In the presence of larger large cardinals, (two measurables), $L[U]$ is not "close to V".

The inner model HOD

OD is the collection of all sets definable with ordinal parameters.

- $\mathrm{Ord} \subseteq \mathrm{OD}$.
- (Lévy-Montague Reflection) Every first-order formula $\varphi(x, a)$ is reflected by unboundedly many V_α.
- A set $a \in \mathrm{OD}$ iff there is an ordinal α such that a is ordinal definable in V_α.
- OD is definable.

Gödel's hereditarily ordinal definable sets HOD

- The "hereditary" condition makes HOD transitive.
- If $V \models \mathrm{ZF}$, then $\mathrm{HOD} \models \mathrm{ZFC}$.

HOD is not canonical

"We can code information into HOD using forcing".

Theorem: If $\mathbb{P}$ is a weakly homogeneous forcing notion[1] and $G \subseteq \mathbb{P}$ is V-generic, then $\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}^V$.

Theorem: Consistently $\mathrm{HOD}^{\mathrm{HOD}} \subsetneq \mathrm{HOD}$.

Proof: Start in L.

- Let g be an L-generic Cohen real.
- $\mathrm{HOD}^{L[g]} = L$ because Cohen forcing is weakly homogeneous.
- Let $\mathbb{P}$ be the product forcing to make GCH to fail at $\aleph_n$ iff the n-th bit of g is 1 and let $H \subseteq \mathbb{P}$ be $L[g]$-generic.

  "Code g into the continuum function."
- $\mathbb{P}$ is weakly homogeneous.
- $V = L[g][H]$.
- $g \in \mathrm{HOD}$, so $\mathrm{HOD} = L[g]$.
- $\mathrm{HOD}^{\mathrm{HOD}} = L$. □

Corollary: HOD can be changed by forcing.

[1] For every $p, q \in \mathbb{P}$, there is an automorphism π of $\mathbb{P}$ such that $\pi(p)$ is compatible to q.

HOD can be close to V

Theorem: (Roguski) V is the HOD of one of its class forcing extensions.

Proof:

- (McAloon) Let $\mathbb{P}$ be the Easton-support Ord-length product to code every set in V into the continuum function and let $G \subseteq \mathbb{P}$ be V-generic.
- $\mathbb{P}$ is weakly homogeneous.
- $\mathrm{HOD}^{V[G]} \subseteq V$.
- $\mathrm{HOD}^{V[G]} = V$. $\square$

Corollary:

- GCH can fail at every regular cardinal in HOD.
- Every known large cardinal is compatible with HOD.

HOD can be far from V

"HOD can be wrong about large cardinals."

Theorem: (Cheng, Friedman, Hamkins)

- It is consistent that every measurable cardinal of V is not even weakly compact in HOD.
- It is consistent that there is a supercompact cardinal in V that is not even weakly compact in HOD.

"HOD can be wrong about successor cardinals."

Theorem: (Cummings, Friedman, Golshani) It is consistent that $\alpha^+ > (\alpha^+)^{\text{HOD}}$ for every infinite cardinal α.

HOD in the presense of large cardinals

Woodin has conjectured that in the presence of large cardinals HOD must be close to V.

Conjecture: (Woodin)

- If there is a supercompact cardinal, then there is a measurable cardinal in HOD.
- If there is an extendible cardinal δ, then $\alpha^+ = (\alpha^+)^{\mathrm{HOD}}$ for all singular cardinals $\alpha > \delta$.

Is V a class forcing extension of HOD?

Theorem: (Vopěnka) Every set of ordinals is set-generic over HOD: If A is a set of ordinals, then there is a partial order $\mathbb{P} \in \mathrm{HOD}$ and $G \subseteq \mathbb{P}$ which is HOD-generic such that $\mathrm{HOD}[A] = \mathrm{HOD}[G]$.

Question: Is V a class forcing extension of HOD?

Theorem: (Hamkins, Reitz) It is consistent that V is not a class forcing extension of HOD.

Theorem: (Friedman) There is a definable class S such that every initial segment of S is in HOD and V is a class forcing extension of (HOD, S).

- $(\mathrm{HOD}, S) \models \mathrm{ZFC}$.
- There is a class partial order $\mathbb{P}$ definable in (HOD, S) and $G \subseteq \mathbb{P}$ which is (HOD, S)-generic such that $\mathrm{HOD}[G] = V$.
  - $\mathbb{P}$ has the Ord-cc: every maximal antichain of $\mathbb{P}$ definable in (HOD, S) is set-sized.
  - $(V, G) \models \mathrm{ZFC}$, but G is not definable over V.

Theorem: (Friedman) V is a class forcing extension of $(L[S], S)$.

The Stability Predicate

"The stability predicate codes elementarity relations between initial segments H_α of V."

Models H_α

- For a cardinal α, H_α is the set of all sets a with $|\text{tc}(a)| < \alpha$.
- (Lévy) For every cardinal α, $H_\alpha \prec_{\Sigma_1} V$.
- For every regular cardinal α, $H_\alpha \models \mathrm{ZFC}^-$.

***n*-good cardinals**

- A cardinal α is n-good if:
  - α is a strong limit.
  - $H_\alpha \models \Sigma_n$-Collection.
- Every strong limit cardinal α is 1-good.
- For every n-good cardinal α with $n \geq 2$, if $H_\beta \prec_{\Sigma_n} H_\alpha$, then β is n-good.

Stability predicate S: triples (α, β, n) such that

- α, β are n-good cardinals.
- $H_\alpha \prec_{\Sigma_n} H_\beta$.

The Stable Core $(L[S], S)$

Observation:

- The collection of all strong limit cardinals of V is definable in $(L[S], S)$.
- If the GCH holds, then the collection of all limit cardinals of V is definable in $(L[S], S)$.

Proof: α is a strong limit cardinal iff $(\alpha, \beta, 1) \in S$ for some β. $\square$

Observation: $L[S] \subseteq \mathrm{HOD}$.

Proof: All initial segments of S are ordinal definable. $\square$

Corollary: (Hamkins, Reitz) It is consistent that S is not definable over HOD.

The Stable Core $(L[S], S)$ (continued)

Some forcing absoluteness

Suppose $\mathbb{P} \in H_\alpha$ is a forcing notion, $G \subseteq \mathbb{P}$ is V-generic, and $\beta > \alpha$ is a cardinal.

- $H_\alpha^{V[G]} = H_\alpha[G]$ (definability of forcing relation).
- $H_\alpha \prec_{\Sigma_n} H_\beta$ iff $H_\alpha[G] \prec_{\Sigma_n} H_\beta[G]$ (definability of forcing relation, ground model is Δ_2-definable).
- $H_\alpha \models \Sigma_n$-Collection iff $H_\alpha[G] \models \Sigma_n$-Collection (definability of forcing relation, ground model is Δ_2-definable).
- $(\alpha, \beta, n) \in S$ iff $(\alpha, \beta, n) \in S^{V[G]}$.
- S and $S^{V[G]}$ agree above the size of $\mathbb{P}$.
- If $\mathbb{P}$ has size smaller than the first strong limit cardinal, then $S = S^{V[G]}$.

Theorem: (Friedman) It is consistent that $L[S] \subsetneq \mathrm{HOD}$.

Proof: Use the "coding universe into a real" forcing. □

$0^\#$ in the Stable Core

Lemma: If $0^\#$ exists, then $0^\# \in L[S]$.

Proof:

- $0^\# = \{\varphi \mid L_{\aleph_\omega^V} \models \varphi(\aleph_1^V, \ldots, \aleph_n^V)\}$.
- Let $\langle \alpha_i \mid i < \omega\rangle \in L[S]$ be the first ω-many strong limit cardinals of V.
- $\varphi(x_1, \ldots, x_n) \in 0^\#$ iff $L_{\alpha_{n+1}} \models \varphi(\alpha_1, \ldots, \alpha_n)$ $\square$.

Measurable cardinals in the Stable Core

Theorem: (Friedman, G., Müller) If there is a measurable cardinal, then $L[U] \subseteq L[S]$.

Proof: Suppose κ is measurable and U is a normal measure on κ.
Let $\mu = U \cap L[U]$ be the unique normal measure on κ in $L[U] = L[\mu]$.

Step 1: Iterate μ to a "simple" measure $\mu_\lambda \in L[S]$.

- Let $\lambda \gg \kappa^+$ be a limit of strong limit cardinals in V.
- Let $j_\lambda : L[\mu] \to L[\mu_\lambda]$ be the λ-th iterated ultrapower by μ.
- In $L[\mu_\lambda]$, μ_λ is a normal measure on $\lambda = j_\lambda(\kappa)$.
- Let $\langle \kappa_\eta \mid \eta < \lambda \rangle$ be the critical sequence of the μ-iteration.
- Let $A_\xi = \{\eta < \lambda \mid \xi \leq \eta \text{ is a strong limit cardinal}\}$ for $\xi < \lambda$ (tails of strong limit cardinals in λ).
- Let $\mathcal{F}$ be the filter on λ generated by the tails A_ξ.
- $\mathcal{F} \in L[S]$, so $L[\mathcal{F}] \subseteq L[S]$.

$L[\mu_\lambda] = L[\mathcal{F}] \subseteq L[S]$.

- $\mu_\lambda \subseteq \mathcal{F}$.
  - $X \in \mu_\lambda$ iff there is $\xi < \lambda$ such that $\{\kappa_\eta \mid \xi \leq \eta < \lambda\} \subseteq X$.
  - $\kappa_\eta = \eta$ for V-cardinals $\eta > \kappa$.
- $\mathcal{F} \cap L[\mu_\lambda] = \mu_\lambda$.
  - $\mathcal{F}$ is a filter and μ_λ is an ultrafilter in $L[\mu_\lambda]$.

Measurable cardinals in the Stable Core (continued)

Proof: (continued)

Step 2: Collapse a well-chosen $X \prec L_\theta[\mu_\lambda]$ to obtain $L_{\bar\theta}[\mu]$ with $\bar\theta > \kappa^+$.

- In $L[S]$, define a sequence $\langle \nu_\xi \mid \xi < \kappa^+\rangle$ of strong limit cardinals of V above λ such that $\text{cf}^V(\nu_\xi) \geq \kappa^+$.
- $j_\lambda(\nu_\xi) = \nu_\xi$ ($\nu_\xi > \lambda$, length of iteration, is a strong limit of cf greater than κ).
- Let θ be above $\sup\langle \nu_\xi \mid \xi < \kappa^+\rangle$.
- Let $X \prec L_\theta[\mu_\lambda]$ be generated by $\kappa \cup \{\nu_\xi \mid \xi < \kappa^+\} \subseteq X$.
  - $\lambda \in X$ (the unique measurable cardinal in $L_\theta[\mu_\lambda]$).
  - $X \subseteq j_\lambda$ " $L[\mu]$ ($\kappa \cup \{\nu_\xi \mid \xi < \kappa^+\} \subseteq j_\lambda$ " $L[\mu] \prec L[\mu_\lambda]$).
- Let N be the Mostowski collapse of X.
- λ collapses to κ (there is nothing in j_λ " $L[\mu]$ between κ and λ).
- $N = L_{\bar\theta}[\nu]$ with $\bar\theta \geq \kappa^+$.
- By uniqueness, $\mu = \nu$. $\square$

Corollary:

- The Stable Core of K^{DJ} is K^{DJ}.
- The Stable Core of $L[\mu]$ is $L[\mu]$.

More measurable cardinals in the Stable Core

Theorem: (Friedman, G., Müller) If $\kappa^{(\xi)}$ for $\xi < \nu < \kappa^{(0)}$ are distinct measurable cardinals with normal measures $U^{(\xi)}$, then $L[\langle U^{(\xi)} \mid \xi < \nu\rangle] \subseteq L[S]$.

Proof: Generalize the one measurable cardinal argument using Kunen's generalized uniqueness. □

Question: Can the Stable Core have a measurable limit of measurable cardinals?

The model $L[\mathrm{Card}]$

Studied by Kennedy, Magidor, and Väänänen.

Theorem: (Kennedy, Magidor, Väänänen)

- If $0^\#$ exists, then $0^\# \in L[\mathrm{Card}]$.
- If there is a measurable cardinal, then $L[U] \subseteq L[\mathrm{Card}]$.
- If $\kappa^{(\xi)}$ for $\xi < \nu < \kappa^{(0)}$ are distinct measurable cardinals with normal measures $U^{(\xi)}$, then $L[\langle U^{(\xi)} \mid \xi < \nu\rangle] \subseteq L[\mathrm{Card}]$.

We generalized their techniques to the Stable Core using strong limit cardinals.

The structure of $L[\mathrm{Card}]$ becomes regular in the presence of large cardinals.

Theorem: (Kennedy, Magidor, Väänänen, Welch) Assume there is (a little more than) a measurable limit of measurables, then in $L[\mathrm{Card}]$:

- There are no measurable cardinals.
- GCH holds.

Coding into the Stability Predicate

Any object that can be added generically to L can exist in the Stable Core.

Theorem: (Friedman, G., Müller) Suppose $\mathbb{P} \in L$ is a forcing notion and $G \subseteq \mathbb{P}$ is L-generic. Then there is a further forcing extension $L[G][H]$ such that $G \in L[S^{L[G][H]}]$.

Corollary: The following can consistently happen in the Stable Core:

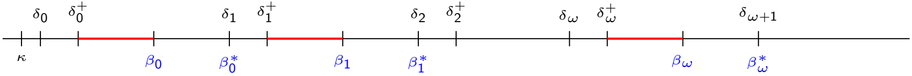
- The GCH fails on a large initial segment of the cardinals.
- An arbitrarily large cardinal of L is countable.
- Martin's Axiom holds.

Proof of Theorem: G can be coded into any "n-th slice" of the Stable Core.

Without loss $G \subseteq \kappa$ for some κ.
Fix $\delta_0 \gg \kappa$. Above δ_0:

- L and $L[G]$ agree on the cardinals, GCH, and the stability predicate S.
- We will define a sequence of "coding pairs" (β_ξ, β^*_ξ) for $\xi < \kappa$.
  - $(\beta_\xi, \beta^*_\xi, n) \in S^L, S^{L[G]}$.
  - In $L[G][H]$, we will have $(\beta_\xi, \beta^*_\xi, n) \in S^{L[G][H]}$ iff $\xi \in G$.
- In L, call a cardinal α n-useful if it is a $\beth$-fixed point such that $H_\alpha \prec_{\Sigma_n} L$.

Coding into the Stability Predicate (continued)



- β_ξ is the least n-useful cardinal above δ_ξ of cofinality δ_ξ^+.
- β_ξ^* is the least n-useful cardinal above β_ξ.
- δ_η is the supremum of the β_ξ^* for $\xi < \eta$.

Coding forcing: full-support product $\mathbb{C} = \Pi_{\xi<\kappa}\mathbb{C}_\xi$.

- If $\xi \in G$, then $\mathbb{C}_\xi$ is trivial.
- If $\xi \notin G$, then $\mathbb{C}_\xi = \mathrm{Coll}(\delta_\xi^+, \beta_\xi)$.
- $\mathbb{C}$ preserves GCH above δ_0.
- $\mathbb{C}$ collapses a cardinal γ iff $\gamma \in (\delta_\xi^+, \beta_\xi]$ with $\xi \notin G$.

Suppose $H \subseteq \mathbb{C}$ is $L[G]$-generic.

- $L[S^{L[G][H]}]$ can see the coding pairs (β_ξ, β_ξ^*) because it can define L.
- If $\xi \notin G$, then $(\beta_\xi, \beta_\xi^*, n) \notin S^{L[G][H]}$.
- If $\xi \in G$, then $(\beta_\xi, \beta_\xi^*, n) \in S^{L[G][H]}$.
  - β_ξ and β_ξ^* are strong limit cardinals in $L[G][H]$.
  - Factor $\mathbb{C} = \Pi_{\eta<\xi}\mathbb{C}_\eta \times \Pi_{\xi<\eta<\kappa}\mathbb{C}_\eta$. Correspondingly factor $H = H_1 \times H_2$.
  - $\Pi_{\eta<\xi}\mathbb{C}_\eta \in H_{\beta_\xi}^{L[G]}$, $\Pi_{\xi<\eta<\kappa}\mathbb{C}_\eta$ is $\leq \beta_\xi^*$-closed.
  - $H_{\beta_\xi}^{L[G]}[H_1] = H_{\beta_\xi}^{L[G]}[H] \prec_{\Sigma_n} H_{\beta_\xi^*}^{L[G]}[H] = H_{\beta_\xi^*}^{L[G]}[H_1]$. □

The GCH can fail everywhere in the Stable Core

Theorem: (Friedman, G., Müller) It is consistent that the GCH fails at all regular cardinals in the Stable Core.

Proof: Start in L. Let $L[G]$ be the class forcing extension in which the GCH fails at all regular cardinals.

- Let $A \subseteq \mathrm{Ord}$ code all subsets of cardinals added by G.
- Define a sequence of "coding pairs" (β_ξ, β_ξ^*) for $\xi \in \mathrm{Ord}$.
- Define the coding forcing Easton-support product $\mathbb{C} = \Pi_{\xi \in \mathrm{Ord}} \mathbb{C}_\xi$.
- $\xi \in A$ if and only if $(\beta_\xi, \beta_\xi^*, n) \in S^{L[G][H]}$. □

Coding into the Stable Core over $L[U]$

Theorem: (Friedman, G., Müller) Suppose $\mathbb{P} \in L[U]$ is a forcing notion and $G \subseteq \mathbb{P}$ is $L[U]$-generic. Then there is a further forcing extension $L[U][G][H]$ such that $G \in L[S^{L[U][G][H]}]$.

Proof: $L[U] \subseteq L[S^{L[U][G][H]}]$. $\square$

Theorem: (Friedman, G., Müller) It is consistent that the Stable Core has a measurable cardinal and the GCH fails on on a tail of regular cardinals.

Big Open Question: Does the structure of the Stable Core become regular in the presence of stronger large cardinals?

Measurable cardinals are not downward absolute to the Stable Core

Theorem: (Kunen) Weakly compact cardinals are not downward absolute.

Proof: Suppose κ is weakly compact.

- Let $\mathbb{P}_\kappa$ be the Easton-support iteration of length κ forcing with $\mathrm{Add}(\xi,1)$ at every inaccessible cardinal $\xi \in V^{\mathbb{P}_\xi}$. Let $G \subseteq \mathbb{P}_\kappa$ be V-generic.
- In $V[G]$, let $\mathbb{Q}$ be the forcing to add a homogeneous κ-Suslin tree. Let $T \subseteq \mathbb{Q}$ be $V[G]$-generic.
- In $V[G][T]$, κ is not weakly compact.
- Let $b \subseteq T$ be a $V[G][T]$-generic branch through T.
- $\mathbb{Q} * \dot{T}$ is forcing equivalent to $\mathrm{Add}(\kappa,1)$.
- κ is again weakly compact in $V[G][T][b]$ ($\mathbb{P}_\kappa * \mathrm{Add}(\kappa,1)$ preserves weak compactness of κ). □

Measurable cardinals are not downward absolute to the Stable Core (continued)

Theorem: (Friedman, G., Müller) It is consistent that κ is measurable in V, but not even weakly compact in the Stable Core.

Proof: Start in $V = L[U]$ where κ is measurable.

- Let $G * T$ be V-generic for $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$.
- κ is not weakly compact in $V[G][T]$.
- Let $\mathbb{C}$ be the coding forcing (high above κ) to code T into the Stable Core. Let $H \subseteq \mathbb{C}$ be $V[G][T]$-generic.
- κ is not weakly compact in $L[S^{V[G][T][H]}]$.
  - $T \in L[S^{V[G][T][H]}]$.
  - $V[G][T][H]$ does not have a branch through T ($\mathbb{C}$ is highly closed).
- Let $b \subseteq T$ be $V[G][T][H]$-generic.
- κ is measurable in $V[G][T][H][b]$.
  - κ is measurable in $V[G][T][b]$ ($\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$ preserves measurability of κ).
  - $V[G][T][b]$ and $V[G][T][H][b]$ have the same subsets of κ.
- $S^{V[G][T][H][b]} = S^{V[G][T][H]}$ (not obvious).
- κ is not weakly compact in $L[S^{V[G][T][H][b]}]$. □

Big Open Question: Are measurable cardinals downward absolute to the Stable Core in the presence of large cardinals?

Another way to code into the Stability Predicate

Start in L.
Let $\mathbb{P}$ be a forcing notion and $G \subseteq \mathbb{P}$ be L-generic. Assume $G \subseteq \kappa$.
In L, high above κ, choose a sequence (β_ξ, β^*_ξ) for $\xi < \kappa$ of widely spaced "coding pairs".
Let $\mathbb{C}$ be the Easton-support product forcing:

$$\mathrm{GCH}\text{ fails cofinally often in } \beta_\xi \text{ iff } \xi \notin G.$$

Let $H \subseteq \mathbb{C}$ be $L[G]$-generic.

- L, $L[G]$, and $L[G][H]$ agree on cardinals above κ.
- $L[S^{L[G][H]}]$ sees the sequence of coding pairs because it can define L.
- Fix some large n.
- If $\xi \notin G$, then $(\beta_\xi, \beta^*_\xi, n) \notin S^{L[G][H]}$ because GCH fails cofinally in β_ξ, but not in β^*_ξ.
- If $\xi \in G$, then $(\beta_\xi, \beta^*_\xi, n) \in S^{L[G][H]}$.
  - Factor $\mathbb{C} = \mathbb{C}_{\text{small}} \times \mathbb{C}_{\text{tail}}$ with $\mathbb{C}_{\text{small}} \in H_{\beta_\xi}$ and $\mathbb{C}_{\text{tail}}$ is $\leq \beta^*_\xi$-closed.
  - Factor $H = H_{\text{small}} \times H_{\text{tail}}$ correspondingly.
  - $H^{L[G]}_{\beta_\xi}[H_{\text{small}}] = H^{L[G]}_{\beta_\xi}[H] \prec_{\Sigma_n} H^{L[G]}_{\beta^*_\xi}[H] = H^{L[G]}_{\beta^*_\xi}[H_{\text{small}}]$.
- $\xi \in G$ iff $(\beta_\xi, \beta^*_\xi, n) \in S^{L[G][H]}$.

Separating $L[\mathrm{Card}]$ and $L[S]$

Theorem: (Friedman, G., Müller) It is consistent that $L[\mathrm{Card}] \subsetneq L[S]$.

Proof: Start in L. Let $g \subseteq \omega$ be L-generic for Cohen forcing.

- Let $\mathbb{C}$ be the forcing to code G into the stability predicate using GCH failure.
- Let $H \subseteq \mathbb{C}$ be $L[g]$-generic.
- $L[\mathrm{Card}^{L[g][H]}] = L$ (L and $L[g][H]$ have same cardinals).
- $g \in L[S^{L[G][H]}]$. $\square$

Note: Same argument works over $L[U]$.

Open Questions

Question: Is it consistent that the Stable Core of the Stable Core is smaller than the Stable Core?

- The analogous result for HOD uses coding.

Question: Is the Stable Core of every canonical inner model the inner model itself?

- $L[S^{K^{DJ}}] = K^{DJ}$.
- $L[S^{L[U]}] = L[U]$.
- Is $L[S^{M_1}] = M_1$? (M_1 is the canonical model for one Woodin cardinal.)

Question: What does the Stable Core look like in the presence of large cardinals?

- Is there a bound on the large cardinals the Stable Core can have? Or:
- Are large cardinals downward absolute to the Stable Core?
- Does the GCH hold?