Filter extension games with mini supercompactness filters

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This is joint work with Tom Benhamou.

Smaller large cardinals and mini measures

Measurable, strongly compact, and supercompact cardinals can be characterized by the existence of certain measures.

Smaller large cardinals κ, such as weakly compact, ineffable, and Ramsey, can be characterized by the existence of certain 'mini measures' on families $\mathcal{A} \subseteq P(\kappa)$ of size κ.

In practice, it suffices to consider families that arise as $P(\kappa)^M$ for a κ-sized transitive $\in$-model M of (a sufficiently large fragment of) set theory.

The 'mini measures' are external to the model M, but we can still carry out the ultrapower construction using functions from M to produce an elementary embedding on M with critical point κ.

Assume onwards that κ is inaccessible.

Small models and ultrafilters

Definition: A transitive $\in$-model M is a weak κ-model:

- $M \models \mathrm{ZFC}^-$ (no powerset)
- $|M| = \kappa$
- $\kappa \in M$

M is a κ-model if additionally $M^{<\kappa} \subseteq M$.

e.g. elementary substructures of H_{κ^+} of size κ

Definition: Suppose M is a weak κ-model. A set $U \subseteq P(\kappa)^M$ is a weak M-ultrafilter:

- ultrafilter on $P(\kappa)^M$
- M-κ-complete - closed under intersections of $<\kappa$-length sequences from M.

U is an M-ultrafilter if it is additionally M-normal - closed under diagonal intersections of κ-length sequences from M.

Definition: A weak M-ultrafilter U:

- good if the ultrapower of M by U is well-founded
- has the countable intersection property if for every $\{A_n \mid n < \omega\} \subseteq U$, $\bigcap_{n<\omega} A_n \neq \emptyset$

  e.g. if $M^\omega \subseteq M$, in particular, if M is a κ-model
- weakly amenable if for every $A \in M$, with $|A|^M = \kappa$, $U \cap A \in M$

Small models and ultrafilters (continued)

Suppose M is a weak κ-model.

Proposition: The following are equivalent.

- There is an M-ultrafilter.
- There is an elementary $j : M \to N$ with $\text{crit}(j) = \kappa$ and $\kappa \in N$. N not necessarily well-founded
  - $H^M_{\kappa^+} \subseteq N$
  - N is well-founded at least up to $(\kappa^+)^M$

Proposition: The following are equivalent.

- There is a weakly amenable M-ultrafilter.
- There is an elementary $j : M \to N$ with $\text{crit}(j) = \kappa$ and $H^M_{\kappa^+} = H^N_{\kappa^+}$.

  N is well-founded beyond $(\kappa^+)^N$

Proposition: We can iterate the ultrapower construction with a weakly amenable M-ultrafilter U.

- If $j : M \to N$ is the ultrapower map, then $j(U) = \{[f]_U \subseteq j(\kappa) \mid \{\alpha < \kappa \mid f(\alpha) \in U\} \in U\}$.
- (Kunen) If U has the countable intersection property, then U is iterable - all the iterated ultrapowers are well-founded.
- (G., Welch) It is consistent that for every $\alpha < \omega_1$, there is a cardinal κ_α with a weak κ_α-model M_α for which there is an M_α-ultrafilter with exactly α-many well-founded iterated ultrapowers.

Small large cardinals and M-ultrafilters

Theorem: (Folklore) The following are equivalent.

- κ is weakly compact.
- Every $A \subseteq \kappa$ is in a weak κ-model M for which there is a good M-ultrafilter.
- Every $A \subseteq \kappa$ is in a κ-model M for which there is an M-ultrafilter.
- Every κ-model has an M-ultrafilter.

Theorem: (Abramson, Harrington, Kleinberg, Zwicker) The following are equivalent.

- κ is ineffable.
- Every $A \subseteq \kappa$ is in a weak κ-model M for which there is a good M-ultrafilter with a stationary diagonal intersection. ($U = \{A_\xi \mid \xi < \kappa\}$ and $\Delta_{\xi<\kappa} A_\xi$ is stationary)
- Every $A \subseteq \kappa$ is in a κ-model for which there is an M-ultrafilter with a stationary diagonal intersection.
- Every κ-model has an M-ultrafilter with a stationary diagonal intersection.

Theorem: (Mitchell) The following are equivalent.

- κ is Ramsey.
- Every $A \subseteq \kappa$ is in a weak κ-model M for which there is a weakly amenable M-ultrafilter with the countable intersection property.

Weak amenability

Definition: (G.) A cardinal κ is:

- 0-iterable if every $A \subseteq \kappa$ is in a weak κ-model for which there is a weakly amenable M-ultrafilter. the M-ultrafilter is not necessarily good
- strongly Ramsey if every $A \subseteq \kappa$ is in a κ-model for which there is a weakly amenable M-ultrafilter.

Theorem: (G.)

- A 0-iterable cardinal is a limit of ineffable cardinals.
- A strongly Ramsey cardinal is a limit of Ramsey cardinals.
- The assertion that every κ-model has a weakly amenable M-ultrafilter is inconsistent.

Extending M-ultrafilters

Definition: Suppose κ is weakly compact.

- weak extension property: For every κ-model M_0, weak M_0-ultrafilter U_0, and κ-model $M_1 \supseteq M_0$, there is a weak M_1-ultrafilter $U_1 \supseteq U_0$.
- extension property: For every κ-model M_0, M_0-ultrafilter U_0, and κ-model $M_1 \supseteq M_0$, there is an M_1-ultrafilter $U_1 \supseteq U_0$.

Theorem: Suppose κ is weakly compact.

- (Keisler, Tarski) The weak extension property holds.
- (G.) The extension property is inconsistent.

Proof: Suppose κ is least weakly compact with the extension property.

- Choose a κ-model $M_0 \prec H_{\kappa^+}$ and an M_0-ultrafilter U_0.
- Given $M_n \prec H_{\kappa^+}$ and U_n, choose $M_{n+1} \prec H_{\kappa^+}$ with $M_n, U_n \in M_{n+1}$ and $U_{n+1} \supseteq U_n$.
- Let $M = \bigcup_{n<\omega} M_n$ and $U = \bigcup_{n<\omega} U_n$. Then U is a weakly amenable M-ultrafilter.
- Let $j : M \to N$ be the ultrapower map. Then $M = H^N_{\kappa^+}$.
- N satisfies that κ is weakly compact with the extension property.
- $M \prec H_{\kappa^+}$ satisfies that there is a weakly compact $\alpha < \kappa$ with the extension property.
- Contradiction! $\square$

What if we choose the M_0-ultrafilter U_0 strategically to make sure that for any $M_1 \supseteq M_0$ we can extend U_0 to U_1?

Filter extension games

(Holy and Schlicht) Suppose $\delta \leq \kappa^+$.

weak game $wG_\delta(\kappa)$

- players: challenger and judge
- game length: δ
- stage 0:
  - challenger: κ-model M_0
  - judge: M_0-ultrafilter U_0
- stage $\gamma > 0$:
  - challenger: κ-model M_γ with $\{\langle M_\xi, U_\xi\rangle \mid \xi<\gamma\} \in M_\gamma$ and $M_\xi \prec M_\gamma$
  - judge: M_γ-ultrafilter $U_\gamma \supseteq \bigcup_{\xi<\gamma} U_\xi$
- winning condition for judge: keep playing

game $G_\delta(\kappa)$

winning condition for judge: $U_\delta = \bigcup_{\xi<\delta} U_\xi$ is a good $M_\delta = \bigcup_{\xi<\delta} M_\xi$-ultrafilter.

strong game $sG_\delta(\kappa)$

winning condition for judge: U_δ has the countable intersection property

Observations:

- U_δ is weakly amenable.
- All games are equivalent if $\mathrm{cof}(\delta) \neq \omega$.
- $G_\delta(\kappa, \theta)$: $\theta \geq \kappa^+$ is regular and challenger plays 'κ-model like' $M \prec H_\theta$ - affects existence of winning strategies only if $\mathrm{cof}(\delta) = \omega$.

Large cardinals from a winning strategy for the judge

Notation: Judge_G - judge has a winning strategy in the game G.

Theorem:

- κ is weakly compact if and only if $\text{Judge}_{G_1(\kappa)}$.
- (Holy, Schlicht) Suppose $2^\kappa = \kappa^+$. κ is measurable if and only if $\text{Judge}_{G_{\kappa^+}(\kappa)}$.
- (Nielsen) If $\text{Judge}_{G_n(\kappa)}$, then κ is Π^1_{2n-1}-indescribable.
- (Abramson, Harrington, Kleinberg, Zwicker) κ is completely ineffable if and only if $\text{Judge}_{wG_\omega(\kappa)}$.
- (Nielsen, Schindler) $\text{Judge}_{G_\omega(\kappa)}$ is equiconsistent with a virtually measurable cardinal.
- (Foreman, Magidor, Zeman) Suppose $2^\kappa = \kappa^+$. If $\text{Judge}_{sG_\omega(\kappa)}$, then there is a uniform normal precipitous ideal on κ.

Definitions: A cardinal κ is:

- ineffable if for every $\vec{A} = \{A_\xi \subseteq \xi \mid \xi < \kappa\}$, there is $A \subseteq \kappa$ such that $\{\xi < \kappa \mid A_\xi = A \cap \xi\}$ is stationary.
- completely ineffable if there is a non-empty upward closed collection $\mathcal{S}$ of stationary subsets of κ such that for every $\vec{A}$ and $S \in \mathcal{S}$, there is $A \subseteq \kappa$ and $\bar{S} \subseteq S$ in $\mathcal{S}$ such that $\bar{S} \subseteq \{\xi < \kappa \mid A_\xi = A \cap \xi\}$.
  - — totally indescribable
- virtually measurable if for every $\theta > \kappa$, there is a transitive M such that some forcing extension has a $j : H_\theta \to M$ with $\text{crit}(j) = \kappa$.
  - — limit of completely ineffable cardinals
  - — can exist in L

Generic measurability

Definition: A cardinal κ is generically measurable if some forcing extension has a $j : V \to M$ with $\text{crit}(j) = \kappa$ and M transitive.

Proposition: κ is generically measurable if and only if some forcing extension has a good V-ultrafilter.

Theorem: (Jech, Prikry) A generically measurable cardinal is equiconsistent with a measurable cardinal.

Variations:

- weak amenability equiconsistent with a measurable
  - good weakly amenable V-ultrafilter
  - iterable weakly amenable V-ultrafilter
- H_θ-ultrafilters equiconsistent with a virtually measurable
  - for every regular $\theta > \kappa$, good H_θ-ultrafilter
  - for every regular $\theta > \kappa$, good weakly amenable H_θ-ultrafilter
- no well-foundedness
  - V-ultrafilter (regular)
  - weakly amenable weak V-ultrafilter (weakly compact)
  - weakly amenable V-ultrafilter (completely ineffable)
- class of forcing equiconsistent with a measurable
  - $P(\kappa)/\mathcal{I}$ for a uniform normal precipitous ideal $\mathcal{I}$
  - δ-closed forcing

Generic measurability (continued)

Definition: A cardinal κ is:

- weakly almost generically measurable with weak amenability (wa) if some forcing extension has a weakly amenable weak V-ultrafilter. weakly compact
- almost generically measurable with weak amenability (wa) if some forcing extension has a weakly amenable V-ultrafilter. completely ineffable
- generically measurable for sets if for every regular $\theta > \kappa$, some forcing extension has a good H_θ-ultrafilter.
- generically measurable for sets with weak amenability (wa) if for every regular $\theta > \kappa$, some forcing extension has a good weakly amenable H_θ-ultrafilter.
- generically measurable with weak amenability (wa) if some forcing extension has a good weakly amenable V-ultrafilter.
- generically measurable with weak amenability (wa) and iterability if some forcing extension has an iterable weakly amenable V-ultrafilter.
- (any of the above notions) by $\mathcal{P}$ (a class of forcings) if the required (weak) V-ultrafilter exists in a forcing extension by some $\mathbb{P} \in \mathcal{P}$.

Generic measurability from a winning strategy for the judge

Theorem:

- κ is weakly almost generically measurable with wa if and only if $\text{Judge}_{G_1(\kappa)}$. weakly compact
- κ is almost generically measurable with wa if and only if $\text{Judge}_{wG_\omega(\kappa)}$. completely ineffable
- (Nielsen, Schindler) If $\text{Judge}_{G_\omega(\kappa,\theta)}$ for every regular $\theta > \kappa$, then κ is generically measurable for sets with wa.
- (Nielsen, Schindler) κ is generically measurable for sets with wa is equiconsistent with $\text{Judge}_{G_\omega(\kappa,\theta)}$ for every regular $\theta > \kappa$.
- (G., Benhamou) If $\text{Judge}_{sG_\omega(\kappa)}$, then κ is generically measurable with wa and iterability (by $\text{Coll}(\omega, H_{\kappa^+})$).
- (Foreman, Magidor, Zeman) Suppose $2^\kappa = \kappa^+$. If $\text{Judge}_{sG_\omega(\kappa)}$, then κ is generically measurable with wa by $P(\kappa)/\mathcal{I}$ for a uniform normal precipitous ideal $\mathcal{I}$.
- (G. Benhamou) For uncountable regular δ, $\text{Judge}_{G_\delta(\kappa)}$ if and only if κ is generically measurable with wa by δ-closed forcing.

Small models and ultrafilters: two-cardinal setting

Assume onwards that λ is a cardinal with $\lambda^{<\kappa} = \lambda$.

Definition: A transitive $\in$-model M is a weak (κ, λ)-model:

- $M \models \mathrm{ZFC}^-$
- $|M| = \lambda$
- $\lambda \in M$
- some bijection $f : \lambda \to P_\kappa(\lambda) \in M$

A weak (κ, λ)-model M is a (κ, λ)-model if $M^{<\kappa} \subseteq M$.

Definition: Suppose M is a weak (κ, λ)-model. $U \subseteq P(P_\kappa(\lambda))^M$ is a weak M-ultrafilter:

- ultrafilter on $P(P_\kappa(\lambda))^M$
- fine - $\{a \in P_\kappa(\lambda) \mid \alpha \in a\} \in U$ for every $\alpha < \lambda$
- M-κ-complete - closed under intersections of $<\kappa$-length sequences from M

U is an M-ultrafilter if it is additionally M-normal - closed under diagonal intersections of λ-length sequences from M. $\Delta_{\xi<\lambda} A_\xi = \{x \subseteq P_\kappa(\lambda) \mid x \in \bigcap_{\xi \in x} A_\xi\}$

Definition: A weak M-ultrafilter U:

- good if the ultrapower of M by U is well-founded
- has the countable intersection property if for every $\{A_n \mid n < \omega\} \subseteq U$, $\bigcap_{n<\omega} A_n \neq \emptyset$
- weakly amenable if for every $A \in M$, with $|A|^M = \lambda$, $U \cap A \in M$

Small models and ultrafilters: two-cardinal setting (continued)

Suppose M is a weak (κ,λ)-model.

Proposition: The following are equivalent.

- There is a weak M-ultrafilter.
- There is an elementary $j: M \to N$ with $\mathrm{crit}(j) = \kappa$ such that in N, there is s with $j\,``\,\lambda \subseteq s$ and $|s| < j(\kappa)$. N not necessarily well-founded

Proposition: The following are equivalent.

- There is an M-ultrafilter.
- There is an elementary $j: M \to N$ with $\mathrm{crit}(j) = \kappa$ such that $j\,``\,\lambda \in N$ and $j(\kappa) > \lambda$.
  - $H^M_{\lambda^+} \subseteq N$
  - N is well-founded at least up to $(\lambda^+)^M$

Proposition: The following are equivalent:

- There is a weakly amenable M-ultrafilter.
- There is an elementary $j: M \to N$ with $\mathrm{crit}(j) = \kappa$ such that $H^M_{\lambda^+} = H^N_{\lambda^+}$

  N is well-founded beyond $(\lambda^+)^N$.

Proposition: We can iterate the ultrapower construction with a weakly amenable M-ultrafilter U. If U has the countable intersection property, then U is iterable.

Two-cardinal versions of weak compactness

Definition: A cardinal κ is:

- (White) nearly λ-strongly compact if every $A \subseteq \lambda$ is in a weak (κ, λ)-model for which there is a weak M-ultrafilter.
- (Schanker) nearly λ-supercompact if every $A \subseteq \lambda$ is in a weak (κ, λ)-model for which there is an M-ultrafilter.

Proposition: (Schanker, White) The following are equivalent:

- κ is nearly λ-supercompact (strongly compact)
- Every $A \subseteq \kappa$ is in a (κ, λ)-model M for which there is a (weak) M-ultrafilter.
- Every (κ, λ)-model M has a (weak) M-ultrafilter.

Observation: If κ is nearly λ-supercompact (strongly compact), then κ is θ-supercompact (strongly compact) for every $2^{\theta^{<\kappa}} \leq \lambda$.

Theorem: (Schanker) It is consistent that κ is nearly λ-supercompact, but not measurable.

Extending M-ultrafilters: two-cardinal setting

Definition:

- weak extension property: Suppose κ is nearly λ-strongly compact. For every (κ, λ)-model M_0, weak M_0-ultrafilter U_0, and (κ, λ)-model $M_1 \supseteq M_0$, there is a weak M_1-ultrafilter $U_1 \supseteq U_0$.
- extension property: Suppose κ is nearly λ-supercompact. For every (κ, λ)-model M_0, M_0-ultrafilter U_0, and (κ, λ)-model $M_1 \supseteq M_0$, there is an M_1-ultrafilter $U_1 \supseteq U_0$.

Theorem:

- (Buhagiar, Džamonja) Suppose κ is nearly λ-strongly compact. Then the weak extension property holds.
- Suppose κ is nearly λ-supercompact. Then the extension property is inconsistent.

What if we choose the M_0-ultrafilter U_0 strategically to make sure that for any $M_1 \supseteq M_0$ we can extend U_0 to U_1?

Filter extension games: two-cardinal setting

Suppose $\delta \leq \lambda^+$.

weak game $wG_\delta(\kappa,\lambda)$

- players: challenger and judge
- game length: δ
- stage 0:
  - challenger: (κ,λ)-model M_0
  - judge: M_0-ultrafilter U_0
- stage $\gamma > 0$:
  - challenger: κ-model M_γ with $\{\langle M_\xi, U_\xi\rangle \mid \xi<\gamma\} \in M_\gamma$ and $M_\xi \prec M_\gamma$
  - judge: M_γ-ultrafilter $U_\gamma \supseteq \bigcup_{\xi<\gamma} U_\xi$
- winning condition for judge: keep playing

game $G_\delta(\kappa,\lambda)$

winning condition for judge: $U_\delta = \bigcup_{\xi<\delta} U_\xi$ is a good $M_\delta = \bigcup_{\xi<\delta} M_\xi$-ultrafilter.

strong game $sG_\delta(\kappa,\lambda)$

winning condition for judge: U_δ has the countable intersection property

non-normal game $(w/s)G^*_\delta(\kappa,\lambda)$: judge plays weak M-ultrafilters

Filter extension games: two-cardinal setting (continued)

Observations:

- U_δ is weakly amenable.
- All (non)-normal games are equivalent if $\text{cof}(\delta) \neq \omega$.
- $(w/s)G_\delta^{(*)}(\kappa,\lambda,\theta)$: $\theta > \lambda$ is regular and challenger plays '(κ,λ)-model like' $M \prec H_\theta$ - affects existence of winning strategies only if $\text{cof}(\delta) = \omega$.

Large cardinals from a winning strategy for the judge

Notation: Judge_G - judge has a winning strategy in the game G.

Theorem:

- The following are equivalent:
  - κ is nearly λ-strongly compact
  - $\text{Judge}_{G_1^*(\kappa,\lambda)}$
  - $\text{Judge}_{wG_\omega^*(\kappa,\lambda)}$
- κ is nearly λ-supercompact if and only if $\text{Judge}_{G_1(\kappa,\lambda)}$.
- Suppose $2^\lambda = \lambda^+$. κ is:
  - strongly compact if and only if $\text{Judge}_{G^*_{\lambda^+}(\kappa,\lambda)}$.
  - supercompact if and only if $\text{Judge}_{G_{\lambda^+}(\kappa,\lambda)}$.
- If $\text{Judge}_{G_n(\kappa,\lambda)}$, then κ is λ-Π^1_{2n-1}-indescribable.
- κ is completely λ-ineffable if and only if $\text{Judge}_{wG_\omega(\kappa,\lambda)}$.
- Suppose $2^\lambda = \lambda^+$. If $\text{Judge}_{sG_\omega(\kappa,\lambda)}$, then there is a uniform normal precipitous ideal on $P_\kappa(\lambda)$.

Definitions: A cardinal κ is:

- λ-ineffable if for every $f : P_\kappa(\lambda) \to P_\kappa(\lambda)$ such that $f(x) \subseteq x$ for all $x \in P_\kappa(\lambda)$, there is $A \subseteq \lambda$ such that $\{x \in P_\kappa(\lambda) \mid A \cap x = f(x)\}$ is stationary.
- completely λ-ineffable if there is a non-empty upward closed collection $\mathcal{S}$ of stationary subsets of $P_\kappa(\lambda)$ such that for every $f : P_\kappa(\lambda) \to P_\kappa(\lambda)$, with $f(x) \subseteq x$ for all $x \in P_\kappa(\lambda)$, and $S \in \mathcal{S}$, there is $A \subseteq \lambda$ and $\bar{S} \subseteq S$, with $\bar{S} \in \mathcal{S}$, such that $\bar{S} \subseteq \{x \in P_\kappa(\lambda) \mid A \cap x = f(x)\}$.

Generic supercompactness

Definition: A cardinal κ is:

- generically λ-strongly compact if some forcing extension has a $j : V \to M$, with $\text{crit}(j) = \kappa$ and M transitive, such that in M, there is s with $j\,"\,\lambda \subseteq s$ and $|s| < j(\kappa)$.
- generically λ-supercompact if some forcing extension has a $j : V \to M$, with $\text{crit}(j) = \kappa$, M transitive, $j\,"\,\lambda \in M$.

Proposition: κ is generically λ-supercompact (strongly compact) if and only if some forcing extension has a good (weak) V-ultrafilter.

Variations:

- weak amenability
  - good weakly amenable V-ultrafilter
  - iterable weakly amenable V-ultrafilter
- H_θ-ultrafilters
  - for every regular $\theta > \lambda$, good H_θ-ultrafilter
  - for every regular $\theta > \lambda$, good weakly amenable H_θ-ultrafilter
- no well-foundedness
  - V-ultrafilter (regular)
  - weakly amenable weak V-ultrafilter (nearly λ-strongly compact)
  - weakly amenable V-ultrafilter (complete λ-ineffable)
- class of forcing
  - $P(P_\kappa(\lambda))/\mathcal{I}$ for a uniform normal fine precipitous ideal $\mathcal{I}$
  - δ-closed forcing

Generic supercompactness (continued)

Definition: A cardinal κ is:

- weakly almost generically λ-strongly compact with weak amenability (wa) if some forcing extension has a weakly amenable weak V-ultrafilter. nearly λ-strongly compact
- almost generically λ-supercompact with weak amenability (wa) if some forcing extension has a weakly amenable V-ultrafilter. completely λ-ineffable
- generically λ-supercompact for sets if for every regular $\theta > \lambda$, some forcing extension has a good H_θ-ultrafilter.
- generically λ-supercompact for sets with weak amenability (wa) if for all regular $\theta > \lambda$, some forcing extension has a good weakly amenable H_θ-ultrafilter.
- generically λ-supercompact with weak amenability (wa) if some forcing extension has a weakly amenable good V-ultrafilter.
- generically λ-supercompact with weak amenability (wa) and iterability if some forcing extension has an iterable weakly amenable V-ultrafilter.
- (any of the above notions) by $\mathcal{P}$ (a class of forcings) if the (weak) V-ultrafilter exists in a forcing extension by some $\mathbb{P} \in \mathcal{P}$.

Generic supercompactness from a winning strategy for the judge

Theorem:

- The following are equivalent:
  - κ is weakly almost generically λ-strongly compact with wa. nearly λ-strongly compact
  - $\text{Judge}_{G_1^*(\kappa,\lambda)}$
  - $\text{Judge}_{wG_\omega^*(\kappa,\lambda)}$
- κ is almost generically λ-supercompact with wa if and only if $\text{Judge}_{wG_\omega(\kappa,\lambda)}$.

  completely λ-ineffable
- If $\text{Judge}_{G_\omega(\kappa,\lambda,\theta)}$ for every regular $\theta > \lambda$, then κ is generically λ-supercompact for sets with wa.
- If $\text{Judge}_{sG_\omega(\kappa,\lambda)}$, then κ is generically λ-supercompact with wa and iterability.
- Suppose $2^\lambda = \lambda^+$. If $\text{Judge}_{sG_\omega(\kappa,\lambda)}$, then κ is generically λ-supercompact with wa by $P(\kappa)/\mathcal{I}$ for a precipitous normal fine ideal $\mathcal{I}$.
- For uncountable regular δ, $\text{Judge}_{G_\delta(\kappa,\lambda)}$ if and only if κ is generically λ-supercompact with wa by δ-closed forcing.

Theorem: The least κ for which there is λ such that κ is almost generically λ-supercompact with wa is not generically λ-supercompact with wa for sets.

Generic supercompactness and winning strategies

Theorem: κ is almost generically λ-supercompact with wa if and only if $\text{Judge}_{wG_\omega(\kappa,\lambda)}$.

Proof: Suppose $\text{Judge}_{wG_\omega(\kappa,\lambda)}$.

- Fix a strategy σ for the judge.
- Let $G \subseteq \text{Coll}(\omega, H_{\lambda^+})$ be V-generic.

In $V[G]$:

- Enumerate $H_{\lambda^+} = \{a_n \mid n < \omega\}$.
- Let $\langle M_0, U_0, M_1, U_1, \ldots, M_n, U_n, \ldots\rangle$ be a play according to σ with $a_n \in M_n \prec H_{\lambda^+}$.
- $\bigcup_{n<\omega} M_n = H_{\lambda^+}$ and $U = \bigcup_{n<\omega} U_n$ is a weakly amenable V-ultrafilter.

Suppose κ is almost generically λ-supercompact with wa.

- Let $V[G]$ be a forcing extension by $\mathbb{P}$ with a weakly amenable V-ultrafilter U.
- Fix $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{U}$ is a weakly amenable V-ultrafilter".
- Winning strategy for judge:
  - challenger: M_0
  - judge: fix $p_0 \leq p$ deciding $U_0 = \dot{U} \cap M_0$ (weak amenability)
  - challenger: M_{n+1}
  - judge: fix $p_{n+1} \leq p_n$ deciding $U_{n+1} = \dot{U} \cap M_{n+1}$. $\square$

Generic supercompactness questions

Questions:

- Do nearly λ-supercompact cardinals have a generic large cardinal characterization?
- Are generically λ-supercompact for sets cardinals equiconsistent with generically λ-supercompact for sets with wa cardinals?

  Generically measurable for sets cardinals are generically measurable with wa for sets in L.
- Can we separate the following notions via equivalence or equiconsistency:
  - generically λ-supercompact for sets with wa
  - generically λ-supercompact with wa
  - generically λ-supercompact with wa and iterability
  - generically λ-supercompact not equivalent to generically λ-supercompact with wa

Precipitous ideal from a winning strategy for the judge: preliminaries

Definition: An ideal $\mathcal{I}$ on $P_\kappa(\lambda)$ is:

- fine if $\{x \in P_\kappa(\lambda) \mid \alpha \notin x\} \in \mathcal{I}$ for all $\alpha < \lambda$.
- κ-complete if it is closed under unions of length $< \kappa$.
- normal if it is closed under diagonal unions of length λ.
- γ-saturated if $P(P_\kappa(\lambda))/\mathcal{I}$ is γ-cc.
- λ-measuring if for every $B \in \mathcal{I}^+$ and $\vec{A} = \{A_\xi \subseteq P_\kappa(\lambda) \mid \xi < \lambda\}$, there is $B \supseteq \bar{B} \in \mathcal{I}^+$ such that for every $\xi < \lambda$, either $\bar{B} \subseteq_{\mathcal{I}} A_\xi$ or $\bar{B} \subseteq_{\mathcal{I}} \kappa \setminus A_\xi$.

Theorem: A fine κ-complete ideal $\mathcal{I}$ is λ-measuring if and only if the generic V-ultrafilter added by $P(P_\kappa(\lambda))/\mathcal{I}$ is weakly amenable.

Definition: Suppose σ is a winning strategy for the judge in a game G_δ.

- Hopeless ideal $\mathcal{I}(\sigma)$: $\{A \subseteq P_\kappa(\lambda) \mid \text{for all } R = \langle \ldots, M_\xi, U_\xi, \ldots\rangle\ A \notin U_\xi\}$, where R is a partial run of length $< \delta$ according to σ. collection of all A that don't make it into any filter on a winning run
- Conditional hopeless ideal $\mathcal{I}(R, \sigma)$: R is a partial run according to σ. $\{A \subseteq P_\kappa(\lambda) \mid \text{for all } R \subseteq R' = \langle \ldots, M_\xi, U_\xi, \ldots\rangle\ A \notin U_\xi\}$, where R' is a partial run of length $< \delta$ according to σ.

Theorem: Suppose σ is a winning strategy for the judge in a game G_δ.

- If G_δ is a non-normal game, then $\mathcal{I}(\sigma)$ and $\mathcal{I}(R, \sigma)$ are κ-complete fine ideals.
- If G_δ is a normal game, then $\mathcal{I}(\sigma)$ and $\mathcal{I}(R, \sigma)$ are λ-measuring normal fine ideals.

More preliminaries: the ideal game

Ideal game $G_{\mathcal{I}}$: $\mathcal{I}$ is an ideal.

- players: I and II
- game length: ω
- stage 0:
  - player I: $X_0 \in \mathcal{I}^+$.
  - player II: $X_0 \supseteq Y_0 \in \mathcal{I}^+$.
- stage n:
  - player I: $Y_{n-1} \supseteq X_n \in \mathcal{I}^+$.
  - player II: $X_n \supseteq Y_n \in \mathcal{I}^+$.
- Winning condition for player II: $\bigcap_{n<\omega} X_n \neq \emptyset$.

Theorem: (Jech) An ideal $\mathcal{I}$ is precipitous if and only if player I doesn't have a winning strategy in $G_{\mathcal{I}}$.

Precipitous ideal from a winning strategy for the judge

Theorem: Suppose $2^\lambda = \lambda^+$ and there is no normal fine λ^+-saturated ideal on $P_\kappa(\lambda)$. If $\text{Judge}_{sG_\omega(\kappa,\lambda)}$, then there is a precipitous λ-measuring normal fine ideal on $P_\kappa(\lambda)$.

Proof:

- Fix an internally approachable sequence $\vec{N} = \langle N_\xi \mid \xi < \lambda^+\rangle$ of (κ, λ)-models $N_\xi \prec H_{\lambda^+}$ such that $H_{\lambda^+} = \bigcup_{\xi<\lambda^+} N_\xi$.
- Define an auxiliary game $sG^{\vec{N}}_\omega(\kappa,\lambda)$, where the challenger plays models N_ξ.
- Fix a winning strategy σ for the judge in $sG^{\vec{N}}_\omega(\kappa,\lambda)$.
- The tree $T(\sigma)$:
  - Fix $\{A_{\langle\xi\rangle} \subseteq P_\kappa(\lambda) \mid \xi < \lambda^+\} \subseteq \mathcal{I}(\sigma)^+$ such that $A_{\langle\xi\rangle} \cap A_{\langle\eta\rangle} \in \mathcal{I}(\sigma)$.
  - Choose a winning run $R_{\langle 0\rangle} = \langle \ldots, N_{\gamma_{\langle 0\rangle}}, U_{\gamma_{\langle 0\rangle}}\rangle$ with $A_{\langle 0\rangle} \in U_{\gamma_{\langle 0\rangle}}$.
  - Given runs $R_{\langle\xi\rangle}$ for $\xi < \xi'$, choose $R_{\langle\xi'\rangle} = \langle \ldots, N_{\gamma_{\langle\xi'\rangle}}, U_{\gamma_{\langle\xi'\rangle}}\rangle$ with $A_{\langle\xi'\rangle} \in U_{\gamma_{\langle\xi'\rangle}}$ and $\gamma_{\langle\xi'\rangle} > \gamma_{\langle\xi\rangle}$ for all $\xi < \xi'$.
  - Level 1: $N_{\gamma_{\langle\xi\rangle}}$-ultrafilters $U^{\langle\xi\rangle} = U_{\gamma_{\langle\xi\rangle}}$ for $\xi < \lambda^+$. $\bigcup_{\xi<\lambda^+} N_{\gamma_{\langle\xi\rangle}} = H_{\lambda^+}$
  - Fix a node $U^{\langle\xi\rangle}$ on level 1.
  - Fix $\{A_{\langle\xi\eta\rangle} \subseteq \kappa \mid \eta < \lambda^+\} \subseteq \mathcal{I}(R_{\langle\xi\rangle}, \sigma)^+$ such that $A_{\langle\xi\eta_1\rangle} \cap A_{\langle\xi\eta_2\rangle} \in \mathcal{I}(R_{\langle\xi\rangle}, \sigma)$.
  - Choose $R_{\langle\xi 0\rangle} = \langle \ldots, N_{\gamma_{\langle\xi\rangle}}, U_{\gamma_{\langle\xi\rangle}}, \ldots, N_{\gamma_{\langle\xi 0\rangle}}, U_{\gamma_{\langle\xi 0\rangle}}\rangle$ extending $R_{\langle\xi\rangle}$ with $A_{\langle\xi 0\rangle} \in U_{\gamma_{\langle\xi 0\rangle}}$.
  - Given runs $R_{\langle\xi\eta\rangle}$ for all $\eta < \eta'$, $R_{\langle\xi\eta'\rangle} = \langle \ldots, N_{\gamma_{\langle\xi\rangle}}, U_{\gamma_{\langle\xi\rangle}}, \ldots, N_{\gamma_{\langle\xi\eta'\rangle}}, U_{\gamma_{\langle\xi\eta'\rangle}}\rangle$ extending $R_{\langle\xi\rangle}$ with $A_{\langle\xi\eta'\rangle} \in U_{\gamma_{\langle\xi\eta'\rangle}}$ and $\gamma_{\langle\xi\eta'\rangle} > \gamma_{\langle\xi\eta\rangle}$ for $\eta < \eta'$.
  - Level 2: $N_{\gamma_{\langle\xi\eta\rangle}}$-ultrafilters $U^{\langle\xi\eta\rangle} = U_{\gamma_{\langle\xi\eta\rangle}}$ for $\xi, \eta < \lambda^+$.

Precipitous ideal from a winning strategy for the judge (proof continued)

- $\vdots$
- Along a branch of $T(\sigma)$, the models and ultrafilters extend.
- Strategy τ for judge in $sG_\omega(\kappa,\lambda)$: play along $T(\sigma)$.
  - challenger: M_0
  - judge: choose ξ_0 such that $M_0 \subseteq N_{\gamma_{\langle \xi_0 \rangle}}$ and play $U_0 = U^{\langle \xi_0 \rangle} \restriction M_0$.
  - challenger: M_1
  - judge: choose ξ_1 such that $M_1 \subseteq N_{\gamma_{\langle \xi_0 \xi_1 \rangle}}$ and play $U_1 = U^{\langle \xi_0 \xi_1 \rangle} \restriction M_1$.
  - $\vdots$
- $\mathcal{I}(\tau)$ is a λ-measuring normal fine ideal on $P_\kappa(\lambda)$.
- $\mathcal{I}(\tau)^+ = \bigcup_{\vec{\xi} \in (\lambda^+)^{<\omega}} U^{\vec{\xi}} \subseteq \mathcal{I}(\sigma)^+$.
- $\mathcal{I}(\tau)$ is precipitous because player II has a winning strategy in $G_{\mathcal{I}(\tau)}$.
  - player I: $X_0 \in \mathcal{I}(\tau)^+$. Then $X_0 \in U^{\langle \xi_0, \ldots, \xi_{m_0} \rangle}$.
  - player II: $Y_0 = X_0 \cap A_{\langle \xi_0 \rangle} \cap A_{\langle \xi_0, \xi_1 \rangle} \cap \cdots \cap A_{\langle \xi_0, \ldots, \xi_{m_0} \rangle}$.
  - player I: $Y_0 \supseteq X_1 \in \mathcal{I}(\tau)^+$. Then $X_1 \in U^{\langle \eta_0, \ldots, \eta_{m_1} \rangle}$ with $m_1 > m_0$.
  - By disjointness of $A_{\vec{\xi}}$ modulo $\mathcal{I}(\sigma)$, $\langle \eta_0, \ldots, \eta_{m_1} \rangle = \langle \xi_0, \ldots, \xi_{m_0}, \ldots, \eta_{m_1} \rangle$.
  - $\vdots$
  - $U = \bigcup_{n<\omega} U^{\langle \xi_0, \ldots, \xi_{m_n} \rangle}$ has the countable intersection property.

    U is the union of judge's moves according to σ
  - $\bigcap_{n<\omega} X_n \neq \emptyset$. $\square$

Happy birthday, Ralf!